Introduction to ECH

always assume Y connected without further mentioning

1. Definition and the Weinstein conjecture

$Y = Y^3$ closed oriented 3-mfd, λ contact form on Y (i.e. $\lambda \wedge d\lambda > 0$ volume form), $\xi := \ker(\lambda)$ contact structure. The Reeb vector field R on Y is defined by

$$\begin{cases} d\lambda(R, \cdot) = 0, \\ \lambda(R) = 1. \end{cases}$$

A Reeb orbit is a closed orbit of R, i.e. a map

$$\gamma : \mathbb{R}/T\mathbb{Z} \longrightarrow Y, \qquad T > 0,$$

s.t. $\dot{\gamma} = R(\gamma)$.

Rmk. A Reeb orbit is either ~~embedded~~ simple or the m-fold cover, $m \geq 2$, of an embedded Reeb orbit. ∎

Def. A Reeb orbit γ is called non-degenerate if 1 is not an eigenvalue of the linearized first return map

$$P_\gamma : (\xi_{\gamma(0)}, d\lambda) \longrightarrow (\xi_{\gamma(0)}, d\lambda).$$

A non-degenerate γ is called elliptic if the eigenvalues of P_γ are on the unit circle, and hyperbolic otherwise.

Def. A contact form λ is called non-degenerate if all its Reeb orbits are non-degenerate. ← generic condition

Assume λ is non-degenerate and fix $\Gamma \in H_1(Y)$. We define a chain complex

acs on symplectization of Y

$$ECC(Y, \lambda, \Gamma, J)$$

$\mathbb{Z}$-coefficients are also possible

freely generated over $\mathbb{Z}/_2\mathbb{Z}$ by finite sets of pairs $\alpha := \{(\alpha_i, m_i)\}$ with:

- The α_i's are pairwise distinct embedded Reeb orbits;
- $m_i \in \mathbb{N}$, $\forall\, i = 1, .., N$;
- $\sum_i m_i [\alpha_i] = \Gamma$;
- $m_i = 1$ whenever α_i is hyperbolic.

Rmk. Why the last condition?

(i) Computation of Taubes Gromov invariant in case of mapping tori (see Section 2.6 of Hutchings' lecture notes) shows that hyperbolic orbits are counted only with weight 1.

(ii) Proof of $\partial^2 = 0$ (truncation procedure). ∎

proof of well-def. of ∂ and of $\partial^2 = 0$ in Barney's talk (6)

The boundary operator ∂ is defined as follows: let $(M := \mathbb{R} \times Y, d(e^s \cdot \lambda))$ be the symplectization of (Y, λ), and let J be a symplectization-admissible acs on M:

- J is $\mathbb{R}$-invariant,

- $J(\partial_s) = R$,
- $J|_\xi : \xi \to \xi$ rotating positively wrt $d\lambda$.

For $\alpha := \{(\alpha_i, m_i)\}$ and $\beta := \{(\beta_j, n_j)\}$ chain complex generator, the coefficient

$$\langle \partial\alpha, \beta \rangle \in \mathbb{Z}/2\mathbb{Z}$$

is a mod 2 count of J-holomorphic curves C (i.e. "$J \circ dC = dC \circ j$") in M (modulo $\mathbb{R}$-translations and "equivalence of currents") s.t.:

(i) $C \xrightarrow[s \to +\infty]{} \sum_i m_i \alpha_i$, $C \xrightarrow[s \to -\infty]{} \sum_j n_j \beta_j$ as currents (more details in Bas' talk (2) and Johanna's talk (5)).

(ii) C has ECH-index 1 (see Jonas' talk (4)).

Rmk. For generic choice of J, ECH-index 1 curves are embedded, except possibly for multiple covers of trivial cylinders $\mathbb{R} \times \{\gamma\}$, γ Reeb orbit. ■

The induced homology is denoted by

one usually writes $ECH_*(Y, \xi, \Gamma)$

$$ECH_*(Y, \lambda, \Gamma, J)$$

and is called embedded contact homology. ECH_* turns out to be independent of J and λ, a direct proof of this is however not available (because of some unsolved technical problems, see Section 5.5 of Hutchings' lecture notes). Currently, the only proof is via the following

Section 2.8 in Hutchings' lecture notes, "Taubes's proof of the Weinstein conjecture in dimension three"

Thm. (Taubes) There is a canonical isomorphism of relatively graded modules

Poincaré dual of Γ

$$ECH_*(Y,\lambda,\Gamma,J) \cong HM^{-*}(Y, s_\xi + PD(\Gamma))$$

statement correct if $s_\xi + PD(\Gamma)$ non torsion, otherwise should replace with $\widehat{HM}^{-*}$ "from version"

Seiberg-Witten Floer cohomology def. by Kronheimer-Mrowka

spin-c structure determined by ξ (set of spin-c structures can be seen as an affine space over $H^2(Y;\mathbb{Z})$)

Corollary ECH is a topological invariant (up to shifting Γ if one changes the contact structure).

Corollary (Weinstein conjecture in dimension 3) Every closed contact 3-mfd. (Y,λ) has a Reeb orbit.

proof.

Suppose (Y,λ) has no Reeb orbits. Then λ is non-degenerate and

empty set of Reeb orbits is a chain complex generator

$$ECH_*(Y,\lambda,\Gamma,J) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \Gamma = 0, \\ 0 & \text{otherwise}. \end{cases}$$

However

$$HM^*(Y, s_\xi + PD(\Gamma))$$

show that there is always such a Γ

is infinitely generated whenever $c_1(\xi) + 2\cdot PD(\Gamma) \in H^2(Y;\mathbb{Z})$

is torsion (see Kronheimer - Mrowka). ξ ∎

Rmk. The fact that ECH is infinitely generated does not imply the existence of ∞-many embedded Reeb orbits. Consider for instance the ellipsoid

$$\partial E(a,b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} = 1 \right\}$$

with contact form

$$\lambda := \frac{1}{2} \cdot \sum_{i=1}^{2} (x_i \cdot dy_i - y_i \cdot dx_i) \Big|_{TE(a,b)} .$$

Then, if $\frac{a}{b} \in \mathbb{R} \setminus \mathbb{Q}$, then there are exactly two simple Reeb orbits (explicit computation of ECH in Jan's talk (7)). ∎

Rmk. By making stronger use of Taubes's isomorphism (among other things) one can prove several refinements of the Weinstein conjecture in dimension 3:

- Cristofaro-Gardiner, Hutchings: $\exists$ always at least two embedded Reeb orbits. {essentially based on the Weyl law
- Cristofaro-Gardiner, Hryniewicz, Hutchings, Liu: i) characterization of contact 3-mfds with exactly two simple Reeb orbits ($Y \cong S^3$ or lens space, the Reeb flow has a global disk-like surface of section, its dynamics is described by a pseudo-rotation, ξ is universally

tight, ...), and ii) if $c_1(\xi)$ is torsion, then either 2 or ∞-many simple Reeb orbits. ■

2. Additional structures

A nice feature of ECH is that it has several additional structures. These are crucially used in the definition of ECH-capacities (and in the applications we will describe in the next section).

The ECH-contact invariant: ECH contains a canonical class which is defined as follows. If λ is non-degenerate, then the empty set $\emptyset$ of Reeb orbits is a generator of the chain complex

$$ECC(Y, \lambda, 0, J).$$

As J-holomorphic curves in $M = \mathbb{R} \times Y$ can be seen as "gradient flow lines of the symplectic action

$$\mathbb{A}(\alpha) := \sum_i m_i \cdot \underbrace{\int_{\alpha_i} \lambda}_{= T_i > 0 \text{ period of } \alpha_i}$$

Used to defined filtered ECH^L, $L \in \mathbb{R}$

we readily see that $\partial \emptyset = 0$, i.e. $\emptyset$ is a cycle.

One can show that the homology class of $\emptyset$ does not depend on J or λ, and thus $\emptyset$ defines a class

$$c(\xi) \in ECH_*(Y, \xi, 0)$$

called the ECH contact invariant.

explicit example for $\mathbb{T}^3$ in Simon's talk (8)

Rmk. $c(\xi)$ can distinguish some contact structures:

- If ξ is overtwisted, then $c(\xi) = 0$.
- $c(\xi) \neq 0$ if (Y, ξ) is strongly symplectically fillable.

The U-map Assume Y is connected. Then

$$U : ECH_*(Y, \xi, \Gamma) \longrightarrow ECH_{*-2}(Y, \xi, \Gamma)$$

is induced by a chain map defined by counting ECH-index 2 curves that pass through a base point $(0, z) \in M$.

for λ non-degenerate (for degenerate case approximation)

The ECH-spectrum Write $\xi = \ker(\lambda)$ and assume $c(\xi) \neq 0$.

We define a sequence of real numbers

$$0 = c_0(Y, \lambda) < c_1(Y, \lambda) \leq c_2(Y, \lambda) \leq \dots \leq +\infty$$

as follows:

are action selectors

defined on ECH^L as well

$$c_k(Y, \lambda) := \inf \{ L > 0 \mid \exists \eta \in ECH^L(Y, \lambda, 0) \text{ s.t. } U^k \eta = [\phi] \}.$$

Rmk. $c_k(Y, \lambda) < +\infty$ iff $c(\xi) \in \operatorname{Im}(U^k)$.

def. works for weak Liouville domains as well and for general case using monotonicity

Let now (X, ω) be a Liouville domain, i.e. an exact symplectic filling of a contact 3-mfd. (Y, λ). The ECH-capacities of (X, ω) are defined by

$$c_k(X, \omega) := c_k(Y, \lambda), \quad \forall k \in \mathbb{N}_0.$$

Rmk. If λ, λ' are contact form on Y s.t.

$$d\lambda = d\lambda' = \omega|_Y$$

then $c_k(Y, \lambda) = c_k(Y, \lambda')$, $\forall k \in \mathbb{N}_0$. In particular, ECH-capacities are well-defined.

Thm. (properties of ECH-capacities)

(i) Monotonicity: If (X, ω) embeds symplectically in (X', ω'), then $c_k(X, \omega) \leq c_k(X', \omega')$, $\forall k \in \mathbb{N}_0$.

(ii) Conformality: $c_k(X, r \cdot \omega) = |r| \cdot c_k(X, \omega)$, $\forall r \neq 0$.

Other properties can be found in Thm 1.3 in Hutchings's lecture notes and in the next Section

As an application we consider the following symplectic embedding problem: for $a, b > 0$ let

$$E(a,b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \,\middle/\, \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} \leq 1 \right\}$$

endowed with the standard symplectic form of $\mathbb{C}^2 \cong \mathbb{R}^4$.

To $E(a,b)$ we associate the sequence

$$N(a,b) = \{ \ell \cdot a + j \cdot b \mid \ell, j \in \mathbb{N}_0 \}$$

of all nonnegative integer linear combinations of a and b, arranged in nondecreasing order.

Thm. (McDuff) $\exists$ symplectic embedding $\operatorname{int}(E(a,b)) \to E(c,d)$ iff $N(a,b) \leq N(c,d)$, i.e. $N(a,b)_k \leq N(c,d)_k$, $\forall k \geq 0$.

To prove the "only if" part of the theorem one can use the monotonicity property of ECH-capacities and

(iii) <u>Ellipsoids</u>: $c_k(E(a,b)) = N(a,b)_k \, , \quad \forall k \in \mathbb{N}_0$.

<u>Rmk</u>. It is a non-trivial problem to determine whether

$$N(a,b) \leq N(c,d).$$

For an account see Thm. 1.2 in Hutchings's lecture notes.

3. The Weyl law and applications

<u>Thm</u>. (Cristofaro-Gardiner, Hutchings, Ramos) If (X,ω) is a Liouville domain with all ECH-capacities finite (e.g. a star-shaped domain in $\mathbb{R}^4$), then

in this case we will see a proof in Alberto's talk (9)

$$\lim_{k\to\infty} \frac{c_k(X,\omega)^2}{k} = 4 \cdot \mathrm{vol}(X,\omega).$$

As an application of the Weyl law we see here the following

<u>Thm</u>. (Irie) Let (Y,λ) be a closed contact 3-mfd. Then, the set

$$\left\{ f \in C^\infty(Y,\mathbb{R}_{>0}) \;\middle/\; \begin{array}{c}\text{Reeb orbits of } (Y, f\cdot\lambda) \\ \text{dense in } Y\end{array} \right\}$$

is residual in $C^\infty(Y,\mathbb{R}_{>0})$ wrt C^∞-topology.

We set $\mathcal{P}(Y,\lambda) := \{\gamma \text{ Reeb orbit of } (Y,\lambda)\}$. The key step of the proof is the following

Lemma (C^∞-closing lemma) Let $\mathcal{U} \subset Y$ be open and non-empty and let $\varepsilon > 0$. Then $\exists\ f \in C^\infty(Y)$ such that $d_{C^\infty}(f, 1) < \varepsilon$ and $\exists\ \gamma \in \mathcal{P}(Y, f \cdot \lambda)$ nondegenerate with $\gamma(\cdot) \cap \mathcal{U} \neq \emptyset$.

proof. under the assumption $c(\xi) \neq 0$ (in general one uses a generalization of the ECH-capacities defined for arbitrary $\sigma \in ECH_*(Y, \xi, \Gamma) \setminus \{0\}$ and of the Weyl law)

Assume $c(\xi) \neq 0$. Take $h \in C^\infty(Y, \mathbb{R}_{\geq 0})$ with $\operatorname{supp}(h) \subset \mathcal{U}$,

$$\|h\|_{C^\infty} := \sum_{\ell=0}^{+\infty} 2^{-\ell} \cdot \frac{\|h\|_{C^\ell}}{1 + \|h\|_{C^\ell}} < \varepsilon \quad \text{and} \quad h \not\equiv 0.$$

Clearly,

$$d(h \cdot \lambda) = dh \wedge \lambda + h \cdot d\lambda,$$

so that

$$(1+h)\lambda \wedge d((1+h)\lambda) = (1+h^2) \cdot \lambda \wedge d\lambda$$

and hence in particular

$$\operatorname{vol}(Y, (1+h)\lambda) > \operatorname{vol}(Y, \lambda).$$

Claim $\exists\ \bar{t} \in [0,1]$, $\gamma \in \mathcal{P}(Y, (1+\bar{t} \cdot h)\lambda)$ with $\gamma(\cdot) \cap \mathcal{U} \neq \emptyset$.

Once the claim is proved one takes

$$f := e^g (1 + \bar{t} \cdot h)$$

with suitable $g \in C^\infty(Y)$ to achieve nondegeneracy.

Assume by contradiction that $\forall\ t \in [0,1]$, $\forall\ \gamma \in \mathcal{P}(Y, (1+th)\lambda)$

$$\gamma(\cdot) \cap \mathcal{U} = \emptyset.$$

Then, $\mathcal{P}(Y,(1+th)\lambda) = \mathcal{P}(Y,\lambda)$, $\forall\, t \in [0,1]$, and hence in particular

$$\mathcal{A}(Y,(1+th)\lambda) = \mathcal{A}(Y,\lambda),$$

where $\mathcal{A}(Y,\lambda) := \{ T_\gamma \mid \gamma \in \mathcal{P}(Y,\lambda)\} \subset (0,+\infty)$ is the action spectrum. Therefore

$$c_k(Y,(1+th)\lambda) \in \mathcal{A}(Y,(1+th)\lambda)_+ = \mathcal{A}(Y,\lambda)_+,$$

$$\mathcal{A}(Y,(1+th)\lambda)_+ = \{\alpha_1 + \dots + \alpha_m \mid \alpha_i \in \mathcal{A},\ m \in \mathbb{N}_0\}$$

and since

continuity of ECH-spectrum

$$t \mapsto c_k(Y,(1+th)\lambda) \quad \text{continuous}$$

and $\mathcal{A}(Y,\lambda)_+$ is a closed zero measure set in $(0,+\infty)$, we deduce that

$$c_k(Y,(1+th)\lambda) = c_k(Y,\lambda) \qquad \forall\, k \in \mathbb{N}_0,\ \forall\, t \in [0,1].$$

$$\overset{\text{Weyl}}{\underset{\text{law}}{\Longrightarrow}} \quad \mathrm{vol}(Y,(1+th)\lambda) = \mathrm{vol}(Y,\lambda). \; \lightning \qquad \blacksquare$$

<u>proof.</u> (Irie's thm)

Take $\mathcal{U} \subset Y$ non-empty and let

$$\mathcal{F}_{\mathcal{U}} := \{ f \in C^\infty(Y,\mathbb{R}_{>0}) \mid \exists\, \gamma \in \mathcal{P}(Y, f\cdot\lambda) \text{ nondeg.} \text{ s.t. } \gamma(\cdot) \cap \mathcal{U} \neq \emptyset \}.$$

Clearly, $\mathcal{F}_{\mathcal{U}}$ is open in $C^\infty(Y,\mathbb{R}_{>0})$, and dense by the C^∞-closing lemma above. The claim follows taking a countable base $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ of open sets in Y and $\bigcap_{i\in\mathbb{N}} \mathcal{F}_{\mathcal{U}_i}$. $\blacksquare$