1) **Review of intersection #'s**

$\underbrace{M^m, N^n}_{\text{compact, possibly w/ bdry}}, W^{m+n}$ oriented

$f: M \to W$, $g: N \to W$ smooth

$f(\partial M) \cap g(N) = f(M) \cap g(\partial N) = \emptyset$

If $x = f(p) = g(q)$ transverse int., i.e.

$$\operatorname{im} df(p) \oplus \operatorname{im} dg(q) = T_x W,$$

we set $\operatorname{sign}(f,p;g,q) := \pm 1$

($+1$, if induced orientation on LHS of (*) matches orientation of $T_x W$).

If $f \pitchfork g$: $\operatorname{int}(f,g) := \sum_{f(p)=g(q)} \operatorname{sign}(f,p;g,q)$

this implies, that intersection pts are discrete in $M \times N$.

<u>Lemma:</u> $f_t, g_t \quad (t \in [0,1])$, s.t.

$$f_t(\partial M) \cap g_t(N) = f_t(M) \cap g_t(\partial N) = \emptyset \quad (*)$$

and $f_0 \pitchfork g_0$, $f_1 \pitchfork g_1$, then

$$\operatorname{int}(f_0, g_0) = \operatorname{int}(f_1, g_1)$$

$\rightsquigarrow$ can define $\operatorname{int}(f,g) := \operatorname{int}(f', g')$,

(f, g not nec. $\pitchfork$)

where $f' \pitchfork g'$ and $f' \simeq f$, $g' \simeq g$

(via homotopies satisfying $(*)$)

If $f: M^n \to W^{2n}$, we set

$$\operatorname{int}(f) := \operatorname{int}(f, f).$$

Isolated intersections:

Suppose $f(p) = g(q)$ is an isol. int., i.e. we find closed disks D_p, D_q, s.t.

$$f(D_p \setminus \{p\}) \cap g(D_q \setminus \{q\}) = \emptyset$$

Define the local int. index

$$\iota(f, p; g, q) := \operatorname{int}(f|_{D_p}, g|_{D_q})$$

(does not depend on D_p, D_q !)

special case:

$M^2 \overset{i}{\hookrightarrow} W^4$ embedded,

$g: N^2 \to W$ has isol. int $g(q) = p \in M \subset W$.

in oriented charts ($q = 0 \in \mathbb{R}^2$, $p = 0 \in \mathbb{R}^4$,

$M = \mathbb{R}^2 \times \{0\}$):

$g(z) = (a(z), b(z))$, b has isol. zero at 0.

$$\rightsquigarrow \iota(\cancel{f}, i, p; g, q) = \operatorname{wind}(b|_{\partial D_\varepsilon})$$

2) Intersections of J-hol. maps

(M^{2n}, J) almost cpl. mfd has canonical orientation: $(v_1, Jv_1, \dots, v_n, Jv_n) > 0$

Let $u:(\Sigma, j) \to (M, J)$, $v:(\Sigma', j') \to (M, J)$ be J-hol, $u(z) = v(\zeta)$ transverse int., (a, ja), $(b, j'b)$ pos. bases for $T_z\Sigma$, $T_\zeta \Sigma'$. Then:

$$(du(a), \underbrace{du(ja)}_{= J du(a)}, dv(b), \underbrace{dv(j'b)}_{= J dv(b)}) > 0,$$

i.e. $\iota(u, z; v, \zeta) = +1$

Thm (local ~~positivity~~ positivity of int.)

u, v as above, $u(z) = v(\zeta)$

$\Rightarrow \exists\, \mathcal{U}_z, \mathcal{U}_\zeta$, s.t. either

$$u(\mathcal{U}_z) = v(\mathcal{U}_\zeta), \text{ or}$$

$$u(\mathcal{U}_z \setminus \{z\}) \cap v(\mathcal{U}_\zeta \setminus \{\zeta\}) = \emptyset \quad \text{and}$$

$$\iota(u, z; v, \zeta) \geq 1$$

$$(= 1 \iff \text{int. is } \pitchfork)$$

proof sketch (simple case):

Suppose $du(z) \neq 0$.

w.l.o.g. u is an embedding and

$\Sigma = \Sigma' = \mathbb{D}$, $z, \zeta = 0 \in \mathbb{D}$.

One can find coord. on M (identifying $u(z) = v(\zeta)$ with $0 \in \mathbb{R}^4$), s.t.

$u(z) = (z, 0)$ and $J(z, 0) = i$

cpl. mult. on $\mathbb{C}^2 \cong \mathbb{R}^4$!

In this coord $v(z) = (a(z), b(z))$, $b(0) = 0$.

Using $\partial_s v + J \partial_t v = 0$ one can show, that

$$\partial_s b + i \partial_t b + C \cdot b = 0,$$

where $C : B_r \longrightarrow \mathbb{R}^{2\times 2}$ $(B_r \subset \mathbb{D})$.

$\Longrightarrow$ $b(z) = \Phi(z)\, h(z)$ on some $B_{r'} \subset B_r$,

similarity principle

where $\Phi(z) \in \mathbb{C}^*$,

$\Phi(0) = 1$,

$\bar{\partial} h = 0$

With h being holom., it follows immediately, that the intersection is isolated.

write $h(z) = a_k z^k + \sum_{i>k} a_i z^i \qquad a_k \neq 0$, $(k \geq 1)$

then: $c(u, 0; v, 0) = \mathrm{wind}(b|_{\partial D_\varepsilon})$

(for some suff. small $\varepsilon < r'$)

$\underset{\substack{\Phi \text{ continuous,} \\ \Phi(0)=1}}{=} \mathrm{wind}(h|_{\partial D_\varepsilon}) = k$ $\square$

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3) Adjunction formula for closed curves

$u : \Sigma \to (M, J)$ simple, J-hol

$D(u) := \{(z, \zeta) : z \neq \zeta, u(z) = u(\zeta)\}$

$S(u) := \{z : du(z) = 0\}$

Thm: $v : (S, j) \to (W, J)$ simple, J-hol. (S, W not nec. compact)
Then the set of non-injective pts. is discrete.

($z \in S$ is injective, if $du(z) \neq 0$ and $u^{-1}(u(z)) = \{z\}$).

$\Longrightarrow$ $D(u)$, $S(u)$ both finite.

<u>Def</u> (Singularity index)

$$\delta(u) := \frac{1}{2} \sum_{(z,\zeta) \in D(u)} \iota(u,z;u,\zeta) + \sum_{z \in S(u)} \delta(u,z)$$

(Here $\delta(u,z) \geq 1$ is the algebraic count of self-intersections of an immersed perturbation of u (locally at z).
It is a hard theorem, to show that this is ≥ 1 !)

<u>Observation:</u> $\delta(u) \geq 0$ and
$\delta(u) = 0 \iff u$ embedding.

Thm (Adj. formula)

$$2\delta(u) = \mathrm{int}(u) + \chi(\Sigma) - c_1([u])$$

$$\left(= A \cdot A + \chi(\Sigma) - c_1(A), \quad A = u_*([\Sigma]) \right)$$

Rmk's:

- $c_1([u]) := \langle c_1(TM, J), u_*[\Sigma] \rangle$

$$= \langle c_1(u^*TM, J), [\Sigma] \rangle$$

$$= c_1(u^*TM)$$ (1st chern number of $\begin{pmatrix} (u^*TM, J) \\ \downarrow \\ \Sigma \end{pmatrix}$)

- If u immersed,

$\begin{matrix} N_u \\ \downarrow \\ \Sigma \end{matrix}$ normal bundle (w.r.t. some J-inv. metric)

$$\rightsquigarrow \quad (u^*TM, J) \cong (T\Sigma, j) \oplus (N_u, J)$$

$$\Longrightarrow \quad c_1([u]) = \underbrace{c_1(T\Sigma)}_{= \chi(\Sigma)} + c_1(N_u).$$

– If $\begin{matrix} L \\ \downarrow \\ \Sigma \end{matrix}$ is a cpl. line bundle,

then $c_1(L) = \mathrm{int}(s_0)$,

$s_0 : \Sigma \to L$ zero section

<u>pf (simple case)</u>:

$S(u) = \emptyset$, ~~inter~~ self-int. are transverse

and u at most $2:1$.

Then: $\delta(u) = \frac{1}{2} \sum_{D(u)} 1 = \frac{1}{2} \# D(u)$.

Take a generic section η of N_u,

non-vanishing at self-int. pts.,

$u_t := \exp_u(t\eta)$.

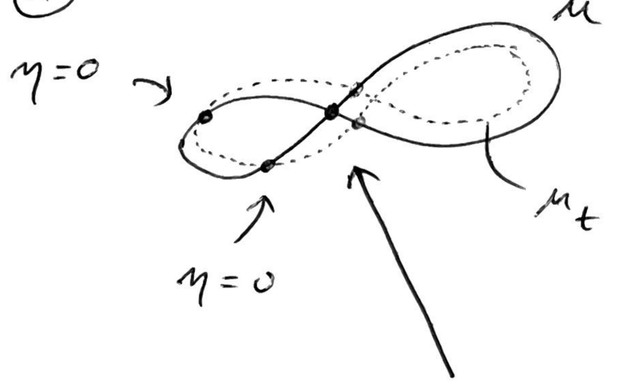
$\Rightarrow$ $\mathrm{int}(u) = \mathrm{int}(u, u_t)$

$\overset{(*)}{=}$ alg. count of zeros of η $+ \# D(u)$

$= c_1(N_u) + 2\delta(u)$

$= c_1([u]) - \chi(\Sigma) + 2\delta(u)$

⊛



For each $\cancel{(z,\varsigma)\in D(u)}$ $u(z)=u(\varsigma)$ we pick up two int. of u and u_t

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4) Relative adjunction formula

$u : \dot{\Sigma} \longrightarrow \mathbb{R}\times Y$ simple J-hol.,

representing $Z \in H_2(Y,\alpha,\beta)$

Thm (rel. adj. formula)

$$2\,\delta(u) = \underbrace{Q_\tau(u)}_{=\,Q_\tau(Z)} + \chi(\dot{\Sigma}) - \underbrace{c_1^\tau(u^*\xi)}_{=\,c_1^\tau(Z)} + w_\tau(u)$$

$$\big(w_\tau(u) \leq q = q(\alpha,\beta)\big).$$

Rmk: Not clear, that $\delta(u)$ makes sense, since self-int. and singular pts. might accumulate near punctures!

We rule this out in the next section.

4.1 Normal form near punctures

Analogy: $(M,g) \xrightarrow{f} \mathbb{R}$ Morse,

$$\dot{\alpha} = -\nabla f \circ \alpha \qquad \alpha(s) \xrightarrow{s\to\infty} p \in \operatorname{crit}(f)$$

$\Rightarrow$ $s \gg 0$: $\alpha(s) = \exp_p h(s)$

$h(s) \in T_p M$.

$$A_p := \nabla(\nabla f)(p) : T_p M \circlearrowleft$$

$\rightsquigarrow \exists!\ v \in T_p M$, $A_p v = \lambda v$ $(\lambda < 0)$,

s.t. $h(s) = e^{\lambda s}(v + \underset{\substack{\downarrow \\ 0}}{r(s)})$

(of course similar for $\alpha(s) \xrightarrow{s\to-\infty} q \in \operatorname{crit}(f)$)

<u>Now</u>: ~~Ass~~

$u = (a, f) : Z_+ := [0, \infty) \times S^1 \to \mathbb{R} \times Y$ J-hol.

(Think of u as a local model of $u : \dot{\Sigma} \to \mathbb{R} \times Y$ at some positive puncture)

$f(s, \cdot) \longrightarrow \gamma$, $\dot{\gamma} = T \cdot R \circ \gamma$

$\rightsquigarrow$ for $s >> 0$ u is close to the orbit cylinder ~~(...)~~, parameterized by $(s,t) \mapsto (Ts, \gamma(t)) =: \tilde{\gamma}(s,t)$.

$\Rightarrow$ up to reparameterizing Z_+:

$$u(s,t) = \exp_{\tilde{\gamma}} h(s,t) = (Ts, \exp_\gamma h(s,t))$$

$$h \in \Gamma(\tilde{\gamma}^* \xi) = \Gamma(N_{\tilde{\gamma}})$$

(w.r.t. metric $g = da^2 + \lambda^2 + d\lambda(\cdot, J\cdot)$)

(can equivalently think of $h \in \Gamma(\gamma^* \xi)$)

We can – very loosely – interpret $s \mapsto f(s, \cdot)$ as a gradient flow line of $\Phi_\lambda : C^\infty(S^1, Y) \to \mathbb{R}$, $\gamma \mapsto \int_\gamma \lambda$, whose critical pts are loops at γ tangent to R. The "Hessian" is the <u>asympt. operator</u>:

$$A_\gamma : \Gamma(\gamma^*\xi) \to \Gamma(\gamma^*\xi)$$

A_γ has discrete spectrum $\subset \mathbb{R}$ consisting of eigenvalues only and if v is a non-trivial eigenvector, then

$v(t) \neq 0 \quad \forall t$.

<u>Thm</u>: For $u : Z_+ \to \mathbb{R} \times Y$ J-hol, asympt. to γ : $\exists!\ v_\lambda \in \Gamma(\gamma^*\xi)$:

non-degenerate

$$A_\gamma v_\lambda = \lambda v_\lambda \quad (\lambda < 0), \text{ s.t.}$$

$$h(s,t) = e^{\lambda s}(v_\lambda(t) + r(s,t))$$

$$r(s,t) \xrightarrow[s \to \infty]{\text{uniformly, as}} 0$$

This can be seen as a special case of u approaching some ~~embedded~~ immersed J-hol curve, in this case $\tilde{\gamma}$.

Here is a generalization:

<u>Thm</u> $u, v : Z_+ \to \mathbb{R} \times Y$ J-hol., asympt. to non-deg γ, then either

$h_u - h_v \equiv 0$ (in which case u and v have to cover the same image), or

$$(h_u - h_v)(s,t) = e^{\lambda s}(v_\lambda(t) + r(s,t))$$
$$\downarrow$$
$$0$$

<u>Cor</u>: $u : \dot\Sigma \to \mathbb{R} \times Y$, $v : \dot\Sigma \to \mathbb{R} \times Y$ J-hol, asympt. cylindrical, w/ non-identical images

$\Rightarrow$ # intersections $< \infty$

<u>Cor</u>: $u : \dot\Sigma \to \mathbb{R} \times Y$ simple, J-hol, asympt. cylindrical

$\Rightarrow u$ is embedded near punctures.

Conclusion: $\delta(u)$ makes sense for simple, asympt. cyl., J-hol maps!

4.2 The relative self-int.

$u : \dot{\Sigma} \to \mathbb{R} \times Y$ J-hol, simple, ...

Fix τ, a unitary trivialization of ξ along each simple, non-deg. Reeb orbit.

For γ such an orbit, we write

$$\tau_\gamma : S^1 \times \mathbb{C} \longrightarrow \gamma^* \xi .$$

This induces trivializations of ~~all~~ ξ along all iterates of simple orbits:

$$\tau_{\gamma^k} : S^1 \times \mathbb{C} \to (\gamma^k)^* \xi$$

$$(t, w) \longmapsto \tau_\gamma(kt, w)$$

For each γ^k we also fix a preffered section:

$$\eta_\infty^{\gamma^k}(t) := \tau_{\gamma^k}(t, 1)$$

(Note: $\mathrm{wind}^\tau(\eta_\infty^{\gamma^k}) = 0$).

Define a perturbation u^τ of u:

Over each embedded end $Z_\pm \subset \dot{\Sigma}$ of u, asympt. to some γ^k, we set

$$u^{\tau,\varepsilon}(s,t) := \exp_u \varepsilon\eta(s,t),$$

where
$$\eta(s,t) \in (N_{u|_{Z_\pm}})_{(s,t)},$$
$$\eta(s,t) \xrightarrow{s\to\pm\infty} \eta_\infty^{\gamma^k}(t)$$

(we implicitly identify $N_{u|_{Z_\pm}}$ with $(\tilde{\gamma}^k)^* \xi$)

$\rightsquigarrow$ $$Q_\tau(u) := \mathrm{int}(u, u^\tau) := \mathrm{int}(u, u^{\tau,\varepsilon})$$

one shows, that for ε suff. small this does not depend on ε.

Rmk: - Q_τ only depends on the homotopy class of τ

- For $Z \in H_2(Y, \alpha, \beta)$ $Q_\tau(Z) := Q_\tau(u)$ is well-defined, where u represents Z.

4.3. Calculation of $Q_\tau(u)$ (simple case)

Suppose again $S(u) = \emptyset$, all self-int. $\pitchfork$ and u at most $2:1$.

Pick some section $\eta \in \Gamma(N_u)$
← normal bundle wrt J-inv. metric

with finitely many zeros,
non-vanishing near double pts and
$\eta(s,t) \to \eta_\infty^{\gamma^k}(t)$ on each end of u.

We set $u^\tau = \exp_u \varepsilon \eta$.

Contributions to $Q_\tau(u) = \mathrm{int}(u, u^\tau)$:

(i) alg. count of zeros of η

$$= c_1^\tau(N_u)$$

(ii) as in the closed case: ~~each double pt (z,ζ) yields two~~
each intersection point $u(z) = u(\zeta)$
gives rise to two positive, $\pitchfork$ intersec.
of u and u^τ $\rightsquigarrow 2\delta(u)$.

(iii) "intersections at ∞"

Call this number $\iota_\infty^\tau(u)$

(each multiply covered end and each pair of punctures with the same asympt. orbit may contribute to $\iota_\infty^\tau(u)$. It can be calculated via winding numbers of eigenvectors appearing in the formula for the asympt. representative. Thus it ~~relates~~ relates to the quantity $w_\tau(u)$ via $\iota_\infty^\tau(u) = -w_\tau(u)$)

Observe now that $c_1^\tau(N_u) + \chi(\dot\Sigma) = c_1^\tau(u^*\xi)$ due to the two splittings

$$(T\dot\Sigma, j) \oplus (N_u, J) \cong (u^*T(\mathbb{R}\times Y), J)$$

$$\cong \underbrace{(u^*\langle \partial_a, R\rangle, J)} \oplus (u^*\xi, J)$$

$$\begin{array}{c} \dot\Sigma \times \mathbb{C} \\ \downarrow \\ \dot\Sigma \end{array} \cong$$