Seminar Generating Functions 16.5.24

Hofer's metric

$$\mathcal{D} = \mathrm{Ham}_c(\mathbb{R}^{2n}) = \{ \phi_H^1 \mid H \in C_c^\infty([0,1] \times \mathbb{R}^{2n}) \}$$

- $\phi_H^1 \circ \phi_K^1 = \phi_{H\#K}^1$, $\quad H\#K(t,x) = H(t,x) + K(t, (\phi_H^t)^{-1}(x))$
- $(\phi_H^1)^{-1} = \phi_{\hat{H}}^1$, $\quad \hat{H}(t,x) = -H(t, \phi_H^t(x))$

$\mathcal{D}$ Lie group with Lie algebra $\mathcal{H} := C_c^\infty(\mathbb{R}^{2n})$

$\|H\| := \max H - \min H$ norm on $\mathcal{H}$

$\|H \circ \varphi\| = \|H\| \quad \forall \varphi \in \mathcal{D} \quad \Rightarrow \quad \|\cdot\|$ induces bi-invariant Finsler str. on $\mathcal{D}$

$\gamma : [a,b] \to \mathcal{D} \qquad \frac{d}{dt}\gamma(t) = X_{H_t}(\gamma(t))$

$$\mathrm{Length}_H(\gamma) := \int_a^b \|H_t\| \, dt$$

$$d_H(\varphi, \psi) := \inf_{\substack{\gamma : [0,1] \to \mathcal{D} \\ \gamma(0) = \varphi,\ \gamma(1) = \psi}} \mathrm{Length}_H \gamma \qquad \text{bi-invariant pseudo-distance}$$

Thm 1 (Hofer 1990) d_H is a distance.

Action selector

$\varphi \in \mathcal{D}$, $x \in \mathrm{Fix}\,\varphi$ $\qquad \varphi = \phi_H^1$

$$A_\varphi(x) := \int_{t \mapsto \phi_H^t(x)} \lambda_0 + \int_0^1 H(t, \phi_H^t(x))\,dt$$ indep. of H

$$\sigma(\varphi) = \{ A_\varphi(x) \mid x \in \mathrm{Fix}\,\varphi \} \subset \mathbb{R}$$
zero measurable compact set

$\exists\ c^+ : \mathcal{D} \to [0, +\infty)$ $\quad C^\infty$- continuous s.t.

(i) $c^+(\varphi) \in \sigma(\varphi)$

(ii) $c^+(\varphi) + c^+(\varphi^{-1}) > 0$ if $\varphi \neq id$

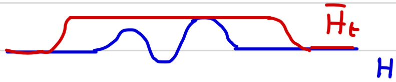
(iii) $H \geq K \implies c^+(\phi_H^1) \geq c^+(\phi_K^1)$

Rmk Both Hofer's and Viterbo's selectors satisfy (i)-(iii)

Prop 1 $\forall \varphi \in \mathcal{D}$ we have

$$d_H(\varphi, id) \geq c_+(\varphi) + c_+(\varphi^{-1})$$

Pf $\varphi = \phi_H^1$.



$H_t \leq \overline{H}_t$ ← no non-constant periodic orbits

$$\sigma(\phi_{\overline{H}}^1) = \{ 0, \int_0^1 \max H_t\,dt \}$$

$$c_+(\phi_H^1) \leqslant c_+(\phi^1_{\overline{H}}) = \int_0^1 \max H_t \, dt$$

$$c_+((\phi_H^1)^{-1}) \leqslant -\int_0^1 \min H_t \, dt . \qquad \square$$

- (ii) implies that d_H is non-degenerate
- $d_H(\varphi, \psi) \geqslant d_V(\varphi, \psi)$ Viterbo's distance

Local properties of d_H near id

Symplectomorphism $\Phi : (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega_0 \oplus (-\omega_0)) \longrightarrow (T^* \mathbb{R}^{2n}, \omega_{can})$

$\Phi(\Delta) = 0\text{-section}$, $\Delta = \{(x,x) \mid x \in \mathbb{R}^{2n}\}$

$\mathbb{R}^{2n} = \mathbb{C}^n$ $\quad \Phi(x, X) = \left(\frac{x+X}{2}, i(x-X)\right)$

$\varphi \in \mathcal{D}$ C^1-close to id $\Rightarrow$ graph φ C^1-close to Δ

$\Rightarrow$ $\Phi(\text{graph } \varphi)$ C^1-close to zero-section

$\Rightarrow$ $\Phi(\text{graph } \varphi) = \text{graph } dS$ with $S \in \mathcal{H}$ C^2-small

Conversely, for any C^2-small $S \in \mathcal{H}$, $\Phi^{-1}(\text{graph } dS)$

is the graph of a $\varphi \in \mathcal{D}$ which is C^1-close to id.

$$\Phi\left(\{(x, \varphi(x)) \mid x \in \mathbb{R}^{2n}\}\right) = \{(y, dS(y)) \mid y \in \mathbb{R}^{2n}\}$$

$$\| $$

$$\left\{\left(\frac{x+\varphi(x)}{2}, i(x-\varphi(x))\right) \mid x \in \mathbb{R}^{2n}\right\}$$

$$\boxed{i\,(x - \varphi(x)) = \nabla S\left(\frac{x+\varphi(x)}{2}\right)} \qquad (*).$$

There exists $\mathcal{S}$ convex C^2-nbh. of 0 in $\mathcal{H}$

and $\mathcal{U}$ C^1-neigh. of id in $\mathcal{D}$ and bijection

$$\Psi : \mathcal{S} \longrightarrow \mathcal{U} \qquad S \longmapsto \varphi \quad \text{defined by } (*)$$

<u>Thm</u> 2 (Bialy–Polterovich 1994) The map

$\Psi : (\mathcal{S}, \|\cdot\|) \longrightarrow (\mathcal{U}, d_H)$ is an isometry.

<u>Consequences</u>:

- Hofer's geometry is flat
- If $\varphi, \psi \in \mathcal{D}$ are C^1-close then $d_H(\varphi,\psi) = d_V(\varphi,\psi)$

$$\varphi = \Psi(S) \qquad d_V(\varphi, id) = c_V^+(\varphi) - c_V^-(\varphi)$$

$$= \max S - \min S = \| S \| = d_H(\varphi, id)$$

- The length structure induced by d_V coincides with Hofer Finsler length.
- Minimizing geodesics wrt d_H between C^1-close points in $\mathcal{D}$ exist but are never unique.

Example:

$$S_t \qquad S_0 = 0, \quad S_1 = S$$

$$\text{Length}\,(S_t)_{t\in[0,1]} = \int_0^1 \| \partial_t S_t \| \, dt = \| S \|$$

$$\text{if} \qquad \frac{d}{dt} \| S_t \| = \| \partial_t S_t \|$$

$$S_t = tS + F_t \qquad \begin{array}{l} F_0 = F_1 = 0 \\ \text{supp } F_t \cap \text{supp } S = \emptyset \\ \max F_t \leq t \max S \\ \min F_t \geq t \min S \\ \max \partial_t F_t \leq \max S \\ \min \partial_t F_t \geq \min S \end{array}$$

$$\| S_t \| = t \| S \|$$

$$\frac{d}{dt} \| S_t \| = \| S \|$$

$$\| \partial_t S_t \| = \| S + \partial_t F_t \| = \| S \|$$

Generating functions versus Hamiltonians

$(S_t)_{t \in [0,1]}$ smooth path in $\mathcal{S}$

$\varphi_t = \Psi(S_t)$ corr. path in $\mathcal{D}$

$$\frac{d}{dt} \varphi_t = X_{H_t}(\varphi_t)$$

Prop $\quad \partial_t S_t(y) = H_t\left(y + \frac{i}{2} \nabla S_t(y)\right) \qquad (HJ)$

pf

$$i(x - \varphi_t(x)) = \nabla S_t\left(\frac{x + \varphi_t(x)}{2}\right)$$

$$-i X_{H_t}(\varphi_t(x)) = \nabla \partial_t S_t\left(\frac{x + \varphi_t(x)}{2}\right) +$$

$$\nabla H_t(\varphi_t(x)) \qquad + \frac{1}{2} \nabla^2 S_t\left(\frac{x + \varphi_t(x)}{2}\right) i \nabla H_t(\varphi_t(x))$$

$$y = \frac{x + \varphi_t(x)}{2} \qquad \varphi_t(x) = y + \frac{i}{2} \nabla S_t(y)$$

$$\nabla \partial_t S_t(y) = \left(1\!\!1 - \frac{1}{2} \nabla^2 S_t(y) i\right) \nabla H_t(\varphi_t(x))$$

$$= \left(1\!\!1 + \frac{i}{2} \nabla^2 S_t(y)\right)^* \nabla H_t(\varphi_t(x)) = \nabla\left(H_t\left(y + \frac{i}{2} \nabla S_t(y)\right)\right)$$

$$\Rightarrow \quad \partial_t S_t(y) = H_t\left(y + \frac{i}{2}\nabla S_t(y)\right)$$

$\square$

<u>Claim</u> $\quad d_H(\Psi(S), id) = \|S\| \quad \forall S \in \mathcal{S}$

$\varphi_t := \Psi(tS) \qquad \varphi_0 = id \;, \quad \varphi_1 = \Psi(S)$

H_t corresponding Hamiltonian

$$S(y) = H_t\left(y + \frac{i}{2} t \nabla S(y)\right)$$

$$\begin{cases} d_H(\Psi(S), id) \leq \text{Length}(\varphi_t)_{t\in[0,1]} = \\ = \int_0^1 \|H_t\| \, dt = \|S\| \\ \geq d_V(\Psi(S), id) = \|S\| \end{cases}$$

$\square$

In the general case : $\quad S_0, S_1 \in \mathcal{S}$

$$S_t := S_0 + t(S_1 - S_0)$$

$\varphi_t = \Psi(S_t) \qquad \frac{d}{dt}\varphi_t = X_{H_t}(\varphi_t) \quad$ with

$$(S_1 - S_0)(y) = H_t\left(y + \frac{i}{2}\left(\nabla S_0 + t(\nabla S_1 - \nabla S_0)\right)(y)\right) \quad (*)$$

$\Rightarrow \quad \| H_t \| = \| S_1 - S_0 \| \quad \forall t \in [0,1] \quad \Rightarrow$

$d_H(\psi(S_0), \psi(S_1)) \leq \text{Length}(\varphi_t) = \| S_1 - S_0 \|$

We need to prove the other inequality.

$\psi_t := \varphi_t \circ \varphi_0^{-1}$ path from id gen. by H_t

By Prop 1 :

$d_H(\psi(S_0), \psi(S_1)) = d_H(\varphi_0, \varphi_1) = d_H(id, \psi_1)$

$\geq c_+(\psi_1) + c_+(\psi_1^{-1})$

Aim $\quad c_+(\psi_1) \geq \max(S_1 - S_0)$

$c_+(\psi_1^{-1}) \geq -\min(S_1 - S_0)$

- H is quasi autonomous : $\exists\, x_{max}, x_{min} \in \mathbb{R}^{2n}$

s.t. $\quad \max H_t = H_t(x_{max}) \quad \forall t \in [0,1]$

$\min H_t = H_t(x_{min}) \quad \forall t \in [0,1]$

Indeed : $x_{\substack{max \\ min}} = y_{\substack{max \\ min}} + \frac{i}{2} \nabla S_0 (y_{\substack{max \\ min}})$ where

$y_{\substack{max \\ min}}$ are a maximizer and a minimizer of $S_1 - S_0$.

Lemma $x \in \mathrm{Fix}\, \psi_\tau$ for some $\tau > 0 \Rightarrow$ $x \in \mathrm{Fix}\, \psi_t$ $\forall t \in [0,1]$ and

$$A_{\psi_t}(x) = t(S_1 - S_0)(y)$$

where $x = y + \frac{i}{2} \nabla S_0(y)$ and y is a critical point of $S_1 - S_0$.

Conversely, for every critical point y of $S_1 - S_0$ the point x defined as above is a fixed point of ψ_t $\forall t \in [0,1]$.

proof

$\psi_\tau(x) = x \Rightarrow \varphi_\tau \circ \varphi_0^{-1}(x) = x$

$\Rightarrow z := \varphi_0^{-1}(x)$ satisfies $\varphi_\tau(z) = \varphi_0(z)$

$$i(z - \varphi_0(z)) = \nabla S_0\left(\frac{z + \varphi_0(z)}{2}\right) \qquad (1)$$

$$\shortparallel$$

$$i(z - \varphi_\tau(z)) = \nabla S_\tau\left(\frac{z + \varphi_0(z)}{2}\right) \qquad (2)$$

$$\nabla S_\tau = \nabla S_0 + \tau(\nabla S_1 - \nabla S_0)$$

$$\Rightarrow \quad y := \frac{z + \varphi_0(z)}{2} \in \operatorname{crit}(S_1 - S_0)$$

$$\Rightarrow \quad \nabla S_t(y) = \nabla S_0(y) \qquad \forall t \in [0,1]$$

$w = \varphi_t(z)$ is the unique solution of

$$i(z - w) = \nabla S_t\left(\frac{z+w}{2}\right)$$

Since $\nabla S_t(y) = \nabla S_0(y)$, $w := \varphi_0(z)$ is a solution of the above equation by (1).

Therefore $\varphi_t(z) = \varphi_0(z) \quad \forall t \in [0,1]$

$$\Rightarrow \quad \psi_t(x) = \varphi_t \circ \varphi_0^{-1}(x) = \varphi_t(z) =$$
$$= \varphi_0(z) = x \qquad \forall t \in [0,1]$$

Moreover :

$$\underset{\substack{\parallel \\ 2y - z}}{\varphi_0(z)} \overset{(1)}{=} z + i \nabla S_0(y)$$

$$\Rightarrow \quad z = y - \frac{i}{2} \nabla S_0(y)$$

$$\Rightarrow \quad x = \varphi_0(z) = y + \frac{i}{2} \nabla S_0(y) \quad \text{as claimed.}$$

$$A_{\psi_t}(x) = \int_0^t H_s(x)\,ds = \int_0^t H_s\left(y + \tfrac{i}{2}\nabla S_0(y)\right)ds$$

$$\overset{(*)}{=} \int_0^t (S_1 - S_0)(y)\,ds = t\,(S_1 - S_0)(y).$$

Conversely, if $y \in \mathrm{crit}(S_1 - S_0)$ then

$$\nabla S_t(y) = \nabla S_0(y) \quad \forall t \in [0,1]$$

By differentiating $(*)$ we find

$$0 = \nabla(S_1 - S_0)(y) = \left(\mathbb{1} + \tfrac{i}{2}\nabla^2 S_0(y) + t(\nabla^2 S_1(y) - \nabla^2 S_0(y))\right)^* .$$

$$\nabla H_t\left(y + \tfrac{i}{2}\nabla S_0(y)\right)$$

$\Rightarrow$ $x := y + \frac{i}{2}\nabla S_0(y)$ is a critical point of H_t $\forall t \in [0,1]$

$\Rightarrow$ $x \in \mathrm{Fix}\,\psi_t$ $\forall t \in [0,1]$ $\square$

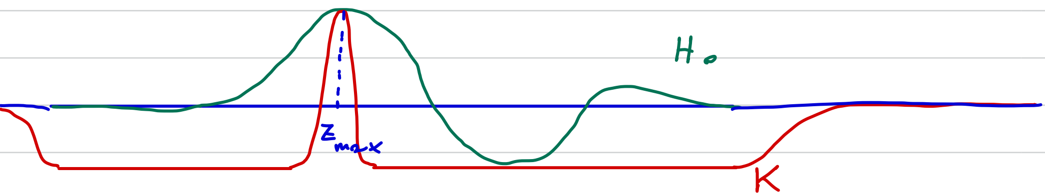
In particular:

$$\boxed{\sigma(\psi_t) = t \cdot \{\text{critical values of } S_1 - S_0\}}$$

Since the function $t \mapsto c^+(\psi_t)$ is continuous, the above fact implies that it is enough to prove the identity

$$c^+(\psi_t) = t \max(S_1 - S_0)$$

for t small. We may assume $\max(S_1 - S_0) > 0$ (otherwise $H \leq 0 \Rightarrow c^+(\psi_t) = 0 \ \forall t$)



$\bigcup_{t \in [0,1]} \operatorname{supp} H_t \subset B_R \subset \operatorname{supp} K$, $K \leq H$

$\max K = \max H_0 = \max(S_1 - S_0)$ achived only at z_{max}

$\nabla^2 K(z_{max}) < \nabla^2 H_0(z_{max})$

The critical values of K are just $\max(S_1 - S_0)$, 0 and $\min K < \min H_0$.

Since $H_t(z_{max}) = H_0(z_{max}) \ \forall t \in [0,1]$, there exists $\tau > 0$ s.t. $K \preccurlyeq H_t \quad \forall t \in [0, \tau]$

If $\varepsilon > 0$ is small enough then $X_{\varepsilon K}$ does not have non-constant 1-periodic orbits.

$\Rightarrow \quad \sigma(\phi^1_{\varepsilon K}) = \{ \varepsilon \min K, 0, \varepsilon \max(S_1 - S_0) \}$

We claim that $c^+(\phi^1_{\varepsilon K}) = \varepsilon \max(S_1 - S_0)$

Indeed, if $c^+(\phi^1_{\varepsilon K}) = 0$ then c^+ would vanish along the isotopy which is obtained by pushing the negative part to K up to zero, contradicting (ii).

If $\varepsilon \leqslant \tau$ then $\varepsilon K \preccurlyeq \varepsilon H_{\varepsilon t} \quad \forall t \in [0, 1]$

$$\Rightarrow \quad c_+(\psi_\varepsilon) = c_+(\phi^1_{\varepsilon H_{\varepsilon t}}) \geqslant c_+(\phi^1_{\varepsilon K})$$
$$= \varepsilon \max(S_1 - S_0)$$

The lower bound for $c_+(\psi_1^{-1})$ follows from this one by considering $\hat{H}_t = -H_t \circ \phi^t_H$. $\square$