TranSurgery

(Reading course on Floer Homology).
Let \((M, \omega)\) be a closed symplectic manifold \(\omega\) positive definite that satisfies \(\tau_\omega(M) = 0\). Fix an almost complex structure \(J\) on \(M\) compatible with \(\omega\), i.e. \(\langle \cdot, \cdot \rangle = \omega(\cdot, \cdot)\) is a Riemannian metric.

For any \(H \in C^\infty(\mathbb{R} \times M, \mathbb{R})\) we denote by \(\Phi_H(t)\) the set of contractible \(1\)-periodic orbit of the Hamiltonian flow generated by \(H\), i.e. the flow of the time dependent vector field \(\dot{X}_H\)

\[ \dot{X}_H = \nabla_H \omega \]

where \(\omega(X_H, \cdot) = -dH\).

The aim of these notes is to show that the set of \(H\) such that

\[ \Phi_H(x^-_t, x^+_t) \]

is a finite dimensional smooth manifold for all \(x^-_t, x^+_t \in \Phi_H\).

is "\(\mathbb{P}, \mathbb{Q}\) generic". We will be more precise in \(I\).

The idea to prove this is the following:

Remark: However, the proof we will give won't follow exactly the order we present now.

Our proof will be \(1, 2, 4, 5, 3, 6\).

1. For any non-degenerate Hamiltonian function \(H\) we construct a Banach manifold \(\mathcal{B}_H\)

\[ \mathcal{B}_H \subset W^{4,1}(\mathbb{R} \times \mathbb{R}^1, M) \text{ and a Banach bundle} \]

\[ E^1 \rightarrow \mathcal{B}_H(x^-_t, x^+_t) \text{ such that} \]

\[ \Phi_H = E^1 \text{ smooth foliation} \]

and \(\mathcal{B}_H(x^-_t, x^+_t) \rightarrow \mathcal{E}^1\) smooth fibration"
2. \[ I(p) \cap \partial \Omega \neq \emptyset \]

If \[ I(p) \subset \partial \Omega \] is a smooth manifold of finite dimension.

3. To show that it happens for generic \( \lambda \) we have to make \( \lambda \) vary. More precisely we will do as follows:

Fix \( \lambda \) near \( 0 \), we define \( \mathcal{F}(\lambda) \subset \mathcal{E}(\mathbb{R}^n, \mathbb{R}) \) a space of deformations of \( \mathcal{F}(\lambda) \) such that \( \lambda \in \mathcal{F}(\lambda), \mathcal{P}(\lambda) = \mathcal{P}(\lambda) \) and \( \mathcal{F}(\lambda) \) is a separable Banach manifold. \( \mathcal{P}(\lambda) \) is a space in \( \mathcal{E}(\mathbb{R}^n, \mathbb{R}) \) and consider:

\[
S : \mathcal{E}(\mathbb{R}^n, \mathbb{R}) \times \mathcal{F}(\lambda) \rightarrow \mathcal{E} \\
(\mu, \lambda) \mapsto S_{\lambda}(\mu) \]

We show that \( \lambda \in \mathcal{F}(\lambda) \) if and only if \( S_{\lambda}(\mu) \) is a regular value of \( S \).

4. \( \lambda \in \mathcal{F}(\lambda) \) is a regular value of \( S \) if and only if \( \lambda \in \mathcal{F}(\lambda) \)

5. We use Sub-Smooth Theorem to conclude that \( \mathcal{F}(\lambda) \) is dense in \( \mathcal{F}(\lambda) \) for \( \mathcal{C}^\infty \) topology.

1) follows almost exclusively from what has been said. 2), 4), 5) are general functional analysis theorems.

Our main task will be to prove 3), especially the pink square, we formalize it in Theorem 3.3 below. The plan will be the following:
I. The setting (The Banach man: fols)
   a. The Sobolev space $W^{s,p}$ and $S^{s,p}(x_-,x_+)$
   b. The space of segregation and the precise statement of the theorem

II. Reformulation of the Theorem
   a. IFT for geopliars of Banach bundle
   b. Reformulation of the Theorem via IFT

III. Proof of the Theorem
   a. Good: General theorem + Transcendence
   b. Proof of the theorem
I. The setting.

a. $\gamma_{t,p}(x,-x_+)$

\[ \text{where } E_{10} \text{ is Fréchet locally } E_{10}^0 := \Gamma^{10}(\pi^* TM) \text{ is the open for} \]

\[ \text{that pops out when we transform the ODE of gradient flow of the} \]

\[ \text{functional } \mathcal{A}_t^1 : M \to \mathbb{R} \text{ with respect to the metric on } \mathcal{A} \text{ defined by} \]

\[ (\cdot) \langle \phi_1, M \rangle = \int_0^1 \omega(g_0(J(x(t)))\phi(t), q(t)) \, dt, \quad \forall \phi, \psi \in \Gamma^{10}(\pi^* TM) \]

to a PDE.

\[ \mathcal{H}_t^1(x,-x_+) = \frac{d}{dt} \mathcal{H}_t^1 \big|_{t=0} \subset \mathcal{C}_{\infty}^0(x,-x_+) \]

\[ \text{where } \mathcal{C}_{\infty}^0(x,-x_+) = \{ u : \mathbb{R} \times \mathbb{R} \to M \mid \lim_{t \to \pm \infty} u(x, t) = x_\pm \} \]

\[ \text{To be able to do analysis we have to work with} \]

\[ \text{Banach manifolds. It is natural to complete } \mathcal{A} \text{ with respect to the metric defined by (1) and go to work in } W^{1,2}(x_-, M). \]

\[ \text{However to complete } \mathcal{C}_{\infty}^0(x,-x_+) \subset \mathcal{C}_{\infty}^0(\mathbb{R}^2, M) \]

given $\dim(\mathbb{R}^2) = 2$, it is natural to work with $W^{1,1}$ norm.

for $p \leq 2$ go that $W^{1,p}(x,-x_+) \subset W^{1,p}(\mathbb{R}^2, M) \subset \mathcal{C}_{\infty}^0(\mathbb{R}^2, M)$

(Also if we define $W^{1,1}$ through an embedding $\mathcal{N}^\infty \hookrightarrow \mathcal{N}^K$ - instead of in terms of distributions - when $p \leq 2$ the definition does not depend on the embedding).

More precisely we will work with:

\[ \gamma_{t,p}(x,-x_+) \subset W^{1,1}(x,-x_+) \text{ where} \]
\( \mathcal{P}^t(x_-,x_+)= \int_{\mathcal{W} \in \mathcal{W}^1(\mathbb{R} \times M, \mathbb{R})} \int_{x \in \mathbb{R}} \mathcal{F}(x_{x_-,x_+},x_{x_-,x_+}) \mathcal{E}(\mathcal{P}(x_{x_-,x_+},x_{x_-,x_+})) \mathcal{D}(x_{x_-,x_+}) \mathcal{G}(x_{x_-,x_+}) \mathcal{H}(x_{x_-,x_+}) \mathcal{I}(x_{x_-,x_+}) \mathcal{J}(x_{x_-,x_+}) \mathcal{K}(x_{x_-,x_+}) \mathcal{L}(x_{x_-,x_+}) \mathcal{M}(x_{x_-,x_+}) \mathcal{N}(x_{x_-,x_+}) \mathcal{O}(x_{x_-,x_+}) \mathcal{P}(x_{x_-,x_+}) \mathcal{Q}(x_{x_-,x_+}) \mathcal{R}(x_{x_-,x_+}) \mathcal{S}(x_{x_-,x_+}) \mathcal{T}(x_{x_-,x_+}) \mathcal{U}(x_{x_-,x_+}) \mathcal{V}(x_{x_-,x_+}) \mathcal{W}(x_{x_-,x_+}) \mathcal{X}(x_{x_-,x_+}) \mathcal{Y}(x_{x_-,x_+}) \mathcal{Z}(x_{x_-,x_+}) \)

**Proposition:** \( \forall \mathcal{P} \in \mathcal{C}^1(\mathbb{R} \times M, \mathbb{R}) \) and \( \forall x_+ \in \mathbb{R} \)
- \( \mathcal{P}^t(x_-,x_+) \) is a smooth Banach manifold: \( \mathcal{P}^t \) with
- \( \mathcal{P}^t_{x_+}(x_-,x_+) = \mathcal{W}^1(\mathbb{R} \times M, \mathbb{R}) \)
- \( \mathcal{H}_{x_+}(x_-,x_+) \subset \mathcal{P}^t_{x_+}(x_-,x_+) \) (exponential decay)
- \( \mathcal{J}_{x_+} : \mathcal{P}^t_{x_+}(x_-,x_+) \to \mathcal{E}_{x_+} \)
- \( u \to \partial_u + \mathcal{J}_{x_+}(\partial_u + x_{x_+}^t(u)) \)

Where \( \mathcal{E}_{x_+} \) Banach bundle \( \mathcal{E}_{x_+} = \mathcal{L}^1(\mathbb{R} \times M, \mathbb{R}) \) is smooth.

We admit this proposition.

**Remark:** Is the extension of \( \mathcal{J}_{x_+} \) to \( \mathcal{B}^1_{x_+} \) in the last proposition can be done by density of \( \mathcal{C}^1_{x_+}(x_-,x_+) \subset \mathcal{B}^1_{x_+}(x_-,x_+) \)? In the way we have done it \( \mathcal{J}_{x_+} \) and \( \mathcal{D}_{x_+} \) designated the weak derivative of \( \mathcal{C}^1_{x_+}(x_-,x_+) \).

Moreover one can use the Calderon-Zygmund inequality to prove elliptic regularity, i.e. if \( u \in \mathcal{B}^1_{x_+}(x_-,x_+) \) then \( \mathcal{J}_{x_+}(u) \in \mathcal{E}_{x_+} \).

In other words:

**Theorem 1:** \( \mathcal{J}_{x_+}^{-1} \mathcal{E}_{x_+} = \mathcal{H}_{x_+}(x_-,x_+) \).
b. Precise statement

For this part we follow mainly Aubin-Damian 8.3.

Since $C^0(\mathbb{R}^k, M)$ does not admit a uniform Banach structure, we will consider for any $H \in C^0(\mathbb{R}^k, M)$ a subspace $\mathcal{H}(H) \subset C^0(\mathbb{R}^k, M)$ that admits a dual.

Consider $E = (E_i)_{i \in \mathbb{N}}$ a sequence $\subset \mathbb{R}^n$. For any $H \in C^0(\mathbb{R}^k, M)$ we define the norm $\|H\|_E := \sum_{k \geq 0} E_k \|H_k\|_{\infty}$ where $H_k$ is defined as follows:

Fix a covering diameter $(U; \phi_i)_{i \in \mathbb{N}}$ of $\mathbb{R}^k$, $\phi_i : \mathbb{R}^k \to \mathbb{R}^n$ $(\phi_{i+1} \subset \phi_i)$ and $\|H_k\|_{\infty} := \max_{i; j, k} \|\phi_{i+1}^{-1}(\phi_i)(x)\|$. We then denote by $\mathcal{P}(\mathbb{R}^k, M)$ the vector space consisting of $H \in C^0(\mathbb{R}^k, M)$, $\|H\|_E < +\infty$. It is a "early" Banach space, i.e. $(\mathcal{P}(\mathbb{R}^k, M), \|\cdot\|_E)$ is complete. Moreover we have the next proposition that we don't prove:

**Proposition:** There exists a sequence $E$ such that $\mathcal{P}(\mathbb{R}^k, M)$ is sense in $C^0(\mathbb{R}^k, M)$ for the $C^0$ topology. (See an explicit sequence $E$.)

Fix grade an $E$ and $H \in C^0(\mathbb{R}^k, M)$. Consider $\mathcal{H}(H) \subset C^0(\mathbb{R}^k, M)$ defined by $H \in \mathcal{H}_E$ if $\|H(t, x) - \phi(t, x)\| < \epsilon$ in neighborhoods of $\phi(t, x)$ for $t \in \mathbb{R}$ and $\|H - \phi(t, x)\|_E < \epsilon$. $\mathcal{H}(H)$ endowed with $\|\cdot\|_E$ is a Banach manifold with $T_H \mathcal{H}(H) := \{H \in \mathcal{H}(H) | \|H - T_H \phi(t, x)\|_E = 0\}$.

Moreover for $\epsilon$ small enough $H \in \mathcal{H}(H)$, $\mathcal{H}(H) \subset \mathcal{P}(\mathbb{H}(H))$. We then denote $\mathcal{H}(H)$.

We can finally state properly the theorem we want to show:

**Theorem:**

Let $H_0$ be a non-negative Hamiltonian function. The set $\mathcal{H}(H_0)$ of $H \in \mathcal{H}(H_0)$ s.t. $H_0(x, t, x^+) \subset \mathcal{P}(\mathbb{R}^k, M)$ is a smooth finite dimensional manifold if $H_0$ is residual in $\mathcal{H}(H_0)$ endowed with the $C^0$ topology.
To explicit the sequence \( E \) of the previous projection we use and admit the next lemma proven in [Austin-Damian].

**Lemma:** Endowed with the \( C^0 \) topology the space \( C^0(\mathbb{R}^2, \mathbb{R}) \) is separable, i.e. it admits a dense sequence.

**Remark:** For the \( C^0 \) topology this result can be deduced from Stone-Weierstrass after embedding \( \mathbb{R}^2 \to [0,1]^4 \).

So consider \( (f^k) \subset C^0(\mathbb{R}^2, \mathbb{R}) \) a dense sequence for the \( C^0 \) topology and put

\[
E^n = \frac{1}{\max_{k \leq n} \| f^k \|_{C^0}}
\]

where \( \| f \|_{C^0} = \sum_{i=0}^n \| f^i \|_{C^0} \) as we defined in the beginning of the section.

## II. Reformulation of the Theorem via IFT

a. IFT for Banach Beurling.
Let $E \xrightarrow{\varphi} \beta$ be a Banach bundle.

Denote by $\sigma_0 : \beta \to E$ the $\varphi$-section, and consider the canonical splitting of $\sigma_0 : E \to T_\beta \oplus E_x$ and denote by $T_x : T_{\sigma_0(x)} E \to E_x$.

Then for any geodesic $\sigma : \beta \to E$ and any $x \in \sigma_0$ for $\sigma_0 (\beta)$ we define the geodesic distance of $\sigma$ at $x$ to be

$$d^{\sigma} (x) : T_\beta \beta \to E_x \quad \sigma \to T_x (d^{\sigma} (\beta)).$$

**Theorem 1:** Let $\sigma : \beta \to E$ be a geodesic. Suppose that for all $x \in \sigma_0$ $\sigma_0_\beta$ is quasisphere and admits a right inverse then $\sigma_0_\beta$ is a Banach manifold. Furthermore $\sigma_0_\beta$ is Fredholm then it is a covering of (finite) dimension equal to the Fredholm index.

**Remark/Idea of the proof:** $d^{\sigma_0_\beta}$ quasisphere for all $x \in \sigma_0_\beta$ and admits right inverse $\Leftrightarrow \text{ker} (T_x \beta) \oplus \text{ker} (T_\beta \beta) = T_\beta \beta = T_{\sigma_0(x)} E$.

The condition about admitting right inverse is superfluous in finite dimension.

- A continuous operator $A : E \to F$ between two Banach space is Fredholm if $\dim \ker A + \infty$ and $\dim F / \text{im} A < + \infty$.

  Index $A = \dim \ker A - \dim (F / \text{im} A)$.

**b. Reformulation of the theorem**

Thanks to the Calderon-Zygmund inequality one can prove that

**Theorem 2.2:** Let $\Gamma$ be a non degenerate strongly harmonic function. Then

$$\forall x, x' \in \mathbb{R}^n$$

$$d^{\Gamma_\beta} (x) \text{ is Fredholm}.$$
Thanks to the previous theorems, one can reformulate the theorem.

Theorem

Let \( H \) be a non-separable Hamiltonian function. The set \( \Omega_{\text{reg}}(H,0) \) of

\[ H \in \mathcal{H}(H,0) \text{ s.t. } \frac{d}{dt} J_t^H(u) \text{ is surjective for } \forall x \in \mathcal{F}(H), \forall \varepsilon \in B \left( x, \varepsilon \right) \]

is a residual subset of \( \mathcal{F}(H,0) \) endowed with the \( C^\infty \) topology.

III. Proof of the Theorem

a. Sub-Single Theorem

We say that \( y \in Y \) is a regular value of \( f : X \to Y \) if \( f^{-1}(y) = \emptyset \) or \( d_x f \) is surjective \( \forall x \in \mathcal{F}(H) \).

Theorem 3.1: Let \( X \) and \( Y \) be separable Banach manifolds, i.e., they admit dense sequences, and \( f : X \to Y \) a smooth Frechet map, i.e., \( d_x f : T_x X \to T_{f(x)} y \) is Frechet for all \( x \in X \).

Then \( y \in \Omega_{\text{reg}}(f) \) if \( y \in Y \) is a regular value of \( f \); it is residual in \( Y \).

We admit this previous theorem. This allows to deduce the following theorem.

Theorem 3.2: Let \( E \to \mathcal{B}_x \mathcal{C} \) be a Banach bundle (everything separable) and

\[ \Sigma : \mathcal{B}_x \mathcal{C} \to E \quad (b,x) \mapsto \Sigma(b,x) = \sigma_x(b) \text{ a smooth section.} \]

a) \( \Sigma(b,x) \in \Sigma^{-1}(0) \), the vertical derivative of \( \Sigma \) at \((b,x)\) is surjective.

b) \( x \in \mathcal{C}, \Sigma(b,\mathcal{C}) = 0 \), the vertical derivative of \( \Sigma \) at \( b \) is Frechet.

Then the set \( \mathcal{C}_{\text{reg}} = \{ x \in \mathcal{C} | d_x \sigma(x) \text{ is surjective (and dense for } b \}) \]

is a residual subset of \( \mathcal{C}_{\text{reg}} \).
Rough idea of the proof of Theorem 3.2:

1) The hypotheses a) + b) allow to make sure that the vertical derivative of $\Sigma$ admits a right inverse at any $(b, z) \in \Sigma^{-1} \setminus \emptyset$.

2) This allows to apply IFT and deduce that $\Sigma^{-1} \setminus \emptyset$ is a manifold.

3) $\pi_2: \Sigma \rightarrow \Sigma$ is Fredholm of some index as the vertical derivative of $\pi_2$ and moreover $\pi_2$ is a regular value of $\pi_2 \iff d\pi_2(x)$ surjective.

4) Use the Fredholm - Surjectivity Theorem to conclude.

b) Proof of Theorem

We would like to apply the previous Theorem 3.2 for the map

$S: \mathbb{R}^r \times (x, \mathbb{H}) \times H_0 \rightarrow \mathbb{E}^n, (u, \mathbb{H}) \mapsto S(u, \mathbb{H}) = \pi^* (u)$

Since we already know that this map satisfies assumption b) of Theorem 3.2 (thanks to Theorem 2.2), it remains to show that it satisfies also assumption a). More precisely,

Theorem 3.3 The vertical derivative of $S$ at any point $(u, \mathbb{H}) \in \Sigma^{-1} \setminus \emptyset$ is surjective.

Indeed, showing Theorem 3.3 then with Theorem 3.2 together with Theorem 2.1 allows to prove Theorem.

Recall that if $E$ is a subvector space of a vector space $E$

we denote by $F_{\text{norm}} = \partial, P \in E^*, \partial (E, R) \mid \partial = 0$.

Lemma: Let $F$ be closed. Then $F = E \iff F_{\text{norm}} = 0$.

Proof: Use Hahn - Banach.

Since $H \in H_0$, it is non-separate, for all $(u, \mathbb{H}) \in \Sigma^{-1} \setminus \emptyset$ $d\pi_2(u)$ is Fredholm and $d\pi_2(u)\pi^* (\mathbb{H})$ is closed.
Thanks to the previous lemma to show the surjectivity of $d^r S(a, H)$ it is enough to show that

$$\text{Im}(d^r S(a, H)) = \left\{ w \in L^p(\R^m, \omega^* TM)^* \mid \int_{\text{Fun} d^r S(a, H)} w = 0 \right\} \subset \Omega^2 = \{ 0 \}$$

Since $(L^p)^* \cong L^q$ is naturally isomorphic to $L^q$ (one we fix a measure on $\R^m$ and a metric on $M$) there $\frac{1}{p} + \frac{1}{q} = 1$ or $a$

$$L^q \rightarrow (L^p)^* \quad w \mapsto \left( p \mapsto \int_{\R^m} \langle w, p \rangle \, d \omega^* dt \right).$$

Using this identification

$$\text{Fun} d^r S(a, H) = \left\{ w \in L^p \left( d^r S(a, H)[\hat{\omega}, \hat{H}], w \right) : \text{d}^{*} \text{d} \omega^* dt = 0 \right\}$$

$$\forall (\hat{\omega}, \hat{H}) \in T_{(a, H)} \left( B \left( \omega^{-1}(x_i) \times \tilde{g}(H_a) \right) \right)$$

We will show that if $w \neq 0$ then $\int_{\R^m} \langle \partial_a w, w \rangle \, d \omega^* dt = + \infty$ contradicting Hölder inequality.

1. First one can see that $d^r S(a, H)(\hat{\omega}, \hat{H}) = d^r \partial_H (a) (\hat{\omega}) - \nabla \hat{H}_t (a),$ so putting $\hat{H} = 0$ we get

$$\int_{\R^m} \langle d^r \partial_H (a) (\hat{\omega}), w \rangle \, d \omega^* dt = 0, \quad \forall \hat{\omega}$$

$$\int_{\R^m} \langle \hat{\omega}, (d^r \partial_H (a))^* w \rangle \, d \omega^* dt = 0, \quad \forall \hat{\omega}$$

As by elliptic regularity that $w$ is smooth. Actually more is true: one can find a fibrewise function $\bar{F} : \R^m \times \R^{2n} \rightarrow \omega^* TM$ such that in fibre coordinates

$$\int_{\R^m} (d^r \partial_H (a))^* w = 0 \iff \partial_a \bar{\omega} + \partial_a F \bar{\omega} = 0 \text{ where } \bar{\omega}(x, t) = \bar{F}^{-1} (w(x, t))$$

$\bar{F}$ the standard complex structure of $\R^{2n}$ and $S : \R^m \times \R^{2n} \rightarrow \text{End}(\R^{2n})$.
So it satisfies a «prefered Cauchy-Picard equation» so it shows some proper-
nes of holomorphic map. In particular \( f(x, t) \in \mathcal{O} \) if and only if it can be proven using Cauchy-Picard principle, we will about it.

2) Now putting \( \dot{w} = 0 \) equation (1) gives

\[
\mathbb{H} \times \mathbb{R} \quad \langle -\nabla \hat{H}_t(u), w \rangle - dSdt = 0 \quad \text{for all } \hat{H} \in \mathcal{T}_t \mathcal{H}(\mathcal{H}_0).
\]

\[
\mathbb{H} \times \mathbb{R} \quad \langle d\hat{H}_t(d), w \rangle dSdt = 0 \quad \text{for all } \hat{H} \in \mathcal{T}_t \mathcal{H}(\mathcal{H}_0) \quad (2).
\]

Claim: This implies that \( f(X, s) \rightarrow \mathbb{R} \),
\[ w(s, t) = \alpha(s) \in \mathbb{R} \quad \text{for all } (s, t) \in \mathbb{R} \times \mathbb{R}. \]

We abdut this opinon for the moment and give a sketch of the proof below.

3) Let show that \( \frac{d}{ds} \int_{t_0}^{t_1} \langle \hat{w}(s, t), \hat{w}(s, t) \rangle dt = 0 \) for any \( t_0, t_1 \).

Indeed first remark that

\[
d^2 \hat{H}_t(u) \left[ d_s w \right] = 0 \quad \text{and} \quad (d^2 \hat{H}_t(u)^t) \left[ w \right] = 0 \quad (3.1).
\]

We already now the second equality, the first one come from: Since \( d^2 \hat{H}_t(u) = 0 \),

\[ \text{this imply that } d^2 \hat{H}_t(u_2) = 0 \text{ where } u_2(s, t) = u_1(s, t) + u_1(s, t). \]

\[ \frac{d}{dt} \hat{H}_t(u_2) = d^2 \hat{H}_t(u_2) (d_s w) = 0 \text{ which implies } (d^2 \hat{H}_t(u_2) (d_s w) = 0.
\]

So using the definication \( \mathcal{O} \) and putting \( \hat{w}(s, t) = \hat{w} \alpha(s, t) \), \( \hat{w}(s, t) = \hat{w} \alpha(s, t). \)

We have that \( d_s \hat{w} + J_0 d_t \hat{w} + \hat{w} = 0 \) and \( d_s \hat{w} + J_0 d_t \hat{w} + \hat{w} = 0 \)

\[ \int d_s \hat{w} + J_0 d_t \hat{w} + \hat{w} = 0 \quad \text{for all } \hat{w}. \]

\[
= \int \left[ -J_0 d_t \hat{w} - \hat{w} \right] dt \quad + \left[ \hat{w}, J_0 d_t \hat{w} + \hat{w} \right] dt
\]
\[ \int \langle \mathcal{J} \mathbb{d} \mathbb{w}, \mathbb{w} \rangle - \langle \mathbb{w}, \mathbb{w} \rangle + \langle \mathbb{w}, \mathbb{J} \mathbb{w} \rangle + \langle \mathbb{w}, \mathbb{J} \mathbb{w} \rangle = \int \frac{1}{2} \langle \mathbb{w}, \mathbb{J} \mathbb{w} \rangle \, dt = 0. \]

4) Suppose that \( \omega \neq 0 \). Then \( \omega : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \). Indeed suppose \( \phi(0) = 0 \), then \( \phi \in \mathcal{D}(\mathbb{R}) \), \( \phi(x) = 0 \) for all \( k \in \mathbb{N} \). Now using again \( \mathcal{J} \mathbb{d}_k \mathbb{w} \) and the trivial fact that \( \mathcal{J} \mathbb{d}_k \mathbb{w} - \mathcal{J} \mathbb{d}_k \mathbb{w} = 0 \), we get by induction that \( \mathcal{J} \mathbb{d}_k \mathbb{w}(x) = 0 \) for all \( k \) and \( x \).

So by 1) this would imply that \( \omega = 0 \). \( \square \)

5) \[ \int \langle \mathcal{J} \mathbb{d} \mathbb{w}, \mathbb{w} \rangle \, dt = \int \langle \mathcal{J} \mathbb{d} \mathbb{w}, \mathbb{w} \rangle \, dt = \int \langle \mathcal{J} \mathbb{d} \mathbb{w}, \mathbb{w} \rangle \, dt = 0. \]

by 2). And so by 3) \[ \int \langle \mathcal{J} \mathbb{d} \mathbb{w}, \mathbb{w} \rangle \, dt = 0. \]

This contradicts our inequality. \( \square \)

It remains to prove the Claim of point 2). To do so we have to focus on two of terms \( \mathcal{R}(u) \) and \( \mathcal{C}(u) \) associated to \( x \in \mathcal{N}_H(\mathbb{R}, \mathbb{R}) \).

\[ \mathcal{C}(u) = \begin{cases} \langle \mathcal{J} \mathbb{d} \mathbb{w}, \mathbb{w} \rangle = 0 \end{cases} \]

\[ \mathcal{R}(u) = \begin{cases} 1. \mathcal{J} \mathbb{d}_k \mathbb{w} = 0 \end{cases} \]

Since \( \mathcal{J} \mathbb{d}_k \mathbb{w} \) and \( \mathcal{J} \mathbb{d}_k \mathbb{w} \) are \( (3.1) \) one can use again Cauchy–

Similarity principle and the principle of analytic continuation to get the next theorem that we admit:
**Theorem 3.4:** If \( x + x^- \) then \( C(a) \) is Sturmian and \( R(a) \) is open and dense.

**Proof of the Claim.** We will proceed as follows:

1) We prove the existence of \( \lambda : C(a)^e \to R \) such that
\[
\omega(x,t) = \lambda(x,t) \partial_x u(x,t) \quad \text{for all } (x,t) \in R \times R.
\]

2) We prove that \( \partial_x \omega = 0 \) so it can be seen as a function \( \lambda : R \to R \).

Both proofs will be by contradiction:

**Proof of 1):**

Suppose by contradiction that \( \omega \) and \( \partial_x u \) are linearly independent at \((x,t_0) \in C(a)^e \).

Then \( f : J \subset \mathbb{R} \times V \to \mathbb{R} \times M \)
\[
(x,r,s,t) \mapsto (\exp_x (r,t) \partial_x w(x,t))
\]

is an embedding for \( \epsilon > 0 \) and \( \theta \) neighborhood of \((x,t_0) \) small enough, indeed
\[
\frac{d}{dr} \mid_{r=0} f(x,r) = \omega(x,t) \quad \text{and} \quad \frac{d}{ds} \mid_{t=0} f(0,s) = \partial_x u(x,t).
\]

Construct \( \mathcal{H} \) with support inside \( U \), a small neighborhood of \( J \), such that
\[
\mathcal{H}(f(x,r,t)) = r \beta(x,t) \quad \text{for } \beta \text{ a positive function on } V.
\]

Then \( \frac{d}{dr} \mid_{r=0} \mathcal{H}(f(x,r,t)) = \beta(x,t) = \mathcal{H}(f(0,x,t)) \omega(x,t) \)
\[
= \mathcal{H}(\omega(x,t)) \omega(x,t).
\]

And so \( \frac{d}{dr} \mathcal{H}(\omega(x,t)) \omega(x,t) = \mathcal{H}(\omega(x,t)) \omega(x,t) \)

/Cheat (see the remark below)
\[ \Phi_t(u(s,t))(w(s,t)) \, dt \geq 0, \quad \text{So this contradicts (3)} \]

Remark: We have cleared for the previous equality, indeed it can happen that for \((s,t) \notin U, \,(t,u(s,t)) \in \mathcal{U}! \) To solve this problem we should first do the same thing, but constructing \( \lambda : u(a) \to \mathbb{R} \), note that \( R(u) \subset C(u)^c \). Indeed in this case one can make sure that for \( u \) small enough and \( U \) small enough \( (t, u(a(s,t))) \in \mathcal{U} \iff (t, u(s,t)) \in \mathcal{D} \).

Moreover in this case we can make sure that \( \mathcal{U} \cap 8 \times \Phi(h) = \emptyset \) since \( (x,t) \in R(u) \Rightarrow u(x,t) = x + f(t) \). And so \( \forall t \in T \Phi(h) \).

Then we have \( \lambda : R(u) \to \mathbb{R} \) \((s,t) \to \langle w, a(s,t) \rangle \).

One can then extend \( \lambda \) to \( C(u)^c \) and since \( R(u) \) is dense we still have that \( w(s,t) = \lambda'(s,t) \, \partial_u u(s,t) \) for all \((s,t) \in C(u)^c \).

**Proof of (2):** Suppose that \( \partial_u u \neq 0 \). 

\[ \Rightarrow \int (s,t_0) \in R(u), \partial_u(x,s_0,t_0) 
eq 0. \]

Then there exists \( k : R \times 8 \to R \) \( \forall t \) such that:

\[ \int_{8 \times R} (\partial_u u) k \, dxt = \int_{8 \times R} (\partial_u u) \lambda \, dxt \quad (\mathcal{I} \Phi). \]

As before we consider \( \Phi_0 \) small enough s.t.:

\( \Phi_0 \subset 8 \times \mathcal{M} \) \( (s,t) \to (t, u(s,t)) \) embedding and \( \mathcal{U} \) a neighborhood of this map s.t. \((t, u(s,t)) \in \mathcal{U} \iff (s,t) \in \mathcal{U} \).
Then we can construct \( \hat{\alpha} : \mathbb{R} \times M \to \mathbb{R} \) supported in \( W \) s.t.
\[
\hat{\alpha}(t, u(x, t)) = k(x, t) \quad \text{for all } (x, t) \in W \quad \text{and extend it arbitrarily.}
\]
So
\[
\frac{d}{dt} \hat{\alpha}(t, u(x, t)) = d\hat{\alpha}^+(u(x, t)) (\partial_x u(x, t)) = \partial_x^+ k(x, t)
\]
\[
\int_{\mathbb{R} \times M} d\hat{\alpha}^+(u(x, t)) (w(x, t)) \, dx \, dt = \int_{\mathbb{R} \times M} d\hat{\alpha}^+(u(x, t)) (w(x, t)) \, dx \, dt
\]
\[
= \int_{\mathbb{R} \times M} d\hat{\alpha}^+(u(x, t)) (\alpha(x, t) \partial_x u(x, t)) \, dx \, dt
\]
\[
= \int_{\mathbb{R} \times M} \alpha(x, t) d\hat{\alpha}^+(u(x, t)) (\partial_x u(x, t)) \, dx \, dt
\]
\[
= \int_{\mathbb{R} \times M} \alpha(x, t) \partial_x k(x, t) \, dx \, dt \neq 0, \quad \text{and then}
\]
contradicting (3). Since \( \mathcal{R}(u) \) is dense in \( C(U) \),
\[
\partial_a^+ \alpha = 0 \quad \text{also in } \mathcal{C}(u). \quad \text{Since } \mathcal{C}(u) \text{ is dense in } C(U), \quad \mathcal{C}(u) \text{ is}
\]
connected \( \implies \alpha \) can be seen as a function \( \mathbb{R} \to \mathbb{R} \). \( \square \)