Seminar Generating Functions 16.5.24

Hofer’s metric

\[ D = \text{Ham}_c (\mathbb{R}^{2n}) = \{ \phi^t_H \mid H \in C^\infty (\mathbb{R}^n) \} \]

\[ \phi^t_H \cdot \phi^t_K = \phi^{t+t}_{H \# K} , \quad H \# K (t, x) = H(t, x) + K(t, \phi^t_H(x)) \]

\[ (\phi^t_H)^{-1} = \phi^{-t}_H , \quad \hat{H}(t, x) = -H(t, \phi^t_H(x)) \]

\[ D \text{ Lie group with Lie algebra } \mathcal{X} : = C^\infty_c (\mathbb{R}^{2n}) \]

\[ \| H \| : = \max H - \min H \text{ norm on } \mathcal{X} \]

\[ \| H \cdot \varphi \| = \| H \| \quad \forall \varphi \in D \Rightarrow \| \cdot \| \text{ induces bi-invariant Finder str. on } D \]

\[ \gamma : [a, b] \rightarrow D \quad \frac{d}{dt} \gamma(t) = X_{H(t)} (\gamma(t)) \]

\[ \text{Length}_{H} (\gamma) : = \int_{a}^{b} \| H(t) \| \, dt \]

\[ d_H (\varphi, \psi) : = \inf \text{ Length}_{H} \gamma \text{ bi-invariant pseudo-distance} \]

\[ \gamma : [0, 1] \rightarrow D \quad \gamma(0) = \varphi , \quad \gamma(1) = \psi \]

Thm1 (Hofer 1990) \[ d_H \text{ is a distance.} \]
**Action selector**

$\phi \in D$, $x \in Fix \phi$ \quad $\phi = \phi'_H$

$A_\phi(x) := \int_0^1 \lambda_0 + \int_0^1 H(t, \phi'_H(x)) \, dt$ \quad \text{indep. of } H

$\sigma(\phi) = \{ A_\phi(x) \mid x \in Fix \phi \} \subset \mathbb{R}$

zero measurable compact set \text{c.f.}

\[ \exists \ C^+ : D \to [0, +\infty) \quad C^\infty - \text{continuous r.b.} \]

\[(i) \quad C^+(\phi) \in \sigma(\phi) \]

\[(ii) \quad C^+(\phi) + C^+(\phi^{-1}) > 0 \quad \text{if} \quad \phi \neq id \]

\[(iii) \quad H > K \quad \Rightarrow \quad C^+(\phi'_H) \geq C^+(\phi'_K) \]

**Rmk** Both Hofer’s and Viterbo’s selectors satisfy (i)-(iii).

**Prop 1** \quad $\forall \phi \in D$ \quad we have

\[ d_H(\phi, \text{id}) \geq C^+(\phi) + C^+(\phi^{-1}) \]

**pf** \quad $\phi = \phi'_H$, $\bar{H}_t \quad H_t < \bar{H}_t \leftrightarrow$ \text{no non-constant}

\[ \sigma(\phi'_H) = \{ 0, \int_0^1 \max H_t \, dt \} \quad \text{periodic orbits} \]
\[ c_+ (\phi_H^t) \leq c_+ (\phi_H^t) = \int_0^1 \max H_t \, dt \]
\[ c_+ ((\phi_H^t)^{-1}) \leq -\int_0^1 \min H_t \, dt . \]

* (ii) implies that \( d_H \) is non-degenerate.

* \( d_H (\varphi, \psi) \geq d_V (\varphi, \psi) \) Viterbo's distance

**Local properties of \( d_H \) near id**

**Symplectomorphism** \( \Phi : (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega \otimes (-\omega)) \rightarrow (T^* \mathbb{R}^{2n}, \omega_{can}) \)

\( \Phi (\Delta) = 0 \text{-section} \), \( \Delta = \{(x, x) \mid x \in \mathbb{R}^{2n}\} \)

\( \mathbb{R}^{2n} = C^n \quad \Phi (x, x) = (\frac{x + x}{2}, i(x - x)) \)

\( \varphi \in \mathcal{D} \) \( C^1 \)-close to id \( \Rightarrow \) graph \( \varphi \) \( C^1 \)-close to \( \Delta \)

\( \Rightarrow \Phi (\text{graph } \varphi) \) \( C^1 \)-close to zero-section

\( \Rightarrow \Phi (\text{graph } \varphi) = \text{graph } dS \) with \( S \in \mathcal{C} \)

\( C^2 \)-small

Conversely, for any \( C^2 \)-small \( S \in \mathcal{C} \), \( \Phi^{-1} (\text{graph } dS) \)
is the graph of a $\Phi \in D$ which is $C'$-close to $id$.

$$
\Phi \left( \left\{ (x, \varphi(x)) \mid x \in \mathbb{R}^m \right\} \right) = \left\{ (y, dS(y)) \mid y \in \mathbb{R}^n \right\}
$$

$$
\left\{ \left( \frac{x + \varphi(x)}{2}, \ i(x - \varphi(x)) \right) \mid x \in \mathbb{R}^m \right\}
$$

\[ i(x - \varphi(x)) = \nabla S \left( \frac{x + \varphi(x)}{2} \right) \quad \text{(*)}. \]

There exists $S$ convex $C^2$-uhb. of $0$ in $X$ and $U$ $C'$-neigh. of $id$ in $D$ and bijection $\Psi: S \rightarrow U$ $S \rightarrow \varphi$ defined by (\text{*}).

**Thm 2 (Biely-Polterovich 1994)** The map $\Psi: (S, \| \cdot \|) \rightarrow (U, d_H)$ is an isometry.

**Consequences:**

- Hofer's geometry is flat.
- If $\varphi, \psi \in D$ are $C'$-close then $d_H(\varphi, \psi) = d_V(\varphi, \psi)$.
\[ \varphi = \psi(s) \]
\[ d_v(\varphi, \text{id}) = c^+(\varphi) - c^-(\varphi) \]
\[ = \max S - \min S = \| S \| = d_H(\varphi, \text{id}) \]

- The length structure induced by \( d_v \) coincides with Hofer-Finsler length.

- Minimizing geodesics with \( d_H \) between \( C' \)-close points in \( D \) exist but are never unique.

**Example:**

\[ S_t : S_0 = 0, \quad S_1 = S \]

Length \((S_t)_{t \in [0,1]}\) = \[ \int_0^1 \| \partial_t S_t \| \, dt = \| S \| \]

if \[ \frac{d}{dt} \| S_t \| = \| \partial_t S_t \| \]

\[ F_0 = F_1 = 0 \]

\[ S_t = tS + F_t \]

\[ \supp F_t \cap \supp S = \emptyset \]

\[ \max F_t \leq t \max S \]

\[ \min F_t \geq t \min S \]

\[ \max \partial_tF_t \leq \max S \]

\[ \min \partial_tF_t \geq \min S \]

\[ \| \partial_t S_t \| = \| S + \partial_t F_t \| = \| S \| \]
**Generating functions versus Hamiltonians**

\((S_t)_{t \in [0,1]} \) smooth path in \( \mathbb{R} \)

\( \varphi_t = \Psi(S_t) \) corr. path in \( \mathcal{D} \)

\[ \frac{d}{dt} \varphi_t = X_{H_t}(\varphi_t) \]

Prop \( \partial_t S_t(y) = H_t(y + \frac{i}{2} \nabla S_t(y)) \) (HJ)

\[ i(x - \varphi_t(x)) = \nabla S_t \left( \frac{x + \varphi_t(x)}{2} \right) \]

\[ - i X_{H_t}(\varphi_t(x)) = \nabla \partial_t S_t \left( \frac{x + \varphi_t(x)}{2} \right) + \]

\[ \nabla H_t(\varphi_t(x)) + \frac{1}{2} \nabla^2 S_t \left( \frac{x + \varphi_t(x)}{2} \right) i \nabla H_t(\varphi_t(x)) \]

\[ \gamma = \frac{x + \varphi_t(x)}{2} \quad \varphi_t(x) = \gamma + \frac{i}{2} \nabla S_t(y) \]

\[ \nabla \partial_t S_t(y) = \left( \mathbb{I} - \frac{i}{2} \nabla^2 S_t(y) i \right) \nabla H_t(\varphi_t(x)) \]

\[ = \left( \mathbb{I} + \frac{i}{2} \nabla^2 S_t(y) \right)^* \nabla H_t(\varphi_t(x)) = \nabla \left( H_t(y + \frac{i}{2} \nabla S_t(y)) \right) \]
Claim \[ d_H(\psi(s), \text{id}) = \| S \| \quad \forall S \in \mathcal{S} \]

\[ \varphi_t := \psi(tS) \quad \varphi_0 = \text{id} \quad \varphi_t = \psi(S) \]

The corresponding Hamiltonian

\[ S(y) = H_t \left( y + \frac{i}{2} t \mathcal{D} S (y) \right) \]

\[ d_H(\psi(s), \text{id}) \leq \text{Length} \left( \varphi_t \right)_{t \in [0,1]} = \int_0^1 \| H_t \| \, dt = \| S \| \]

\[ \geq d_V(\psi(s), \text{id}) = \| S \| \]

In the general case: \( S_0, S_1 \in \mathcal{S} \)

\[ S_t := S_0 + t \left( S_1 - S_0 \right) \]

\[ \varphi_t = \psi(S_t) \quad \frac{d}{dt} \varphi_t = \mathcal{X}_H(\varphi_t) \quad \text{with} \]

\[ (S_1 - S_0)(y) = H_t \left( y + \frac{i}{2} \left( \mathcal{D} S_0 + t \left( \mathcal{D} S_1 - \mathcal{D} S_0 \right) \right)(y) \right) \]
\[ \| H_t \| = \| S_1 - S_0 \| \quad \forall t \in [0, 1] \Rightarrow \]
\[ d_H (\psi(S_0), \psi(S_1)) \leq \text{Length} (\varphi_t) = \| S_1 - S_0 \| \]

We need to prove the other inequality.

\[ \psi_t := \varphi_t \circ \varphi_0^{-1} \quad \text{path from id gen. by } H_t \]

By Prop 1:

\[ d_H (\psi(S_0), \psi(S_1)) = d_H (\varphi_0, \varphi_1) = d_H (\text{id}, \psi_t) \]

\[ \geq c_+ (\psi_1) + c_+ (\psi_1^{-1}) \]

Aim \[ c_+ (\psi_1) \geq \max (S_1 - S_0) \]

\[ c_+ (\psi_1^{-1}) \geq -\min (S_1 - S_0) \]

* \( H \) is quasi autonomous \( \exists x_{\max}, x_{\min} \in \mathbb{R} \)

s.t. \[ \max H_t = H_t (x_{\max}) \quad \forall t \in [0, 1] \]

\[ \min H_t = H_t (x_{\min}) \quad \forall t \in [0, 1] \]

Indeed: \( \frac{x_{\max} - x_{\min}}{2} \text{ is } S_0 (\max(x_{\min})) \) where

\[ x_{\max}, x_{\min} \text{ are a maximizer and a minimizer} \]
Lemma \( x \in \text{Fix } \Psi_x \) for some \( x > 0 \) \Rightarrow

\[ x \in \text{Fix } \Psi_t \ \forall t \in [0, 1] \text{ and } \]

\[ A_{\Psi_t}(x) = t (S_1 - S_0)(y) \]

where \( x = y + \frac{i}{2} \nabla S_0(y) \) and \( y \)

is a critical point of \( S_1 - S_0 \).

Conversely, for every critical point \( y \) of \( S_1 - S_0 \) the point \( x \) defined as above

is a fixed point of \( \Psi_t \ \forall t \in [0, 1] \).

**Proof**

\[ \Psi_x(x) = x \Rightarrow \varphi_x \circ \varphi_x^{-1}(x) = x \]

\[ \Rightarrow x := \varphi_x^{-1}(x) \text{ satisfies } \varphi_x(x) = \varphi_x(2) \]

\[ i(2 - \varphi_x(2)) = \nabla S_0 \left( \frac{2 + \varphi_x(2)}{2} \right) \quad (1) \]

\[ \nabla S_0 + z \left( \nabla S_1 - \nabla S_0 \right) \]
\[
Y := \frac{Z + \varphi_0(z)}{2} \in \text{crit}(S_1 - S_0)
\]
\[
\Rightarrow \nabla S_t(Y) = \nabla S_0(Y) \quad \forall t \in [0, 1]
\]
\[
w = \varphi^+_t(z) \text{ is the unique solution of }\]
\[
i(Z - w) = \nabla S_t\left(\frac{Z + w}{2}\right)
\]
Since \( \nabla S_t(Y) = \nabla S_0(Y) \), \( w := \varphi_0(z) \)

is a solution of the above equation by (1).

Therefore \( \varphi_t(z) = \varphi_0(z) \quad \forall t \in [0, 1] \)

\[
\Rightarrow \psi_t(x) = \varphi_t \circ \varphi_0^{-1}(x) = \varphi_t(z) = \varphi_0(z) = x \quad \forall t \in [0, 1]
\]

Moreover:

\[
(1)
\]
\[
\varphi_0(z) = Z + i \nabla S_0(Y)
\]
\[
\Rightarrow 2Y - Z
\]
\[
\Rightarrow Z = Y - \frac{i}{2} \nabla S_0(Y)
\]
\[
\Rightarrow x = \varphi_0(z) = Y + \frac{i}{2} \nabla S_0(Y) \quad \text{as claimed.}
\]
\[ A_{\psi_t}(u) = \int_0^t H_s(u) \, ds = \int_0^t H_s(y + \frac{i}{2} \nabla \varphi(y)) \, ds \]

\[ = \int_0^t (S_1 - S_0)(y) \, ds = t (S_1 - S_0)(y). \]

Conversely, if \( y \in \text{crit}(S_1 - S_0) \) then

\[ \nabla S_t(y) = \nabla S_0(y) \quad \forall t \in [0, 1] \]

By differentiating (*) we find

\[ 0 = \nabla (S_1 - S_0)(y) = (1 + \frac{i}{2} \nabla^2 \varphi(y) + t (\nabla^2 S_1(y) - \nabla^2 S_0(y))) \]

\[ \nabla H_t(y + \frac{i}{2} \nabla S_0(y)) \]

\[ \Rightarrow x := y + \frac{i}{2} \nabla S_0(y) \text{ is a critical point of } H_t \quad \forall t \in [0, 1] \]

\[ \Rightarrow x \in \text{Fix } \psi_t \quad \forall t \in [0, 1] \]

In particular:

\[ \sigma(\psi_t) = t \cdot \{ \text{critical values of } S_1 - S_0 \} \]
Since the function $t \mapsto c^+(\psi_b)$ is continuous, the above fact implies that it is enough to prove the identity

$$c^+(\psi_b) = b \max (S_1-S_0)$$

for $t$ small. We may assume $\max (S_1-S_0) > 0$ (otherwise $H = 0 \Rightarrow c^+(\psi_b) = 0 \forall t$).

$U \supp H_b \subset B_R \subset \supp K$, $K \leq H \forall t \in [0,1]$.

$$\max K = \max H_0 = \max (S_1-S_0) \text{ achieved only at } z_{\text{max}}$$

$$\nabla^2 K(z_{\text{max}}) < \nabla^2 H_0(z_{\text{max}})$$

The critical values of $K$ are just $\max (S_1-S_0)$ and $\min K < \min H_0$.

Since $H_b(z_{\text{max}}) = H_0(z_{\text{max}}) \forall t \in [0,1]$,
there exists $\varepsilon > 0$ s.t. $K = H_{\varepsilon} \forall \varepsilon \in [0, \varepsilon]$

If $\varepsilon > 0$ is small enough then $X_{EK}$ does not have non-constant $1$-periodic orbits.

$$\Rightarrow \sigma(\phi_{EK}^1) = \{ \varepsilon \min K, 0, \varepsilon \max(S_1 - S_0) \}$$

We claim that $C^+(\phi_{EK}^1) = \varepsilon \max(S_1 - S_0)$

Indeed, if $C^+(\phi_{EK}^1) = 0$ then $C^+$ would vanish along the isotopy which is obtained by pushing the negative part to $K$ up to zero, contradicting (ii).

If $\varepsilon \leq \varepsilon$ then $E_{K} \leq \varepsilon H_{\varepsilon} \forall \varepsilon \in [0, 1]$

$$\Rightarrow C^+(\psi_{\varepsilon}) = C^+(\phi_{\varepsilon}^1) \geq C^+(\phi_{EK}^1)$$

$$= \varepsilon \max(S_1 - S_0)$$

The lower bound for $C^+(\psi_{\varepsilon}^1)$ follows from this one by considering $\hat{H}_{\varepsilon} = -H_{\varepsilon} \circ \phi_{H}^1$. □