J-holomorphic curves, currents, and transversality

Recap of last week We defined

$$ECC(Y, \lambda, \Gamma, J)$$

chain complex generated by finite sets $\{(\alpha_i, m_i)\}$, where α_i are distinct simple Reeb orbits, (Y, λ) contact 3-mfd, $\Gamma \in H_1(Y)$, $\sum_i m_i [\alpha_i] = \Gamma$, and boundary operator ∂ is defined by a mod 2 count of "currents" connecting generators.

Today

- What is a current?
- Moduli space of J-holomorphic curves and currents
- Transversality and index of ∂_J.

1. What is a current?

symplectitation-compatible acs

finite set of punctures

Def. A J-holomorphic map is a smooth map

$$u: (\mathring{S}, j) \longrightarrow (W := \mathbb{R} \times Y, J)$$

where S is a closed Riemann surface, $\mathring{S} = S \setminus \{v_1, \dots, v_n\}$, with $du \circ j = J \circ du$. For γ Reeb orbit of (Y, λ) we say that u has a positive end at γ if there is a puncture s.t. in a nbhd of the puncture there complex coordinates $(\sigma, \tau) \in [0, +\infty) \times \mathbb{R}/T\mathbb{Z}$ with $j \partial_\sigma = \partial_\tau$ and $\lim_{\sigma \to +\infty} \pi_{\mathbb{R}} u(\sigma, \tau) = +\infty$, $\lim_{\sigma \to +\infty} \pi_Y u(\sigma, \tau) = \gamma(\tau)$

uniformly in τ (negative ends are defined similarly).

<u>Def.</u> A J-holomorphic map is called <u>irreducible</u> if the domain is connected, and <u>somewhere injective</u> if it is an injective immersion at some point in its domain.

<u>Rmk.</u> A somewhere injective J-hol. map has a discrete set of points where it is not immersive and injective.

We assume wlog that a J-holomorphic map has <u>no</u> removable punctures.

<u>Thm.</u> Let $u: \dot{S} \to W$ be a non-constant J-hol. map with all positive and negative punctures on Reeb orbits. Then $\exists$ a compact Riemann surface (S', j') and a branched holomorphic map $\phi: S \to S'$ respecting punctures and a J-hol. somewhere injective map $v: \dot{S}' \to W$ s.t. $u = v \circ \phi$.

<u>Def.</u> A <u>J-holomorphic curve</u> is an equivalence class of J-maps

$$u: (\dot{S}, j) \to (W, J)$$

quotienting by biholomorphisms $S \to S'$ respecting punctures.

If u is somewhere injective we will abuse notation and write $\dot{S}$ instead

A <u>J-holomorphic current</u> from $\alpha = \{(\alpha_i, m_i)\}$ to $\beta = \{(\beta_j, n_j)\}$ is a set $\mathcal{C} := \{(C_k, d_k)\}$ where C_k are irreducible somewhere injective J-hol. curves in W, d_k are positive integers, C_k is positively asymptotic to some of the α_i and negatively asymptotic to some β_j and (possibly multiply covering α_i / β_j) s.t. $\sum_k d_k \cdot \underbrace{\# C_{k,i}}_{\text{pos. asymptotic to } \alpha_i} = m_i$ (similar for n_j)

Def. A holomorphic current is <u>somewhere injective</u> if $d_k = 1$ for all k, and an <u>embedding</u> if additionally each C_k is embedded and pairwise disjoint.

Want to relate currents to elements of relative homology.

Let $H_2(Y, \alpha, \beta)$ be 2-chains Σ in Y with boundary

$$\partial \Sigma = \sum m_i [\alpha_i] - \sum n_j [\beta_j]$$

modulo 3-cycles. This space is affine over $H_2(Y)$. Notice that a current $\mathcal{C}$ defines an element of $H_2(Y, \alpha, \beta)$.

<u>Idea:</u> $H^2(Y, \alpha, \beta)$ with real coefficients can be nicely topologized using deRham's theorem and looking at 2-forms. This defines a dual topology on $H_2(Y, \alpha, \beta)$.

2. Moduli spaces and Fredholm operators

Want to look at the space of J-hol. curves C in W which are somewhere injective and irreducible.

$E \to B$ Banach bundle

Fredholm

Thm. For a generic choice of J, there is a Fredholm operator $\bar\partial_J : \boxed{B \to E}$ whose linearization is transverse to the zero-section of E, whose preimage of $\mathbb{O}_E$ consists of all these curves and whose index is given by

we are in dim 4

$$\operatorname{Ind}(C) := -\underbrace{\chi(C)}_{\text{Euler characteristic of the domain}} + 2 \cdot c_\tau^1(C) + \sum_{i=1}^{k} CZ_\tau(\gamma_i^+) - \sum_{j=1}^{\ell} CZ_\tau(\gamma_j^-)$$

"trivialization of ξ at the ends"

Here $c_\tau^1(C)$ is the algebraic count of zeros of a generic section $\phi : C \to \xi|_C$ which is constant and non-vanishing near the punctures.