Continuation and computation of Floer Homology

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June 2023

1 Introduction

This talk covers chapter 9 of Will Merry's notes, about the independence of Floer homology on the choice of regular (H, J) and the isomorphism of Floer homology with the (cellular/simplicial/Morse) homology of the base space. Recall we originally wanted to prove the following theorem:

Theorem 1.1 (Theorem 2.25 in the notes)**.** *Let (Q, ω) be a closed symplectically aspherical manifold. Let $H \in \mathbb{C}^\infty(S^1 \times Q)$ be non-degenerate. Then there is a chain complex $(CF_*(H), \partial)$ associated to H generated by contractible one-periodic orbits,*

$$CF_*(H) := \sum_{x \in P_1^\circ(H)} \mathbb{Z}_2\langle x \rangle.$$

The homology of this complex is called the Floer homology of H and is denoted $HF_*(H)$.
Moreover the Floer homology is canonically independent of the choice of H, and hence we can define the Floer homology of (Q, ω) to be $HF_* := HF_*(H)$ for any non-degenerate $H \in \mathbb{C}^\infty(S^1 \times Q)$.
Finallly, there is a canonical isomoprhism

$$HF_*(Q, \omega) \cong H_{*+n}(Q; \mathbb{Z}_2).$$

Remark 1.2. Recall that non-degenerate meant that the linearization of every one-periodic contractible orbit did not have 1 as an eigenvalue. △

What have we proved so far?

Definition 1.3. The set $\mathcal{HJ}_{\text{reg}}$ are all pairs in $C^\infty(S^1 \times Q) \times \mathcal{J}(Q, \omega)$ such that H is non-denerate and for any two $x^+, x^- \in P_1^\circ(H)$ and any $u \in \mathcal{M}(x^-, x^+)$, the vertical derivative

$$D^v \overline{\partial}_{J,H}(u)$$

is surjective. We call such a pair regular. △

- Pierre showed that this set is of second category (the Transversality Theorem), and hence dense (so non-empty).
- Non-degeneracy of H implies that the chain complex is finitely generated.
- The fact that the vertical derivative for $u \in \mathcal{M}(x^-, x^+)$ was Fredholm with index $\mu_{CZ}(x^-) - \mu_{CZ}(x^+)$ (Luca), together with the regularity of (H, J) and the implicit function theorem, meant that $\mathcal{M}(x^-, x^+)$ was a manifold of dimension equal to the index (which was explained by Manuel).
- Michael proved that $\underline{\mathcal{M}}(x^-, x^+)$ was also 'compact up to breaking', and together with Simon showed that if the index difference is 2, that this has the structure of a manifold, 1-dimensional

with boundary, and that the boundary consists precisely of pairs of index difference 1 orbits. I.e. if $\mu_{CZ}(x^-) = \mu_{CZ}(x^+) + 2$, then

$$\partial \underline{\mathcal{M}} = \bigcup_{y \in P_1^\circ(H)} \underline{\mathcal{M}}(x^-, y) \times \underline{\mathcal{M}}(y, x^+).$$

Here the union runs over all y with 'middle' index.

- The boundary operator was defined as $\#_2 \underline{\mathcal{M}}(x^-, x^+)$. This was well defined, since this set was a zero-dimensional manifold (index difference 1), and the above result of the previous lecture showed that $\partial \circ \partial = 0$.

So it remains to prove that for any other regular pair $(\tilde{H}, \tilde{J})$, the resulting homology is the same as (isomorphic to that of) (H, J), and the isomorphism to the ordinary (Morse) homology of Q. On this last part we will spend only a few words (if time permits).

2 Continuation maps

How do we compare the homology of two regular pairs? Observe that $C^\infty(S^1 \times Q) \times \mathcal{J}(Q, \omega)$ is a product of path-connected spaces, hence path-connected. We will construct a chain map between the chain complexes of two pairs, using a path connecting them. Let us start with some definitions.

Definition 2.1. Let $(H^\pm, J^\pm)$ be two regular pairs. An asymptotically constant path connecting (H^-, J^-) to (H^+, J^+) is a a smooth path

$$\chi : \mathbb{R} \to C^\infty(S^1 \times Q) \times \mathcal{J}(Q, \omega), \quad \chi(s) = (H^s, J^s)$$

such that

$$\chi(s) = \begin{cases} (H^-, J^-), & s \leq T, \\ (H^+, J^+), & s \geq T. \end{cases}$$

△

By the path-connectedness, such paths always exist χ. Let $x^- \in P_1^\circ(H^-)$ and $x \in P_1^\circ(H^+)$. We define the Banach bundle $\mathcal{E}^p \to \mathcal{B}^{1,p}(x^-, x^+)$ in the same way that Pierre did: the base is contained in $W^{1,p}(\mathbb{R} \times S^1, Q)$, with the additional property that $u : \mathbb{R} \times S^1 \to Q$ is 'asymptotic' to x^- and x^+ as $s \to \pm\infty$ (i.e. there is a formula involving the exponential map around either orbit expressing this convergence). The sections of $\mathcal{E}$ over u are maps in $L^p(\mathbb{R} \times S^1, u^*TQ)$.

Definition 2.2. For an asymptotically constant path χ (as above), define a section

$$\overline{\partial}_\chi : \mathcal{B}^{1,p}(x^-, x^+) \to \mathcal{E}^p$$

by setting

$$\overline{\partial}_\chi(u) := \partial_s u + J^s(u)(\partial_t u) - J^s(u) X_{H_t^s}(u).$$

We write $\mathcal{N}_\chi(x^-, x^+)$ for the zero set of $\overline{\partial}_\chi$. △

Assume that $x^\pm$ have CZ-index k, and that $\mathcal{N}_\chi(x^-, x^+)$ is in fact a zero dimensional manifold for every x^+ with index k whose parity is denoted $n_\chi(x^-, x^+)$. Then the **continuation homomorphism of** χ, $\Phi_\chi : CF_k(H^-) \to CF_k(H^+)$ is defined on generators as

$$\Phi_\chi \langle x^- \rangle = \sum_{y \in P_1^\circ(H^+), \mu_{CZ}(x^+) = k} n_\chi(x^-, x^+) \langle x^+ \rangle,$$

and extended by linearity to a map between the chain complexes.

This definition should remind you of how the boundary operator for $H^\pm$ was defined. To verify the assumption about $\mathcal{N}_\chi(x^-, x^+)$, the following definition makes sense:

Definition 2.3. Let χ be an asymptotically constant path between regular pairs $(H^\pm, J^\pm)$. We say that χ is regular if the vertical derivative $D^v\overline{\partial}_\chi(u)$ is surjective for all $u \in \mathcal{N}_\chi(x^-, x^+)$ and all $x^\pm \in P_1^\circ(H^\pm)$. △

We will need to work our way through all the steps (transversality, Fredholm operator + index calculation, compactness, and gluing) to prove the follwing theorem, which is the main result

Theorem 2.4 (Continuation maps)**.** *Let $(H^\pm, J^\pm)$ be two regular pairs, then there exists a regular asymptotically constant path connecting them. For any such path χ, there is a well-defined chain map*

$$\Phi_\chi : CF_*(H^-) \to CF_*(H^+), \quad \Phi_\chi \circ \partial_{J^-} = \partial_{J^+} \circ \Phi_\chi,$$

inducing a map on homology (denoted ϕ_χ).
Moreover, if (H^0, J^0) is a third regular pair and χ^0 and χ^1 are regular asymptotically constant paths connecting (H^-, J^-) to (H^0, J^0) to (H^+, J^+) (in that order), then the induced maps satisfy

$$\phi_\chi = \phi_{\chi^1} \circ \phi_{\chi^0}.$$

Finally, if $(H^-, J^-) = (H^+, J^+)$ and χ is a constant path, then this path is regular and the chain map is the identity.

We remark one more thing (which you can verify yourself but is omitted from the notes): if χ is a regular as. const. path, then the 'inverse' path $\chi^{-1}(t) := \chi(-t)$ is also regular. The following theorem most of what we wanted to prove!

Corollary 2.5. *Suppose $(H^\pm, J^\pm)$ are two regular pairs.*

1. *For any regular asymptotically constant path χ connecting (H^-, J^-) to (H^+, J^+), the induced map on homology depends only on the pair of regular pairs, not on χ.*
2. *The induced map on homology is an isomorphism.*

The proof of the corollary with the above theorem (and the interjected fact about χ^{-1}) is now a great exercise.

2.1 Mumble mumble 'similar as before' mumble mumble

In this section we will comment on what needs to be said to prove Theorem 2.4. You should have in mind that we are looking at the zeroes of the section $\overline{\partial}_\chi$ instead of $\overline{\partial}_J$, and we are trying to prove statements about the space $\mathcal{N}(x^-, x^+)$ instead of $\mathcal{M}(x^-, x^+)$.

2.1.1 Fredholm and index

The first statement is analoguous to what Luca, Pierre, and Manuel had together proved: let χ be an asymptotically constant path between two regular pairs (not necessarily regular). Then:

Theorem 2.6. *The vertical derivative $D^v\overline{\partial}_\chi$ is a Fredholm operator of index $\mu_{CZ}(x^-) - \mu_{CZ}(x^+)$.*

What did we need to prove this again (roughly)? Recall that we first made our problem a lot easier by choosing a 'symplectic trivilization', which eventually allowed us to identify $\overline{\partial}_J u$ with

$$\partial_s u + J_0 \partial_t u + Su, \quad u : \mathbb{R} \times S^1 \to \mathbb{R}^{2n},$$

where $S : \mathbb{R} \times S^1 \to L(\mathbb{R}^{2n})$ was a zero-th order operator, which was asymptotically symmetric (i.e. went to the linearization of the limit orbits multiplied with the a.c.s.). This made life a lot easier analytically, and we proved that this operator, denoted D_S, was Fredholm wit the right index, by breaking it into three pieces; the two asymptotic pieces (whence the CZ-index entered the picture) and one defined on $[-T, T] \times S^1$, which was compact (and hence didn't influence the index). We can essentially apply the same proof again now: you can argue that there is again a trivialization as above, where we now also shuffle the s-dependence of J into the S-term, and since the path is asymptotically constant, this 'breaking into three parts' also still works!

That concludes the first mumbling. It is important to note that, if χ is now a regular path, then $\mathcal{N}(x^-, x^+)$ are manifolds of dimension equal to the index. Since J^s is not s-independent, there is no translation $\mathbb{R}$-action on this moduli space, hence we shall not want to pass a quotient space like $\underline{\mathcal{M}}$. This also has the following important implication: ϕ_χ will make sense as a degree 0-map (unlike the boundary map, which has degree -1).

2.1.2 Transversality

The second statement is the analogue of the Transversality Theorem that Pierre proved, which shows that regular paths are dense (and in particular, exist!).

Theorem 2.7. *Suppose that χ (or rather (H^s, J^s)) is an asymptotically constant path. For any $\epsilon > 0$ there exists a regular asymptotically regular path $(\tilde{H}^s, J^s)$ connecting the asymptotes of χ, such that $\|H^s - \tilde{H}^s\|_{C^\infty(S^1 \times Q)} < \epsilon$ for all $s \in \mathbb{R}$.*

What did we need to prove this again (roughly)? The proof was split in roughly two parts. First there was a part which proved that if $x^- \neq x^+$, then the set of 'regular' points in $\mathbb{R} \times S^1$ of $u : \mathbb{R} \times S^1 \to Q$ was open and dense. Here regular meant that the point was injective, did not fall onto one of the limit orbits, and that the s-derivative didn't vanish. This was hard, and required proving some form of 'analytic continuation' argument (which involved a whole lot of machinery, like Carleman's similarity principle), and a trick where we put X_H equal to zero, by 'pulling back our operator operator along the flow of H'. These arguments should hold as before, and I won't comment more. The second part was showing that, given this first fact, the vertical derivative $D^v\partial_{J,H}$ was surjective, or equivalently that the annhilator was trivial. That was, if

$$\int_{\mathbb{R}} \int_{S^1} \left\langle D^v\sigma(u, H)[\hat{u}, \hat{H}], \hat{w} \right\rangle_J dtds = 0 \quad \forall (\hat{u}, \hat{H}) \in W^{1,p}(\hat{u}^*TQ) \times T_H\mathcal{H}^k(H_0),$$

then $\hat{w} = 0$. Here $D^v\sigma(u, H)[\hat{u}, \hat{H}] = D^v\partial_{J,H}(u)[\hat{u}] - \nabla\hat{H}_t(u)$. Pierre had to work quite hard to prove this: he wanted to show that $\hat{w}$ had to be a function valued multiple of $\partial_s u$, and that this function was non-zero everywhere and only dependent on $t \in S^1$. With the first fact and expressing the vertical derivative in terms of $(u, H, \hat{u}, \hat{H})$, we could show that this implied that $\partial_s u$ had to identically vanish if $\hat{w}$ was non-zero. Fortunately, the prove simplifies a lot this time: since H is now also s-dependent, all the second part of the proof can be done in the 'straigh-forward' measure theory way: arguing by contradiction, if $\hat{w}$ is not zero at some point, we can choose $\hat{u} = 0$ and $\hat{H}$ such that the pairing in the integral is strictly positive. Hence, no additional arguments need to be made.

That concludes the second mumbling.

2.1.3 Compactness, gluing, and chain map

The third statement is about compactness. We shall simply not state it, it is obvious what it should be: for a given pair $x^\pm \in P_1^\circ(H^\pm)$ define

$$\mathcal{N}_\chi^\# := \{u \in C^\infty(\mathbb{R} \times S^1, Q) | u(s, \cdot) \in \mathcal{L}Q, \forall s \in \mathbb{R}, \overline{\partial}_\chi(u) = 0\},$$

and define the (possibly infinite) energy of a curve in this set as

$$\mathbb{E}(u) := \int_{-\infty}^{\infty} \langle\langle \partial_s u, \partial_s u \rangle\rangle ds.$$

The statement we want to prove is that the subset of $\mathcal{N}_\chi^\#$ with finite energy is compact (even if we take all possible pairs $x^\pm$ together) up to broken trajectories. Since u is a zero of $\overline{\partial}_\chi$,

$$\mathbb{E}(u) = -\int_{-\infty}^{\infty} \int_{S^1} \langle \partial_s u, J^s(\partial_t u - X_{H_t^s}(u(s,t)))\rangle dt ds.$$

Note that H is now dependent on s (unlike when we were studying $\mathcal{M}(x^-, x^+)$), so it follows that

$$\begin{aligned}\mathbb{E}(u) &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial s}\mathbb{A}(u(s,\cdot))ds - \int_{-\infty}^{\infty}\int_{S^1} \frac{\partial H_t^s}{\partial s}(u(s,t))dtds \\ &\leq \mathbb{A}_{H^-}(x^-) - \mathbb{A}_{H^+} + \Delta(H^s).\end{aligned}$$

I.e. that last term now appears additionally, from the dependence of H on s. Here we used that χ is asymptotically constant, i.e. there exists $T > 0$ such that H_t^s does not vary in s for $|s| \geq T$, and of course

$$\Delta(H^s) := \sup_{s\in[-T,T]} \sup_{t\in S^1} \sup_{q\in Q} \left|\frac{\partial H_t^s}{\partial s}(q)\right|.$$

Recall the crucial estimate of Michael: there existed a constant $C > 0$ such that for all $u \in \mathcal{M}$ (all possible pairs of orbits), one had $\sup_{(s,t)\in\mathbb{R}\times S^1} |\nabla u(s,t)|_J \leq C$. It relied on a similar inequality as the above one, and the fact that there were finitely many $x^\pm$. So we can repeat all of Michael's results, adding only this additional $\Delta(H^s)$ term, to prove the compactness result. Note that here we have really used the asymptotically constant condition. This almost concludes the mumbling for compactness. What we should also talk about is the 'convergence of a subsequence up to broken trajectories'. Again, almost nothing is changed, but it is good to remark that the asymptotically constant condition is used: taking a cylinder u, making a translation $u(s,t) \mapsto u(s+\tau,t)$ for $\tau \to \pm\infty$ allows you to conclude that subsequences of $\mathcal{N}_\chi$ converge up to subsequence to a possibly broken trajectory, and that crucially these broken trajectories are still trajectories of H^- or H^+ (depending on which direction you take τ).

About the gluing I will say nothing. Suffice to say, we can again conclude that is the 'converse' to the convergence into broken orbits, and we obtain the following theorem

Theorem 2.8. *Suppose that $x^\pm \in P_1^\circ(H^\pm)$ respectively with $CZ(x^-) = CZ(x^+) + 1$. Then the boundary of the compactification $\overline{\mathcal{N}_\chi(x^-,x^+)}$ of the one-dimensional manifold $\mathcal{N}_\chi(x^-,x^+)$ can be identified with*

$$\left(\bigcup_{y^-\in P_1^\circ(H^-), CZ(y^-)=CZ(x^-)-1} \underline{\mathcal{M}}(x^-,y^-)\times\mathcal{N}_\chi(y^-,x^+)\right)\cup$$
$$\left(\bigcup_{y^+\in P_1^\circ(H^+), CZ(y^+)=CZ(x^-)} \mathcal{N}_\chi(x^-,y^+)\times\underline{\mathcal{M}}(y^+,x^+)\right).$$

So recall how we defined Φ_χ as a count of elements in $\mathcal{N}_\chi(x^-,x^+)$ if they had the same CZ-index. With the above theorem, you can now prove (exercise) that Φ_χ is indeed a chain map, and hence descends to a map on homology.

We can now also conclude the following (which was on the todo list): if $(H^-,J^-) = (H^+,J^+)$ and χ is the constant path, then ϕ_χ is the identity map. Indeed, in this case $\mathcal{N}(x^-,x^+) = \mathcal{M}(x^-,x^+)$, and since they have equal CZ-index, this space could only be non-empty if $x^- = x^+$ and the moduli space contained only trivial cylinders. We don't really use anything we haven't proved before, other than that ϕ_χ is well-defined.

The final things on the to-do list are showing that ϕ_χ only depends on the asymptotes, and the 'composition' property. These are, again, variations on the same theme: you can construct a 'regular path of path' between two regular paths χ^0 and χ^1, and a moduli space which should connect the moduli spaces $\mathcal{N}_{\chi^i}$. Again, not much is changing, and you can prove that there exist such regular paths of paths (and that they are dense), and that the moduli space will be 'compact up to broken cylinders' where you can give the moduli space a manifold structure with boundary, if the CZ index difference is minimal. You can then show that Φ_{χ^0} and Φ_{χ^1} are chain homotopic, which precisely means they induce the same map on homology. The composition property is another variation. Please consult the notes for concrete statements.