Ruhr-Universität Bochum

Faculty of Mathematics
Algebraic Combinatorics

Polytopal Realization of Multi-Associahedra

Thesis to obtain the academic degree of
Master of Science

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Bochum, the 23rd of October 2024

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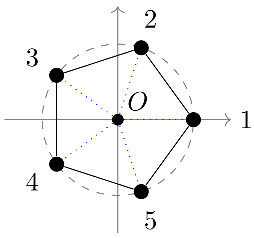
1 Introduction

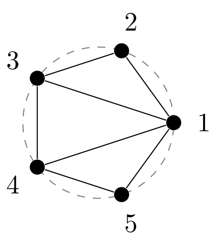
This thesis deals with the question of whether or not, respectively to what extent, a geometric realization (as a polytope) of a certain kind of structural object (the multi-associahedron) exists. The multi-associahedron is linked to the theory of finite reflection groups in which we will make a detour to elaborate this connection. We will mainly focus on two different techniques for geometric realization based on the two articles *Fan realizations of subword complexes and multi-associahedra using Gale duality* [BCL14] published in 2014 and *Realizations of multiassociahedra via rigidity* [RS22b] published in 2022. Comparing the techniques and their results will give us an overview about the difficulty of the topic and the progress towards solving this question, which is presented at the end of this thesis. First we will explain what the structural object in question shall be and, furthermore, what our understanding of a geometric realization is. This will be the content of the next two subsections.

1.1 Triangulations of the n-gon and Associahedra

Let $n \in \mathbb{N}$ be a positive integer and consider a set of n points in the plane. More precisely, we let these points lie equidistantly distributed on the unite circle and number them by $\{1, \ldots, n\}$, where i and $i + 1$, and n and 1 lie directly next to each other. By connecting consecutive points we obtain an object called the regular polygon with n vertices and edges. It has the property of being convex (thus every straight line between two points lies completely within the area bounded by the n points), equiangular (all angles of the form $\angle(i, O, i + 1)$ are equal in measure and O denotes the origin) and equilateral (the length of the sides are equal in measure). In short, we will call this the n-gon. For $n = 5$ we obtain the following picture:



Let us now triangulate the 5-gon, thus, draw straight, non-crossing lines between the points in such way, that the resulting areas are solely triangles. For example, by connecting the vertices 1 and 3, and 1 and 4, which we denote as the edges $[1, 3]$ and $[1, 4]$, we obtain the following triangulation:



In total, there are five different triangulations of the 5-gon, which are listed in Table 1.1.

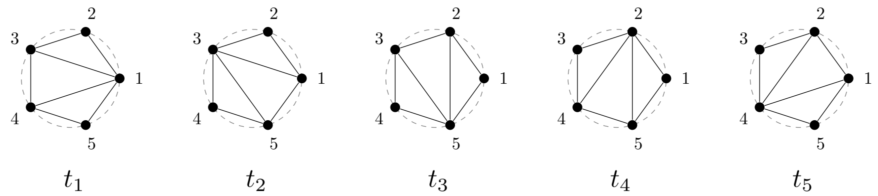
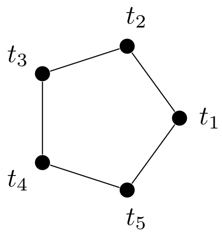


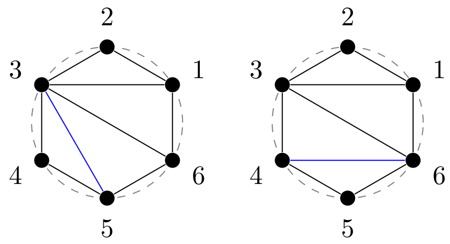
Table 1.1: The five triangulations of the 5-gon labeled by t_i. Remark, that we obtain the triangulation t_2 from the triangulation t_1 by 'flipping' the edge $[1, 4]$ to the edge $[3, 5]$. This continues to work between adjacent triangulations by flipping certain edges. In later sections we will go into further details about this procedure.

Using the flip of edges described in the caption of Table 1.1 we consider the following. Taking the triangulations as vertices and connecting two vertices whenever the two corresponding triangulations are connected by the flip of an edge, yields the following picture (after positioning the triangulations in a nice way)



where for example the triangulation t_1 and t_5 are connected since the first is obtained from the latter by flipping the edge $[1, 3]$ to the edge $[2, 4]$ and backwards. The resulting object is again a pentagon, which is not always the case as we will see in a moment. By identifying the vertices of our 'new' pentagon with points in $\mathbb{R}^2$ on the unite circle, in the same manner as we did before, we can talk about a geometric object which has the property of being polytopal (we will go into further details later and give the precise definition, for now imagine a polytope to be an object in $\mathbb{R}^n$ which is 'bounded' in some kind of sense and has flat surfaces). The structural object is the 'network' of different triangulations which are somehow connected by flips of edges and the geometric realization is the way we described just before. We call the 'new' pentagon the 2-dimensional associahedron.

Lets consider a slightly bigger example and look at triangulations of the 6-gon. There are in total 14 different triangulations, for example the following two which are, again, connected by the flip of an edge.



By doing the same geometric realization as we did for the triangulations of the pentagon, we obtain the 3-dimensional realization pictured in Figure 1.1 which is again a polytope. Remark, that these geometric realizations are not unique.

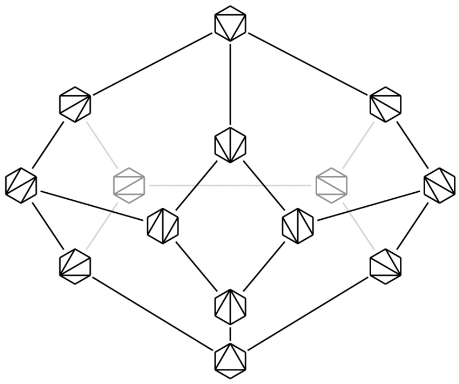


Figure 1.1: The 3-dimensional associahedron in $\mathbb{R}^3$. The picture is taken from [FZ03].

In the course of the thesis we will discuss the polytopality of the associahedron of an arbitrary dimension, thus, we will look at different versions of the next theorem. Remark, that this statement was proven some time ago. For more details concerning the history of the associahedron and its timeline we refer to [Ceb12], from which Theorem 1.1 is taken from aswell.

Theorem 1.1 ([Ceb12])**.** *The n-dim. associahedron* Asso(n) *corresponds to the triangulations of a convex $(n+3)$-gon, has $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$ many vertices and is a simple polytope.*

1.2 k-Triangulations and Multi-Associahedra

The realization of the associahedron as a polytope is well known for quiet some time. Our main goal is to look at the generalization of triangulations, which we will discuss now. Remember the rule for triangulating the n-gon: Drawing non-crossing edges by connecting the vertices of the n-gon. The obvious way to generalize this is to allow a certain number of crossings in the following manner: Consider k-many edges in the n-gon. We say that they form a k-crossing, if they are all pairwise crossing. Furthermore, we call a k-triangulation of the n-gon a maximal $(k+1)$-crossing free set of edges (thus, adding any additional edge would result in the existence of a $(k+1)$-crossing).

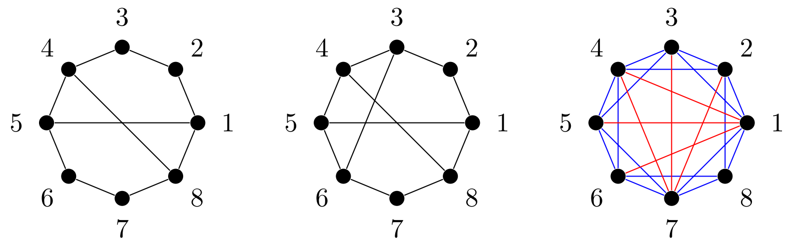
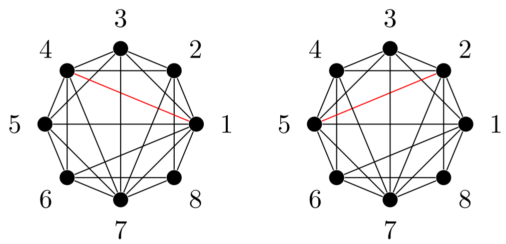


Table 1.2: An example of a 2-triangulation of the 8-gon. On the left we see the crossing of two edges. In the middle is an example of a 3-crossing. On the right is a maximal 3-crossing free set of edges, thus, a 2-triangulation of the 8-gon.

In Table 1.2 is an example for such a multi-triangulation. Remark, that there are certain edges which will never be part of a 3-crossing and are part of any 2-triangulation. We will go into further details about the classification of edges which are important (red) and not-important (blue) for a k-crossing in a later section of the thesis.

Now, staying with the example in Table 1.2, just like we were able to obtain the triangulations from one another by flipping an edge we can obtain a different 2-triangulation of the 8-gon by flipping an edge from the given one. Lets take the edge $[1, 4]$ and flip it to the edge $[2, 5]$. This yields another 2-triangulation, which is depicted below. Remark, that not every flip yields a 3-crossing free set of edges!



Again, we can take the different 2-triangulations of the 8-gon and connect them, whenever the differ by a flip. The question is, whether or not this is doable in such a manner and dimension, that the geometric realization is polytopal. For this particular example the answer is yes, although we can not give a picture since it is realized in $\mathbb{R}^7$. The multi-associahedron $\triangle(n, k)$ is the structure based on connecting k-triangulations of the n-gon by flipping edges. This will be the object of our interest in this thesis.

We will discuss the multi-associahedron after introducing polytopes and talking about Theorem 1.1 in Section 3.

2 Polytopes

First, we want to go into details about our main geometric object - polytopes and how to realize them. For our purpose we only talk about bounded, convex polytopes and will always (as it is consent in the literature) omit the words bounded and convex. Beginning with the main definitions and looking at some examples and important properties, our main goal is to give an understanding of what a polytope is and to show an example of how a structure like the associahedron can be realized as a polytope.

2.1 Definitions and Examples

In this subsection we will work very closely with the chapters one, two and seven of [Zie95]. Remark, that we do not want to explore the theory of polytopes in all its depth and refer for that to the cited reference. Let K be a finite subset of $\mathbb{R}^d$. Remember that the convex hull of K is defined as

$$\operatorname{conv}(K) = \bigcap_{\substack{K \subseteq K' \subseteq \mathbb{R}^d \\ K' \text{ convex}}} K'$$

and a subset K is called convex, if any line segment between arbitrary points of K is completely included in K. One can show that for $K = \{x_1, \dots, x_n\}$ the following equality holds, giving us a more useful and geometric interpretation.

$$\operatorname{conv}(K) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \,\middle|\, \lambda_i \geq 0 \,, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

We are now able to give the definition of one kind of polytope based on the convex hull of a set of points.

Definition 2.1. A **$\mathcal{V}$-polytope** is the convex hull of a finite set of points in some $\mathbb{R}^d$.

A polytope can thus be determined by a finite set of points. On the other hand the geometric intuition leads to another approach which uses halfspaces as boundaries.

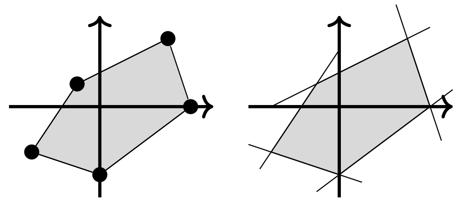
Definition 2.2. An **$\mathcal{H}$-polyhedron** is an intersection of finitely many closed halfspaces in some $\mathbb{R}^d$. Define an **$\mathcal{H}$-polytope** as a bounded $\mathcal{H}$-polyhedron meaning, that it does not contain any ray of the form $\{x + ty \,|\, t \geq 0\}$ for any vector $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d \setminus \{0\}$.

Remark 2.1. Saying, that a polyhedron P is an intersection of finitely many closed halfspaces in some $\mathbb{R}^d$ is the same as to say that there is a system of m inequalities that the points in P have to satisfy, thus, the polyhedron P can be presented in the form

$$P = P(A, z) = \{x \in \mathbb{R}^d \,|\, Ax \leq z\}$$

where $A \in \mathbb{R}^{m \times d}$ and $z \in \mathbb{R}^m$.

Example 2.1. The following pictures illustrate the definition of a $\mathcal{V}$-polytope and an $\mathcal{H}$-polytope



where

$$P = \operatorname{conv}\left(\left\{\begin{pmatrix}3/2\\3/2\end{pmatrix}, \begin{pmatrix}-1/2\\1/2\end{pmatrix}, \begin{pmatrix}-3/2\\-1\end{pmatrix}, \begin{pmatrix}0\\-3/2\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}\right\}\right)$$

respectively

$$P = P\left(A = \begin{pmatrix}-2 & 4\\-6 & 4\\-2 & -6\\3 & -4\\3 & 1\end{pmatrix}, z = \begin{pmatrix}3\\5\\9\\6\\6\end{pmatrix}\right).$$

Looking at the given example and considering the definitions it might become clear that the definition of a $\mathcal{V}$- and an $\mathcal{H}$-polytope define the same geometric object. In fact, they are equivalent.

Theorem 2.1. *A subset $P \subseteq \mathbb{R}^d$ is a $\mathcal{V}$-polytope $P = \operatorname{conv}(V)$ for some $V \subseteq \mathbb{R}^d$ if and only if it is an $\mathcal{H}$-polytope $P = P(A, z)$ for some $A \in \mathbb{R}^{m\times d}$, $z \in \mathbb{R}^m$.*

Definition 2.3. A **polytope** P is a set of points in $\mathbb{R}^d$ which can be presented either as a $\mathcal{V}$- or an $\mathcal{H}$-polytope.

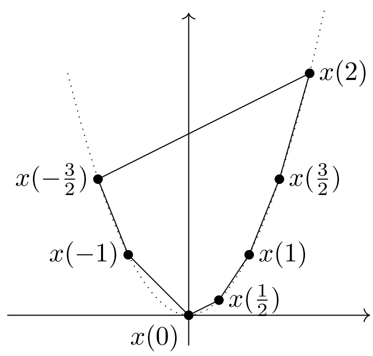
Example 2.2. The moment curve in $\mathbb{R}^d$ is defined by the map

$$x : \mathbb{R} \to \mathbb{R}^d,\ t \mapsto x(t) = (t, t^2, \ldots, t^d)^T.$$

We define the **cyclic polytope** as the convex hull

$$C_d(t_1, \ldots, t_n) = \operatorname{conv}\{x(t_1), \ldots, x(t_n)\}$$

of $n > d$ distinct points $x(t_i)$, with $t_1 < \cdots < t_n$, on the moment curve. The points $x(t_i)$ are the vertices of the polytope and the combinatorial equivalence class of the polytope does not depend on the specific choice of the t_i (this is more or less obvious from the fact that the t_i are always strictly increasing). Thus, we can use the notation $C_d(n)$ and speak about 'the' cyclic polytope. The cyclic polytope will appear again, when we talk about the more simple cases of realizing certain multi-associahedra as polytopes. We look at a quick example for the cyclic polytope $C_2(7)$



We will now introduce the necessary definitions to describe a polytope more clearly, thus, in terms of its vertices, edges, ridges and facets. This terminology will appear again in a later section of this thesis when we will talk about simplicial complexes.

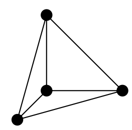
Definition 2.4. Let $P \subseteq \mathbb{R}^d$ be a polytope and consider the linear inequality $cx \leq c_0$, where $x \in \mathbb{R}^d$ and $c^T \in \mathbb{R}^d$. We call this inequality **valid** for P if it is satisfied for all points $x \in P$. Furthermore, we define a **face** of P to be any set of the form

$$F = P \cap \{x \in \mathbb{R}^d \mid cx = c_o\}$$

for a valid inequality $cx \leq c_0$ for P. The **dimension** of a face is the dimension of its affine hull.

For the valid inequalities $0x \leq 0$ and $0x \leq 1$ we can see that P itself, respectively the empty set, are faces of P. The faces of dimensions 0, 1, $\dim(P) - 2$ and $\dim(P) - 1$ are called **vertices**, **edges**, **ridges** and **facets**, respectively. We denote the vertex set by $\mathrm{vert}(P)$.

Example 2.3. Consider the 2-simplex in $\mathbb{R}^3$, i.e., the convex hull of the three standard basis vectors and 0. Its facets are the triangular shaped sides and the ridges are the lines connecting the vertices.



In the following we collect some simple and basic facts about faces of polytopes, which are good to know for understanding the nature of them.

Proposition 2.2. *Let $P \subset \mathbb{R}^d$ be a polytope and $V = \mathrm{vert}(P)$ be the set of vertices. Let F be a face of P.*

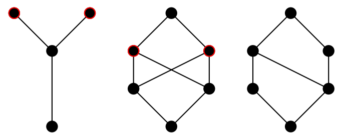
1. *The face F is a polytope with $\mathrm{vert}(F) = F \cap V$.*
2. *Every intersection of faces of P is a face of P.*
3. *The faces of F are exactly the faces of P that are contained in F.*
4. *$F = P \cap \mathrm{aff}(F)$.*

To be able to give a combinatorial description of polytopes we have to talk about *partially ordered sets* (posets). We only give the necessary terminology to understand the definition of a *face lattice* of a polytope and its properties.

Definition 2.5. Let S be a finite set with an reflexive, transitive and antisymmetric relation $\leq$. We then call $(S, \leq)$ a **poset**. We call a poset **bounded**, if it has an unique minimal and an unique maximal element, respectively. A poset is a **lattice**, if it is bounded, every two elements $x, y \in S$ have an unique minimal upper bound in S and every two elements $x, y \in S$ have an unique maximal lower bound in S.

There is a graphical representation of posets as graphs in the plane using *Hasse diagrams*: The vertices correspond to elements of S and there is an increasing path from x to y whenever $x \leq y$ holds.

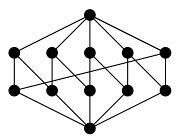
Example 2.4. The following examples show a not bounded poset (there is no unique maximal element), a bounded poset which is not a lattice (e.g. the two vertices on the second level do not have an unique minimal upper bound) and a lattice poset.



Definition 2.6. The **face lattice** of a polytope P is the poset $L = L(P)$ of all faces of P, partially ordered by inclusion.

Thanks to Proposition 2.2 we know that this definition makes sense. Even more, one can show that the face lattice is indeed a lattice poset, justifying the terminology. Furthermore, we can now call two polytopes **combinatorially equivalent**, whenever their face lattices are isomorphic.

Example 2.5. The following is the face lattice of the convex pentagon.



A common tool for describing polytopes, or creating them, are fans, which we will introduce now. Before doing so, we need the notion of a cone.

Definition 2.7. Let $Y = \{y_1, \dots, y_k\} \subseteq \mathbb{R}^d$ be an arbitrary finite set of points. We define its **conical hull** as the set

$$\text{cone}(Y) = \{\sum_{i=1}^{n} \lambda_i y_i \mid \lambda_i \geq 0\}.$$

More generally, a **cone** is a nonempty set of vectors that with every finite set of vectors also contains all their linear combinations with nonnegative coefficients.

Remark, that a cone always contains the origin by definition. Just like for polytopes there is a characterizing theorem for **polyhedral cones**.

Theorem 2.3. *A cone $C \subseteq \mathbb{R}^d$ is a finitely generated combination of vectors $C = \text{cone}(Y)$ for some $Y \in \mathbb{R}^{d \times n}$, if and only if it is a finite intersection of closed linear halfspaces $C = P(A, 0)$ for some $A \in \mathbb{R}^{m \times d}$.*

Remark 2.2. With the notation $C = P(A, 0)$ we do not mean that the cone is a polytope, since it is never bounded in the needed sense (all halfspaces run through the origin).

We are now able to give the notation of a fan. Since this term is based on cones and faces of cones we remark that the definition of a face of a polytope can be naturally extended to polyhedral cones.

Definition 2.8. A **fan** in $\mathbb{R}^d$ is a family $\mathcal{F} = \{C_1, \dots, C_n\}$ of nonempty polyhedral cones with the following two properties

1. Every nonempty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$.
2. The intersection of any two cones in $\mathcal{F}$ is a face of both.

Furthermore, we call $\mathcal{F}$

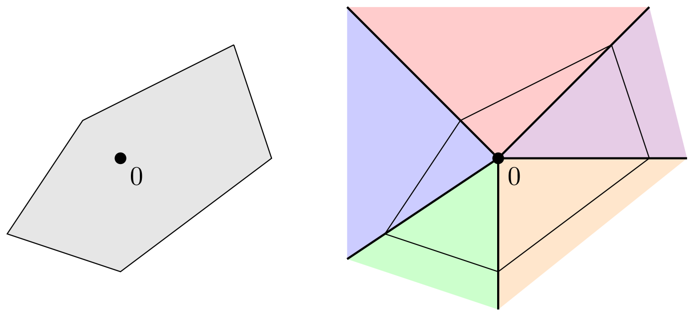
- **complete**, if $\bigcup \mathcal{F} = \bigcup_{i=1}^n C_i = \mathbb{R}^d$.
- **pointed**, if $\{0\} \in \mathcal{F}$.
- **simplicial**, if all cones are simplicial cones, i.e., cones spanned by linearly independent vectors.

By definition the simplicial cones are automatically pointed.

Example 2.6. Let P be a polytope in $\mathbb{R}^d$ with $0 \in \operatorname{relint}(P)$. Define the **face fan** of P as the set of all cones spanned by proper faces of P, thus

$$\mathcal{F}(P) = \{\operatorname{cone}(F) \mid F \in \operatorname{L}(P) \setminus P\}.$$

$\mathcal{F}$ is a pointed fan in $\operatorname{lin}(P)$, i.e., its union is the linear hull $\operatorname{lin}(P)$. It is a complete fan in $\mathbb{R}^d$, if P is a d-polytope with $0 \in \operatorname{int}(P)$.



Above is on the left side a polytope in $\mathbb{R}^2$ and on the right side its face fan. The cones corresponding to the facets of the polytope are colored and the cone corresponding to the vertices, called **rays**, are the thick lines starting at the origin.

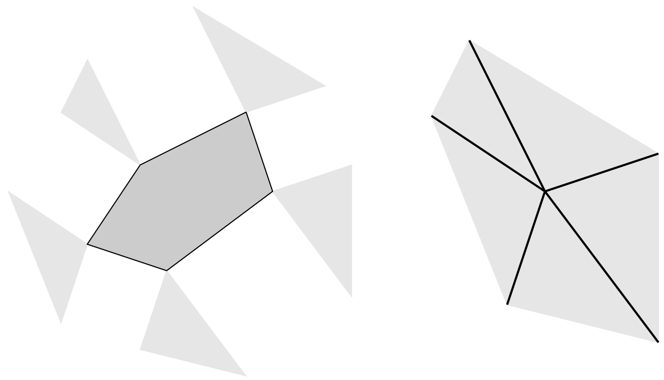
Example 2.7. Denote by $(\mathbb{R}^d)^*$ the dual space of $\mathbb{R}^d$ and let P be a nonempty polytope in $\mathbb{R}^d$. For the **normal fan** of P take the cones of those linear functions, which are maximal on a fixed face of P. That is, for every nonempty face F of P we define

$$N_F = \left\{ c \in (\mathbb{R}^d)^* \,\middle|\, F \subseteq \{x \in P \mid cx = \max_{y \in P} cy\} \right\}$$

and furthermore

$$\mathcal{N}(P) = \{N_F \mid F \in L(P) \setminus \emptyset\}.$$

$\mathcal{N}(P)$ is a complete fan in $(\mathbb{R}^d)^*$. If P is d-dimensional, then the fan is pointed, since then $\{0\} = N_P$ is in the fan.



On the left is a polytope and one the right its normal fan. For this, we have identified $\mathbb{R}^2$ with $(\mathbb{R}^2)^*$ via the standard euclidean scalar product, which accounts for the right angles in the figure.

2.2 Many Associahedra

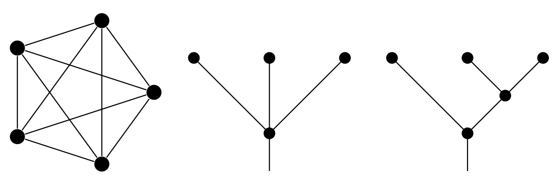
Our plan for this subsection is to present two realizations of the associahedron as polytopes. We will not prove any of this and refer to the corresponding articles and books, respectively, as the goal is to show that the formulation **the** realization of the associahedron is far from being right.

In the following we will present a realization of the *Stasheff polytope* by *Loday* using *planar binary trees*. The results are summarized from [Lod02].

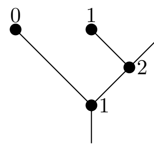
Definition 2.9. An **undirected graph** $G = (V, E)$ consists of a set of vertices V and a set of edges E, where an edge $e = \{v, w\} \in E$ connects two vertices $v, w \in V$. A **path** in G is a sequence of edges joining vertices, in which all vertices and edges are distinct. We call G a **tree**, if any two vertices in V can be connected by an unique path. A vertex which is connected to only one other vertex is called a **leave**. A tree in which every vertex is connected to at most two other vertices is called **binary**.

We will visualize graphs in the plane by identifying the vertices with points in $\mathbb{R}^2$ and drawing lines between two points if the corresponding vertices of the graph are connected by an edge.

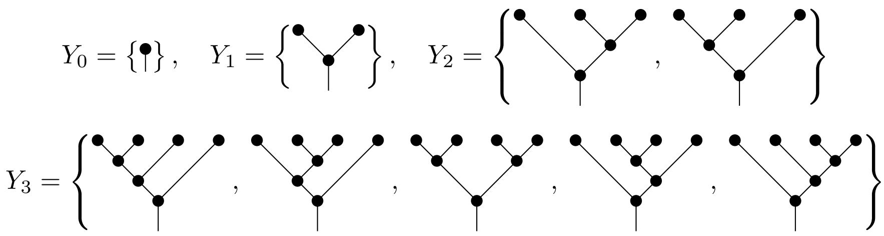
Example 2.8. A graph (called the *complete graph* on five vertices) which is not a tree, a tree that is not binary and a binary tree.



As a convention we will depict trees as in the previous example, thus, we arrange the vertices in such way that the leaves are at the top of the picture. Considering binary trees we give a labelling of the leaves and the internal vertices (the non-leaves) as follows: Label the leaves from left to right by $0, 1, 2, \dots$. Then label the internal vertices by $1, 2, \dots,$ where the ith vertex is the one inbetween the leaves $i-1$ and i. This looks like this:



Now consider the set Y_n of binary trees with $n+1$ leaves and n internal vertices. For $n = 0, 1, 2, 3$ they look like this:



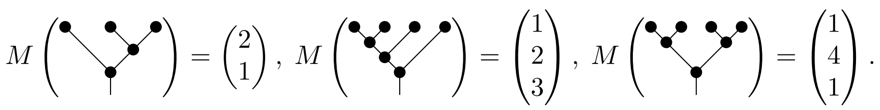
To a binary tree $t \in Y_n$ we associate the vector $M(t) \in \mathbb{R}^n$ given by

$$M(t) = (a_1 b_1, \dots, a_i b_i, \dots, a_n b_n)^T$$

where

a_i = number of leaves on the left side of the ith internal vertex, and
b_i = number of leaves on the right side of the ith internal vertex.

These vectors are very easy to compute, for example



Theorem 2.4. *The n-dimensional associahedron is the convex hull of the vectors given by binary trees in Y_{n+1}, thus,*

$$\mathrm{Asso}(n) = \mathrm{conv}\{M(t) \mid t \in Y_{n+1}\}.$$

The Loday realization is a special case of the realization by *Postnikov*. Let $e_1, \dots, e_{n+1} \in \mathbb{R}^{n+1}$ be the standard basis vectors and define

$$\Delta_{[i,\dots,j]} = \mathrm{conv}\{e_i, \dots, e_j\} \subset \mathbb{R}^{n+1}.$$

For any positive vector $\mathbf{a} = \{a_{ij} \in \mathbb{R}^{>0} \mid 1 \le i \le j \le n+1\}$ the **Postnikov associahedron** is the set

$$\mathrm{Post}_n(\mathbf{a}) = \sum_{1 \le i \le j \le n+1} a_{ij} \Delta_{[i,\dots,j]}.$$

Proposition 2.5 (Section 8.2 in [Pos05]). $\mathrm{Post}_n(\boldsymbol{a})$ *is an n-dimensional associahedron. In particular, for* $a_{ij} = 1$ *this yields the realization of Loday.*

There are more realizations of the associahedron apart from the ones shown here and not all of them are combinatorially equivalent. Furthermore, there are different techniques of showing that the given constructions indeed form a polytope and realize the associahedron, such that there is no simple answer to the question of how we can realize a structure like the multi-associahedron as a polytope. We will discuss two approaches concerning the multi-associahedron at a later point of the thesis.

3 k-Triangulations

This section offers a more detailed discussion about multi-triangulations, which may have first appeared in [CP01] in 1992. Pilaud and Santos ([PS08]) found a very beautiful and satisfying way of exploring their structure using *k-stars* in 2008. Their definitions, results and methods are the content of the first three subsections.

3.1 Introducing the Star

We will begin by giving a proper definition of a multi-triangulation, which we already saw in the introduction. For this, we need the notion of a complete graph and the crossing of edges in it.

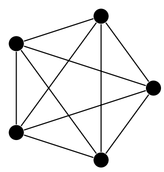
Definition 3.1. Consider n-points on the unit circle which we call vertices of the n-gon. By connecting each pair of vertices by an edge we obtain the **complete graph** in n points, denoted by K_n. If v_1 and v_2 are two vertices, we denote the edge connecting them by $[v_1, v_2]$. If v_1, v_2, v_3 and v_4 are distinct vertices lying in counterclockwise order on the unit circle, we say that the edges $[v_1, v_3]$ and $[v_2, v_4]$ cross.

In Example 3.1 is the complete graph K_5 pictured.

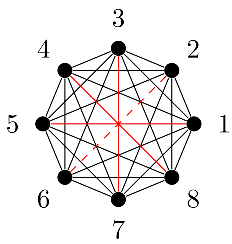
Definition 3.2. A **k-triangulation** of the n-gon is a maximal set of edges of the complete graph K_n, such that no $k+1$ many of them mutually cross.

Next to the explicit example we gave in the first section, there are the trivial cases considering certain pairs of integers (n, k).

Example 3.1. In the case $k = 1$ the 1-triangulations are the normal triangulations of the n-gon. If $n \leq 2k+1$ the graph K_n does not contain $k+1$ mutually intersecting edges. Thus, K_n is the unique k-triangulation. For the case where $n = 2k+2$ there are $k+1$ many k-triangulations obtained from K_n by deleting one of the diagonals $[i, i+k]$.



It is not possible to have a 3-crossing in the pentagon. Hence, K_5 is the unique 2-triangulation of the 5-gon.



By choosing three of the four diagonals (in red) of the 8-gon and adding all other 'non diagonal' edges we obtain a 3-triangulation.

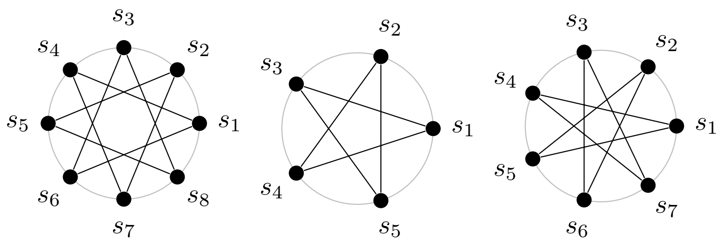
We will now go into more details about how k-triangulations are composed and especially, which edges play a role, which do not and how we differentiate between them. The following definition builds the foundation for the object of our interest in this section.

Definition 3.3. Let $p, q \in \mathbb{Z}$ be coprime. A **star polygon** of type $\{\frac{p}{q}\}$ is a polygon of the following form: Given a set $V = \{s_j \,|\, j \in \mathbb{Z}_p\}$ of p points on the unit circle, connect them with the set $E = \{[s_j, s_{j+q}] \,|\, j \in \mathbb{Z}_p\}$ of edges of length q.

Since triangles are used to decompose the n-gon we will introduce the following generalization for k-triangulations. In fact, we will see with much satisfaction that this seems to be the key element for understanding k-triangulations.

Definition 3.4. A k**-star** is a star polygon of type $\{\frac{2k+1}{k}\}$.

Example 3.2. The following polygons are a star polygon of type $\{\frac{8}{3}\}$, a 2-star and a 3-star.



The next theorem will summarize the most important results of this section. It is important to understand the nature of flipping edges, which we already mentioned in the introduction, and the role k-stars play in k-triangulations.

Theorem 3.1. *Let T be a k-triangulation of the n-gon with $n \geq 2k+1$. Then the following properties hold:*

1. *T contains exactly $n - 2k$ many k-stars.*
2. *Each edge of T belongs to zero, one or two k-stars, depending on whether its length is smaller, equal or greater than k.*
3. *Any common edge f of two k-stars R and S of T can be flipped to another edge e, so that $T \triangle \{e, f\}$ is a k-triangulation. Moreover, the edges e and f depend only on $R \cup S$, not the rest of T.*

To prove these statements we need some further notation and definitions. From now on we ignore the trivial cases and let $k \geq 1$ and $n \geq 2k+1$. Denote V_n to be the set of vertices of the convex n-gon.

Definition 3.5. For $u, v, w \in V_n$ we denote $u \prec v \prec w$ if they appear in counterclockwise order on the circle. We define the **cyclic interval** as the set $\mathtt{J}u, v\mathtt{K} = \{w \in V_n \,|\, u \preceq w \preceq v\}$, and similarly the other intervals $\mathtt{K}u, v\mathtt{J}$, $\mathtt{J}u, v\mathtt{J}$ and $\mathtt{K}u, v\mathtt{K}$.

Now that we have a notion for an order of vertices on the circle we can give an explicit definition for k-stars in a given set of vertices.

Definition 3.6. Let $s_0 \prec s_1 \prec \cdots \prec s_{2k} \prec s_0$ be vertices of V_n. We then define the corresponding k-**star** S to be the set of edges $\{[s_j, s_{j+k}] \mid j \in \mathbb{Z}_{2k+1}\}$. We can cyclically label the vertices of S in

1. **circle order** in cyclic order around the circle, or
2. **star order** by tracing the edges of S.

In Table 3.1 is an illustration of the two different labels of a 2-star. The next definition introduces the notation to differentiate between edges that play a crucial role in a k-triangulation and those who do not. Take a look at the 3-triangulation of the 8-gon in Example 3.1. For an edge $[e, f]$ to be able to cross pairwise with three other edges, there must be at least three vertices 'between' e and f. Every other edge that does not satisfy this condition will never be able to form a 4-crossing and is thus part of every 3-triangulation.

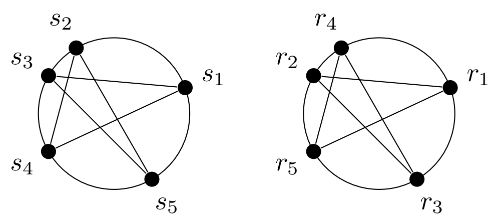


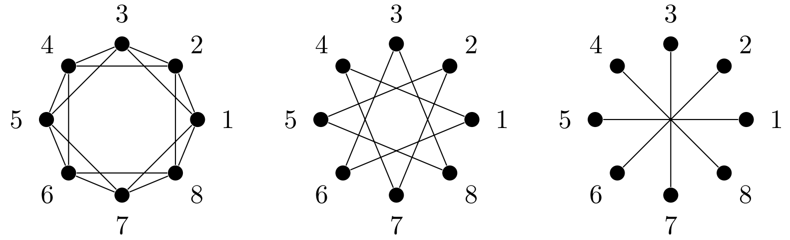
Table 3.1: The first is the circle order of the vertices of a 2-star and the latter depicts the star order.

Definition 3.7. For $u, v \in V_n$, let $[u, v]$ denote the edge connecting the vertices $u \neq v$. Define its length $|u - v|$ to be the minimal number of vertices inbetween them including the start vertex, thus, $|u - v| = \min\{|\llbracket u, v \rrbracket|, |\llbracket v, u \rrbracket|\}$. We call an edge $[u, v]$

- k-**relevant**, if $|u - v| > k$.
- k-**boundary**, if $|u - v| = k$.
- k-**irrelevant**, if $|u - v| < k$.

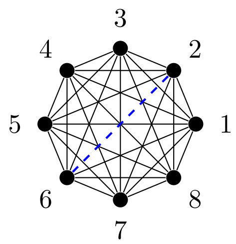
Remark 3.1. Although the k-boundary edges are only 'almost' relevant and do not contribute to a $(k + 1)$-crossing they are very important for the theory, especially for k-stars.

Example 3.3. Lets consider 3-triangulations of the 8-gon. We can distinguish between the 3-irrelevant, 3-boundary and 3-relevant edges of K_8 and as we can see only the 3-relevant edges form a 4-crossing.



Definition 3.8. Let $E \subseteq K_n$ be a subgraph. A pair of edges $\{[u, v], [v, w]\}$ in E, where $u \prec v \prec w$, is called an **angle** $\angle(u, v, w)$, if for each vertex t inbetween u an w the edge $[v, t]$ is not in E. For an angle $\angle(u, v, w)$ and a vertex t inbetween u and w we call the edge $[v, t]$ a **bisector** of $\angle(u, v, w)$. An angle is k-**relevant**, if its edges are either k-relevant or k-boundary edges.

Example 3.4. The edges $[3, 6]$ and $[3, 7]$ form a 3-relevant angle, whereas the edges $[2, 7]$ and $[2, 8]$ do not form a 3-relevant angle, but an angle (since $[2, 8]$ is a 3-irrelevant edge). The edge $[2, 6]$ (which is not in the 3-triangulation) is a bisector of the angle $\angle(3, 6, 1)$.



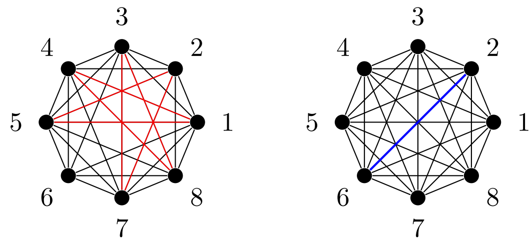
3.2 The Art of k-Stars

Now that we have the necessary definitions we begin to study the mutual positions of k-stars in k-triangulations.

Lemma 3.2. *Let $E \subseteq K_n$ be $(k+1)$-crossing free and let R and S denote two k-stars of E.*

1. *Any angle of S is also an angle of E and is k-relevant.*
2. *For any vertex t not in S there is an unique angle $\angle(u, v, w)$ in S that is bisected by $[v, t]$.*

In the figure below is our 3-triangulation of the 8-gon and a 3-star contained in it. As we can see, every angle of the star is an angle of the 3-triangulation and the edge $[2, 6]$ is the bisector of the unique angle $\angle(5, 2, 7)$.



Proof. Let $V = \{s_j \mid j \in \mathbb{Z}_{2k+1}\}$ be the set of vertices of S in star order and suppose that E contains an edge $[s_j, t]$ where $s_j \in V$ and $t \in \rrbracket s_{j+1}, s_{j-1} \llbracket$. Then the following set forms a $(k+1)$-crossing:

$$\{[s_{j+1}, s_{j+2}], [s_{j+3}, s_{j+4}], \ldots, [s_{j-2}, s_{j-1}], [s_j, t]\}.$$

To be more precise, the k edges of the form $[s_m, s_{m+1}]$ always form a k-crossing in a k-star (for visualization look at Example 3.2 and Table 3.1) and the edge $[s_j, t]$

crosses every one of those by definition. Thus, every angle of S must be an angle of E. Since any edge of S separates the other vertices of S into two parts of size k and $k-1$ it is at least k-boundary and thus the angle $\angle(s_{j-1}, s_j, s_{j+1})$ is k-relevant.
The second part of the lemma is obvious by definition of bisector and k-stars (every vertex in between two vertices of S bisects an angle formed on the 'opposite side' of this vertex). □

Corollary 3.3. *The two k-stars R and S in Lemma 3.2 can not share any angle.*

Proof. The previous lemma induces that the knowledge of an arbitrary angle $\angle(s_{j-1}, s_j, s_{j+1})$ of S permits the recovery of all the k-star, since the vertex s_{j+2} is the unique vertex such that $\angle(s_j, s_{j+1}, s_{j+2})$ is an angle of E (the first possible angle). □

Remark 3.2. Since R and S have $2k+1$ edges, they can not share more than k edges.

Corollary 3.4. *For any edge $[u, v]$ of E, the number of vertices of S between u and v and the number of vertices between v and u are different.*

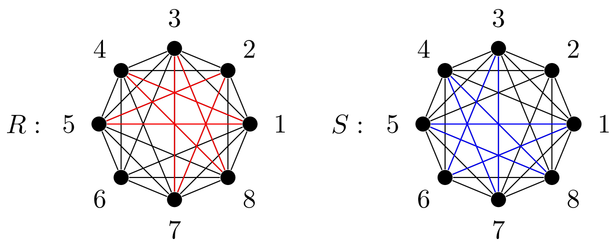
Proof. Suppose that the number of vertices on both sides is the same. Since S has $2k+1$ vertices, one of the vertices of $[u, v]$ is a vertex of S, lets say it is u. But then $[u, v]$ is either an edge in S, which can not be, or $[u, v]$ is a bisector of the angle of u, thus $[u, v]$ can not be in E. □

The previous corollary makes the following definition valid, which we will use from now on.

Definition 3.9. Let $[u, v]$ be an edge of E. We say that S lies on the **positive side** of the oriented edge from u to v, if there are more vertices of S in $\rrbracket v, u\llbracket$ than in $\rrbracket u, v\llbracket$. Otherwise we say that S lies on the **negative side**. Furthermore, we say that S is **contained in an angle** $\angle(u, v, w)$ of E, if it lies on the positive side of both edges $[u, v]$ and $[v, w]$ oriented in their order of appearance.

Remark 3.3. We will omit the addition of the orientation and assume, that the orientation is defined by the order of appearance of the vertices in the notation of the edge and angle, respectively.

Example 3.5. Look at the following two 3-stars in our 3-triangulation of the 8-gon.



They both share the edges $[1, 5]$, $[3, 7]$ and $[4, 8]$, but no angle (in fact, the angles of R at the shared edges are always on the opposite side than the angles of S). Furthermore, we can observe that R lies on the positive side of the edge $[3, 7]$, whereas S lies on its negative side. The angle $\angle(3, 6, 1)$ contains R.

Lemma 3.5. *Let $\angle(u, v, w)$ be an angle of E containing the k-star S. Then either v is a vertex of S and $\angle(u, v, w)$ is an angle of S, or v is not a vertex of S and $\angle(u, v, w)$ has a common bisector with an angle of S.*

Proof. Let v be a vertex of S and let $\angle(x, v, y)$ be the angle of S at v. Since $\angle(u, v, w)$ contains S, we have $w \preceq y \prec x \preceq u$. Then we can conclude that $x = u$ and $y = w$, for $\angle(x, v, y)$ is an angle of S and $\angle(u, v, w)$ is an angle of E.

Now suppose that v is not a vertex in S. Using Lemma 3.2 we have an unique angle $\angle(x, y, z)$ of S containing v. We want to show, that $\angle(u, v, w)$ contains y, such that $[v, y]$ is a common bisector. Assuming that $y \in \mathtt{K}u, v\mathtt{J}$ we know that $\mathtt{K}u, v\mathtt{J}$ contains all the $k + 1$ vertices of S between y and z (because each edge of a k-star divides by definition its vertices in sets of $k - 1$ and k vertices and $z \in \mathtt{K}v, y\mathtt{J}$). Since S lies on the positive side of $[u, v]$, this can not be. The same follows for the case $y \in \mathtt{K}v, w\mathtt{J}$. If $y = u$ or $y = w$, then $[u, v]$ or $[v, w]$ is a bisector of $\angle(x, y, z)$, which can not be either, since it is an angle (of S). □

Theorem 3.6. *Every pair of k-stars whose union is $(k + 1)$-crossing free have an unique common bisector.*

Proof. Let R and S be two k-stars whose union is $(k + 1)$-crossing free and vertices r_j and s_j in star order. It is easy to see from the definitions, that if S lies on the negative side of the edge $[r_{j-1}, r_j]$, then it lies on the positive side of the edge $[r_j, r_{j+1}]$ (all vertices in $\mathtt{J}r_{j-1}, r_j\mathtt{K}$ also lie in $\mathtt{K}r_j, r_{j+1}\mathtt{J}$). But since the number of vertices of R is odd, there is an index $j \in \mathbb{Z}_{2k+1}$ such that S lies on the positive side of $[r_{j-1}, r_j]$ and $[r_j, r_{j+1}]$, thus, in the angle $\angle(r_{j-1}, r_j, r_{j+1})$. Using Lemma 3.5, we have found a common bisector of R and S.

To prove the uniqueness, suppose that e and f are two common bisectors of R and S and label the vertices so that $e = [r_0, s_0]$. Let $f = [r_a, s_b]$ for some $a, b \in \mathbb{Z}_{2k+1} \setminus \{0\}$. Note, that a and b have the same parity, thus, by symmetry we can assume that $a = 2\alpha$ and $b = 2\beta$ for $1 \leq \beta \leq \alpha \leq k$. But then the set

$$\{[r_{2i}, r_{2i+1}] \,|\, 0 \leq i \leq \alpha - 1\} \cup \{[s_{2j}, s_{2j+1}] \,|\, \beta \leq j \leq k\}$$

forms a $(k+1+\alpha-\beta)$-crossing which contradicts the assumption that the union of R and S is $(k + 1)$-crossing free. □

The common bisector of the 3-stars R and S in Example 3.5 is the edge $[2, 6]$. The next Lemma plays an important role in understanding how k-triangulations are influenced by the flipping of particular edges.

Lemma 3.7. *Let f be a common edge of R and S and e their common bisector. Then $E \triangle \{e, f\}$ is a $(k + 1)$-crossing free subset of E, where $\triangle$ denotes the symmetric difference.*

The proof of this Lemma requires the two technical Lemmas 3.6 and 3.7 in [PS08]. They make statements about the parallelism of corresponding edges of R and S, how they separate the k-stars and about the positioning of the common bisector in a k-crossing that crosses it. Since both proofs are proven by contradiction using the construction of $(k + 1)$-crossings, we will not go into detail here and refer to [PS08] for the details.

3.3 Flipping Edges

In the following part we will examine an important property of k-triangulations that connects them with subword complexes (which we will study in the next section): The flip of edges. From now on let T be a k-triangulation of the n-gon. We begin by stating the foundamental result of this subsection.

Theorem 3.8. *Any k-relevant angle of T belongs to an unique k-star contained in T.*

To prove this theorem we need the next definition, which will help us in constructing the proof.

Definition 3.10. Let $\angle(u, v, w)$ be a k-relevant angle of T and let $f = [a, b]$ and $e = [c, d]$ be two edges of T that intersect the angle, thus,

$$u \prec \genfrac{}{}{0pt}{}{a}{c} \prec v \prec \genfrac{}{}{0pt}{}{b}{d} \prec w.$$

We say that e is v-**farther** than f, if $a \preceq c$ and $d \preceq b$. We generalize this: Let E and F be two $(k-1)$-crossings that intersect $\angle(u, v, w)$ with their edges being labelled $e_i = [a_i, b_i]$ and $f_i = [c_i, d_i]$ for $1 \le i \le k-1$, respectively, and satisfying

$$u \prec \genfrac{}{}{0pt}{}{a_1}{c_1} \prec \ldots \prec \genfrac{}{}{0pt}{}{a_{k-1}}{c_{k-1}} \prec v \prec \genfrac{}{}{0pt}{}{b_1}{d_1} \prec \ldots \prec \genfrac{}{}{0pt}{}{b_{k-1}}{d_{k-1}} \prec w.$$

We say E is v-**farther** than F, if for every $1 \le i \le k-1$ the edge e_i is v-farther than f_i. Furthermore, we say that E is **maximal**, if there is no $(k-1)$-crossing intersecting $\angle(u, v, w)$, which is v-farther.

In Table 3.2 is a visualization of this definition. Let us now start with the proof of the theorem above.

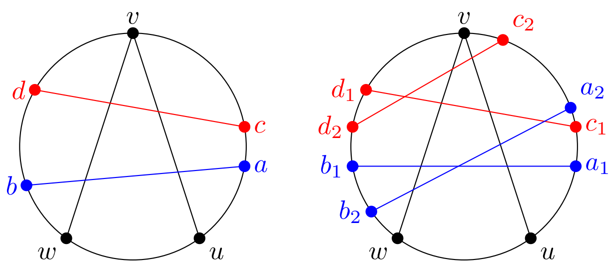


Table 3.2: $[a, b]$ is v-further than $[c, d]$, as is $\{[a_i, b_i]\}$ than $\{[c_i, d_i]\}$.

Proof of Theorem 3.8. This proof is accompanied by Table 3.3. Let $\angle(u, v, w)$ be a k-relevant angle of T. Assume for the moment that $[u, v{+}1] \in T$. Since $[u, v]$ is at least k-boundary and at most k-relevant and $v \prec v+1$ (thus $[u, v+1]$ is at least k-relevant and at most k-boundary) we can deduce, that $\angle(v+1, u, v)$ is a k-relevant angle of T. Certainly, if $\angle(v+1, u, v)$ is contained in a k-star S of T, then so is $\angle(u, v, w)$, since it is the next possible angle.

Moreover, if $n > 2k+1$ ($n = 2k+1$ being trivial), T can not contain all edges

$$\{[u+i, v+i] \,|, 0 \le i \le n-1\} \text{ and } \{[u+i, v+i+1] \,|\, 0 \le i \le n-1\}.$$

Consequently, we can assume that $[u, v+1] \notin T$.
Thus, we have a k-crossing E of the form $e_i = [a_i, b_i] \in T$ preventing $[u, v+1]$ and satisfying

$$u \prec a_1 \prec \cdots \prec a_k \prec v+1 \prec b_1 \prec \cdots \prec b_k \prec u.$$

Since $[u, v] \in T$ we have $a_k = v$ (because of the maximality) and since $\angle(u, v, w)$ is an angle we must have $v+1 \prec b_k \preceq w$. Thus, the set $\{e_1, \ldots, e_{k-1}\}$ forms a $(k-1)$-crossing that intersects $\angle(u, v, w)$, which we assume to be v-maximal. For a better visualization look at Table 3.3.

Next, we will prove that $[u, b_1], [a_1, b_2], \ldots, [a_{k-2}, b_{k-1}], [a_{k-1}, w] \in T$, such that the points $u, a_1, \ldots, a_{k-1}, v, b_1, \ldots, b_{k-1}$ will be the vertices of a k-star of T containing $\angle(u, v, w)$. We will do this in the following two steps, where the second step exists for reiterating the argument for the other edges:

1. Prove that $\angle(a_1, b_1, u)$ is an angle of T.
2. Show that $e_2, \ldots, e_{k-1}, [v, w]$ forms a $(k-1)$-crossing intersecting the angle $\angle(a_1, b_1, u)$ and is b_1-maximal.

First Step: Suppose $[u, b_1] \notin T$, thus, there exists a k-crossing F in T preventing this edge. Let $F = \{f_i = [c_i, d_i] \,|\, 1 \le i \le k\}$, where

$$u \prec c_1 \prec \cdots \prec c_k \prec b_1 \prec d_1 \prec \cdots \prec d_k \prec u.$$

First note that $v \prec d_k \preceq w$. Indeed, if $d_k \in]\!]w, u[\![$ then we must have $c_k \neq v$, since $\angle(u, v, w)$ is an angle. Thus, either $c_k \in]\!]u, v[\![$ and the set $F \cup \{[u, v]\}$ is a $(k+1)$-crossing, or $c_k \in]\!]v, b_1[\![$ and $E \cup \{[c_k, d_k]\}$ is a $(k+1)$-crossing, which both leads to a contradiction. Consequently, we must have $b_1 \prec \cdots \prec d_{k-1} \prec w$. Now, let $l = \max\{1 \le i \le k-1 \,|\, b_i \prec d_i \prec w\}$. We can then deduce that for any $1 \le i \le l$ we have $u \prec c_i \preceq a_i$, since inductively $\{e_1, \ldots, e_i\} \cup \{f_i, \ldots, f_k\}$ does not form a $(k+1)$-crossing (the e_i and f_i are in T, thus $\{e_1\} \cup \{f_1, \ldots, f_k\}$ must be $(k+1)$-crossing free, which is only possible if e_1 is parallel to f_1 [see Table 3.3] and therefore $u \prec c_1 \preceq a_1$). Thus, for any $1 \le i \le l$ we have

$$u \prec c_i \preceq a_i \prec v \prec b_i \prec d_i \prec w$$

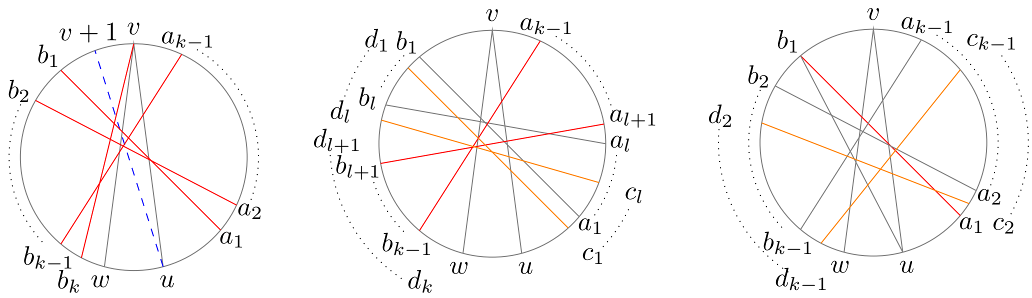
such that f_i is v-farther than e_i. Furthermore, the two chains

$$u \prec c_1 \prec \cdots \prec c_l (\preceq a_l) \prec a_{l+1} \prec \cdots \prec a_{k-1} \prec v$$

and

$$v \prec d_1 \prec \cdots \prec d_l \overset{\text{def. } l}{\prec} b_{l+1} \prec \cdots \prec b_{k-1} \prec w.$$

hold. Thus $\{f_1, \ldots, f_l, e_{l+1}, \ldots, e_{k-1}\}$ is a $(k-1)$-crossing that is v-farther than $\{e_1, \ldots, e_{k-1}\}$, which is a contradiction and we obtain $[u, b_1] \in T$.
Finally we can show that $\angle(a_1, b_1, u)$ is an angle. Suppose the opposite. Then there exists $a_0 \in]\!]u, a_1[\![$, such that $[a_1, a_o] \in T$. But then the $(k-1)$-crossing $\{[a_0, b_1], e_2, \ldots, e_{k-1}\}$ is v-farther than $\{e_1, \ldots, e_{k-1}\}$, which is a contradiction.



(a) (b) (c)

Table 3.3: (a) The k-crossing E; (b) The first step: $[u, b_1] \in T$; (c) The second step: $\{e_1, \dots, e_{k-1}, [v, w]\}$ is b_1-maximal.

Second Step: Proving this statement will be very similar to the technique above. Suppose that $F = \{f_i = [c_i, d_i] \,|\, 2 \le i \le k\}$ is a $(k-1)$-crossing that intersects $\angle(a_1, b_1, u)$, is b_1-farther than $\{e_2, \dots, e_{k-1}, [v, w]\}$ and satisfies

$$a_1 \prec c_2 \prec \cdots \prec c_k \prec b_1 \prec d_2 \prec \cdots \prec d_k \prec u.$$

Again, we can note that $b_k \preceq d_k \preceq w$ for the same reasons we discussed earlier: If $d_k \in \mathsf{K} w, u \mathsf{J}$, then $c_k \neq v$ since $\angle(u, v, w)$ is an angle. Thus, either $c_k \in \mathsf{K} a_1, v \mathsf{J}$ and $F \cup \{[u, v], e_1\}$ is a $(k+1)$-crossing, or $c_k \in \mathsf{K} v, b_1 \mathsf{J}$ and $E \cup \{[c_k, d_k]\}$ is a $(k+1)$-crossing. Hence, we have $b_1 \prec d_2 \prec \cdots \prec d_{k-1} \prec w$.
Furthermore, for any $2 \le i \le k-1$, f_i is $\angle(a_1, b_1, u)$-farther than e_i such that

$$a_1 \prec c_i \preceq a_i \prec b_1 \prec b_i \preceq d_i \prec u.$$

In particular, we have $a_1 \prec c_{k-1} \preceq a_{k-1} \prec v$ and get

$$u \prec a_1 \prec c_2 \prec \cdots \prec c_{k-1} \prec v \prec b_1 \prec d_2 \prec \cdots \prec d_{k-1} \prec w$$

such that $\{e_1, f_2, \dots, f_{k-1}\}$ is v-farther than $\{e_1, \dots, e_{k-1}\}$, which is a contradiction.
This ends the proof and

$$S = \{[v, u], [u, b_1], [b_1, a_1], [a_1, b_2], \dots, [a_{k-2}, b_{k-1}], [b_{k-1}, a_{k-1}], [a_{k-1}, w], [w, u]\}$$

is by construction the unique k-star in T containing the angle $\angle(u, v, w)$. $\square$

Theorem 3.8 enables us to tell exactly the connection between the different edges of T and the k-stars it contains.

Corollary 3.9. *Let e be an edge of T.*

1. *If e is k-relevant, then it belongs to exactly two k-stars in T (one on each side).*
2. *If e is a k-boundary edge, then it belongs to exactly one k-star.*
3. *If e is k-irrelevant, then it does not belong to any k-star.*

Proof. Let $e = [a, b]$ be a k-relevant edge. Consider without loss of generality the vertex a. In a there are exactly two k-relevant angels, one 'to the left' of e and one 'to the right' of e, with either a k-boundary edge in a, or a different k-relevant edge (there is always at least [in the sense of 'at most' a k-relevant edge] a k-boundary edge lying on each side of e). Using Theorem 3.8 we obtain the two unique k-stars containing e. If e is k-boundary, it can only form one k-relevant angle in a, with either another k-boundary edge, or with a k-relevant edge. Finally, a k-irrelevant edge can neither be part of a k-relevant angle, nor of any k-star by definition. □

Corollary 3.10. *Let T be a k-triangulation.*

1. *For any k-star S in T and for any vertex r not in S there is an unique k-star R in T such that r is a vertex of the common bisector of R and S.*
2. *Any k-relevant edge which is not in T is the common bisector of an unique pair of k-stars of T.*

Proof. For the first part, let $\angle(u, s, v)$ be the unique angle of S which contains r (Lemma 3.2). Let $\angle(x, r, y)$ be the unique angle of T containing s (Lemma 3.5). Using Theorem 3.8, the angle $\angle(x, r, y)$ belongs to a unique k-star R in T. The common bisector must be $[r, s]$ and it is unique because of Theorem 3.6 (the union of two k-stars in the same k-triangulation is $(k + 1)$-crossing free) and so is R.
Now, let $e = [r, s] \notin T$ be a k-relevant edge and let $\angle(x, r, y)$ and $\angle(u, s, v)$ denote the unique angles of T which contain s and r, respectively. Again, Theorem 3.8 gives us an unique k-star R containing $\angle(x, r, y)$ (respectively S containing $\angle(u, s, v)$) and their unique common bisector is e. □

We can observe that the previous Corollary 3.10 gives us two bijections between

1. 'vertices not used in S' and 'k-stars in T that are not S', and
2. 'k-relevant edges not used in T' and 'pairs of k-stars in T

which we can use in combination with Corollary 3.9 for double counting to obtain the next corollary, the proof of which we will not discuss.

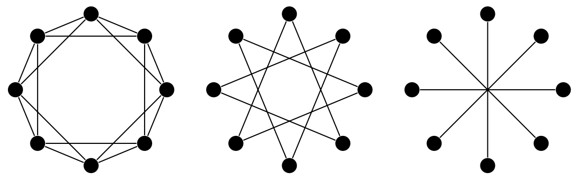
Corollary 3.11. 1. *Any k-triangulation of the n-gon contains exactly $n-2k$ many k-stars, $k(n - 2k - 1)$ many k-relevant edges and $k(2n - 2k - 1)$ edges overall.*

2. *The k-triangulations are exactly the $(k + 1)$-crossing free subsets of K_n of cardinality $k(2n - 2k - 1)$.*

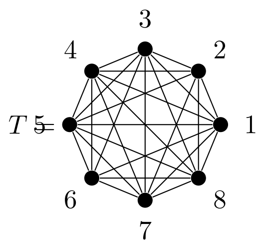
The corollarys yield an important property of k-triangulations. Let T be a k-triangulation of the n-gon and let f be a k-relevant edge of T. Thanks to Corollary 3.9, there are exactly two k-stars R and S containing f. Using Theorem 3.6, let e be the common bisector of R and S (their union is trivially $(k + 1)$-crossing free). Due to Lemma 3.7 we know that $T' := T \triangle \{e, f\}$ is a $(k + 1)$-crossing free subset of K_n. Moreover, T' is maximal: Suppose that T' is properly contained in a k-triangulation $\tilde{T}$. With the same arguments from above, $\tilde{T} \triangle \{e, f\}$ is a $(k + 1)$-crossing free subset which properly contains T, which is not possible because of the maximality of T.

We will call this process the **flipping** of the edge f and by doing so we obtain new k-triangulations of the n-gon.

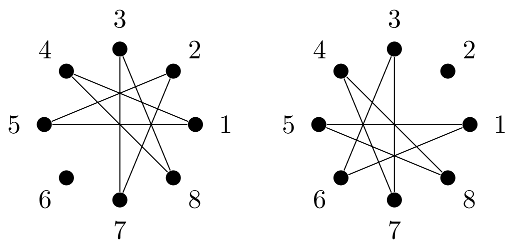
Example 3.6. At the end of this subsection we summarize our gained knowledge by considering our previous example for a 3-triangulation of the 8-gon. First, we can distinguish between the 3-irrelevant, 3-boundary and 3-relevant edges of K_8.



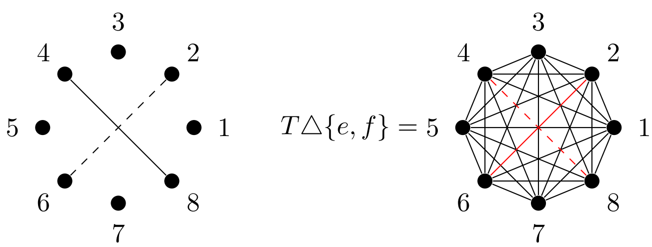
To obtain a 3-triangulation we have to choose three 3-relevant edges, e.g., $[1,5]$, $[3,7]$ and $[4,8]$. Let T be induced by these edges, thus



Since $n = 8$ and $k = 3$ and using Corollary 3.11, we know that T contains exactly two 3-stars, which are the following:



Furthermore, they share the common 3-relevant edge $f = [4,8]$ and the unique common bisector is $e = [2,6]$. The flip from f to e yields a different 3-triangulation of the 8-gon.



3.4 Multi-Associahedra

This subsection contains the definition of our main object of interest. We will shortly introduce the basic concept of an *abstract simplicial complex* without discussing it too much. In the section about subword complexes we will give further insight in the topic. For further details and a more topological point of view we refer to [Wac06].

Definition 3.11. Let V be a set and $\Delta \subseteq \mathcal{P}(V)$ be a collection of nonempty finite subsets of V. We call Δ an **abstract simplicial complex**, if the following two conditions are satisfied:

1. For every **vertex** $v \in V$ the singleton $\{v\}$ is in Δ.
2. For every **face** $\sigma \in \Delta$ and every nonempty subset $\tau \subseteq \sigma$, the **face** τ **of** σ is also in Δ.

The maximal faces of Δ (i.e., faces that are not subsets of other faces) are called **facets**. For a face $\sigma \in \Delta$ we define its **dimension** as $\dim(\sigma) = |\sigma| - 1$ and ∞, if it is not finite. The dimension of the complex Δ is the largest dimension of any of the faces it contains. Finally, we call an abstract simplicial complex **pure**, if it is of finite dimension and every facet has the same dimension.

Remark 3.4. There are four comments to be made.

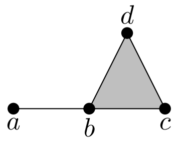
1. We will omit the word abstract and only talk about simplicial complexes, although the connection with geometric simplicial complexes based on points, lines, triangles, tetrahedra and higher dimensional simplices is apparent (as we will see).
2. In our studies we only consider finite dimensional complexes, thus, the vertex set V is finite.
3. The object of our interest, the multi-associahedron, is a simplicial complex. In fact, we already saw examples of 1- and 2-dimensional simplicial complexes in the introduction.
4. Just like for polytopes there exists the definition of a face lattice for simplicial complexes. Of cause, if the face lattice of simplicial complex is isomorphic to the face lattice of a polytope, this complex is realizable as a polytope.

Example 3.7. We can determine a full simplicial complex by just giving the facets, since every face of a facet is a face of the complex and any face of the complex is a subset of a facet. Look at the following two examples:

1. Let Δ be the simplicial complex induced by the facets $\{a, b\}$ and $\{b, c, d\}$ (thus, the vertex set is $V = \{a, b, c, d\}$). Then we can determine the simplicial complex to be

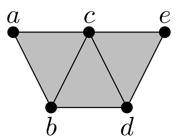
$$\Delta = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}.$$

   We can visualize it by using the singletons as vertices and connecting them by an edge whenever they form a face (indirectly, this is the geometric simplicial complex, but for the moment we will not identify the vertices with points in $\mathbb{R}^n$). This would look as follows



and we shaded the 'triangle' $\{a, b, c\}$ since it forms a 2-face (i.e., a face of dimension 2) of Δ and not only consist of the 1-faces that form its sides.

2. Using the visualization above, the simplicial complex formed by the facets $\{a, b, c\}$, $\{b, c, d\}$ and $\{c, d, e\}$ looks like this



3. The complex in 2. is pure, whereas the one in 1. is not.

Definition 3.12 (e.g., [PS08])**.** For the two positive integers $n, k \in \mathbb{N}$ define the **multi-associahedron** with parameters (n, k) to be the simplicial complex whose faces are $(k+1)$-crossing free subsets of the n-gon. Thus, the facets are k-triangulations and the ridges are exactly the flips between the different k-triangulations. We denote the complex by $\triangle_{n,k}$.

Remark 3.5. There are two things to point out.

1. By definition, for $k = 1$ the k-triangulations of the n-gon are exactly the standard triangulations and the multi-associahedron is the normal associahedron. Furthermore, we know that $\triangle_{n,1}$ is always realizable as a polytope.

2. Thanks to Corollary 3.11 we can deduce that $\triangle_{n,k}$ is a pure simplicial complex of dimension $k(2n - 2k - 1) - 1$.

Finally, we can present the conjecture on which this thesis is based on in a reasonable manner. This may have been first formulated by [Jon05] in 2005 and found several connections with other areas of mathematics, for example *Coxeter theory.*

Conjecture 3.12 ([Jon05])**.** *For arbitrary $n, k \in \mathbb{N}$, the multi-associahedron $\triangle_{n,k}$ is realizable as a polytope.*

3.5 Realizing the First Polytopes

We are now in the position to prove Conjecture 3.12 for some cases of (n, k), most of them trivial. We will do this by identifying the facets, thus k-triangulations, of $\triangle_{n,k}$ with facets of known polytopes, which yields an isomorphism between their face lattices. The results are summarized from Section 8 in [PS08].

Remark, that this kind of realization differs from the realizations we saw in the introduction of this thesis, where we identified the facets of $\triangle_{n,1}$ with points in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Instead, we will realize certain multi-triangulations as the *boundary complex* of a polytope.

Definition 3.13 ([BP09])**.** Let $P \subset \mathbb{R}^d$ be a polytope. We define the **boundary** of P as the set of proper faces of P, i.e.,

$$\partial(P) = \{F \in \mathrm{L}(P) \mid F \neq P\}.$$

Furthermore, we only need to take the k-relevant edges into account and thus identify faces of $\triangle_{n,k}$ only if they contain all k-irrelevant and k-boundary and at least one k-relevant edge. Thus, the vertices of the polytope are in correspondence with exactly one k-relevant edge.

Definition 3.14 ([BP09])**.** Let Δ be a simplicial complex and P a polytope. We say that P is a **realization** of Δ, if its boundary complex $\partial(P)$ is isomorphic to Δ, i.e., if $\phi : \Delta \to \partial(P)$ is a bijection which respects inclusion (thus, their face lattices).

We will implicitly use this definition for the upcoming realizations.

Corollary 3.13. *For $k = 1$, the multi-associahedron is polytopal.*

Proof. This is Theorem 1.1, since 2-crossing free sets are exactly triangulations of the n-gon. Thus, the multi-associahedron $\triangle_{n,1}$ is the $(n-3)$-dimensional associahedron. □

The next two results follow from the considerations in Example 3.1.

Corollary 3.14. *The multi-associahedron $\triangle_{2k+1,k}$ is realizable as a polytope.*

Proof. There is only one unique k-triangulation. Thus, the realizing polytope is a point. □

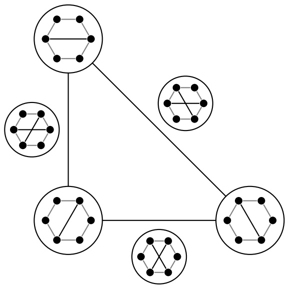
Corollary 3.15. *The multi-associahedron $\triangle_{2k+2,k}$ is realizable as a polytope.*

Proof. There are $k+1$ many k-triangulations. The realizing polytope is the following set, called the k-**simplex**:

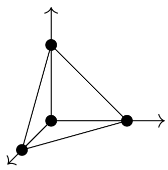
$$\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \in \mathbb{R}^k \;\middle|\; x_i \geq 0 \text{ for all } 1 \leq i \leq k,\ \sum_{i=1}^{k} x_i = 1 \right\}.$$

This is obviously a polytope (it is the convex hull of the k standard basis vectors of $\mathbb{R}^k$ and the origin) with $k+1$ many facets and the facets correspond to the facets of $\triangle_{2k+2,k}$, thus, they are combinatorially equivalent. □

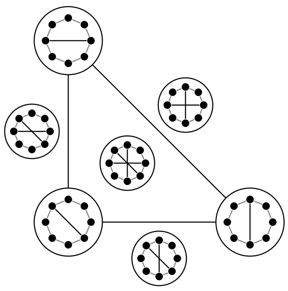
Example 3.8. For $k = 2$, the 2-simplex is a triangle whose facets are its sides.



For $k = 3$, the 3-simplex is a tetrahedron, thus, the following polytope with 4 triangular facets and the ridges are the lines connecting the vertices.



Identifying each of the 4 facets with one of the 4 facets of $\triangle_{8,3}$ yields for example the following picture for one of the facets:

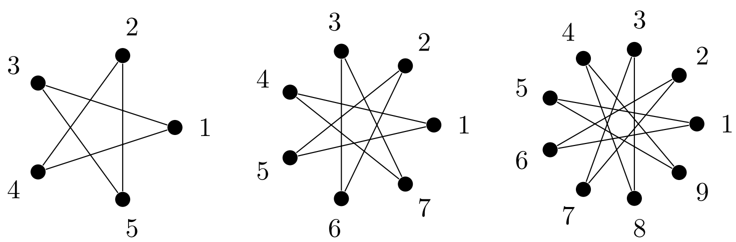


The next and last result is less obvious to prove.

Corollary 3.16. *The multi-associahedron $\triangle_{2k+3,k}$ is realizable as a polytope.*

Before going into the proof we look at examples of how k-relevant edges behave in a $(2k+3)$-gon.

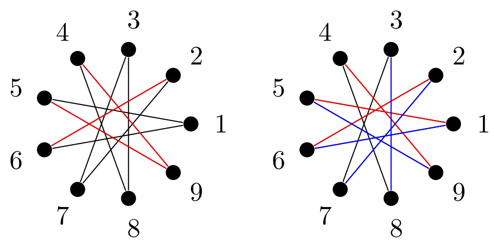
Example 3.9. As we can see, the set of k-relevant edges form $(k+1)$-stars in the corresponding $(2k+3)$-gons. For $k = 1, 2, 3$ this looks as follows:



Furthermore, the set of k-relevant edges form a cycle of length $2k+3$. For the general case, the set (i.e. the k-star) is

$$S = \{[1, k+2], [1, k+3], [2, k+3], [2, k+4], \dots, [k+1, 2k+3], [k+2, 2k+3]\}.$$

Proof of Corollary 3.16. By Corollary 3.11 we know that a k-triangulation of the $(2k+3)$-gon consists of $2k$ many k-relevant edges, thus, we can obtain a k-triangulation by removing three edges from the set S in the example above. But not every removal yields a $(k+1)$-crossing free subset. We claim that this only happens, if by removing the edges the length of paths of k-relevant edges in the triangulation is even, thus, we decompose the cycle set S into subsets of even cardinality. By path we mean consecutive k-relevant edges in star order. We give a brief example of what we mean. For $k = 3$, the first removal (marked by red edges) does yield a 4-crossing free set, whereas the second does not (marked by blue edges).



In the first picture, the edges were decomposed in the paths

$$\{[6, 1], [1, 5]\} \text{ and } \{[4, 8], [8, 3], [3, 7], [7, 2]\}$$

whereas in the second picture in the paths

$$\{[5, 9]\}, \{[4, 8], [8, 3], [3, 7], [7, 2]\} \text{ and } \{[6, 1]\}.$$

Observe, that m consecutive edges contribute with $\frac{m}{2}$ (m even), respectively $\frac{m+1}{2}$ (m odd), edges to a $(k+1)$-crossing, since consecutive edges share one vertex. It is easy to see that the set of $2k+3$ edges S can be decomposed into subsets with cardinalities pictured in Table 3.4 and that only those which contain uneven numbers generate $(k+1)$-crossings.

Crossing	Cardinality of Subsets		Cardinality of Subsets			Crossing
k	2k		2k-2	1	1	k+1
k+1	2k-1	1	2k-3	2	1	k+1
k	2k-2	2	2k-4	3	1	k+1
k+1	2k-3	3	2k-4	2	2	k
k	2k-4	4	2k-5	4	1	k+1
...	...	...	...	...	...	...

Table 3.4: The possible cardinalities of the decomposed subsets of S and the crossing they form.

We now conclude that $\triangle_{2k+3,k}$ is realizable as the cyclic polytope $C_{2k}(2k+3)$, introduced in Example 2.2 of Section 2, by using the next theorem.

Theorem 3.17 (Gale's eveness condition, [Zie95])**.** *Let $n > d \geq 2$. We will use $[n]$ to denote the set $\{1, \ldots, n\}$ and choose real parameters $t_1 < \cdots < t_n$. A subset $S \subseteq [n]$ of cardinality d forms a facet of $C_d(n)$ if and only if the following 'eveness condition' is satisfied: If $i < j$ are not in S, then the number of $k \in S$ between i and j is even, thus*

$$2 \,\Big|\, \#\{k \in S \,|\, i < k < j\} \text{ for } i, j \notin S.$$

With our considerations from above we can therefore close the proof. □

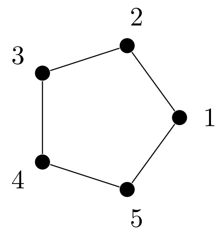
4 Subword Complexes

Now that we introduced multi-associahedra and stated the conjecture we will look at the connection with subword complexes, which are introduced in this section. Before we go into further details we give a leading example.

Example 4.1. Lets consider the symmetric group $\mathfrak{S}_3$. It is well known that it is generated by the transpositions $S = \{s_1, s_2\}$ where $s_i = (i, i+1)$. The set S will be, talking about sub*word* complexes, our alphabet. For the moment we rename the simple transitions to $s_1 = a$ and $s_2 = b$. Now we look at the word $Q = (b, a, b, a, b)$, which is a (in this case 5-) tuple of letters from our alphabet S. In this word we search for all 'subwords' of the form bab and aba, taking into account the positions of the letters (in fact, considering bab and aba as elements of $\mathfrak{S}_3$ by just composing the transpositions, they 'decode' the same permutation). We mark the letters of such a subword by $\times$ and mark the letters not used by $\circ$ to obtain the following tabular:

b	a	b	a	b
$\times$	$\times$	$\times$	$\circ$	$\circ$
$\times$	$\circ$	$\circ$	$\times$	$\times$
$\times$	$\times$	$\circ$	$\circ$	$\times$
$\circ$	$\times$	$\times$	$\times$	$\circ$
$\circ$	$\circ$	$\times$	$\times$	$\times$

By regarding the letters of Q as vertices and connecting two vertices by an edge whenever they occur in the same complement of a subword (thus, both corresponding letters are marked with $\circ$ in the same row), we obtain the following pentagon, where the number of the vertex corresponds to the position of the letter in Q.



What we just described and visualized is a simplicial complex, whose vertices are the letters of Q and facets are given by the complements of the subword we searched for in Q. We will call this a subword complex, which is in this case identical with the multi-associahedron $\triangle_{5,1}$.

We give one more example to establish the connection between multi-associahedra and the word-structure just introduced.

Example 4.2. We consider the same alphabet, word, subword and tabular from the last example. In addition, we consider the convex 5-gon (not the subword complex from the last example!) and give a list of its ordered diagonals,i.e., we give all proper diagonals in the pentagon with starting vertex 1, then all diagonals with starting vertex 2 and so on until we have all diagonals of the pentagon. This ordered list is the following:

$$q_1 = [1, 3], q_2 = [1, 4], q_3 = [2, 4], q_4 = [2, 5], q_5 = [3, 5].$$

We now connect the position i of Q with the ith ordered diagonal. Considering the rows of the tabular above and drawing the two with $\circ$ marked diagonals will give us a triangulation of the pentagon. In fact, doing this for each row will give us all triangulations. This procedure yields the facets of the multi-associahedron $\triangle_{5,1}$. An extended example can be found at the end of this section.

These two examples are exemplary for the statements that we will examine in this section. Before we can go into further details and exact definitions we have to introduce finite reflection groups.

4.1 Finite Reflection Groups

In this subsection we will work very closely with [Hum90]. Due to its very good structure and detailed proofs, we omit most of them and consider the statements only. The goal of this section is to give a grasp of what finite reflection groups are, what properties their elements have and how they interact.

Let V be real euclidean space and $\langle\cdot,\cdot\rangle$ a positive definite symmetric bilinear form on V. For $\alpha \in V$ define the **reflection** of α as the linear operator

$$s_\alpha : V \to V,\ \beta \mapsto \beta - \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$$

which maps α to its negative and fixes the hyperspace $H_\alpha := (\mathbb{R}\alpha)^\perp$ of vectors in V perpendicular to α with respect to $\langle\cdot,\cdot\rangle$. Thus, reflections form a subgroup of the orthogonal group $\mathrm{O}(V)$. A subgroup of $\mathrm{O}(V)$ is called a **finite reflection group**, if it is generated by reflections.

Example 4.3. Let $\mathfrak{S}_n$ be the symmetric group acting on $\mathbb{R}^n$ by permuting the standard basis vectors $e_1,\dots,e_n$. Thus, reflections are exactly transpositions (i,j) sending $e_i - e_j$ to its negative. It is known, that the transpositions $(i,i+1)$, $1 \le i \le n-1$ generate $\mathfrak{S}_n$ making it a reflection group.

To study finite reflection groups and their action and geometry on V we introduce the following definition.

Definition 4.1. Let $\Phi \subset V$ be a finite set of nonzero vectors which we call **roots**. We call Φ a **root system** with associated reflection group W, if for all $\alpha \in \Phi$ the following conditions are satisfied:

1. $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$
2. $s_\alpha\Phi = \Phi$

We then define W to be the group generated by all reflections s_α, $\alpha \in \Phi$.

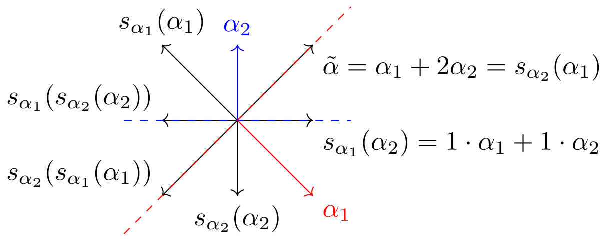
From now on let Φ be a root system with associated reflection group W. For exploring the action of W on the vector space we introduce the following total ordering on V. Let $\{v_1,\dots,v_n\}$ be a basis of V. We then order the vectors in lexicographic order, i.e.,

$$\sum_{i=1}^n a_i v_i < \sum_{i=1}^n b_i v_i \Leftrightarrow a_k < b_k \text{ for the least index } k \text{ for which } a_k \neq b_k$$

and we will call $v \in V$ **positive** if $0 < v$. For this purpose, fix a basis of V.

Definition 4.2. A subset $\Pi \subset \Phi$ is called a **positive system**, if all roots are positive relative to the total ordering of V. Hence we have $\Phi = \Pi \sqcup (-\Pi)$. A subset $\Delta \subset \Phi$ is called a **simple system**, if Δ is a basis for the $\mathbb{R}$-span of Φ in V and if each root is a linear combination of Δ with coefficients all of the same sign. We call the elements in the respective subset **positive** and **simple** roots.

Example 4.4. The following picture shows an example for a root system in $\mathbb{R}^2$. Beginning with the two roots α_1 and α_2 we obtain the full root system by using the reflections induced by α_1 and α_2.



Furthermore we have a longest root $\tilde{\alpha}$ regarding the simple system $\Delta = \{\alpha_1, \alpha_2\}$ and Δ induces the positive system $\Phi = \{\alpha_1, \alpha_2, \tilde{\alpha}, s_{\alpha_1}(\alpha_2)\}$ and in fact, as we will see in a moment, these two sets define each other uniquely.

The following theorem gives us the existence and uniqueness of simple systems.

Theorem 4.1.

1. *For every simple system Δ in Φ there is a unique positive system containing Δ.*
2. *Every positive system Π in Φ contains an unique simple system.*

The choice of neither a positive nor a simple system is unique. For instance, by using a simple system Δ and an element $w \in W$ we obtain another simple system $w\Delta$ with corresponding positive system $w\Pi$. Luckily, positive and simple systems determine each other uniquely.

Theorem 4.2. *Any two positive, respectively simple, systems in Φ are conjugate under W.*

We defined W to be the group generated by all reflections induced by all roots in the root system Φ. Since each root $\alpha \in \Phi$ can be written as a linear combination of simple roots the question is, whether W is generated by **simple reflections**, thus reflections s_α for $\alpha \in \Pi$.

Theorem 4.3. *For a fixed simple system Δ, W is generated by the simple reflections induced by Δ.*

Our next goal is to find an efficient presentation of W as an abstract group. For this we will need to study the ways in which an arbitrary $w \in W$ can be written as a product of simple reflections.

Definition 4.3. Let $w = s_1 \dots s_r$ be an element in W, where $s_i = s_{\alpha_i}$ for some $\alpha_i \in \Delta$. Define the **length** $l(w)$ of w to be the smallest r for which such an expression exists.

It turns out that the length of an element of the reflection group coincides with the number of positive roots sent to negative roots by this element, which we denote by $n(w)$. To be more precise:

$$\text{For } w \in W \text{ we have } l(w) = n(w) = |\Pi \cap w^{-1}(-\Pi)|.$$

The following two results are crucial in understanding how reduced expressions for group elements can be obtained.

Theorem 4.4 (Delete Condition). *Let Δ be a simple system and $w = s_1 \dots s_r$ an arbitrary not reduced expression of $w \in W$, thus $n(w) < r$. Then there exist indices $1 \le i < j \le r$ satisfying*

$$w = s_1 \dots \hat{s_i} \dots \hat{s_j} \dots s_r$$

and the hat denotes omission.

Thus, a reduced expression can be obtained by pairwise omitting simple reflections. This can be reformulated as follows.

Theorem 4.5 (Exchange Condition). *For an arbitrary expression $w = s_1 \dots s_r$ and a simple reflection $s = s_\alpha$, such that $l(ws) < l(w)$, there exists an index i for which*

$$ws = s_1 \dots \hat{s_i} \dots s_r \Leftrightarrow w = s_1 \dots \hat{s_i} \dots s_r s.$$

In particular, w has a reduced expression ending in s if and only if $l(ws) < l(w)$.

The Exchange Condition will be the central concept of subword complexes in the next subsection. After exploring reduced expressions, we search for the longest element of W. For this, recall that because of Theorem 4.2, W acts transitive on positive and simple systems. In fact, the action is simple transitive after all, which follows from the theorem in Chapter 1.8 in [Hum90].
By definition, a set Π is a positive system whenever $-\Pi$ is (just switch the signs). Thus, there exists an element $w_\circ \in W$ which maps Π and $-\Pi$ to each other. Furthermore, it is the longest possible element in W, since we must have $l(w_\circ) = n(w_\circ) = |\Pi|$ and with that it must be unique.

We will now give the efficient presentation of W mentioned earlier. For this, we introduce the integer $m(\alpha, \beta)$, denoting the order of $s_\alpha s_\beta \in W$ for roots $\alpha, \beta \in \Phi$.

Theorem 4.6. *For a fixed simple system Δ in Φ, the finite reflection group W is generated by the set $S := \{s_\alpha \mid \alpha \in \Delta\}$, subject only to the relations*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = \mathrm{id}_V .$$

Example 4.5. These relations are also called **braid relations** for the following reason. Consider the symmetric group $\mathfrak{S}_4$ as a reflection group as introduced in Example 4.3. It is generated by the transpositions $s_i = (i, i+1)$ for $i = 1, 2, 3$. Furthermore, they are subject to the relations $(s_i s_{i+1})^3 = \mathrm{id}$ and $(s_i s_j)^2 = \mathrm{id}$ for not consecutive i, j. These equations are equivalent to

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ and } s_i s_j = s_j s_i,$$

justifying the expression 'braid relation'. We call an application of these equations a **braid move**, e.g., the transformation of the element $s_1s_2s_3s_2 \in \mathfrak{S}_4$ to $s_1s_3s_2s_3$ to $s_3s_1s_2s_3$. They define the same element but differ in their representation in simple reflections.

Thus we do not only know that, by definition, the reflection groups are generated by reflections of the form s_α for $\alpha \in \Phi$, but also that the generating set is much smaller and subject to special relations. One question remains though: Are all reflections in W of the form s_α? The answer is yes.

Proposition 4.7. *Every reflection in W is of the form s_α for some $\alpha \in \Phi$.*

Now that we have this result, we can give one last characterization for the reflections.

Proposition 4.8. *For $\alpha \in \Delta$ and $w \in W$ we have $ws_\alpha w^{-1} = s_{w\alpha}$. In particular, if T is the set of all reflections s_α for $\alpha \in \Phi$, we have*

$$T = \bigcup_{w \in W} wSw^{-1}.$$

This will be of need when we define the Bruhat ordering on W at a later point. Before we start with the next subsection we end this one with the study of the subgroup structure of W. For this we will look at subgroups generated by sets of simple reflections.

Definition 4.4. Let W be generated by the set of simple reflections S corresponding to the simple system Δ. For any subset $I \subset S$, define W_I to be the subgroup of W generated by all $s_\alpha \in I$ with corresponding simple roots $\Delta_I := \{\alpha \in \Delta | s_\alpha \in I\}$. We call subgroups obtained in this way **parabolic subgroups**. Furthermore, define Φ_I to be the intersection of Φ with the $\mathbb{R}$-span V_I of Δ_I in V.

The following proposition shows, that parabolic subgroups behave in a convenient way with our definitions so far.

Proposition 4.9. *Let $I \subset S$ and W_I, Δ_I and Φ_I as above.*

1. *Φ_I is a root system in $V_I \subset V$, with simple system Δ_I and corresponding reflection group W_I.*
2. *Viewing W_I as a reflection group, with length function relative to Δ_I, we have $l = l_I$ on W_I.*
3. *Define $W^I := \{w \in W \,|\, l(ws) > l(w) \text{ for all } s \in I\}$. For all $w \in W$ there is a unique $u \in W^I$ and a unique $v \in W_I$ such that $w = uv$ and $l(w) = l(u) + l(v)$. Furthermore, u is the unique element of smallest length in the coset wW_I.*

4.2 Classification and Coxeter Systems

In the following we will determine all possible finite reflection groups using their Coxeter graph. Though this classification might not be very trivial using group theory, the proof using Coxeter graphs is very easy and satisfying and can be found in the second chapter in [Hum90]. Afterwards we will shortly introduce Coxeter systems, which turn out to be the same as finite reflection groups. The definition of the Coxeter graph makes use of the fact that finite reflection groups are generated in the way of Theorem 4.6.

Definition 4.5. Let W be a finite reflection group determined by the simple system Δ and positive integers $m(\alpha, \beta)$, $\alpha, \beta \in \Delta$. Construct the **Coxeter graph** Γ as follows:

- The vertex set is the set of simple roots in Δ.
- Join the two vertices corresponding to α, β by an edge if $m(\alpha, \beta) \geq 3$.
- Label the edges with the corresponding integer. By convention we will omit the label 3.

Remark 4.1. Since simple systems are conjugate, the Coxeter graph does not depend on the choice of the simple system Δ.

Given a Coxeter graph Γ with vertex set $S = \{s_1, \ldots, s_n\}$ we can associate a symmetric $n \times n$ matrix A by setting

$$a_{ij} := -\cos \frac{\pi}{m(s_i, s_j)}$$

and thus define a bilinear form $x^t A y$. By using the standard terminology we call a Coxeter graph **positive definite**, respectively **positive semidefinite**, whenever the corresponding bilinear form is. Furthermore we say that a Coxeter graph is of **positive type**, if it is positive definite or semidefinite.

Example 4.6. Direct computation yields, that the graphs in Table 4.1 are of positive type.

Theorem 4.10. *The graphs in Table 4.1 are the only connected Coxeter graphs of positive type.*

Remark 4.2. To each graph of positive type exists a corresponding finite reflection group (see chapters five and six in [Hum90]).

At the end of the introduction of finite reflection groups we introduce Coxeter systems. Instead of considering the group of reflections in an euclidean vector space V, we introduce the *Coxeter group* as a group subject only to relations of the form in Theorem 4.6.

Definition 4.6. A **Coxeter system** is a pair (W, S) consisting of a **Coxeter group** W and a set of generators $S \subset W$, which are solely determined by the relations

$$(ss')^{m(s,s')} = e,$$

where e is the neutral element of W and the integers $m(s, s')$ for $s, s' \in S$ are defined as

$$m(s, s) = 1,\ m(s, s') = m(s', s) \geq 2.$$

If no relation occurs, we set $m(s, s') = \infty$.

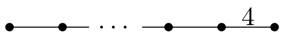
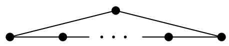
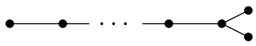
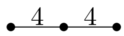
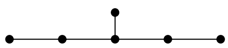
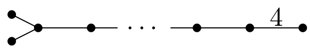
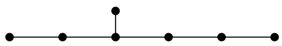
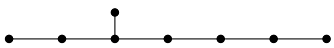
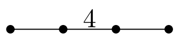
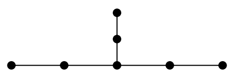
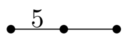
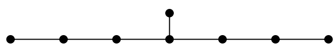
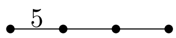
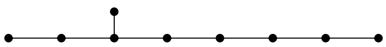
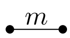
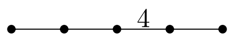
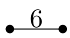
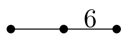
Group	Coxeter graph	Group	Coxeter graph
A_n		$\tilde{A}_1$	
B_n		$\tilde{A}_n$	
D_n		$\tilde{B}_2 = \tilde{C}_2$	
E_6		$\tilde{B}_n$	
E_7		$\tilde{C}_n$	
E_8		$\tilde{D}_n$	
F_4		$\tilde{E}_6$	
H_3		$\tilde{E}_7$	
H_4		$\tilde{E}_8$	
$I_2(m)$		$\tilde{F}_4$	
G_2		$\tilde{G}_2$	

Table 4.1: In the left part are positive definite and in the right part positive semidefinite graphs with their respective Coxeter groups.

Example 4.7. Finite reflection groups are examples for Coxeter systems. In fact, we will be particularly interested in the Coxeter group A_n in Table 4.1 which we already examined in Example 4.5 for $n = 3$. To be more precise, the Coxeter group A_n can be identified with the symmetric group $\mathfrak{S}_{n+1}$.

Remark 4.3. We will only be interested in finite Coxeter groups and as just explained only in the case A_n. Furthermore, all statements we have for finite reflection groups exist for Coxeter groups as well. For more details about Coxeter groups we refer to the fifth chapter of [Hum90].

4.3 The Subword Complex

We will now give the definition of subword complexes and discuss its properties. Most results of this subsection are taken from [KM03]. Let from now on (W, S) be a finite Coxeter system. Our main goal will be to show the following statement, which shows the very-easy-to-classify nature of subword complexes.

Theorem 4.11. *The subword complex is a simplicial sphere homeomorphic to either a sphere or a ball, i.e., there is a bijective, continuous function with continuous inverse function, that maps the complex to a sphere, respectively ball, with appropriate dimension.*

Eventually, we will learn what a subword complex is, that it is a simplicial complex by nature and how we can classify them. Both of the next definitions introduce the language of this subsection.

Definition 4.7. A **word** Q of size m of S is a m-tuple of simple reflections in S, thus $Q = (s_1, \dots, s_m)$ where the $s_i \in S$ are arbitrary (in this case, we do not mean $s_i = s_{\alpha_i}$!). An ordered subsequence P of Q is called a **subword**. For an element $w \in W$ we say that a subword P

1. **represent** w, if the ordered product of the generating elements in P is a reduced expression for w.
2. **contains** w, if some subword of P represents w.

The notion of words and subwords build the foundation for the object of our interest in this section.

Definition 4.8. Let Q be a word in S and $w \in W$. The **subword complex** $\Delta(Q, w)$ is the set of subwords $Q \setminus P$ whose complements P contain w.

The following lemma shows the nice structure subword complexes have.

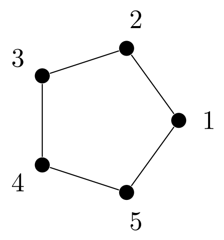
Lemma 4.12. *For a word Q and an element $w \in W$ the subword complex $\Delta(Q, w)$ is a pure simplicial complex whose facets are the subwords $Q \setminus P$ such that P represents w.*

Proof. Let Q be a word which contains at least one reduced expression of $w \in W$ (otherwise the subword complex would be empty). It is apparent, that the facets are given by subwords which represent w (by considering complements of subwords containing w, they only become smaller the longer the subword is). Moreover, every subword containing a subword that represents w corresponds to a face being a subset of the corresponding facet, and the vertices of the subword complex are exactly the different letters in Q, whose complement contains a reduced expression of w. Thus, the subword complex is a simplicial complex. Since all reduced expressions of w have the same length (otherwise their inversion set would be different, which would be a contradiction) it is a pure simplicial complex. □

Example 4.8. We will repeat the example we gave at the beginning of this section with the appropriate language. Consider the Coxeter system $W = \mathfrak{S}_3$ and $S = \{s_1, s_2\}$ where $s_i = (i, i+1)$. Let the word be $Q = (s_2, s_1, s_2, s_1, s_2)$ and $w = s_2 s_1 s_2 = s_1 s_2 s_1$. We determine the facets of $\Delta(Q, w)$ by finding all subwords in Q, which represent w while tracking the positions of letters in Q used in these subwords. We obtain the following tabular:

s_2	s_1	s_2	s_1	s_2
×	×	×	◦	◦
×	◦	◦	×	×
×	×	◦	◦	×
◦	×	×	×	◦
◦	◦	×	×	×

Since Q is a word of length five, we will have five vertices corresponding to each position in the word. Two vertices are connected by an edge, if they occur in the same complement of a subword representing w (marked with $\circ$). Thus, the subword complex $\Delta(Q, w)$ is the following pentagramm:



In the example, the importance of the exchange condition in Theorem 4.5 becomes clear: For a certain $\times$ in a row there is an unique other $\times$ we can choose instead. These are the ridges between facets.

Definition 4.9. Let Δ be a simplicial complex and $F \in \Delta$ a face.

1. The **deletion** of F from Δ is the set $\mathrm{del}(F, \Delta) = \{G \in \Delta \mid G \cap F = \emptyset\}$.
2. The **link** of F in Δ is the set $\mathrm{lk}(F, \Delta) = \{G \in \Delta \mid G \cap F = \emptyset\, , G \cup F \in \Delta\}$.

Both, the link and the deletion of a face are simplicial complexes themselves. In correspondence with the next definition we will see examples for both.

Definition 4.10. We call a simplicial complex Δ with vertex set V **vertex-decomposable** if it is pure and one of the following conditions are satisfied:

1. $\Delta = \emptyset$ or the only facet of Δ is V, or
2. for some vertex $v \in \Delta$, both complexes $\mathrm{del}(v, \Delta)$ and $\mathrm{lk}(v, \Delta)$ are vertex-decomposable and every facet of $\mathrm{del}(v, \Delta)$ is a facet of Δ.

This definition is inductively well-defined, since the deletion and the link of a vertex are of lower dimension than the starting complex.

Example 4.9. Let $V = \{a, b, c, d, e\}$ be the set of vertices and let Δ be generated by the facets $\{a, b\}$, $\{b, c, d\}$ and $\{c, d, e\}$. We will show that this simplicial complex is vertex-decomposable.

1. Since $\Delta \neq \emptyset$ and V is not a facet choose $v = a$. Then $\mathrm{lk}(a, \Delta) = \{b\}$ is vertex-decomposable. On the other hand we need to check that the complex $\Delta' := \mathrm{del}(a, \Delta)$, which is generated by the facets $\{b, c, d\}$ and $\{c, d, e\}$, is vertex-decomposable.
2. Again Δ' is neither empty nor is $\{b, c, d, e\}$ a facet. Hence choose $v' = b$. Then, $\mathrm{lk}(b, \Delta') = \{c, d\}$ is vertex-decomposable and $\mathrm{del}(b, \Delta') = \{c, d, e\}$ is a facet. Furthermore, $\mathrm{del}(b, \Delta')$ is a facet of Δ'. Thus, $\mathrm{del}(b, \Delta')$ is vertex-decomposable and so are Δ' and thus Δ.

Remark, that Δ is not a pure complex.

The definition of vertex-decomposable often comes with the requirement for the complex to be pure. This has mainly to do with the next definition and their correspondence.

Definition 4.11. A **shelling** of a pure simplicial complex Δ is an ordered list of its facets $F_1, \dots, F_t$ such that

$$\left(\bigcup_{i=1}^{k-1} \hat{F}_i \right) \cap \hat{F}_k$$

is a pure subcomplex generated by codimension 1 faces of F_k for each $k \leq t$, where $\hat{F}$ denotes the set of faces of F.

Shellability for simplicial complexes is an important property for our main result of this section. Without going into details we will explain why: Shellable simplicial complex are homotopy equivalent to the wedge of spheres (corresponding to spanning simplices) or they are contractible. For a more detailed summary we refer to [Wac96]. Furthermore, shellable and vertex-decomposable complexes are corresponding as follows.

Lemma 4.13 ([BP79])**.** *Let Δ be a pure simplicial complex. If Δ is vertex-decomposable, then Δ is shellable.*

Since further details into shellable complexes and the proofs of some of the upcoming results would need the introduction of topological definitions and methods, we refer to [KM03], [Wac96] and [BP79] for further details.

Theorem 4.14. *Subword complexes are vertex-decomposable, hence shellable.*

Proof. Let $\Delta = \Delta(Q, w)$ be a subword complex. We will prove this by induction on the length m of Q. The case $m = 1$ is trivial. Suppose $Q = (s, s_2, s_3, \dots, s_m)$. By induction, it suffices to show that $\mathrm{lk}(s, \Delta)$ and $\mathrm{del}(s, \Delta)$ are subword complexes. By definition, both of them consist of subwords of $Q' = (s_2, \dots, s_m)$. First, we show that $\mathrm{lk}(s, \Delta) = \Delta(Q', w)$. Suppose we have a face $F \in \mathrm{lk}(s, \Delta)$. Then $F \cup \{s\} \in \Delta$, which means that $Q \setminus (F \cup \{s\})$ contains a reduced expression of w, which is equivalent to say that $Q' \setminus F$ contains a reduced expression of w, thus, $F \in \Delta(Q', w)$. The other inclusion follows the same way around.
Second, we show that the deletion of s is either its link or the complex $\Delta(Q', sw)$. This depends on whether or not sw is longer or shorter than w. In the first case, no reduced expression for w can start with s. Thus, for a face $F \in \mathrm{del}(s, \Delta)$, we have $F \in \Delta, s \in Q \setminus F$ and $Q \setminus F$ contains a reduced expression of w, implying that $Q' \setminus F$ contains a reduced expression of w, thus, $\mathrm{del}(s, \Delta) = \Delta(Q', w)$. On the other hand, if sw is shorter than w there is a reduced expression of w starting with s. Let $F \in \mathrm{del}(s, \Delta)$, thus, $F \in \Delta$ and $s \in Q \setminus F$.

- If $Q \setminus F$ contains a reduced expression of w starting with s, then $Q' \setminus F$ contains a reduced expression of sw.
- If $Q \setminus F$ contains a reduced expression of w not starting with s, then $Q' \setminus F$ contains a reduced expression of sw by the exchange condition.

Thus, in both cases we have $\mathrm{del}(s, \Delta) = \Delta(Q', sw)$.

□

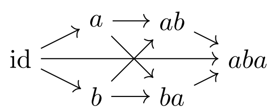
We will now introduce the necessary lemmas to show the main result of this subsection. Before we start to formulate and prove them, we need to introduce a partial order on the Coxeter group W. Let T be the set of all reflections as in Proposition 4.8.

Definition 4.12. For $u, w \in W$ denote

- $u \xrightarrow{t} w$, if $ut = w$ for a $t \in T$.
- $u \to w$, if there exists a $t \in T$ such that $u \xrightarrow{t} w$.
- $u \leq w$, if there exist $u_0, \dots, u_k \in W$ where $u = u_0 \to u_1 \to \cdots \to u_k = w$.

We call '$\leq$' the **Bruhat ordering** on W. The **Bruhat graph** is a directed graph with nodes in W and edges given by the second point above.

Example 4.10. Consider the Coxeter group W of type A_2 given by $S = \{a, b\}$. Then the set of all reflections is $T = \{a, b, aba = bab\}$. The Bruhat graph is the following:



The following two results should give a good understanding of the Bruhat order. For a more detailed discussion we refer to [Hum90].

Theorem 4.15. *Let $w = s_1 \dots s_r$ be an arbitrary reduced expression. Then we have $w' \leq w$ if and only if w' is a subexpression of this reduced expression, i.e.,*

$$w' = s_{i_1} \dots s_{i_q} \text{ for } 1 \leq i_1 < \cdots < i_q \leq r.$$

Proposition 4.16. *Let $w' < w$. Then there exist $w_0, \dots, w_m \in W$ such that $w' = w_0 < \cdots < w_m = w$ and $l(w_i) = l(w_{i-1}) + 1$ for $1 \leq i \leq m$.*

The next definition contains the greedy algorithm for the length of words. It will play an important role in the main result of this section.

Definition 4.13. For a word Q and a letter $s \in S$ let Q' be the word obtained by adding s at the end of Q. The **Demazure product** of Q' is recursively defined by

$$\delta(Q') = \begin{cases} ws, & l(ws) > l(w) \\ w, & l(ws) < l(w) \end{cases}$$

where w denotes the Demazure product of Q and $\delta(\emptyset) = \text{id}$.

Example 4.11. Let $Q = (s_2, s_1, s_1, s_2, s_1, s_2)$. Then we compute its Demazure product as follows: Starting with the first letter we only add the consecutive one if the length of product of the letters increases, thus,

$$s_2 \to s_2 s_1 \to s_2 s_1 \to s_2 s_1 s_2 \to s_2 s_1 s_2 \to s_2 s_1 s_2 = \delta(Q).$$

We now start with giving two lemmas exploring the correspondence of the Demazur product and elements of the Coxeter group. Let $\leq$ and $<$ be the Bruhat order on W and $Q = (s_1, \dots, s_m)$ a word of length m. We denote the omission of s_i from Q to obtain a word of length $m - 1$ by $Q \setminus s_i$.

Lemma 4.17. *Let P be a word and let $w \in W$.*

1. *$\delta(P) \geq w$ if and only if P contains w.*
2. *If $\delta(P) = w$, then every subword of P containing w has Demazure product w.*
3. *If $\delta(P) > w$, then P contains a word T representing an element $w' > w$ satisfying $|T| = l(w') = l(w) + 1$.*

Proof. We will use the first point to prove the other two. By definition every word contains its Demazure product. Let $P' \subseteq P$ be a subword of P containing w. Thus, we have $w = \delta(P) \geq \delta(P') \geq w$ which forces $\delta(P') = w$ proving the second point.

For the third point, choose any element $w' \in W$ such that $l(w') = l(w) + 1$ and $w < w' \leq \delta(P)$ (Proposition 4.16). Such a group element certainly exists, since w can not be the unique longest element, because otherwise $\delta(P)$ must have already been w. By definition of the Bruhat ordering the inequality $w < w' \leq \delta(P)$ holds.

Now to the remaining equivalence. Suppose $w' = \delta(P) \geq w$ and let $P' \subseteq P$ be the subword obtained by reading P in order, omitting any reflections along the way that do not increase its representing elements length. Thus, P' represents w' and contains w by definition. On the other hand, if $T \subseteq P$ represents w, then use induction on the length of P as follows. Let $s \in S$ be the last reflection in the list P, such that $\delta(P)s < \delta(P)$, thus, $\delta(P \setminus s) = \delta(P)$ or $\delta(P \setminus s) = \delta(P)s$.

- If $ws > w$, then $T \subseteq P \setminus s$ and $w \leq \delta(P \setminus s) \leq \delta(P)$.
- If $ws < w$ and $T' \subset T$ represents ws, then $T' \subseteq P \setminus s$ and $ws \leq \delta(P \setminus s)$. Since $ws < w$ it follows, that $w \leq \delta(P)$.

□

Lemma 4.18. *Let T be a word and $w \in W$ such that $|T| = l(w) + 1$.*

1. *There are at most two elements $s \in T$ such that $T \setminus s$ represents w.*
2. *If $\delta(T) = w$, then there are two distinct $s \in T$ sucht that $T \setminus s$ represents w.*
3. *If T represents $w' > w$, then $T \setminus s$ represents w for exactly one $s \in T$.*

Proof. The case $|T| \leq 2$ is obvious. For the first part, suppose that there are reflections $s_1, s_2, s_3 \in T$, such that $T \setminus s_i$ represents w. Without loss of generality we can assume $T = (T_1, s_1, T_2, s_2, T_3, s_3, T_4)$ and using the assumption we have

$$T_1 T_2 s_2 T_3 s_3 T_4 = T_1 s_1 T_2 T_3 s_3 T_4 \iff T_2 s_2 = s_1 T_2$$

by cancelling the same elements from the left and right of the first equation. Using this and the assumption again we obtain

$$w = T_1 s_1 T_2 s_2 T_3 T_4 = T_1 T_2 s_2 s_2 T_3 T_4 = T_1 T_2 T_3 T_4$$

which contradicts the assumption, that $l(w) = |T| + 1$.

Now suppose $\delta(T) = w$. Since $|T| = l(w)+1$, there is a reflection $s \in T$, such that $T = (T_1, s, T_2)$, (T_1, T_2) represents w and $t_1 > t_1 s$, where T_1 represents t_1. Using the third part of the lemma, omitting some reflection s' from T_1 yields a reduced expression for $t_1 s$, while $(T_1 \setminus s', s, T_2)$ must represent w.

The last part is exactly the Exchange condition or Proposition 4.16. □

Lemma 4.19. *Let Δ be a shellable simplicial complex in which every codimension 1 face is contained in at most two facets. Then Δ is a topological manifold-with-boundary that is homeomorphic to either a ball or a sphere. The facets of the topological boundary of Δ are the codimension 1 faces of Δ contained in exactly one facet of Δ.*

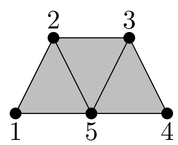
We will not proof this lemma which is Proposition 4.7.22 in [Bjö+00]. The next theorem is the main result we present in this section. It gives us a very easy-to-check property of subword complexes.

Theorem 4.20. *A subword complex $\Delta(Q,w)$ is either a ball or a sphere. A face $Q \setminus P$ is in the boundary of $\Delta(Q,w)$ if and only if $\delta(P) \neq w$.*

Proof. The first point in Lemma 4.18 shows that every codimension 1 face of $\Delta(Q,w)$ is contained in at most two facets, while the shellability of the complex is shown in Theorem 4.14. Thus, the conditions of Lemma 4.19 are satisfied, proving the first sentence of the statement.
Let $Q \setminus F$ be a face and $\delta(F) \neq w$. Then, by part one of Lemma 4.17, $\delta(F) > w$. Choosing T as in the third part of Lemma 4.17, we have by the third part of Lemma 4.18 that $Q \setminus T$ is a codimension 1 face contained in exactly one facet of $\Delta(Q,w)$. Thus, using Lemma 4.19, we conclude $Q \setminus F \subseteq Q \setminus T$ is in the boundary of $\Delta(Q,w)$.
If $\delta(F) = w$, the second part of Lemma 4.17 and Lemma 4.18 say that every codimension 1 face $Q \setminus T \in \Delta(Q,w)$ containing $Q \setminus F$ is contained in two facets of $\Delta(Q,w)$. Lemma 4.19 says each such $Q \setminus T$ is in the interior of $\Delta(Q,w)$, whence $Q \setminus F$ must itself be an interior face. □

Corollary 4.21. *The subword complex $\Delta(Q,w)$ is a sphere if $\delta(Q) = w$ and a ball otherwise.*

Example 4.12. The Demazur product of the word $Q = (s_2, s_1, s_2, s_1, s_2)$ in Example 4.8 is $\delta(Q) = s_2 s_1 s_2 = w$, hence $\Delta(Q,w)$ is a sphere (which we already saw as the pentagramm). Considering the subword complex with the same word Q but with $w' = s_2 s_1$ we can already tell, that $\Delta(Q,w')$ is homeomorphic to a ball. Indeed, the resulting simplicial complex is the following:



The next result shows that every spherical subword complex is always isomorphic to a subword complex with the longest element in the Coxeter group.

Theorem 4.22 (Theorem 3.7. [CLS13])**.** *Every spherical complex $\Delta(Q,w)$ is isomorphic to the complex $\Delta(Q', w_\circ)$ for some word Q' such that $\delta(Q') = w_\circ$.*

Proof. Since $\Delta(Q,w)$ is spherical we must have $\delta(Q) = w$. Now let R be a reduced word for $w^{-1}w_\circ = \delta(Q)^{-1}w_\circ$ and define the word Q' to be the concatenation of Q and R. Thus, we must have $\delta(Q') = w_\circ$, since $w_\circ$ is contained in Q', and every reduced expression of $w_\circ$ in Q' must contain all the letters in R. Since every reduced expression of $w_\circ$ in Q' is given by reduced expressions of w in Q together with all the letters in R the complexes $\Delta(Q,w)$ and $\Delta(Q', w_\circ)$ are isomorphic. □

Remark 4.4. In the same article [KM03] the authors stated the question on whether or not any spherical subword complex can be realized as a polytope. As we will see in a moment, this question is very closely related to Conjecture 3.12 and motivated by the question of when simplicial spheres are realizable as polytopes in general, since there are examples of simplicial complexes homeomorphic to spheres which are not polytopal. An example for a simplicial 3-sphere and its non-polytopality can be found in [BG87].

Now that we have established and classified subword complexes we can give their connection to multi-associahedra. This is the content of the next result.

Theorem 4.23 (Theorem 2.1. [Stu11])**.** *Consider the word*

$$Q_{n,k} = (s_{n-k-1}, \dots, s_1, s_{n-k-1}, \dots, s_2, \dots, s_{n-k-1}, s_{n-k-2}, s_{n-k-1})$$

and the element

$$w_{n,k} = [1, \dots, k, n-k, n-k-1, \dots, k+1] \in \mathfrak{S}_{n-k}.$$

Then $\triangle_{n,k} = \Delta(Q'_{n,k}, w_{n,k})$*, where* $Q'_{n,k}$ *is obtained from* $Q_{n,k}$ *by deleting all letters* s_i *for* $1 \leq i \leq k$*.*

Thus, the multi-associahedron is vertex-decomposable and shellable. The question is, speaking about subword complexes, whether or not the multi-associahedron is homeomorphic to a sphere, or a ball. The answer was given in the same article and uses the fact, that $w_{n,k}$ is the longest element in the parabolic subgroup generated by the simple reflections in $Q'_{n,k}$ ($\delta(Q'_{n,k}) \leq w_{n,k}$) and that $Q'_{n,k}$ contains a reduced expression for $w_{n,k}$ as a suffix ($\delta(Q'_{n,k}) \geq w_{n,k}$ by part one of Lemma 4.17).

Corollary 4.24 (Corollary 2.2. [Stu11])**.** *The multi-associahedron is a triangulated sphere.*

Remark 4.5. Since the connection between subword complexes and multi-associahedra is in type A we will not consider any other Coxeter group, as already mentioned.

In Subsection 2.1 in [CLS13] there is another description of the bijection between subword complexes and multi-associahedra, which we will quickly consider for a better understanding.

We want to obtain the multi-associahedron $\triangle_{n,k}$. For this, let $\mathfrak{S}_{m+1}$ be the symmetric group and consider the m simple transitions $s_i = (i, i+1)$, where $m = n - 2k - 1$. The k-relevant diagonals of the convex n-gon are in bijection with positions of letters in the word

$$Q = (s_m, \dots, s_1, \dots, s_m, \dots, s_1, s_m, \dots, s_1, s_m, \dots, s_2, \dots, s_m, s_{m-1}, s_m)$$

where the sequence $(s_m, \dots, s_1)$ in the beginning is repeated k-times. If the vertices of the n-gon are cyclically labeled by the integers from 1 to n, the bijection sends the ith letter of Q to the ith k-relevant diagonal in lexicographic order. A collection of k-relevant diagonals forms a facet of $\triangle_{n,k}$ if and only if the complement of the corresponding subword in Q forms a reduced expression for the permutation $[m+1, \dots, 2, 1] \in \mathfrak{S}_{m+1}$. We will look at the following example next to the one we gave at the beginning of this section. Lets look at a final example in addition to the one we gave at the beginning.

Example 4.13. We will look at 2-triangulations of the 7-gon, thus, $n = 7, k = 2$ and $m = 2$. Our word is $Q = (s_2, s_1, s_2, s_1, s_2, s_1, s_2)$ and the corresponding element $w = s_2 s_1 s_2 = s_1 s_2 s_1$.

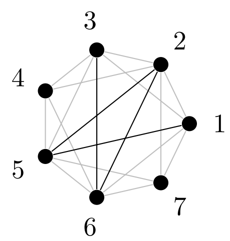
Q	s_2	s_1	s_2	s_1	s_2	s_1	s_2
Diagonal	q_1	q_2	q_3	q_4	q_5	q_6	q_7
t_1	×	×	×	○	○	○	○
t_2	×	×	○	○	×	○	○
t_3	×	×	○	○	○	○	×
t_4	×	○	○	×	×	○	○
t_5	×	○	○	×	○	○	×
t_6	×	○	○	○	○	×	×
t_7	○	×	×	×	○	○	○
t_8	○	×	×	○	○	×	○
t_9	○	×	○	○	×	×	○
t_{10}	○	○	×	×	×	○	○
t_{11}	○	○	×	×	○	○	×
t_{12}	○	○	×	○	○	×	×
t_{13}	○	○	○	×	×	×	○
t_{14}	○	○	○	○	×	×	×

Table 4.2: The reduced expressions of $w = s_2 s_1 s_2 = s_1 s_2 s_1$ in the given word $Q = (s_2, s_1, s_2, s_1, s_2, s_1, s_2)$. The rows yield the different 2-triangulations of the 7-gon.

In Table 4.2 are the facets of the subword complex $\Delta(Q, w)$ and thus of $\triangle_{7,2}$ listed. Furthermore, looking at the 7-gon and listing its 2-relevant diagonals yields

$$q_1 = [1, 4], q_2 = [1, 5], q_3 = [2, 5], q_4 = [2, 6], q_5 = [3, 6], q_6 = [3, 7], q_7 = [4, 7].$$

Looking at the row t_6 we have to draw the diagonals q_2, q_3, q_4 and q_5 to obtain the corresponding facet of $\triangle_{7,2}$.



5 Gale Duality

In the last two sections of this thesis we want to show some of the results of the research towards the Conjecture 3.12. The goal of this section is to give complete simplicial fan realizations for any spherical subword complex of type A_n for $n \leq 3$ and of multi-associahedra $\triangle_{n,k}$ for $n \leq 2k+4$. It is yet unclear whether these are normal fans of polytopes. The construction of the fans work for any subword complex of type A_n and yield complete and simplicial fans for $\triangle_{9,2}$ and $\triangle_{11,3}$, but which are not obtainable as normal fans of a polytope. The results and definitions of this section are taken from [BCL14].

Remark 5.1. Since we know that every spherical subword complex is isomorphic to a subword complex of the form $\Delta(Q, w_\circ)$, we will simply write $\Delta(Q)$.

5.1 Counting Matrices

After giving the necessary definitions we will state the main results of this section.

Definition 5.1. Let Q be a word and $M \in \mathbb{R}^{(r-N)\times r}$, where N denotes the length of the longest element $w_\circ$. We define a natural collection of cones $\mathcal{F}_{Q,M}$ in $\mathbb{R}^{r-N}$ as follows: The rays are given by the column vectors of M and its cones are spanned by the columns corresponding to faces of $\Delta(Q)$.

Let $S = \{s_1, \ldots, s_n\}$ be the set of simple transitions and c a Coxeter element, i.e., the product of the simple transitions in S in some arbitrary sequence. Define $P^m = c^m = (p_1, \ldots, p_{\tilde{r}})$ and choose m such that it contains the word Q as a subword. The number $\tilde{r} = mn$ denotes the number of letters in P_m. Our main ingredient is the following matrix, whose entries count the number of reduced expressions of c in P_m containing the letter p_i in position i, after restricting to standard parabolic subgroups, which were introduced earlier in Definition 4.4. In the following, fix a positive system Φ^+ with corresponding simple system Δ.

Definition 5.2. The **counting matrix** $D_{c,m} \in \mathbb{R}^{N\times\tilde{r}}$ is a matrix, whose rows correspond to positive roots and columns to the position $1 \leq j \leq \tilde{r}$ of the letters of $P_m = c^m$. Given $\alpha \in \Phi^+$ and $1 \leq j \leq \tilde{r}$, denote by $S_\alpha \subseteq S$ the subset of generators whose corresponding simple roots are used in the unique decomposition of α in Δ and by c_α the restriction of c to the generators in S_α. The entry $d_{\alpha,j}$ of $D_{c,m}$ is the number of reduced expressions of c_α in P using the letter p_j in position j. In particular: $p_j \notin S_\alpha \Rightarrow d_{\alpha,j} = 0$.

Example 5.1. Lets consider an example in A_2. Let $c = (s_1, s_2)$ and consider $P_4 = c^4 = (s_1, s_2, s_1, s_2, s_1, s_2, s_1, s_2)$. Fix $\Phi^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$ in this order. The positive roots determine the rows of the counting matrix $D_{c,4}$ in the following way: The ith row counts all reduced expressions of the restriction of c to the alphabet induced by the simple roots, which appear in the ith positive root in Φ^+. Since the first positive root is the simple root α_1, which corresponds to the simple transition s_1, the restriction is $c_{\alpha_1} = (s_1)$. Now, the columns determine which position in P_4 will be fixed to obtain the reduced expression for c_{α_1}. Since every second position in P_4 contains s_2, the corresponding entries are 0.

The second row corresponds to the positive root $\alpha_1 + \alpha_2$ which is obtained by adding the two simple roots α_1 and α_2. Thus, $c_{\alpha_1+\alpha_2} = c$ and for the entry in column i we have to count all possibilities to find c in P_4 while fixing the letter in position i. For example, fixing the third letter s_1 gives us three reduced expressions for c by choosing one of the three s_2 to the right of the third position. Thus, $d_{2,3} = 3$. The middle row is demonstrated more clearly in Table 5.1. The counting matrix is the following:

$$D_{c,4} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 1 & 3 & 2 & 2 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Reduced expressions in Q fixing position p_i								Number of reduced expressions
$\boxed{s_1}$	s_2	s_1	s_2	s_1	s_2	s_1	s_2	4
s_1	$\boxed{s_2}$	s_1	s_2	s_1	s_2	s_1	s_2	1
s_1	s_2	$\boxed{s_1}$	s_2	s_1	s_2	s_1	s_2	3
s_1	s_2	s_1	$\boxed{s_2}$	s_1	s_2	s_1	s_2	2
s_1	s_2	s_1	s_2	$\boxed{s_1}$	s_2	s_1	s_2	2
s_1	s_2	s_1	s_2	s_1	$\boxed{s_2}$	s_1	s_2	3
s_1	s_2	s_1	s_2	s_1	s_2	$\boxed{s_1}$	s_2	1
s_1	s_2	s_1	s_2	s_1	s_2	s_1	$\boxed{s_2}$	4

Table 5.1: An example of how the counting matrix works for the row corresponding to the root $\alpha_1 + \alpha_2$. The boxed letters are the fixed positions p_i and the colored letters contribute to the counter of $c_{\alpha_1+\alpha_2}$.

By choosing a sufficiently large m we can embed any word Q in $P_m = c^m$ via $\varphi : [r] \to [\tilde{r}]$ by sending positions in Q to positions in P_m, where we denote $[r] = \{1, \ldots, n\}$. For any such embedding we will construct a complete simplicial fan realization of the subword complex $\Delta(Q)$.

Definition 5.3. The **restricted matrix** D_φ is the restriction of $D_{c,m}$ to the columns $\varphi(1), \ldots, \varphi(r)$ corresponding to the positions of the letters of Q embedded in P_m.

The next definition contains the main ingredient of the technique used in this section. There is a lot more to say about Gale duality and other coherent concepts, but we will not go into more details here and just adopt the term.

Definition 5.4. Let A be a full rank matrix. We call a matrix B a **Gale dual matrix** of A, if the rows of B form a basis for the kernel of A. From linear algebra it is clear that B is determined up to linear transformation of the rows.

Example 5.2. We will continue the example from above. Consider the word

$$Q = (s_1, s_2, s_2, s_1, s_1)$$

and let φ be the embedding of Q in P_4 by mapping the positions in Q to the first possible position in P_4. The resulting restricted matrix is

$$D_\varphi = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 4 & 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

A Gale dual matrix of D_φ is

$$M_\varphi = \begin{pmatrix} -1 & -2 & 2 & 1 & 0 \\ -1 & -3 & 3 & 0 & 1 \end{pmatrix}.$$

We will often use this notation for the Gale dual matrix corresponding to the restricted matrix.

In the following theorem we will give the main result of this section, which yields infinitely many complete simplicial fan realizations of certain subword complexes. Remark, that computations in [BCL14] showed that no configuration makes these fans the normal fan of a polytope.

Theorem 5.1. *Let $\Delta(Q)$ be a spherical subword complex of type A_n with $n \leq 3$ and let φ be an embedding of Q into c^m. The fan $\mathcal{F}_{Q,M_\varphi}$ is a complete simplicial fan realization of $\Delta(Q)$.*

In the cases where φ is the natural embedding and $Q = c^m$ we get explicit realizations.

Corollary 5.2. *Let $c = (s_2, s_1, s_3)$ be a Coxeter element of type A_3 and $Q = c^m$ with $m \geq 3$. The fan $\mathcal{F}_{Q,M_{213,m}}$ is a complete simplicial fan realization of $\Delta(Q)$ for the matrix $M_{213,m}$ below.*

$$M_{213,m}^T = \left(\begin{array}{c|c|c} \multicolumn{3}{c}{-I_{3m-6}} \\ \hline B_{213,m-2} & \cdots & B_{213,1} \end{array}\right) \in \mathbb{R}^{(3m)\times(3m-6)}$$

where the entries depend on the functions $S(i) = i^2$ and $T(i) = i(i+1)/2$ and

$$B_{213,i} = \begin{pmatrix} S(i+1) & -T(i) & -T(i) \\ 2T(i) & -T(i-1)+1 & -T(i) \\ 2T(i) & -T(i) & -T(i-1)+1 \\ -S(i+1)+1 & T(i) & T(i) \\ -2T(i) & T(i-1) & T(i) \\ -2T(i) & T(i) & T(i-1) \end{pmatrix}.$$

Remark 5.2. There are explicit forms for the counting matrix and its Gale dual matrix like in Corollary 5.2. For example, we can give the following counting matrix and its Gale dual in type A_2 with $c = (s_1, s_2)$ and rows in the order corresponding to $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$:

$$D_{c,m} = \begin{pmatrix} 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 \\ m & 1 & m-1 & 2 & \dots & m-1 & 1 & m \\ 0 & 1 & 0 & 1 & \dots & 1 & 0 & 1 \end{pmatrix}$$

$$M_{c,m} = \left(\begin{array}{c|c} -I_{2m-3} & \begin{array}{c} E_1 \\ \hline \vdots \\ \hline E_{m-2} \\ \hline -1 \quad 1 \quad 1 \end{array} \end{array}\right), \quad E_i = \begin{pmatrix} -m+1 & 1 & m-i \\ m-1 & 0 & -m+i+1 \end{pmatrix}$$

Since their computation is purely counting positions of simple reflections in the word Q and transformation of linear equations, we will not prove this representation.

We can also express these results in terms of multi-associahedra: Consider $\triangle_{n,k}$ where $n = 2k + 4$. Then there are $3k + 6$ many k-relevant diagonals which are in correspondence with the columns of $M_{213,k+2}$ as follows:

- cyclically label the vertices of the n-gon from 1 up to n
- note the k-relevant diagonals in lexicographic order
- the first diagonal corresponds to the last column of $M_{213,k+2}$
- the other k-relevant diagonals correspond to the other columns in lexicographic order

Let $\mathcal{F}_k$ be the simplicial fan in $\mathbb{R}^{3k}$ whose rays are the columns of $M_{213,k+2}$ and whose cones are spanned by the column vectors corresponding to faces of $\triangle_{2k+4,k}$.

Corollary 5.3. *$\mathcal{F}_k$ is a complete simplicial fan realization of $\triangle_{2k+4,k}$.*

Remark 5.3. In [BCL14] is also a fan realization for spherical subword complexes of type B.

5.2 The Graph of Reduced Expressions

We will now go into the details of the proofs of the results summarized before. One ingredient will be the graph of reduced expressions of an element in W and the sign function on this graph. We will discuss both, but will not go into detail about corresponding proofs and refer to [BCL14]. Remember the term *braid relation* which we discussed in Example 4.5.

Definition 5.5. Let (W, S) be a finite Coxeter system and let $w \in W$ be an arbitrary element. We define the **graph of reduced expressions of** w as follows:

- vertices: reduced expressions of w
- edges: if the vertices are related by a single braid relation
- label an edge with the pair of indices $\{i, j\}$ of the braid move m_{ij}

We denote the graph of w by $G(w)$.

Example 5.3. Consider $W = \mathfrak{S}_4$ and let $w_\circ$ be the longest element in type A_3. There are 16 different reduced words for $w_\circ$ for example 323123, where we denote the simple reflections in short by $i = s_i = (i, i + 1)$. Since we have the braid relation $323 = 232$ we know that the vertices 323123 and 232123 are connected via an edge labeled $\{2, 3\}$.

Remark 5.4. It is known that for any finite Coxeter system (W, S) and every element $w \in W$ the graph $G(w)$ is connected, e.g., Theorem 3.3.(ii) in [BB06].

Definition 5.6. We call two pairs of indices $\{i, j\}$ and $\{i', j'\}$ **conjugated**, if there exists a $w \in W$ such that $s_{i'} = w^{-1} s_i w$ and $s_{j'} = w^{-1} s_j w$. Furthermore, we say that they are in the same **automorphism class**, if $\{i', j'\}$ is the image of $\{i, j\}$ under an automorphism of W.

The first result we will discuss is that $G(w)$ is a mega bipartite graph, that is, the graph can be divided into two disjoint and independent sets (bipartite) and any graph obtained by contracting the edges corresponding to a specific set of braid relations is bipartite as well (mega). To give a precise statement we will need one more definitions.

Definition 5.7. Let $Z = \{\{i,j\} \mid 1 \leq i < j \leq n\}$. We call Z **stabled**, if for every $\{i,j\} \in Z$ and every of its images $\{i',j'\}$ via an automorphism of W, the pair $\{i',j'\} \in Z$. For every stabled subset Z, let $G^Z(w)$ be the graph obtained from $G(w)$ by contracting all the edges corresponding to braid relations m_{ij} for $\{i,j\} \notin Z$.

Special cases are where Z consists of pairs $\{i,j\}$, where m_{ij} is even (respectively odd). Thus, $G^{\text{even}}(w)$ (resp. $G^{\text{odd}}(w)$) is the graph obtained from $G(w)$ by contracting all edges corresponding to non-even (resp. non-odd) braid relations. See Table 5.2 for an illustration. We explain this by considering the following example.

Example 5.4. Let $W = \mathfrak{S}_4$ be of type A_3. There are the three braid relations $m_{12} = m_{23} = 3$ and $m_{13} = 2$, thus, we have the sets $Z^{even} = \{\{1,3\}\}$ and $Z^{odd} = \{\{1,2\},\{2,3\}\}$. Remember, that the conjugation of transpositions is subject to the rule $w(i,j)w^{-1} = (w(i), w(j))$. But there is no permutation w that conjugates s_1 to s_1 and in the same time s_3 to s_2, since for the first condition we would need the identity on 1 and 2, but for the second condition we would need w to permute 2 and 3, and 3 and 4, which is not possible. In the same way we can see that $\{1,3\}$ and $\{2,3\}$ are not conjugated. On the other hand, the permutation $(1,2,3,4)$ conjugates $\{1,2\}$ and $\{2,3\}$. Thus, Z^{even} and Z^{odd} are stabled. In general, in type A_n any pair $\{i,i+1\}$ is conjugated to any other pair $\{j,j+1\}$ and any pair $\{k,l\}$, where $l-k>1$, is conjugated to any other similar pair.

The following two statements summarize the results we need to know about the graph of reduced expressions and stabled subsets.

Theorem 5.4. *Let (W,S) be a finite Coxeter system and $w \in W$. Then, for any stabled set Z the graph $G^Z(w)$ is bipartite. In particular, $G(w), G^{even}(w)$ and $G^{odd}(w)$ are bipartite.*

The next definition introduces the sign function on the graph of reduced expression. The sign of a reduced expression will be important when we will talk about the signature matrix, a key definition that appears in the next subsection.

Definition 5.8. The **sign function** on reduced expressions of $w_\circ$ is a map

$$\text{sign} : \{\text{reduced expressions of } w_\circ\} \longrightarrow \{-1,+1\}$$

such that if w, w' are connected by a braid move m_{ij}, then

$$\text{sign}(w') = (-1)^{m_{ij}-1}\text{sign}(w).$$

Since the graph of reduced expressions is connected, the defined function is unique up to global multiplication with -1. The following lemma gives another method to compute the sign of a reduced expression.

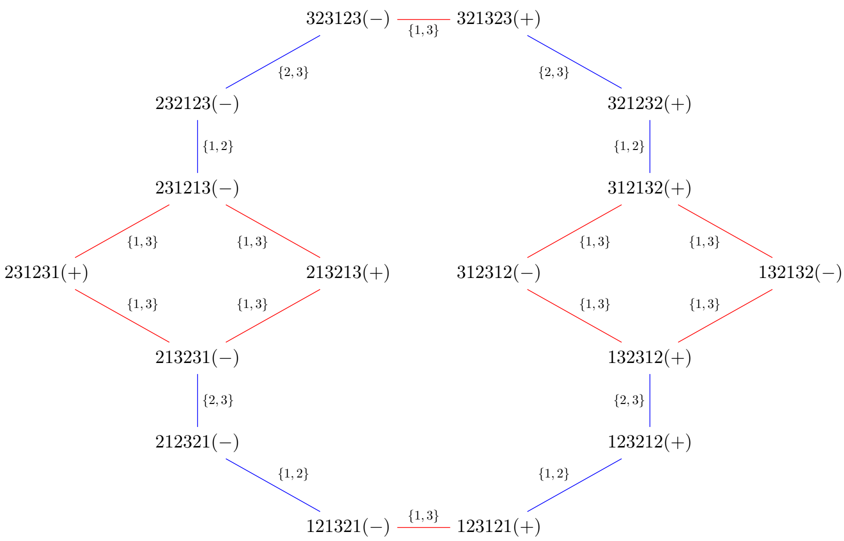


Table 5.2: The graph of reduced expressions of the longest element in type A_3 and their sign in round brackets. The blue edges are corresponding to odd braid relations and the red edges are corresponding to even braid relations. The contraction of either stabled subsets yields a bipartite graph.

Lemma 5.5. *The sign function on reduced expressions of $w_\circ$ is the unique map, up to global multiplication with -1, such that if the elements $w = w_1 \dots w_N$ and $w' = w'_1 \dots w'_N$ are two reduced expressions of $w_\circ$ connected by a flip (thus, $w \setminus w_i = w' \setminus w'_j$), then $\operatorname{sign}(w') = (-1)^{i-j} \operatorname{sign}(w)$.*

Example 5.5. One possible way of defining the sign in type A_3 is marked in Table 5.2. We will continue to use these signs in our further studies of examples in this type. In Remark 3.7 in [BCL14] there is a more detailed explanation about the correspondence of the sign function in type A_n and multi-permutations.

Remark 5.5. The proofs of the statements up to this moments would need the introduction of Coxeter complexes, Coxeter arrangements and other objects. In order not to prolong the discussion about the graph of reduced expressions and its sign function unnecessarily and to focus on our main topic we omit the proofs and refer to Section 3 of [BCL14] for more details.

5.3 Signature Matrices

We will now introduce signature matrices for a pair consisting of a word Q and an element $w \in W$. From now on let $Q = (q_1, \dots, q_r)$ be a word containing at least one reduced expression of $w_\circ$ and let $N := l(w_\circ)$.

Definition 5.9. A matrix $M \in \mathbb{R}^{N \times r}$ is a **signature matrix** of type W for the pair $(Q, w_\circ)$, if for every reduced expression w of $w_\circ$ in Q the inequality $\operatorname{sign}(w) \det(w) > 0$ holds, where $\det(w)$ denotes the determinant of the matrix M restricted to the column corresponding to w.

Example 5.6. Consider type A_2. The word $Q = (s_1, s_2, s_1, s_2)$ contains both reduced expressions for $w_\circ$. Computing the corresponding counting matrix, using the positive system $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$ in this order, yields

$$D_{(s_1,s_2),2} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Since the sign of both reduced expressions is positive (they differ by the braid move $m_{12} = 3$), it suffices to compute the determinant of the restricted matrices

$$M_{s_1s_2s_1} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } M_{s_2s_1s_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

which both are 1. Thus, the counting matrix $D_{(s_1,s_2),2}$ is a signature matrix of type A_2 for the pair $(Q, w_\circ)$.

The Coxeter signature matrix plays a fundamental role in the concept of obtaining fan realizations of subword complexes. We will see in a moment that finding a complete simplicial fan realization of $\Delta(Q)$ is almost equivalent to finding a Coxeter signature matrix for the pair $(Q, w_\circ)$.

Proposition 5.6. *Let W be a Coxeter group of type A_n, where $n \leq 3$, and let c be a Coxeter element. For $Q = c^m$, the counting matrix $D_{c,m}$ is a signature matrix for the pair $(Q, w_\circ)$.*

Proof. We will inspect each case one by one.
Case A_1: There is only one reduced expression whose sign is 1. Since the counting matrix is the $(1 \times m)$-matrix $(1 \ldots 1)$, the corresponding determinant is always 1.
Case A_2: There are exactly two reduced expressions: $s_1s_2s_1$ and $s_2s_1s_2$. This is Example 5.6. The case for the other Coxeter element $c = (s_2, s_1)$ is nearly the same with the difference, that the positive root system needs to be in a different order (we have to swap the order of α_1 and α_2, since otherwise the first and last row of the counting matrix above would be swapped and thus the determinant would be -1).
Case A_3: In this case we have 16 different reduced expression (see Table 5.2). The proof is done by brute force computation. We will only give an example and refer for the rest to the proof in Section 4 of [BCL14], who used the computer software Sage to compute every case.
Lets consider the Coxeter element $c = (s_1, s_2, s_3)$ and consider the positive system $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2, \alpha_2 + \alpha_3, \alpha_3\}$ with their corresponding rows in this order. It is easy to verify that we have to look at the word $Q = c^4$ to be able to compute the determinant of every of the 16 reduced expressions of $w_\circ$. The corresponding counting matrix is the following:

$$D_{c,4} = \left(\begin{array}{cccccccccccc} s_1 & s_2 & s_3 & s_1 & s_2 & s_3 & s_1 & s_2 & s_3 & s_1 & s_2 & s_3 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & 0 & 3 & 2 & 0 & 2 & 3 & 0 & 1 & 4 & 0 \\ 10 & 4 & 1 & 6 & 6 & 3 & 3 & 6 & 6 & 1 & 4 & 10 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 3 & 2 & 0 & 2 & 3 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

We can now compute the determinants of the restriction of $D_{c,4}$ to the appropriate columns (remember that we do this by picking the first fitting columns for the corresponding reduced expression of $w_\circ$, e.g., if $w = 321323 (= s_3s_2s_1s_3s_2s_3)$ we restrict the counting matrix to the columns 3,5,7,9,11 and 12). After computing the matrices we can compare the signs of the reduced expressions (for that look at Table 5.2) with that of their determinant. Let us look at an example.
For the reduced expression $w = 312312$ the corresponding restricted matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 2 & 3 \\ 1 & 6 & 6 & 3 & 3 & 6 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 3 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and its determinant is -1. Since the sign of w is negative, their product is $1 > 0$, as requested. □

Remark 5.6. The authors claim, without explanation, that the proposition does not hold for $n \geq 4$.

Our main goal for now will be to reformulate the problem of finding fan realizations of subword complexes in terms of Coxeter signature matrices. For this, let $Q = (q_1, \ldots, q_r)$ contain at least one reduced expression of $w_\circ$ and let $N = l(w_\circ)$. Let $M \in \mathbb{R}^{(r-N)\times r}$ be of full rank and $M^G \in \mathbb{R}^{N\times r}$ be a Gale dual matrix with associated fan $\mathcal{F}_{Q,M}$.

Theorem 5.7. *$\mathcal{F}_{Q,M}$ is a complete simplicial fan realization of the spherical subword complex $\Delta(Q)$ if and only if*

- *(S) M^G is a Coxeter signature matrix for the pair $(Q, w_\circ)$ and*
- *(I) there is a facet of $\Delta(Q)$ for which the interior of its associated cone is not intersected by any other cone.*

The proof of this theorem follows directly from the next two lemmas, the first of which we will not prove.

Lemma 5.8. *$\mathcal{F}_{Q,M}$ complete simplicial fan if and only if the following three conditions are satisfied:*

- *(B) The vectors associated to a facet of $\Delta(Q)$ form a basis of R^{r-N} (Basis).*
- *(F) If I and J are two adjacent facets that differ by a flip (thus, $I\backslash\{i\} = J\backslash\{j\}$) then the vectors associated to i and j lie in opposite sites of the hyperplane generated by the vectors associated to the intersection $I \cap J$ (Flip).*
- *(I) There is a facet for which the interior of its associated cone is not intersected by any other cone (Injectivity).*

Lemma 5.9. *The Conditions (B) and (F) are satisfied if and only if M^G is a Coxeter signature matrix for the pair $(Q, w_\circ)$.*

Sketch of the proof. First, we can reformulate the the conditions (B) and (F) in Lemma 5.8 for Gale dual matrices, i.e., they are satisfied for a matrix M, if and only if M^G satisfies the following conditions:

1. The vectors associated to the complement of a facet of $\Delta(Q)$ form a basis of $\mathbb{R}^N$.
2. If I and J are two adjacent facets that differ by a flip $I \setminus \{i\} = J \setminus \{j\}$, then the vectors associated to i and j lie in the same side of the hyperplane generated by vectors associated to the complement of $I \cup J$.

The first point implies that for every reduced expression $w \subset Q$ of $w_\circ$ we have $\det(w) \neq 0$. By using Lemma 5.5 and setting the sign and determinant of $w_1 \dots w_N \subset Q$ to be positive, the second point implies that the sign and determinant of w is determined by $\operatorname{sign}(w)\det(w) > 0$. Conversely, these inequalities imply both conditions. □

Remark 5.7. Although condition (I) in Theorem 5.7 is in general difficult to prove, the condition (S) seems to be the most important one.

5.4 Proof of Theorem 5.1

We are now in the position to prove the main result: Theorem 5.1. For this, let $\Delta(Q)$ be a spherical subword complex of type A_n with $n \leq 3$ and let φ be an embedding of Q into c^m. We will prove this result in two steps. The first one will be to prove that it is sufficient to consider the case where $Q = c^m$ and the embedding φ is the trivial embedding. The second step will be the proof of this explicit case.

The proof of the first step is based on the following two lemmas, the first of which is a standard result that we will not prove.

Lemma 5.10. *Let I be a cone in a complete simplicial fan $\mathcal{F}$. The projection of the link $\operatorname{lk}(I, \mathcal{F})$ to the orthogonal space of I is a complete simplicial fan realizing $\operatorname{lk}(I, \mathcal{F})$.*

Lemma 5.11. *Let $Q = c^m$ and φ be the trivial embedding of c^m into itself. If the fan $\mathcal{F}_{Q,M_\varphi}$ is complete, then the fan $\mathcal{F}_{Q',M_{\varphi'}}$ is complete for any embedding φ' of a word Q' into c^m.*

Proof. The idea for the proving this lemma is to obtain the fan associated to Q' as a projection of the fan associated to Q and using Lemma 5.10. We will do this by giving appropriate choices of M_φ and M'_φ and since different choices of Gale dual matrices only effect the fans by linear transformations (by the definition of the fans and Gale dual matrices), and particularly do not affect their completeness, the result will follow.

Throughout the proof we will denote the length of $Q = c^m$ by $\tilde{r}$. Furthermore we will assume that Q' contains at least one reduced expression of $w_\circ$ (otherwise its associated fan would be empty and there would be nothing to prove).

Let $I \subset [\tilde{r}]$ be the face of $\Delta(Q)$ containing the positions in c^m which are not in $\varphi'(Q')$. Then there is a natural isomorphism (just like in the proof of Theorem 4.14)

$$\Delta(Q') \cong \operatorname{lk}(I, \Delta(Q)) \cong \operatorname{lk}(I, \mathcal{F}_{Q,M_\varphi}).$$

Now let I' be a facet of $\Delta(Q)$ containing I. Since the vectors associated to facets form a basis, we can assume that the restriction of M_φ to the columns with indices in I' is the identity matrix (otherwise multiply its inverse with M_φ from the left - the result would still be a Gale dual matrix of D_φ). Since $I \subset I'$, the columns of M_φ with indices in I are certain canonical basis vectors $\{e_{i_1}, \ldots, e_{i_k}\}$, where $k = |I|$. Now let M' be the matrix obtained from M_φ by removing the columns with indices in I and the rows with indices in $\{i_1, \ldots, i_k\}$. We claim, that M' is a Gale dual matrix $M_{\varphi'}$ of $D_{\varphi'}$. This follows from the following two points:

1. $D_{\varphi'}$ is obtained from D_φ by removing the columns with indices in I (by definition of the restricted counting matrix).
2. M_φ has a zero entry in every position which is a column in I and a row not in $\{i_1, \ldots, i_k\}$ (since its the restriction of the identity).

Now, taking $M_{\varphi'} = M'$, we deduce that $\mathcal{F}_{Q',M_{\varphi'}}$ is the projection of $\operatorname{lk}(I, \mathcal{F}_{Q,M_\varphi})$ to the orthogonal space of the cone corresponding to I. Using Lemma 5.10 and the isomorphism above we can say that $\mathcal{F}_{Q',M_{\varphi'}}$ is a complete simplicial fan realizing $\Delta(Q')$. □

We will now finish the proof of Theorem 5.1 by inspecting the case $Q = c^m$.

Proof of Theorem 5.1. To prove this case we follow the steps in Theorem 5.7. The signature condition is equivalent to the statement in Proposition 5.6. Since φ is the trivial embedding we have $M_\varphi^G = D_{c,m}$, which is a Coxeter signature matrix for $(Q, w_\circ)$ as desired. For the injectivity condition we need to prove that there is a cone whose interior is not intersected by any other cone. We will do this, again, by brute force inspection of each case.

Type A_1: The counting matrix in this case is the same as in the proof of Proposition 5.6. Its kernel is given by the span

$$\mathbb{R} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \mathbb{R} \cdot \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Thus the Gale dual matrix defines the rays of the corresponding fan via m rays given by the $m-1$ basis vectors in $\mathbb{R}^{m-1}$. Thus, the maximal cones correspond to subsets of $m-1$ rays and the subword complex is isomorphic to the boundary of an $m-1$ dimensional simplex.

Type A_2: Let us first consider the Gale dual matrix, which has the following form, which can be obtained from its corresponding counting matrix (compare to Remark 5.2):

$$M_{12,m} = \left(\begin{array}{c|c} -I_{2m-3} & \begin{array}{c} E_1 \\ \hline \vdots \\ \hline E_{m-2} \\ \hline -1 \quad 1 \quad 1 \end{array} \end{array} \right), \quad E_i = \begin{pmatrix} -m+1 & 1 & m-i \\ m-1 & 0 & -m+i+1 \end{pmatrix}$$

To prove this case we fix the cone C^* corresponding to the negative orthant (this is corresponding to the facet of $\Delta(Q)$ whose complement is the reduced expression $s_2s_1s_2$ of $w_\circ$ in the last three positions of Q, thus, the rays are given by the negative identity matrix $-I_{2m-3}$). We will show, that the interior of C^* is not intersected by any other cone. Let C be a cone corresponding to a subword of c^m whose complement is a reduced expression of $w_\circ$. There are three possible cases for C that we have to check.

The corresponding facet uses two of the last three positions of Q. Using Lemma 5.8 we can deduce that, since M_φ^G is a Coxeter signature matrix and thus two adjacent cones lie in opposite sides of the hyperplane spanned by their intersection (which is a negative basis vector), the interior of C^* and C is not intersecting .

The corresponding facet uses one of the last three positions of Q. In this case, C uses all negative basis vectors except for two and two of the last three columns of $M_{12,m}$. Denote by $v_1, v_2 \in \mathbb{R}^2$ the restrictions of these columns to the coordinates corresponding to the negative basis vectors not used in C. We will now explain why the cone spanned by v_1 and v_2 does not intersect the negative orthant in $\mathbb{R}^2$, which is equivalent to say that C does not intersect C^*. We have to differentiate the cases where the letter in the last three positions of Q in the reduced expression is the next-to-last s_2, the last s_1 or the last s_2, thus, which of the last three columns is not in the fan. In each case, we provide a vector $v \in \mathbb{R}^2$ with non-negative entries whose inner product with v_1 and v_2 is non-negative and its hyperplane orthogonal to v separates the negative orthant and the cone spanned by v_1 and v_2.

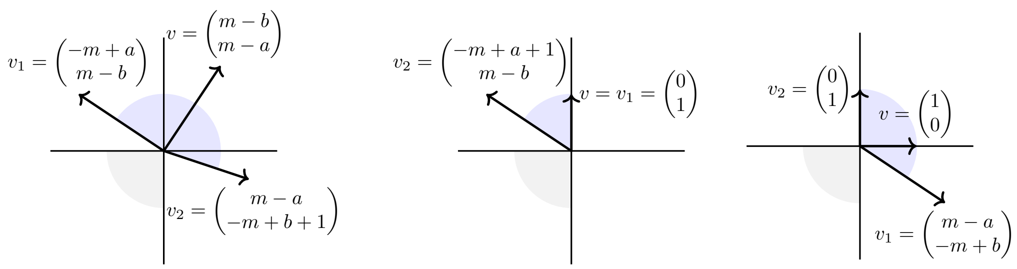


Table 5.3: The three cases in type A_2 using two negative basis vectors.

Note, that the two negative vectors used in the reduced expression have to be corresponding to different simple reflections, thus, they can not be both corresponding to s_1 nor to s_2. This implies that the restricted vectors can only be of the form given in Table 5.3. The first picture shows the case, where the reduced expression uses the last s_1, in the middle the case where the next-to-last s_2 is used and on the right the final case. The parameters $a \leq b \leq m$ denote the copy of c in which the negative basis vectors that are not used in C are taken in the power c^m counted from left to right.

The corresponding facet uses none of the last three positions of Q. In this case, all of the last three columns are used. The procedure is the same as before, but in $\mathbb{R}^3$. Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be the restriction of the last three columns of $M_{12,m}$ to the three negative basis vectors that are not used in C.

There are two possible cases corresponding to the two reduced expressions of $w_\circ$:

$$\begin{pmatrix} -m+a & 1 & m-a \\ m-b & 0 & -m+b+1 \\ -m+c & 1 & m-c \end{pmatrix} \text{ and } \begin{pmatrix} m-a & 0 & -m+a+1 \\ m+b & 1 & m-b \\ m-c & 0 & -m+c+1 \end{pmatrix}$$

where the vectors v_1, v_2, v_3 are given by the columns. Again, the four parameters $a \leq b \leq c \leq m$ denote the copy of c in which the negative basis vectors that are not used in C are taken in the power c^m counted from left to right. In both cases, the hyperplane orthogonal to the vector

$$v = \begin{pmatrix} 0 \\ m-c \\ m-b \end{pmatrix}$$

separates the negative orthant and the cone spanned by v_1, v_2 and v_3, since v has non-negative entries and the inner product of v with every v_i is non-negative.

Type A_3: This case is considered in exactly the same way as in the case for type A_2. Since there are more reduced expressions for $w_\circ$ and the Gale dual matrix for the counting matrix is larger, the consideration of the different cases is just more extensive. We refer for the details to the proof in Section 6 of [BCL14]. □

6 Matroids and Rigidity

In this section we will look at more recent results concerning our conjecture. The article [RS22b] builds the foundations for the statements and techniques we will discuss here. It was first published in 2022, eight years after the previous article [BCL14] and using completely different methods. We will first present the results of this research and the idea behind them. Furthermore, we will introduce matroids, which are in the focus of the concepts in this section. The last subsection, however, deals with the polytopality of $\triangle_{8,2}$, which has already been proven in 2009 in [BP09] with similar techniques.

Indeed, we will present new polytopal realizations of multi-associahedra. To lower the expectations, there are not too many...

Theorem 6.1. *For $(n,k) \in \{(9,2),(10,2),(10,3)\}$ the multi-associahedron is a polytopal sphere. Furthermore, we can realize $\triangle_{13,4}$ as a complete simplicial fan.*

On the other hand we will also show the limitations of the techniques used.

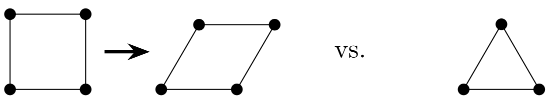
Theorem 6.2. *If $k \geq 3$ and $n \geq 2k+6$ then no choice of points will realize $\triangle_{n,k}$ as a fan using the methods in this section.*

The question at hand is, what these methods are and how they are connected to multi-associahedra. This will be the content of the next subsection.

Remark 6.1. In this section we will work with the reduced multi-associahedron denoted by $\overline{\triangle}_{n,k}$, thus, the simplicial complex we obtain by omitting the kn many k-irrelevant and k-boundary edges of the k-triangulations (compare with Corollary 3.11). It was proven in [Jon05] that this is, not to our surprise, a shellable sphere of dimension $k(n-2k-1)-1$.

6.1 The Idea

The methods that will be used come from rigidity theory. Consider the square with hinges as vertices. This object is rather flexible, as it can be tilted to a parallelogram by 'pressing' against the upper part of one of its sides. But there are structures which are not flexible at all, for example the triangle with hinges as vertices. No force can bend the triangle, no matter where the force is applied. In general, we will look at matrices which can be understood as encodements of forces along the edges of a graph. We call them *rigidity matrices* and they will give us the vectors for our polytopal realizations.



Let us talk about the connection of this theory with k-triangulations and multi-associahedra. We know, that the number of edges in a k-triangulation equals $k(2n-2k-1)$. This coincides with the rank of so called *abstract rigidity matroids* of dimension $2k$ on n elements, and such a matroid corresponds to a *rigidity matrix* mentioned earlier. The idea is, that for any given choice of points $p_1, \ldots, p_n \in \mathbb{R}^{2k}$ in *general position* the rows of their *rigidity matrix* gives a real vector configuration of the desired rank $k(2n-2k-1)$. The question is, if using these vectors as rays yield a fan and whether or not this fan is polytopal. Based on previous research, the authors of [RS22b] decided to use this strategy for points along the moment curve $\{(t, t^2, \ldots, t^{2k}) \mid t \in \mathbb{R}\}$.

6.2 Introducing Matroids

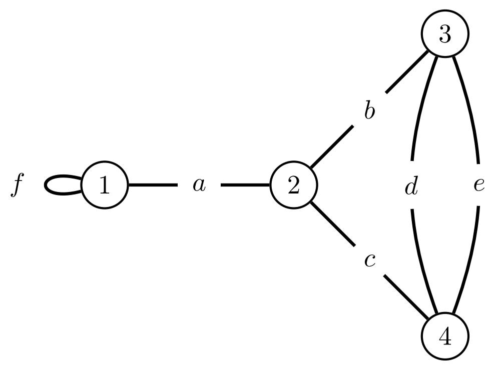
We will now start with the basic definitions of this section, where the first definition is that of a matroid. The content of this subsection is based on [RS22b] and is expanded by [Bjö+00].

Definition 6.1. A **finite matroid** $(E, \mathcal{I})$ is defined by a finite set of elements E and a non-empty family $\mathcal{I}$ of subsets of E, called the independent sets of the matroid, and the following properties:

- The empty set is independent, thus $\emptyset \in \mathcal{I}$.
- Every subset of an independent set is independent itself.
- If $I_1, I_2 \in \mathcal{I}$, such that $|I_1| < |I_2|$, then there exists an element $x \in I_2 \setminus I_1$ with $I_1 \cup \{x\} \in \mathcal{I}$.

A maximal independent set $\mathcal{B} \in \mathcal{I}$ is called **basis** for the matroid, whereas a minimal dependent subset of E (that is, a not independent set whose proper subsets are independent) is called a **circuit**. The **rank** of the matroid is the size of its largest independent set.

Example 6.1. Consider the undirected graph $G = (V, E)$ with its set of vertices $V = \{1, 2, 3, 4\}$ and set of edges $E = \{a, b, c, d, e, f\}$, visualized as follows:



The matroid $M = (E, \mathcal{I})$ contains all the cycle-free subgraphs of G, thus

$$\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{b,c\}, \{b,d\}, \{b,e\}, \{c,d\}, \{c,e\}, \{a,b,c\}, \{a,b,d\}, \{a,b,e\}, \{a,c,d\}, \{a,c,e\}\},$$

where a cycle is a path whose first and last vertex are equal. The last five sets are bases of the matroid, the set $\{b, c, d\}$ is a circuit and M has rank 3.

We can extend this example to arbitrary graphs, which justifies the following definition.

Definition 6.2. Let (V, E) be a (multi-) graph. A **graphic matroid** $(E, \mathcal{I})$ has E as its elements and a set of edges is independent whenever it is a forest (thus, it does not contain simple cycles).

Now that the connection between matroids and graphs is build, we introduce the concept of rigidity matrices and matroids.

Definition 6.3. Let $G = (V, E)$ be an undirected graph of n labeled vertices and m edges, embedded into $\mathbb{R}^d$, i.e., every vertex v_j corresponds to a vector $(x_1^{(j)}, \dots, x_d^{(j)})^T$. We define its **rigidity matrix** with m rows and nd columns as follows:

- The entry in row e and column (v, i) is zero, if and only if v is not an endpoint of e.
- If edge e has vertices v and u as endpoints, then the value of the entry is the difference between the ith coordinates of v and u.

The **rigidity matroid** is the matroid whose elements are the edges E and where a set of edges is independent if it corresponds to a set of rows of the rigidity matrix that is linearly independent.

Example 6.2. Continuing Example 6.1 and embedding G in $\mathbb{R}^2$ by

$$1 \mapsto v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 2 \mapsto v_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, 3 \mapsto v_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, 4 \mapsto v_4 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

yields the following rigidity matrix

$$\begin{pmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 & 0 & 0 \\ 0 & 0 & x_2 - x_4 & y_2 - y_4 & 0 & 0 & x_4 - x_2 & y_4 - y_2 \\ 0 & 0 & 0 & 0 & x_3 - x_4 & y_3 - y_4 & x_4 - x_3 & y_4 - y_3 \\ 0 & 0 & 0 & 0 & x_3 - x_4 & y_3 - y_4 & x_4 - x_3 & y_4 - y_3 \\ x_1 - x_1 & x_2 - x_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $v_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$. Inserting the numbers results in the matrix

$$\begin{pmatrix} -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 2 & 2 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The set of edges $\{a, b, c, d\}$ is a basis for the corresponding rigidity matroid, the set $\{d, e\}$ is a circuit and the matroid has rank 4.

The rigidity matrix defined above has similarly defined relatives, which we will introduce now. Although each of them has its own history and applications we will not go into further details, but they will be important in our later studies. For now, denote by $\mathbf{p} = (p_1, \dots, p_n)$ a configuration (i.e. an ordered set) of n points in $\mathbb{R}^d$.

Definition 6.4. For a configuration $\mathbf{p}$ of n points we define their **bar-and-joint** rigidity matrix as the following $\binom{n}{d} \times nd$ matrix

$$R(\mathbf{p}) = \begin{pmatrix} p_1 - p_2 & p_2 - p_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ p_1 - p_3 & 0 & p_3 - p_1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ p_1 - p_n & 0 & 0 & 0 & \dots & 0 & 0 & p_n - p_1 \\ 0 & p_2 - p_3 & p_3 - p_2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & p_{n-1} - p_n & p_n - p_{n-1} \end{pmatrix}.$$

Instead of looking at the difference of coordinates of points in $\mathbb{R}^d$ we have the difference of complete points in the respective entries of the matrix. This matrix is best to read row by row as follows: Consider the set of ordered edges

$$\binom{[n]}{2} = \{[i,j] \mid 1 \leq i < j \leq n\}.$$

These edges yield the complete graph in n vertices K_n. Then, each of the edges $[i,j] \in \binom{[n]}{2}$ has its own row in the matrix and rows can be considered labeled by edges in K_n. Thus, speaking about vertices as hinges or joints and forces along edges of a graph, this information can be decoded in this matrix by interpreting the coefficients of linear dependencies among the rows of $R(\mathbf{p})$ as forces along the edges and the resultant force on every vertex cancels out (i.e. we would have a 0-block corresponding to every vertex, just as in the last row of the rigidity matrix in Example 6.2, which we omit). An important property of $R(\mathbf{p})$ is the next lemma.

Lemma 6.3 (Theorem 11.1.4 in [Whi96])**.** *If $n \geq d$ and the points $\boldsymbol{p}$ affinely span $\mathbb{R}^d$ then the rank of $R(\boldsymbol{p})$ equals*

$$dn - \binom{d+1}{2} = \frac{d}{2}(2n - d - 1).$$

For $d = 2k$ this is exactly the number of edges of a k-triangulation of the n-gon.

We will now introduce the notion of rigidity and extend our set of matroids accordingly.

Definition 6.5. Let $E \subseteq \binom{[n]}{2}$ be a subset of edges. We say that E is

- **self-stress-free** or **independent**, if the rows of the to E restricted matrix $R(\mathbf{p})|_E$ are linearly independent, and
- **rigid** or **spanning**, if they have the same rank as $R(\mathbf{p})$.

We call the matroid of rows of $R(\mathbf{p})$ the **bar-and-joint matroid** of $\mathbf{p}$ and denote it $\mathcal{R}(\mathbf{p})$.

Example 6.3. Continuing Example 6.2, the basis $\{a, b, c, d\}$ is self-stress-free and rigid, whereas $\{a, b, c, d, e\}$ is rigid but not self-stress-free.

We will now introduce similar rigidity matrices and their matroids, which will be used to prove the results of Theorem 6.1 and Theorem 6.2.

Definition 6.6. The **hyperconnectivity** matroid of the configuration of the points $\mathbf{p} = (p_1, \ldots, p_n)$ in $\mathbb{R}^d$ is the matroid of rows of

$$H(\mathbf{p}) = \begin{pmatrix} p_2 & -p_1 & 0 & \ldots & 0 & 0 \\ p_3 & 0 & -p_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_n & 0 & 0 & \ldots & 0 & -p_1 \\ 0 & p_3 & -p_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & p_n & -p_{n-1} \end{pmatrix}$$

which we denote by $\mathcal{H}(\mathbf{p})$.

Definition 6.7. For points $\mathbf{q} = (q_1, \ldots, q_n)$ in $\mathbb{R}^2$ and a parameter $d \in \mathbb{N}$, the d-dimennsional **cofactor rigidity** matroid of the points $\mathbf{q}$ is the matroid of rows of

$$C_d(\mathbf{q}) = \begin{pmatrix} c_{12} & -c_{12} & 0 & \ldots & 0 & 0 \\ c_{13} & 0 & -c_{13} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_{1n} & 0 & 0 & \ldots & 0 & -c_{1n} \\ 0 & c_{23} & -c_{23} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & c_{n-1,n} & -c_{n-1,n} \end{pmatrix}$$

which we denote by $\mathcal{C}_d(\mathbf{q})$. For $q_i = (x_i, y_i), q_j = (x_j, y_j)$ the vector $c_{ij} \in \mathbb{R}^d$ is defined as

$$c_{ij} = \big((x_i - x_j)^{d-1}, (y_i - y_j)(x_i - x_j)^{d-2}, \ldots, (y_i - y_j)^{d-1}\big)$$

and for $d = 2$ we have $C_2(\mathbf{q}) = R(\mathbf{q})$.

Remark 6.2. The matroids $\mathcal{R}(\mathbf{p})$ and $\mathcal{C}_d(\mathbf{q})$ are invariant under affine transformation of the points, and $\mathcal{H}(\mathbf{p})$ under linear transformation.

From now on we will often (and already have) assume the points to be lying in **general position**, i.e. no $d+1$ of the n points lie in an affine hyperplane. Furthermore, the points lying in general position implies that the rank of the three matrices $R(\mathbf{p}), H(\mathbf{p})$ and $C_d(\mathbf{q})$ and thus the rank of their corresponding matroids equals $dn - \binom{d+1}{2}$ ([RS22b]).

Remark 6.3. Matroids that are in correspondence with a set of vectors are called *linear matroids.* To avoid confusion and to avoid going into further detail, we will omit this term as it is not relevant to us.

We will now explore the details about our interpretation of rigidity and forces along edges using the dependencies of rows in our rigidity matrices. For this we need to introduce *oriented matroids*, although, by following the first two chapters in [Bjö+00], we will not introduce them axiomatically but in the context of directed graphs, as it is sufficient for us.

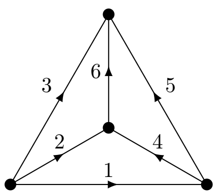
Definition 6.8. Let $G = (V, E)$ be an undirected graph with simple cycle (only the first and last vertex are equal) set $C \subseteq E$. By adding an orientation to the edges of G we can divide a simple cycle $X \in C$ in a set of positive edges X^+, whose orientation is the original one, and in a set of negative edges X^-, whose assigned orientation is reversed. We write $X = (X^+, X^-)$ and call this a **signed circuit** of G. The signed circuits of G form a collection which we denote by

$$\mathcal{C} = \{X = (X^+, X^-) \mid X \text{ is a signed circuit of } G\}.$$

The **oriented matroid** $\mathcal{M}_G$ is the pair $(E, \mathcal{C})$.

Since we do not give a more detailed definition, the following example should help understanding and visualizing oriented matroids.

Example 6.4. Consider the complete graph K_4 with an orientation of the edges as indicated by the arrows.



The edges (without their orientation) $\{1, 2, 5, 6\}$ form a simple cycle in K_4. Then we have the following two signed circuits for this simple cycle, which now respects the orientation of the edges:

$$X = (X^+ = \{2, 6\}, X^- = \{1, 5\}) \quad \text{and} \quad X = (X^+ = \{1, 5\}, X^- = \{2, 6\}).$$

Remark 6.4. There are several properties that come with signed circuits. In fact, by looking at them in a more general context, they form one possible set of axioms that define oriented matroids. They are the following:

1. $\emptyset$ is not a signed circuit.
2. If X is a signed circuit, then so is $-X$, where $-$ denotes the reversal of the orientations.
3. No proper subset of a circuit is a circuit.
4. If X_0 and X_1 are circuits with $X_1 \neq -X_0$ and $e \in X_0^+ \cap X_1^-$, then there is a third circuit $X \in \mathcal{C}$ with $X^+ \subseteq (X_0^+ \cup X_1^+) \setminus \{e\}$ and $X^- \subseteq (X_0^+ \cup X_1^-) \setminus \{e\}$.

The axioms are reminiscent of the axioms of a matroid, although they are for minimal non-dependend subsets, that is, circuits. For a more detailed discussion about oriented matroids and the different set of axioms we refer to [Bjö+00].

We give an example for the fourth axiom based on our previous example.

Example 6.5. Look again at the situation of Example 6.4 and consider the two signed circuits

$$X_0 = (\{2, 6\}, \{1, 5\}) \quad \text{and} \quad X_1 = (\{3\}, \{2, 6\}).$$

The edge $e = \{2, 6\}$ satisfies the condition in 4, such that the signed circuit is $X = (\{3\}, \{1, 5\})$.

Let us now return to our original situation and answer the question of how rigidity matroids and oriented matroids are connected. The answer lies in our interpretation of forces along edges.

Definition 6.9. Consider a finite set $E = \{v_1, \dots, v_n\}$ of vectors that span a vector space of dimension d. A minimal linear dependence of these vectors is the equation

$$\sum_{i=1}^{n} \lambda_i v_i = 0, \quad \lambda_i \in \mathbb{R}$$

such that only a minimal number of the λ_i's is not 0. We consider the sets of indices $\{i \,|\, \lambda_i \neq 0\}$ corresponding to the minimal linear dependencies as the circuits of the (unoriented) matroid. By considering the signed set $X = (X^+, X^-)$ given by

$$X^+ = \{i \,|\, \lambda_i > 0\}, \quad X^-\{i \,|\, \lambda_i < 0\}$$

for all the minimal dependencies among the v_i, we obtain the corresponding oriented matroid $\mathcal{M} = (E, \mathcal{C})$ of the vector configuration E.

Remark 6.5. Note, that the coefficients of a minimal linear dependence are unique up to a common scalar. Thus, for a given circuit of the matroid, there are exactly two signed circuits in $\mathcal{M}$, namely the positive and the negative one. Furthermore, the just defined oriented matroid is indeed an oriented matroid, satisfying the axioms in Remark 6.4.

Hence, all the rigidity matroids that we defined earlier define oriented matroids in the manner just described.

Example 6.6. Consider the columns of the rigidity matrix in Example 6.2 and label them by v_i. We have for example the minimal linear dependencies

$$v_1 + v_3 - v_4 + 2v_7 = 0 \quad \text{and} \quad v_5 - v_6 - v_7 - v_8 = 0$$

and thus the signed circuits $X = (\{1, 3, 7\}, \{4\})$ and $X = (\{5\}, \{6, 7, 8\})$.

Remark 6.6. It is obvious, that for defining an oriented matroid based on a matrix the choice of using rows or columns as the configuration of vectors is irrelevant. However, columns and rows do not define the same matroid.

We will now look at the correspondence between the bar-and-joint, hyperconnectivity and cofactor rigidity matrices. In fact, their rows define the same oriented matroid by choosing points $\mathbf{p}$ and $\mathbf{p}'$ along the moment curve and points $\mathbf{q}$ along the parabola.

Theorem 6.4 ([CS23])**.** *Let $t_1 < \cdots < t_n \in \mathbb{R}$ be real parameters and define the configurations of points by*

$$p_i = (1, t_i, \dots, t_i^{d-1})^T \in \mathbb{R}^d, \quad p'_i = (t_i, t_i^2, \dots, t_i^d)^T \in \mathbb{R}^d, \quad q_i = (t_i, t_i^2)^T \in \mathbb{R}^2.$$

Then, the three matrices $H(\boldsymbol{p})$, $R(\boldsymbol{p}')$ and $C_d(\boldsymbol{q})$ can be obtained from one another by multiplying on the right by a regular matrix and then multiplying rows by some positive scalars. In particular, the rows of the three matrices define the same oriented matroid.

Remark 6.7. For each of the three presented matroids and for every dimension d, we can find an unique most free matroid that can be obtained by sufficiently generic choices of points. We will call them *generic* bar-and-joint, hyperconnectivity and cofactor matroids. In [RS22b] the conjecture was stated, that k-triangulations of the n-gon are bases in the generic bar-and-joint matroid. In [RS22a] they proved this statement for the generic hyperconnectivity matroid. This implies, that at least the individual cones have the right dimension and are simplicial, which is a necessary condition for realizing the multi-associahedron.

Thus, we can translate k-triangulations as graphs to (graphical) matroids and, by embedding the graph in the appropriate dimension, to oriented matroids.

6.3 Obstructions for Realizablity

In this section we will use the cofactor rigidity to show that it does not realize the reduced complex $\overline{\triangle}_{n,k}$ for $n \geq 2k+6$ and $k \geq 3$. The authors in [RS22b] argued that this is the most natural setting, since the combinatorics of multi-associahedra comes from crossings in the complete graph embedded with vertices in convex position in the plane.

Before we start with going into the results we adjust our definition of the cofactor matrix. Consider a vector configuration $\mathbf{Q} = (Q_1, \ldots, Q_n)$ with the 3-dimensional vectors $Q_i = (X_i, Y_i, Z_i) \in \mathbb{R}^3 \setminus \{0\}$. These vectors Q_i generate rays of

$$\operatorname{cone}(\mathbf{Q}) = \left\{ \sum_{i=1}^{n} \lambda_i Q_i \mid \lambda_i \geq 0 \right\}.$$

We usually assume that $\mathbf{Q}$ is in general position (every three of its vectors form a linear basis) and sometimes that it is also in convex position, thus,

- all the rays in $\operatorname{cone}(\mathbf{Q})$ are different, and
- the cyclic order of $Q_1, \ldots, Q_n$ equals their order as rays of $\operatorname{cone}(\mathbf{Q})$.

Based on this configuration let us redefine the vectors c_{ij} in Definition 6.7 in terms of the vectors Q_i as follows. Let

$$x_{ij} = X_i Z_j - Z_i X_j \quad \text{and} \quad y_{ij} = Y_i Z_j - Z_i Y_j$$

and define $c_{ij} = (x_{ij}^{d-1}, y_{ij} x_{ij}^{d-2}, \ldots, y_{ij}^{d-1})$. Now define the cofactor matrix $C_d(\mathbf{Q})$ and its matroid of rows $\mathcal{C}_d(\mathbf{Q})$ exactly as in Definition 6.7. Furthermore, we obtain the original definition of $C_d(\mathbf{q})$ as a special case for $Z_i = 1$ for all i and $q_i = (X_i, Y_i)$. We only assume $\mathbf{Q}$ to be a configuration in dimension three for the results on cofactor rigidity.

Proposition 6.5. *Let $\boldsymbol{Q} = (Q_1, \ldots, Q_n)$ be a vector configuration. Then,*

1. *the column-space of $C_d(\boldsymbol{Q})$, hence the oriented matroid $\mathcal{C}_d(\boldsymbol{Q})$, is invariant under a linear transformation of $\boldsymbol{Q}$, and*
2. *the matroid $\mathcal{C}_d(\boldsymbol{Q})$ is also invariant under rescaling (that is, multiplication by non-zero scalars) of the vectors Q_i. If the scalars are all positive or d is odd, then the same holds for the oriented matroid.*

Sketch of the proof. For $Q \in \mathbb{R}^3 \setminus \{0\}$ consider the set $C_{d-1}^{d-2}(Q)$ of all three-variate polynomials in $\mathbb{R}[X,Y,Z]$ that are homogeneous of degree $d-1$ and such that all their partial derivatives up to order $d-2$ vanish at Q. It is easy to see that for a fixed $Q_i = (X_i, Y_i, Z_i)$ and a vector of variables $Q_j = (X, Y, Z)$ the entries in the vector c_{ij} form a basis for $C_{d-1}^{d-2}(Q_i)$: First of all they satisfy the homogeneous and derivative condition (by definition and using the chain rule for differentiating and then plugging in Q_i) and they form a basis since the homogeneous terms are gradually mixed up to the maximal possible amount. For a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ and a fixed Q_i we can now define the linear map

$$\tilde{L}_i : C_{d-1}^{d-2}(L(Q)) \to C_{d-1}^{d-2}(Q), \quad \tilde{L}_i(f) = f \circ L.$$

For the matrix $M_i \in \mathbb{R}^{d \times d}$ of $\tilde{L}_i$ in the appropriate bases let $M \in \mathbb{R}^{dn \times dn}$ be the block-diagonal matrix consisting of the M_i. We then have

$$C_d(L(\mathbf{Q})) = C_d(\mathbf{Q})M^{-1}$$

proving the first point (since the M_i are invertible and so is M). The second point follows from the fact that multiplying Q_i by a scalar λ_i corresponds to multiplying the rows of edges using i by the scalar λ_i^{d-1} and the matroid of rows is invariant under this transformation for a positive rescaling factor or for d odd, since the sign is then of no importance because of the even exponents. □

We will now introduce a method to obtain a matroid in n points by deleting the last point of a matroid in $n+1$ points. Before we give the next result, we need the notion of a contraction of a set of vectors to an independent subset.

Definition 6.10. For a vector configuration $\mathcal{V} \subset \mathbb{R}^D$ and an independent subset $I \subset \mathcal{V}$ we define the **contraction** of $\mathcal{V}$ at I as the image of $\mathcal{V} \setminus I$ under the quotient linear map $\mathbb{R}^D \to \mathbb{R}^D/\text{lin}(I)$.

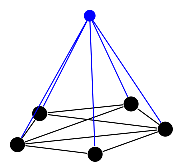
Proposition 6.6 (Coning Theorem)**.** *Let $\boldsymbol{Q} = (Q_1, \dots, Q_{n+1})$ be a vector configuration in general position in $\mathbb{R}^3$. Then, the matroid $\mathcal{C}_d(Q_1, \dots, Q_n)$ is the contraction of the matroid $\mathcal{C}_{d+1}(\boldsymbol{Q})$ at $\binom{[n]}{2}$. If the vectors are in convex position, the same is true for the oriented matroid.*

Before we will give the idea of the proof we want to discuss an interesting implication of this proposition, justifying its name. For this we need the following definition.

Definition 6.11. Let $G = ([n], E)$ be a graph with vertex set $[n]$. Then the **cone** $G * \{n+1\}$ over G is defined as the graph with vertex set $[n+1]$ and with edges

$$E * \{n+1\} = E \cup \{[i, n+1] \mid i \in [n]\}.$$

Example 6.7. The coning of the graph G visualized in black and its coning, where the new edges and vertex are marked in blue.



Corollary 6.7. *A graph G with vertex set $[n]$ is d-rigid when realized on the configuration $(Q_1, \dots, Q_n)$ if and only if its cone $G * \{n+1\}$ is $(d+1)$-rigid on $(Q_1, \dots, Q_n, Q_{n+1})$.*

Thus, this corollary gives us control about when rigidity translates between addition or omission of vertices of graphs, which we will need in the proof of the next theorem.

Sketch of the proof for the Coning theorem. By assuming $Q_{n+1} = (0,1,0)^T$ and using the assumption that the configuration is in general position, the rest of the coordinates can be restricted up to certain (in-) equalities for the X_i (and Z_i). Furthermore, the vectors in the matrix $c_{i,n+1}$ have a very easy form, such that the contraction of the elements $i, n+1$ in the matroid $\mathcal{C}_{d+1}(\mathbf{Q})$ can be performed in the matrix $C_{d+1}(\mathbf{Q})$ through omission of the last columns corresponding to Q_{n+1} and the rows of the form $\{i, n+1\}$ with their corresponding columns. The resulting matrix coincides, up to row-dependent multiplication with scalars, with $C_d(Q_1, \dots, Q_n)$. That these scalars do not affect the matroid is implied by the assumptions on the coordinates. □

The result of the Coning Theorem can be extended by looking at the deletion or addition of an intermediate Q_i. This is proved by relabelling the points cyclically such that the point i becomes $n+1$, then using the Coning Theorem and finally relabelling the points back to their original labels. Remark, that relabelling does change the sign of rows in the matrix. That this is no problem is explained in more details in [RS22b].

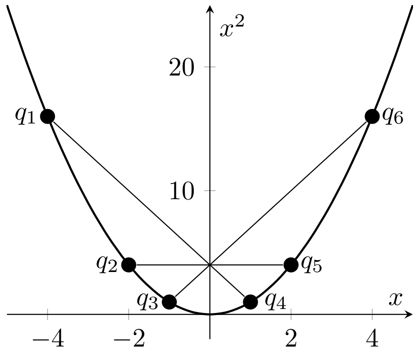
Proposition 6.8. *Let $\mathbf{Q} = (Q_1, \dots, Q_{n+1})$ be a configuration in general position in $\mathbb{R}^3$. Then the oriented matroid $\mathcal{C}_d(Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_{n+1})$ is obtained by contracting the elements $\{i, j\}$ with $j \in [n+1] \setminus \{i\}$, and reorienting the elements $\{j, k\}$ with $1 \leq j < i < k \leq n+1$.*

We will now begin to consider k-triangulations in the context of oriented matroids. For this, let $\mathbf{q}$ be a configuration of points in the plane. First, we will show that the cofactor rigidity does not always realize $\overline{\triangle}_{n,k}$ by considering the 3-triangulation of the 9-gon $K_9 \setminus \{[1,6],[3,7],[4,9]\}$, which is dependent in $\mathcal{C}_6$ under certain circumstances.

Theorem 6.9. *Consider the graph $G = K_6 \setminus \{[2,5],[3,6]\}$ embedded with six points $\mathbf{q}$ in general position. Then, G is spanning in $\mathcal{C}_3$, hence it contains a unique dependence and this dependence may not vanish at any edge other than $[1,4]$.*

Proof. Let $G' = G \setminus \{[i,j]\}$ for an edge $[i,j]$ different from $[1,4]$ and assume without loss of generality that $i \notin \{1,4\}$. Then the graph $G' \setminus i$ is the complete graph K_5 without one edge. For an example look at Table 6.1. In [CJT22] it was proven that K_5 is a circuit in $\mathcal{C}_3$ for any choice of points in general position, thus, $G' \setminus i$ is independent by definition of a circuit and thus a basis. By Corollary 6.7 we have that G' is a basis too. But then, by definition of a basis, the graph G is spanning and contains a circuit which does not vanish at the edge $[i,j]$. □

Remark 6.8. Whether or not the dependence vanishes at $[1,4]$ is dependent on the choice of points $\mathbf{q}$. In Section 3.2 of [RS22b] is a more detailed discussion about this. The main result is this: The graph $K_6 \setminus \{[1,4],[2,5],[3,6]\}$ is a circuit, if the configuration $\mathbf{q} = (q_1, \ldots, q_6)$ lies in *Desargues position*, i.e., if the points lying on the parabola satisfy that the direct lines between q_1 and q_4, q_2 and q_5 and between q_3 and q_6 are concurrent. For example, the configuration displayed below is in Desargues position.



We can now prove that there are positions where the rows of the cofactor matrix do not realize $\overline{\triangle}_{9,3}$ as a basis collection.

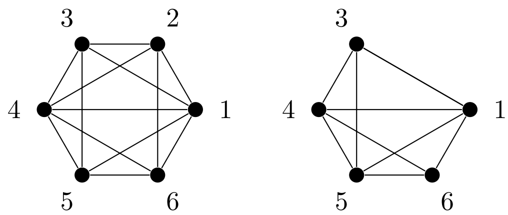
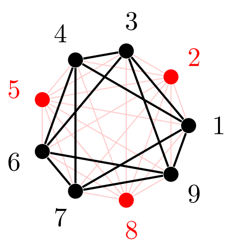


Table 6.1: The graph G and the graph obtained by removing the vertex 2. This graph is the complete graph on 5 vertices without the edge $[3,6]$.

Theorem 6.10. *The graph* $K_9 \setminus \{[1,6],[3,7],[4,9]\}$ *is a 3-triangulation of the 9-gon, but it is dependent in the rigidity matroid* $\mathcal{C}_6(q_1, \ldots, q_9)$ *if the lines through the points* $[q_1, q_6], [q_3, q_7]$ *and* $[q_4, q_9]$ *are concurrent.*

Proof. We start with the graph $K_6 \setminus \{[1,6],[3,7],[4,9]\}$ with vertices labelled $\{1,3,4,6,7,9\}$. Then its coning at the three vertices 2, 5 and 8 is the graph $K_9 \setminus \{[1,6],[3,7],[4,9]\}$. This is visualized below, where the first graph is in black and the coning is marked in red.



Since the graph in black is the graph considered in Theorem 6.9 without the edge $[1, 4]$, the statements follows from Proposition 6.8, Theorem 6.9 and Remark 6.8. □

In the end of this subsection we will show that cofactor rigidity does not only fail to realize the multi-associahedron in some cases. The main result is the next theorem.

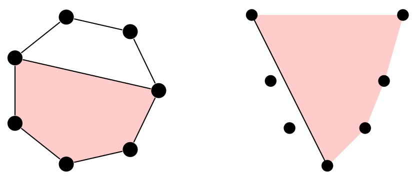
Theorem 6.11. *If $k \geq 3$ and $n \geq 2k + 6$ then no choice of points $\boldsymbol{q}$ in $\mathbb{R}^2$ in convex position makes $C_{2k}(\boldsymbol{q})$ realize the multi-associahedron $\overline{\triangle}_{n,k}$ as a fan. The same happens for bar-and-joint rigidity with any choice of points along the moment curve.*

The plan is as follows: We first assume $n = 2k + 3$ and characterize when exactly cofactor rigidity C_{2k} does realize $\overline{\triangle}_{n,k}$ as a complete fan. Afterwards, we can use this characterization and restrict the configuration of $2k + 6$ points to those of $2k + 3$ points to force a contradiction. The details of the proofs will not be discussed here, but we will talk about their concepts.

Definition 6.12. We call a k-triangulation of the $(2k + 3)$-gon **octahedral**, if its three missing edges have six distinct endpoints.

The triangulation in Theorem 6.10 is octahedral.

Remark 6.9. A k-relevant edge of the $(2k + 3)$-gon leaves k vertices on one side and $k + 1$ on the other. This also happens if we consider the corresponding points of a convex configuration $\mathbf{q}$. We call the half-plane on the side with $k + 1$ points **big half-plane**.



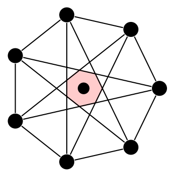
Theorem 6.12. *Let $\boldsymbol{q} = (q_1, \ldots, q_{2k+3})$ be a configuration in convex position in $\mathbb{R}^2$. The following are equivalent:*

1. *$C_{2k}(\boldsymbol{q})$ realizes $\overline{\triangle}_{2k+3,k}$ as a complete fan.*
2. *For every octahedral k-triangulation T, the big half-planes defined by the three edges not in T have non-empty intersection.*
3. *The big half-planes of all relevant edges have non-empty intersection.*

The proof is based on similar considerations about the length of paths that are created by removing three edges from the complete graph K_{2k+3} just like in the proof of Corollary 3.16, in addition with some topological properties which we did not discuss here. The theorem has a very interesting and nice implication.

Corollary 6.13. *For every k there are configurations $\boldsymbol{q}$ such that $C_{2k}(\boldsymbol{q})$ realizes $\overline{\triangle}_{2k+3,k}$ as a fan. For example the vertices of a regular $(2k + 3)$-gon.*

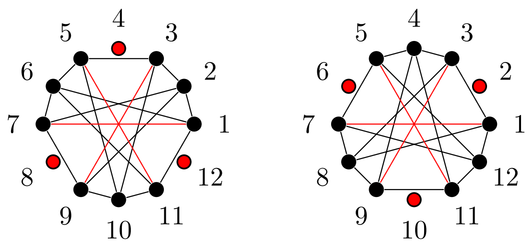
Proof. Since the center of any $(2k+3)$-gon lies in the interior of the intersection of all big half-planes this is true by 3.



$\square$

We are now in the position to discuss the proof of Theorem 6.11 by using our characterization of under which circumstances $\overline{\triangle}_{2k+3,k}$ is realized by $C_{2k}(\mathbf{q})$ to produce a contradiction.

Sketch of the proof of Theorem 6.11. Let $\mathbf{q} = (q_1, \dots, q_n)$ be a configuration in convex position. We will only show the case $n = 2k + 6$ since all cases greater follow from Lemma 2.3 in [RS22b], which states that the reduced multi-associahedra have the property of being monotone, that is, complexes of lower values for n and k appear as links of certain subsets of a complex of higher dimension. Thus, showing the contradiction for the smallest possible case yields a contradiction for the higher cases.
Now let $I_1 = [n] \setminus \{4, k+5, k+9\}$ and $I_2 = [n] \setminus \{2, 6, k+7\}$ such that $\mathbf{q}|_{I_1}$ and $\mathbf{q}|_{I_2}$ are configurations with $2k+3$ points. Here is an example for $k = 3$:



Now the Lemma 3.13 in [RS22b] yields a contradiction, since for $\mathbf{q}|_{I_1}$ to be able to realize $\overline{\triangle}_{2k+3,k}$ the points $(q_1, q_3, q_5, q_{k+4}, q_{k+6}, q_{k+8})$ (the endpoints of the removed edges) need to have another orientation than for $\mathbf{q}|_{I_2}$ in order to realize $\overline{\triangle}_{2k+3,k}$. Thus, one of the two does not realize $\overline{\triangle}_{2k+3,k}$ and by the same Lemma 2.3 in [RS22b] we conclude that $\mathbf{q}$ does not realize $\overline{\triangle}_{2k+6,k}$. $\square$

6.4 Positive and Experimental Results

In this last subsection we will present the positive results on the realizability of multi-associahedra via rigidity theory and the experimantal results of the authors in [RS22b], which produced three new realizations as polytopes and some non-polyotpal fan realizations.

Corollary 6.14. *For $n = 2k + 2$, any choice of points $q_1, \dots, q_{2k+2} \in \mathbb{R}^2$ in convex position realizes $\overline{\triangle}_{2k+2,k}$ as a polytopal fan.*

Sketch of the proof. The proof is based on the fact that all k-triangulations are K_{2k+2} without a diameter. Furthermore, since K_{2k+2} is a circuit, all the k-triangulations are bases. The rest follows from considerations of assigned signs, which yields the necessary topological property for any configuration to realize $\overline{\triangle}_{2k+2,k}$ as a fan. But since we know that these multi-associahedra are realizable as the k-simplex, this fan must be polytopal. □

The next result can be obtained by using Theorem 6.12.

Corollary 6.15. *For $k = 2$ and $n = 7$, any choice of $q_1, \ldots, q_7 \in \mathbb{R}^2$ in convex position realizes $\overline{\triangle}_{7,2}$ as a fan.*

Definition 6.13. The matrix $H(p_1, \ldots, p_n)$ in the statement of Theorem 6.4 is called the **polynomial d-rigidity matrix** with parameters $t_1, \ldots, t_n$. We denote it by $P_d(t_1, \ldots, t_n)$ and its corresponding matroid by $\mathcal{P}_d(t_1, \ldots, t_n)$.

The first computational experiments were done using equispaced parameters. Since all computations were done by computer, we omit further details on the proofs.

Lemma 6.16. *Let $\boldsymbol{t} = \{1, 2, \ldots, n\}$ be equispaced parameters.*

1. *$P_4(\boldsymbol{t})$ realizes $\overline{\triangle}_{n,2}$ as a complete fan for all $n \leq 13$.*
2. *$P_4(\boldsymbol{t})$ realizes $\overline{\triangle}_{n,2}$ as the normal fan of a polytope if and only if $n \leq 9$.*
3. *The positions $\boldsymbol{t} = (-2, 1, 2, 3, 4, 5, 6, 7, 9, 20)$ for $P_4(\boldsymbol{t})$ realize $\overline{\triangle}_{10,2}$ as the normal fan of a polytope.*

Remark 6.10. By Remark 6.2 and the fact that an affine transformation in the space of parameters produces a linear transformation in the rows of $P_{2k}(\mathbf{t})$, we can take without loss of generality $\mathbf{t} = (1, 2, \ldots, n)$ as a representative for equispaced parameters.

As a final result the authors provide one more polytopal realization and three new fans.

Lemma 6.17. *For the same positions $\boldsymbol{t}$ as in the third point of Lemma 6.16, the matrix $P_6(\boldsymbol{t})$ realizes $\overline{\triangle}_{10,3}$ as the normal fan of a polytope. Furthermore, equispaced positions along the circle realize $\overline{\triangle}_{n,k}$ as a fan for the pairs $(11, 3), (12, 4)$ and $(14, 3)$.*

6.5 Realizing $\triangle_{8,2}$

At the end of this section we will consider one more proof of polytopality, which was already mentioned in the introduction. Five years after Conjecture 3.12 was formulated by Jonsson, in 2009 Bokowski and Pilaud found a polytopal realization for the 2-triangulations of the 8-gon and described its *space of symmetric realizations* completely. They did this by realizing $\triangle_{8,2}$ as a *symmetric oriented matroid polytope* and then as a *symmetric polytope under the dihedral group*. Remark, that the dihedral group $\mathbb{D}_n$ is the symmetry group of the n-gon and thus acts naturally on $\triangle_{n,k}$. By following the article [BP09] we consider their approach.

Definition 6.14. The **space of symmetric realizations** of $\triangle_{8,2}$ consists of all polytopes P whose boundary complex $\partial(P)$ is isomorphic to $\triangle_{8,2}$, and such that the natural action of the dihedral group $\mathbb{D}_8$ on $\triangle_{8,2}$ defines an action on P by isometry.

For the next definition, let Δ be a simplicial complex realized by the polytope P via the isomorphism $\phi : \Delta \to \partial(P)$ as in Definition 3.14, and let G be a group that acts on Δ by

$$G \times \Delta \to \Delta,\ (g, E) \mapsto gE.$$

This induces an action of G on $\partial(P)$ by

$$G \times \partial(P) \to \partial(P),\ (g, F) \mapsto gF = \phi(g\phi^{-1}(F)).$$

As mentioned at the beginning of this subsection the group G will be the dihedral group $\mathbb{D}_8$.

Definition 6.15. We say that P is a **symmetric realization** under G, if its action is symmetric, i.e., if for any $g \in G$ the application

$$\partial(P) \to \partial(P),\ F \mapsto gF$$

is an isometry of P.

Let us now talk about symmetric oriented matroids. For this, let $P \subset \mathbb{R}^d$ be a symmetric realization under G of Δ and let V be its set of vertices. For any $v \in V$ denote by $\vec{v} = (v, 1)$. Furthermore, for any $v_0, \dots, v_d \in V$ we denote by $\sigma(v_o, \dots, v_d)$ the orientation of the simplex spanned by $v_0, \dots, v_d$, i.e., the sign of the determinant of $(\vec{v}_i)_{0 \le i \le d}$.

Definition 6.16. The application $\sigma : V^{d+1} \to \{-1, 0, 1\}$ is called the **symmetric oriented matroid** associated to P.

This map satisfies four properties, which we will not mention here but are important for the computations in the proofs of the results. Before we continue, we want to briefly elaborate the connection of σ with oriented matroids.

Remark 6.11. Consider the matroid defined through minimal linear dependencies of the vectors $\vec{v}_i$. If $C = \{\vec{v}_0, \dots, \vec{v}_{d+1}\}$ is a circuit and $\{\vec{v}_0, \dots, \vec{v}_d\}$ is a basis, we can sign C with positive elements

$$C^+ = \{\vec{v}_i \mid (-1)^i \sigma(\vec{v}_0, \dots, \vec{v}_{i-1}, \vec{v}_{i+1} \dots, \vec{v}_{d+1}) = 1\}$$

and negative elements in the complement. Thus, the map σ gives rise to oriented bases of an oriented matroid. In general, these maps are called *chirotopes* and they can be used to define oriented matroids. For more details we refer to Chapter 3.5 of [Bjö+00].

Now, label the 2-relevant edges on the octagon with letters, where the capital letters mark the longest edges, thus, the diagonals:

$$a = [1, 4], b = [2, 5], \dots, e = [5, 8], f = [1, 6], \dots, I = [1, 5], \dots, L = [4, 8].$$

To determine the space of symmetric realizations the authors continued in two steps.

1. Enumerate all possible symmetric oriented matroids realizing $\triangle_{8,2}$.
2. Use this information to study the symmetric polytopes realizing $\triangle_{8,2}$.

For the first step it was sufficient to look for maps $\sigma : \{a, \ldots, L\}^7 \to \{-1, 0, 1\}$ satisfying the four properties. The authors enumerated all possibilities by computer.

Proposition 6.18. *There are exactly 15 symmetric oriented matroids realizing $\triangle_{8,2}$.*

For the second step, assume that $P \subset \mathbb{R}^6$ is a polytope realizing $\triangle_{8,2}$ and which is symmetric under the action of $\mathbb{D}_8$. Let $a, \ldots, L$ denote its vertices corresponding to the 2-relevant diagonals and define $\vec{a}, \ldots, \vec{L}$ as before. At last, consider the matrix $M = (\vec{a} \ldots \vec{L})$, which can be transformed to the matrix

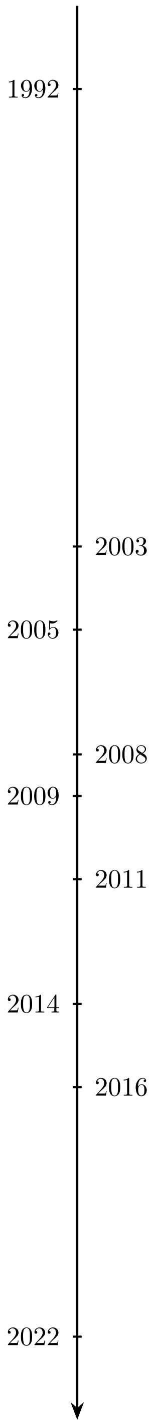
$$\tilde{M} = N^{-1}M = \begin{pmatrix} I_3 & T & 0_{3\times 4} \\ 0_{4\times 3} & B & I_4 \end{pmatrix}.$$

This matrix can be determined up to the submatrices T (which can be determined uniquely) and B (which can be determined up to two values). The matrix N is invertibel by a previous computation and its entries satisfy certain inequalities which describe the realization space completely.

Proposition 6.19. *The space of symmetric realizations of $\triangle_{8,2}$ is four dimensional.*

A concrete realization of $\triangle_{8,2}$ in $\mathbb{R}^7$ can be obtained by taking the convex hull of the column vectors of $N^{-1}M$.

7 History and Summary



The first appearance of multi-triangulations was in the work of Capoyleas and Pach in **1992** ([CP92]). In **2003** Knutson and Miller introduced subword complexes for arbitrary Coxeter groups. They showed that they are shellable spheres or balls and conjectured the polytopality of spherical complexes ([KM03]). Shortly after, in **2005**, Jonsson proved that the reduced multi-associahedra $\overline{\triangle}_{n,k}$ are shellable simplicial spheres of dimension $k(n-2k-1)-1$ and stated the Conjecture 3.12 ([Jon05]). Pilaud and Santos examined k-triangulations by using k-stars in **2008** and proved the trivial cases of polytopality of $\triangle_{n,k}$ for $n \leq 2k+3$ ([PS08]), although they might have been mentioned in earlier works. The first non-trivial case was proved by Bokowski and Pilaud in **2009**. They determined the space of symmetric realizations of $\triangle_{8,2}$ completely ([BP09]). Stump made the connection between subword complexes and multi-triangulations in **2011** ([Stu11]), such that the Conjecture 3.12 is a positive answer in type A to the conjecture of Knutson and Miller. In **2014**, Bergeron, Ceballos and Labbé gave fan realizations for $n \leq 2k+4$ and $n \leq 11$ for $k=3$ via Gale duality. Manneville realized $\triangle_{n,2}$ as a fan for $n \leq 13$ in his article in **2016**, which we did not discuss here ([Man16]). Finally, Ruiz and Santos gave the three new cases of polytopality $\overline{\triangle}_{9,2}, \overline{\triangle}_{10,2}$ and $\overline{\triangle}_{10,3}$ and a fan realization for $\overline{\triangle}_{13,4}$ using rigidity in **2022** ([RS22b]). In the same year they showed that k-triangulations are bases in the generic hyperconnectivity matroid of dimension $2k$ in the article *Multitriangulations and tropical Pfaffians* ([RS22a]).

The timeline should give a good overview about the most important statements in this thesis concerning the fundamental conjecture. Although we do not claim the completeness of all results up to today, in 2024, we summarize the positive complementary results discussed in this thesis in Table 7.1.

Multi-Associahedron	Realization
$\triangle_{n,1}$	Associahedron
$\triangle_{2k+1,k}$	Point
$\triangle_{2k+2,k}$	k-Simplex
$\triangle_{2k+3,k}$	$C_{2k}(2k+3)$
$\triangle_{8,2}$	Symmetric polytope in $\mathbb{R}^7$
$\overline{\triangle}_{9,2}$	Points on the moment curve via rigidity in $\mathbb{R}^4$
$\overline{\triangle}_{10,2}$	Points on the moment curve via rigidity in $\mathbb{R}^4$
$\overline{\triangle}_{10,3}$	Points on the moment curve via rigidity in $\mathbb{R}^4$
$\triangle_{n,k}$ for $n \leq 2k+4$	Complete simplicial fan via Gale duality
$\overline{\triangle}_{11,3}$	Fan realization for equispaced points
$\overline{\triangle}_{14,3}$	Fan realization for equispaced points

Table 7.1: The summary of the results in this thesis.

Next to the positive results we should also mention the negative results. In both the articles [BCL14] and [RS22b] we have seen obstructions for the realizability of multi-associahedra using the respective techniques. The use of Coxeter signature matrices does not work for type A_n, where $n \geq 4$, and to our knowledge there is no general Coxeter signature matrix which solves this problem yet. Even then, the polytopality of the corresponding multi-associahedra would still have to be proven. Furthermore, no choice of points in convex position in the plane makes cofactor or bar-and-joint rigidity realize $\overline{\triangle}_{n,k}$ as a fan if $k \geq 3$ and $n \geq 2k+6$. A point that also requires consideration is that until now, there is no proof of a general case except the trivial ones in Subsection 3.5 and that the new singular cases of polytopality have been proven by computer. It remains to be seen, if future approaches using presented techniques ore other methods will be able to prove Conjecture 3.12, or maybe show that it is not true in its generality.

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