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# On combinatorial formulas for Schur and Macdonald polynomials

A thesis submitted for the degree of
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# Introduction

This thesis concerns Macdonald polynomials, which generalize several interesting functions, including the *Schur functions* and, more generally, *Weyl characters*. For a fixed Weyl group $W$ associated to a crystallographic root system, the Macdonald polynomials are defined as a unique basis of a certain ring of $W$-invariant polynomials. When $W$ is the symmetric group $S_n$, they become a basis for the ring of symmetric functions.

Macdonald polynomials were introduced in 1987 by Macdonald as a $\mathbb{Q}(q,t)$-basis, which is orthogonal with respect to a certain inner product. Both, the polynomials and the inner product, are defined with respect to the two variables $q$ and $t$. If we fix $q$ or $t$ to particular values, or if we consider $W = S_n$, the Macdonald polynomials reduce to other well-known polynomials.

In the case $W = S_n$, they become the *Macdonald symmetric functions* $P_\lambda$, indexed by partitions $\lambda$, and form a basis for the ring of symmetric functions $\mathbb{Q}(q,t)[x]^{S_n}$.

At $q = t$, the Macdonald polynomials reduce to the *Weyl characters*, which are the *Schur functions* for $W = S_n$. At $t = 1$, they become the *orbit sums*, or the *monomials* for $W = S_n$, which is the standard basis for the ring of symmetric functions.
Further special cases of the Macdonald polynomials for $W = S_n$ are the *Hall-Littlewood symmetric functions* at $q = 0$, and the *Jack polynomials*.

There is a remarkable combinatorial formula for Macdonald polynomials, due to Ram and Yip, via *alcove walks* on an affine Weyl arrangement. Lenart gives a specialized formula for the case $W = S_n$ using *nonattacking fillings* of tableaux.

In this thesis, we provide both combinatorial formulas and show how to derive Lenart's from the formula of Ram and Yip.
In Section 1, we introduce the necessary background on root systems and Weyl groups. We define Macdonald polynomials and present the mentioned specializations in Section 2, and in Section 3, we introduce alcove walks and present the formula of Ram and Yip.

In the subsequent sections, we turn our attention to the case $W = S_n$. In Section 4, we explain how the Macdonald polynomials specialize to a basis for the ring of symmetric functions. We then discuss Lenart's formula in Section 5. In the last section, Section 6, we present several examples of Macdonald polynomials using both formulas.

# 1 Root systems and Weyl groups

Macdonald polynomials are defined for every root system. Thus, we first give definitions and some properties for root systems and Weyl groups. For this, we follow [8] and [3], where most of the notation is from [8].

Let $V$ be a real vector space with inner product $\langle\,,\rangle$ and finite dimension $n$. For each nonzero vector $\alpha \in V$ we define the ***reflection*** as the linear operator on $V$ given by

$$s_\alpha \lambda = \lambda - 2\frac{\langle \lambda\,, \alpha\rangle}{\langle \alpha\,, \alpha\rangle}\alpha.$$

We denote $\alpha^\vee = 2\frac{\alpha}{\langle \alpha,\alpha\rangle}$, thus we can write $s_\alpha \lambda = \lambda - \langle \lambda\,, \alpha^\vee\rangle\alpha$.
The reflection $s_\alpha$ is an orthogonal transformation: For every $\lambda, \mu$ in $V$ we have

$$\begin{aligned}\langle s_\alpha\lambda\,, s_\alpha\mu\rangle &= \langle\lambda\,,\mu\rangle - 2\frac{\langle\mu\,,\alpha\rangle}{\langle\alpha\,,\alpha\rangle}\langle\lambda\,,\alpha\rangle - 2\frac{\langle\lambda\,,\alpha\rangle}{\langle\alpha\,,\alpha\rangle}\langle\mu\,,\alpha\rangle + 2\frac{\langle\lambda\,,\alpha\rangle}{\langle\alpha\,,\alpha\rangle}\cdot 2\frac{\langle\mu\,,\alpha\rangle}{\langle\alpha\,,\alpha\rangle}\langle\alpha\,,\alpha\rangle \\ &= \langle\lambda\,,\mu\rangle - 4\frac{\langle\lambda\,,\alpha\rangle\langle\mu\,,\alpha\rangle}{\langle\alpha\,,\alpha\rangle} + 4\frac{\langle\lambda\,,\alpha\rangle\langle\mu\,,\alpha\rangle}{\langle\alpha\,,\alpha\rangle} \\ &= \langle\lambda\,,\mu\rangle.\end{aligned}$$

Moreover, let $H_\alpha$ be the hyperplane perpendicular to $\alpha$, then $s_\alpha$ is the orthogonal reflection in $H_\alpha$. To verify this, note that $s_\alpha$ fixes $H_\alpha$ pointwise and maps $\alpha$ to its negative $-\alpha$. Since the orthogonal complement of $H_\alpha$, as subspace of $V$, is $\mathbb{R}\alpha$, and since $V$ is the direct sum of $H_\alpha$ and its complement, we can write any $\lambda$ in $V$ as $\lambda = c\alpha + h_\alpha$, where $c$ is a real number and $h_\alpha$ is a vector in $H_\alpha$. We have

$$s_\alpha\lambda = s_\alpha(c\alpha + h_\alpha) = s_\alpha(c\alpha) + s_\alpha(h_\alpha) = -c\alpha + h_\alpha.$$

**Definition 1.1.** Let $\Phi$ be a finite nonempty set of nonzero vectors in $V$. We call $\Phi$ a ***root system*** in $V$ if the vectors in $\Phi$ - called ***roots*** - span the vector space $V$ and if they satisfy:

(1) $s_\alpha(\Phi) = \Phi$ for each $\alpha \in \Phi$,

(2) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$,

(3) $\langle \alpha\,, \beta^\vee\rangle \in \mathbb{Z}$ for each pair $\alpha, \beta \in \Phi$.

The ***rank*** of the root system $\Phi$ is the dimension of $V$. We call $\alpha^\vee$ the ***coroot*** to the root $\alpha$. The ***Weyl group*** $W$ of the root system $\Phi$ is the subgroup of $O(V)$ generated by all reflections $s_\alpha$ for $\alpha \in \Phi$.

*Remark.* Definition 1.1 defines a crystallographic root system, which omits additional conditions compared to other authors definitions of a root system. Moreover, we assume that the root systems in this thesis are irreducible, i.e., that they cannot be written as combination of two distinct root systems. In Section 1.2, we will present all crystallographic root systems classified by their *Cartan types*.

Let us have a closer look at the number of roots of a root system $\Phi$. Definition 1.1 just requires $\Phi$ to be a spanning set of $V$. Thus, we can ask for a basis of $V$ consisting of $n$ roots of $\Phi$. In fact, we can say more. Every root system $\Phi$ contains a basis $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi$ of $V$ such that all roots of $\Phi$ can be written as integer linear combination $\sum_{i=1}^n m_i\alpha_i$ and, moreover, the coefficients $m_i$ are either all nonnegative integers or all nonpositive integers. We call such a basis $\Delta$ a ***simple system*** for $\Phi$ and its elements ***simple roots***. According to this, call a root a ***positive root*** or a ***negative root*** depending on the sign of the coefficients $m_i$. The sets of positive roots $\Phi^+$ and negative roots $\Phi^-$ are called a ***positive***, respectively, a ***negative system***. It is clear, that $\Phi = \Phi^+ \cup \Phi^-$ since each root $\alpha$ comes with its 'negative' $-\alpha$ and exactly one must belong to the positive system.

A positive system $\Phi^+$ depends on the choice of the simple roots. But every other set of positive roots is of the form $w\Phi^+$ for a unique element $w$ of the Weyl group $W$ which is shown in [3, Section 1.4]. Moreover, there are no two simple systems which construct the same positive system, so we can fix a set of positive simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ from now on without loss of generality.

The following theorem uses such a fixed simple system to generate the Weyl group by a more manageable, finite set of reflections.

**Theorem 1.1.** *The Weyl group $W$ is generated by the simple reflections $s_1, s_2, \ldots, s_n$.*

The proof uses properties which might be useful to understand the action of the Weyl group on the associated root system. Therefore, we will give an idea of how to proof Theorem 1.1, although it is a well known fact.

*Proofidea.* We follow the proof of [3, Section 1.5]. Let $W' = \langle s_1, \ldots, s_n \rangle$ be the subgroup of $W$ which is claimed to be $W$. The main part of the work are the proofs of

$$W'\Delta = \Phi \tag{1.1}$$

and

$$ts_\alpha t^{-1} = s_{t\alpha} \quad \text{for any } t \in O(V) \ . \tag{1.2}$$

For (1.1), we refer to [3]. The conjecture (1.2) is a straightforward computation.
Let $s_\beta$ be any generator of $W$. Since $s_\beta = s_{-\beta}$ and $\Phi = \Phi^+ \cup \Phi^-$, we consider $\beta$ to be a positive root. By (1.1), there is a $w \in W$ and a simple root $\alpha$ such that $\beta = w\alpha$. Then, (1.2) implies $s_\beta = ws_\alpha w^{-1} \in W'$. Thus, every generator of $W$ can be written as word of the simple reflections $s_i$, which proves $W = W'$. □

Given a group $G$ with set of generators $T$, any group element $g \in G$ can be written as a product of elements in $T$. The product is called a ***word*** for $g$ and a word of the smallest possible number of factors is a ***reduced*** word. The ***length*** of $g$ is then the number of factors in a reduced word, and it is denoted by $\ell(g)$. The length of the identity element is 0.
For an element $w \in W$ consider a reduced word for $w$ as a product of the generators $s_1, \ldots, s_n$, say $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$. The ***length*** $\ell(w)$ of $w$ is then $\ell$ and we define

$$\det(w) = (-1)^{\ell(w)} \ .$$

Denote the longest element of $W$ by $\omega_0$. By [3, Section 1.8], it is unique and satisfies

$$\omega_0^{-1} = \omega_0 \qquad \text{and} \qquad \ell(\omega_0 w) = \ell(\omega_0) - \ell(w) \quad \text{for all } w \in W. \tag{1.3}$$

This will be useful in Section 3.2.

**Definition 1.2.** Define a lattice $P$ in $V$ by

$$P = \left\{ \lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi \right\} .$$

We call $P$ the ***weight lattice*** and its elements ***weights***.

The Weyl group $W$ acts on the weight lattice since $W$ permutes the hyperplanes $H_\alpha$ perpendicular to positive roots $\alpha$. Of course, the weight lattice is spanned by the dual basis, which leads to the following definition.

**Definition 1.3.** The ***fundamental weights*** $\omega_1, \ldots, \omega_n$ are the dual basis to the simple coroots, i.e., they are defined by

$$\langle \omega_i \,, \alpha_j^\vee \rangle = \delta_{ij} \ .$$

The elements of the set

$$P^+ = \left\{ \lambda \in V \mid \langle \lambda \,, \alpha^\vee \rangle \in \mathbb{N}_0 \text{ for all } \alpha \in \Phi^+ \right\}$$

are the ***dominant weights***. Requiring positive integers in $P^+$, instead of nonnegative, defines $P^{++}$, the set of ***dominant regular weights***.

Note, that we can define the set of weights, dominant weights and dominant regular weights equivalently by

$$P = \sum_{i=1}^{n} \mathbb{Z}\omega_i \ , \qquad P^+ = \sum_{i=1}^{n} \mathbb{N}_0\omega_i \ , \qquad P^{++} = \sum_{i=1}^{n} \mathbb{N}\omega_i \ .$$

Likewise to the fundamental weights, the roots form a lattice

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$$

called the ***root lattice***, which is also $W$-invariant. Moreover, we define $Q^+ = \sum_{i=1}^{n} \mathbb{N}_0\alpha_i$ as in [8, Section 2.1]. Denote $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q^+$. This is a partial order on $Q$ and we want a similar partial order on weights. Because of Definition 1.1 (3), each simple root is a weight, so the root lattice $Q$ is a sublattice of the weight lattice $P$. But given a root $\lambda = \sum_{i=1}^{n} \lambda_i\alpha_i \in Q^+$, in general, it is not a dominant weight, since $\langle \alpha_i \,, \alpha_j^\vee \rangle$ might be a negative integer for suitable simple roots $\alpha_j$. Thus, define a partial order on weights by

$$\lambda \geq \mu \quad \text{ if and only if } \quad \lambda - \mu \in Q^+. \tag{1.4}$$

*Remark.* In many references, the definitions and notations concerning to root systems and Weyl groups are motivated in the theory of Lie algebras. A root of a complex semisimple Lie algebra $\mathfrak{g}$ is a linear operator of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This leads to the notation $\mathfrak{h}_\mathbb{R}^*$ (instead of $V$) for the real vector space in which we define roots as vectors. The weight lattice $P$ is then often denoted by $\mathfrak{h}_\mathbb{Z}^*$, such that, for example, the structure

$$V = \mathbb{R} \otimes_\mathbb{Z} \mathfrak{h}_\mathbb{Z}^*$$

is visible from the notation.

**Definition 1.4.** Define the ***dual root system*** $\Phi^\vee$ to a root system $\Phi$ to be the set of its coroots $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$.

Indeed, $\Phi^\vee$ is a root system in $V^\vee \cong V$. Let $W^\vee$ denote its Weyl group. Since we consider a coroot $\alpha^\vee$ as scalar multiple of $\alpha$, the root system $\Phi^\vee$ has the same Weyl group as $\Phi$, meaning

$$w\alpha^\vee = (w\alpha)^\vee$$

for any $w \in W$. The set of simple coroots $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ forms a simple system in $\Phi^\vee$ and it determines the positive system $(\Phi^\vee)^+ = \{\alpha^\vee \mid \alpha \in \Phi^+\}$ in $\Phi^\vee$. Unless otherwise defined, we fix these systems as the simple system and positive system of $\Phi^\vee$.

Recall, that any root can be written as a linear combination of the simple roots. The ***height*** of a root $\alpha = \sum_{i=1}^n m_i\alpha_i$ is the integer $\mathrm{ht}(\alpha) = \sum_{i=1}^n m_i$ and it is not surprising to call the root with the largest height the ***highest root***, commonly denoted by $\varphi$.

**Lemma 1.2.** *The height of a coroot $\gamma^\vee$ in $\Phi^\vee$ is*

$$\mathrm{ht}(\gamma^\vee) = \langle \gamma^\vee , \rho \rangle$$

*where*

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

*is called the* **Weyl vector***.*

*Proof.* Let $\gamma^\vee = \sum_{i=1}^n m_i\alpha_i^\vee$ be a coroot in the root system $\Phi^\vee$. We claim that $\rho = \sum_{i=1}^n \omega_i$. This yields

$$\langle \gamma^\vee , \rho \rangle = \sum_{i=1}^n m_i \langle \alpha_i^\vee , \rho \rangle = \sum_{i=1}^n m_i = \mathrm{ht}(\gamma^\vee) \ .$$

Let us prove, that the Weyl vector $\rho$ equals the sum of the fundamental weights. Definition 1.3 implies, that $w := \sum_{i=1}^n \omega_i$ is uniquely determined by:

$$\langle w , \alpha^\vee \rangle = 1 \qquad \text{for all } \alpha \in \Delta \ . \tag{1.5}$$

Fix a simple root $\alpha$ and consider $s_\alpha(\rho)$. By definition, this is

$$s_\alpha(\rho) = \rho - \langle \rho , \alpha^\vee \rangle \alpha \ . \tag{1.6}$$

On the other hand, we have

$$s_\alpha(\rho) = \frac{1}{2}\left( \sum_{\beta \in \Phi^+ \setminus \{\alpha\}} s_\alpha(\beta) \right) + \frac{1}{2} s_\alpha(\alpha) = \frac{1}{2}\left( \sum_{\beta \in \Phi^+ \setminus \{\alpha\}} s_\alpha(\beta) \right) - \frac{1}{2}\alpha \ . \tag{1.7}$$

The set $\Phi^+ \setminus \{\alpha\}$ is permuted by $s_\alpha$ – this holds for every choice of positive roots and simple root $\alpha$ (see [3, Section 1.4]). Thus, (1.7) reduces to

$$s_\alpha(\rho) = \rho - \alpha \ .$$

Because of (1.6), this implies

$$\langle \rho , \alpha^\vee \rangle = 1 \qquad \text{for all } \alpha \in \Delta \ ,$$

so the Weyl vector $\rho$ satisfies the condition (1.5) which defines $w = \sum_{i=1}^n \omega_i$ uniquely. This completes the proof. □

The proof of Lemma 1.2 implies, that any dominant regular weight $\mu \in P^{++}$ can be written as sum $\rho + \nu$ for a dominant weight $\nu \in P^+$. In particular, we get the bijection

$$\begin{array}{ccc} P^+ & \to & P^{++} \\ \mu & \mapsto & \rho + \mu \end{array} .$$

### 1.1 Fundamental domains for the action of $W$

Consider the hyperplanes $H_\alpha$ for positive roots $\alpha \in \Phi^+$, which determine the reflections $s_\alpha$. The connected components of

$$V \setminus \bigcup_{\alpha \in \Phi^+} H_\alpha$$

are called ***chambers***. The ***fundamental chamber*** is defined by

$$C = \left\{ \lambda \in V \mid \langle \lambda , \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Phi^+ \right\}.$$

We can define $C$ equivalently as

$$C = \left\{ \lambda \in V \mid \langle \lambda , \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Delta \right\} \tag{1.8}$$

since each positive root can be expressed as a linear combination of some simple roots with positive integer coefficients. This implies, that $C$ is bounded by the hyperplanes $H_{\alpha_i}$ perpendicular to the simple root $\alpha_i \in \Delta$. To prove that $C$ is a chamber, we prove connection first. Let $\lambda, \mu \in C$ and consider any vector on the line between them, that is,

$$\nu = \lambda + t(\mu - \lambda) \qquad \text{for any } t \in (0, 1)\ .$$

For any positive root $\alpha$, we have

$$\langle \nu , \alpha^\vee \rangle = \langle \lambda , \alpha^\vee \rangle + t \langle \mu - \lambda , \alpha^\vee \rangle = (1-t) \langle \lambda , \alpha^\vee \rangle + t \langle \mu , \alpha^\vee \rangle > 0.$$

Thus, $C$ is connected.
Note a third equivalent definition for the fundamental chamber

$$C = \sum_{i=1}^{n} \mathbb{R}_{>0} \omega_i$$

and denote the closure of $C$ by

$$\overline{C} = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} \omega_i.$$

Define a partial order on $V$ by

$$\lambda \leq \mu \quad \text{if and only if} \quad \mu - \lambda \in \sum_{i=1}^{n} \mathbb{R}_{\geq 0} \alpha_i.$$

It still remains to prove that the fundamental chamber $C$ is indeed a chamber. This will follow from corollary 1.5 – which follows from Theorem 1.4 – where we show that $C$ is a fundamental domain.

In this thesis, a ***fundamental domain*** for the action of $W$ on $V$ is a subset $D$ of $V$ satisfying the following:

(i) $V = \bigcup_{w \in W} w(\overline{D})$ and

(ii) for each $\lambda \in \bigcup_{w \in W} w(D)$, there is a unique $\mu \in D$ such that $w\lambda = \mu$ for a $w \in W$.

Note, that a more common definition of a fundamental domain requires $D$ to be a closed set and that we can extend the definition for any group which acts on $V$.

The following lemma is from [3, Section 1.12] and verify condition (i) and a part of (ii) for $D = \overline{C}$.

**Lemma 1.3.** *For each $\lambda \in V$, there is an element $\mu \in \overline{C}$ in the $W$-orbit $W\lambda$ and it holds $\lambda \leq \mu$.*

*Proof.* Let $\lambda \in V$. Choose a maximal element $\mu$ of the set $\{\nu \in W\lambda \mid \lambda \leq \nu\}$, which contains at least $\lambda$. For each simple reflection $s_\alpha$ applies

$$s_\alpha \mu = \mu - 2\frac{\langle \mu , \alpha \rangle}{\langle \alpha , \alpha \rangle}\alpha \in W\lambda.$$

Since $\mu$ is a maximal element, $\langle \mu , \alpha \rangle \geq 0$ must hold for every simple root $\alpha$. By (1.8), this implies $\mu \in \overline{C}$. □

**Theorem 1.4.** *The closure of the fundamental chamber $\overline{C}$ is a fundamental domain for the action of $W$ on $V$.*

*Proof.* Let $\lambda, \mu \in \overline{C}$ be in the same $W$-orbit, say $w\lambda = \mu$ for a $w \in W$. Since $W \subset O(n)$, we have

$$0 < \langle \lambda , \alpha^\vee \rangle = \frac{2}{\langle \alpha , \alpha \rangle}\langle w^{-1}\mu , \alpha \rangle = \frac{2}{\langle \alpha , \alpha \rangle}\langle \mu , w\alpha \rangle = \langle \mu , w\alpha^\vee \rangle$$

for each positive root $\alpha \in \Phi^+$. This forces $w(\Phi^+) = \Phi^+$ because $\mu \in \overline{C}$. By [3, Section 1.8], this holds if and only if $w = \mathrm{id}$, so $\lambda = \mu$ holds. Together with Lemma 1.3 the proof is complete. □

**Corollary 1.5.** *The fundamental chamber $C$ is a fundamental domain for the action of $W$ on $V$.*

*Proof.* This follows directly from Theorem 1.4. □

**Corollary 1.6.** *Each weight has exactly one dominant weight in its $W$-orbit.*

*Proof.* This holds since $P$ is $W$-invariant and $P^+ = \sum_{i=1}^n \mathbb{N}_0\omega_i \subset \overline{C}$. □

## 1.2 Realization of root systems

The root systems are classified by types, that is, up to isomorphism any root system belongs to one type. The number of these types is finite; we differ between Type $A$, $B$, $C$, $D$, $E$, $F$ and $G$, where not every Type exist for every rank $n$. More precisely, any root system is exactly one of the following:

$$A_n\ (n \geq 1),\ B_n\ (n \geq 2),\ C_n\ (n \geq 3),\ D_n\ (n \geq 4),\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$$

where the index is the rank of the system. The Types $E$,$F$ and $G$ are the exceptional Types, since they just exist in finitely many dimensions. Note, that some Types are also defined for a lower rank $n$ than given in the list above. But then, they are isomorphic to another mentioned Type. For instance, $A_1 \cong B_1 \cong C_1$ and $B_2 \cong C_2$.

For the details of the finite classification, we refer to [3]. In this thesis, we are just interested in one representative per Type. Each root system has a standard realization, that is a common choice of the vector space $V$ and the roots $\Phi$ with a fixed simple system $\Delta$. Indeed, there are differences in those choices up to the authors, but they are all very similar and of course isomorphic. The examples in this section will follow the constructions in [3, Section 2.10]. For the sake of completeness, we give a realization for every mentioned Type, but later we will focus on Type $A$ and rank 2 root systems. Moreover, we will define Type $C_n$ for $n \geq 2$ to visualize the isomorphic rank 2 root systems $C_2$ and $B_2$.

Let $\langle \, , \rangle$ be the standard scalar product on $\mathbb{R}^n$, i.e., $\langle \varepsilon_i \, , \varepsilon_j \rangle = \delta_{ij}$ for the standard basis $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathbb{R}^n$. Although it suffices to choose either the positive or the simple system, we will give both.

**Type $A_{n-1}$ for $n \geq 2$** Let $V \subset \mathbb{R}^n$ be the vector space consisting of all vectors $\lambda \in \mathbb{R}$ such that their coefficients in $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$ sum up to zero. Let the positive system be

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i \leq j\}.$$

Then, the simple roots are

$$\alpha_1 := \varepsilon_1 - \varepsilon_2, \ \alpha_2 := \varepsilon_2 - \varepsilon_3, \ldots, \ \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n.$$

The Weyl group is the symmetric group $S_n$.

We verify, that this defines a root system with Weyl group $S_n$. Conditions (2) and (3) of Definition 1.1 follow directly from the definition of the inner product $\langle \, , \rangle$ and the implicit assumption $\Phi = \Phi^+ \cup (-\Phi^+)$.
To prove condition (1) and to understand the Weyl group, consider a vector $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$ and a simple root $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. The simple reflection $s_i$ can be identified by the transposition $(i, \ i+1) \in S_n$:

$$s_i \lambda = \lambda - \langle \lambda \, , \alpha_i^\vee \rangle \alpha_i = \lambda - (\lambda_i - \lambda_{i+1}) \alpha_i = \lambda_1 \varepsilon_1 + \cdots + \lambda_{i+1} \varepsilon_i + \lambda_i \varepsilon_{i+1} + \cdots + \lambda_n \varepsilon_n.$$

Analogously, $s_\alpha$ for $\alpha = \varepsilon_i - \varepsilon_j$ can be identified by the transposition $(i, \ j) \in S_n$. The Weyl group permutes the standard basis of $\mathbb{R}$, thus it permutes the roots and is the symmetric group $S_n$.

Consider the vectors $v_i = \varepsilon_1 + \cdots + \varepsilon_i \in \mathbb{R}^n$ for each $1 \leq i \leq n$. These vectors satisfy

$$\langle v_i \, , \alpha_j^\vee \rangle = \langle v_i \, , \alpha_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

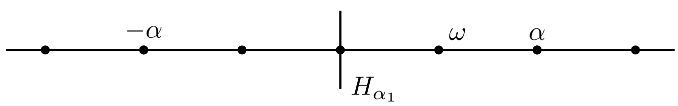
In particular, adding a scalar multiple of $v_n$ to any vector $v \in \mathbb{R}^n$ will not change the value of $\langle v \, , \alpha_j^\vee \rangle$. Thus, the fundamental weights are

$$\omega_i = v_i - \frac{i}{n} v_n \in V$$

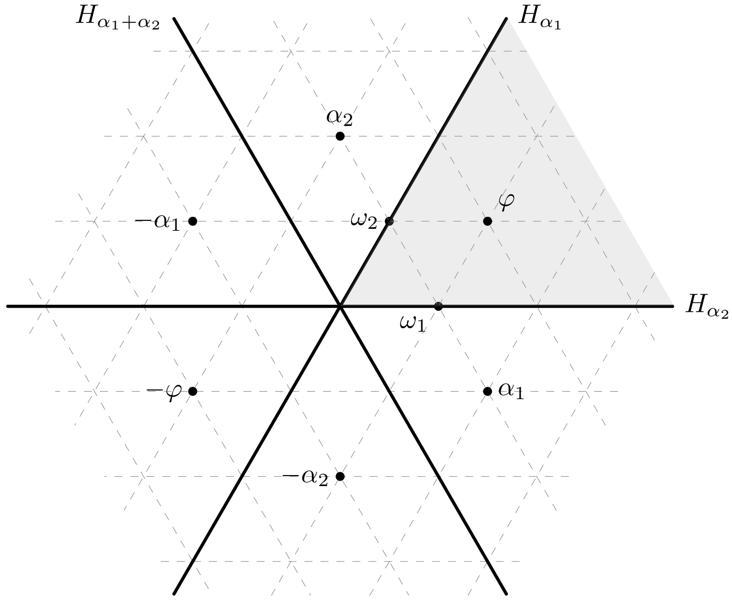
for $1 \leq i \leq n-1$.

The highest root in Type $A$ is $\varphi = \sum_{i=1}^{n-1} \alpha_i = \varepsilon_1 - \varepsilon_n$ with height $\mathrm{ht}(\varphi) = n - 1$.

**Example.** We start with the only root system existing – up to isomorphism – of rank 1. The root system $A_1$ has a single simple root $\alpha = \varepsilon_1 - \varepsilon_2$. The roots in $A_1$ are $\alpha$ and $-\alpha$ and the fundamental weight is $\omega = \frac{1}{2}\alpha$, so the weight lattice is $P = \mathbb{Z}\omega = \frac{1}{2}\mathbb{Z}\alpha$.
We draw the vector space as horizontal line and the hyperplane $H_\alpha$, which is just a point, as a vertical line through this point. The drawn points are the weights – the roots and the fundamental weights are labeled.



**Example.** The rank 2 root system of Type $A$ has simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$. The third positive root is the highest root $\varphi = \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_n$. The fundamental weights are $\omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$ and $\omega_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$. They determine the weight lattice $P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$:




The weights – pictorially the intersection points of the dashed lines – in the gray area are the dominant weights. They are non regular if and only if they do not lay on the hyperplane $H_{\alpha_1}$ or $H_{\alpha_2}$.

**Type $B_n$ for $n \geq 2$** Let $V = \mathbb{R}^n$ be the vector space and choose the positive system

$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \mid i \leq j\} \cup \{\varepsilon_i \mid i = 1, \dots, n\} \ .$$

Then,

$$\alpha_1 := \varepsilon_1 - \varepsilon_2,\ \alpha_2 := \varepsilon_2 - \varepsilon_3,\ \dots,\ \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n \text{ and } \alpha_n := \varepsilon_n$$

are the simple roots. The Weyl group $W$ is $S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$.

Just as for Type $A$, we check if this determines a root system, and again, we only verify, that this satisfies Definition 1.1(1). For simplicity, denote

$$\Phi_1^+ = \{\varepsilon_i \pm \varepsilon_j \mid i \leq j\} \ , \quad \Phi_2^+ = \{\varepsilon_i \mid i = 1, \dots, n\} \quad \text{and} \quad \Phi_i = \Phi_i^+ \cup (-\Phi_i^+)$$

for $i = 1, 2$. First, consider a positive root $\alpha \in \Phi_1^+$. We claim, that $s_\alpha(\Phi_1) = \Phi_1$. For any root $\varepsilon_i \in \Phi_2^+$, we have

$$s_\alpha \varepsilon_i = \varepsilon_i - \langle \varepsilon_i \,, \alpha \rangle \alpha = \begin{cases} \varepsilon_k & \text{if } \alpha = \varepsilon_k \pm \varepsilon_i \text{ for any } k \neq i, \\ \varepsilon_i & \text{else.} \end{cases}$$

Vice versa, we have

$$s_{\varepsilon_i} \alpha = \alpha - 2\langle \alpha \,, \varepsilon_i \rangle \varepsilon_i = \begin{cases} \varepsilon_k \pm \varepsilon_i & \text{if } \alpha = \varepsilon_k \mp \varepsilon_i \text{ for any } k \neq i, \\ \alpha & \text{else.} \end{cases}$$

Now, fix a root $\varepsilon_i \in \Phi_2^+$, then

$$s_{\varepsilon_i}\varepsilon_j = \varepsilon_j - \langle \varepsilon_j\,, \varepsilon_i^\vee\rangle \varepsilon_i = \varepsilon_j - 2\langle \varepsilon_j\,, \varepsilon_i\rangle \varepsilon_i = \begin{cases} \varepsilon_j & \text{if } i = j, \\ -\varepsilon_j & \text{else,} \end{cases}$$

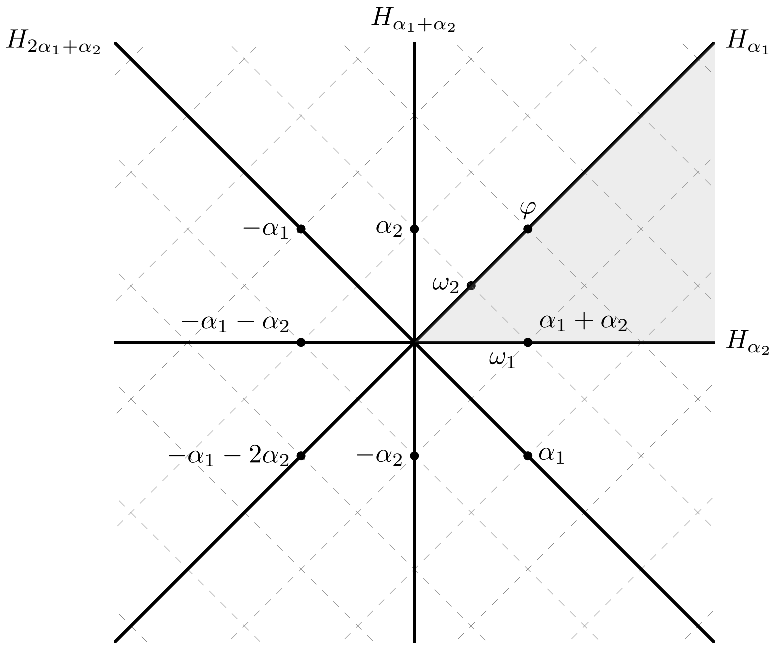
for any root $\varepsilon_j \in \Phi_2^+$. Since the reflections are linear operators on $V$, we have $s_\alpha(\Phi) = \Phi$ for any $\alpha \in \Phi$.

The simple reflections $s_1, \dots, s_n$ permute and change signs of the standard basis, thus the Weyl group $W$ is $S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$.
The fundamental weights are defined by

$$\omega_i = \sum_{j=1}^{i} \varepsilon_j \quad \text{for } 1 \le i < n \qquad \text{and} \qquad \omega_n = \frac{1}{2}\sum_{j=1}^{n} \varepsilon_j .$$

The highest root is $\varphi = \alpha_1 + 2\sum_{i=2}^n \alpha_i = \varepsilon_1 + \varepsilon_2$ with height $\mathrm{ht}(\varphi) = 2n - 1$.

**Example.** The rank 2 root system of Type $B$ has simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2$. The missing two positive roots are $\alpha_1 + \alpha_2 = \varepsilon_1$ and the highest root $\varphi = \alpha_1 + 2\alpha_2 = \varepsilon_1 + \varepsilon_2$. The fundamental weights are $\omega_1 = \varepsilon_1$ and $\omega_2 = \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2$.




As in the previous example, the weights are pictorially the intersection points of the drawn lines. Weights in the gray area are dominant and they are regular if and only if they do not lie on any hyperplane.

**Type $C_n$ for $n \ge 2$** Type $C_n$ is the dual system to Type $B_n$. Therefore, the vector space is again $V = \mathbb{R}^n$ and we choose the positive system

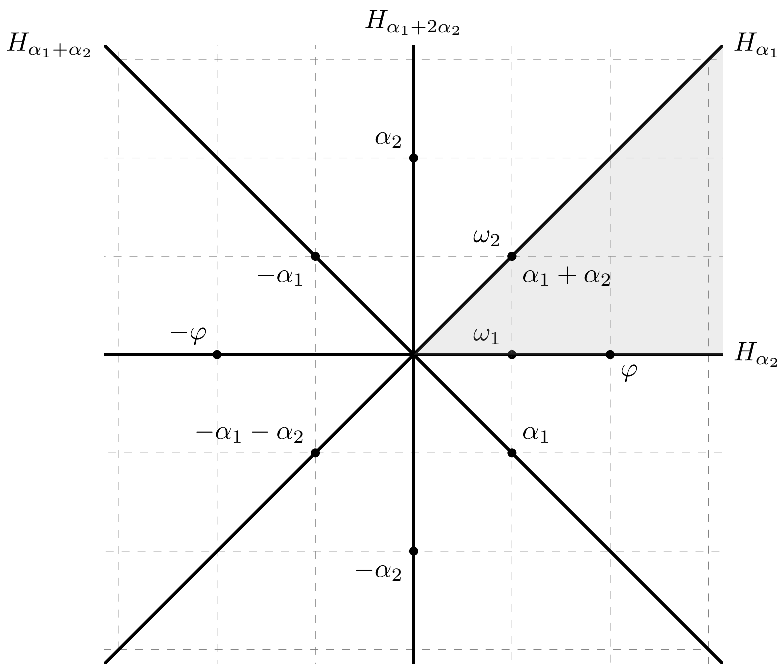
$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \mid i \le j\} \cup \{2\varepsilon_i \mid i = 1, \dots, n\} .$$

Accordingly, the simple roots are

$$\alpha_1 := \varepsilon_1 - \varepsilon_2,\ \alpha_2 := \varepsilon_2 - \varepsilon_3,\ \dots,\ \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n \text{ and } \alpha_n := 2\varepsilon_n \ .$$

The Weyl group $W$ is the same as in Type $B_n$.

**Example.** The rank 2 root system of Type $C$ – which is isomorphic to $B_2$ – has simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = 2\varepsilon_2$. The missing two positive roots are $\alpha_1 + \alpha_2 = \varepsilon_1 + \varepsilon_2$ and the highest root $\varphi = 2\alpha_1 + \alpha_2 = 2\varepsilon_1$. The fundamental weights are $\omega_1 = \varepsilon_1$ and $\omega_2 = \varepsilon_1 + \varepsilon_2$.




**Type $D_n$ for $n \geq 4$** We stay in $V = \mathbb{R}^n$ and choose positive roots

$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \mid i < j\}\,.$$

The simple roots are

$$\alpha_1 := \varepsilon_1 - \varepsilon_2,\ \alpha_2 := \varepsilon_2 - \varepsilon_3,\ \ldots,\ \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n \quad \text{and} \quad \alpha_n := \varepsilon_{n-1} + \varepsilon_n\ .$$

The fundamental weights are

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \text{ for } 1 \leq i \leq n-2\ ,$$
$$\omega_{n-1} = \frac{1}{2}\left(\sum_{i=1}^{n-1} \varepsilon_i\right) - \frac{1}{2}\varepsilon_n \quad \text{and} \quad \omega_n = \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i\ .$$

and the highest root is $\varphi = \alpha_1 + \alpha_{n-1} + \alpha_n + \sum_{i=2}^{n-2} \alpha_i$ with height $\operatorname{ht}\varphi = 2(n-1) - 1$.

**Types $E_6$, $E_7$ and $E_8$** We first give a realization for $E_8$. The root system is

$$\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j\} \cup \left\{\frac{1}{2}\sum_{i=1}^{8} \pm\varepsilon_i \mid \text{even number of positive signs}\right\},$$

where the choices of the signs in $\pm$ in the first set are independent of each other. These 240 roots span the vector space $V = \mathbb{R}^8$. Fix the simple roots

$$\alpha_1 = \frac{1}{2}\left(\varepsilon_1 + \varepsilon_8 - \sum_{i=2}^{7} \varepsilon_i\right), \quad \alpha_2 = \varepsilon_1 + \varepsilon_2 \quad \text{and}$$
$$\alpha_i = \varepsilon_{i-1} - \varepsilon_{i-2} \quad \text{for } i = 3, \ldots, 8.$$

A realization of the root system of Type $E_6$ can be obtained by fixing $\alpha_1, \ldots, \alpha_6$ as the simple roots and the vector space as their span. For $E_7$, we just add the simple root $\alpha_7$.

The highest root of $E_8$ in this realization is

$$\varphi = \varepsilon_7 + \varepsilon_8 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$$

with height $\mathrm{ht}(\varphi) = 33$.
The highest root of $E_7$ is

$$\varphi' = \varepsilon_8 - \varepsilon_7 = 2\alpha_1 + \alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$$

with height $\mathrm{ht}(\varphi') = 15$, and of $E_6$ is

$$\varphi'' = \frac{1}{2}\left(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^{5} \varepsilon_i\right) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

with height $\mathrm{ht}(\varphi'') = 11$.

**Type $F_4$** Let $V = \mathbb{R}^4$ and choose 24 roots

$$\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j\} \cup \{\pm\varepsilon_i \mid i = 1, \ldots, n\} \cup \left\{\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\right\}$$

where the choices of the signs in $\pm$ are independent of each other. The four simple roots then are

$$\alpha_1 := \varepsilon_2 - \varepsilon_3\ , \quad \alpha_2 := \varepsilon_3 - \varepsilon_4, \quad \alpha_3 := \varepsilon_4 \quad \text{and} \quad \alpha_4 := \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\ .$$

The highest root is $\varphi = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \varepsilon_1 + \varepsilon_2$ with height $\mathrm{ht}(\varphi) = 11$.
The fundamental weights are

$$\omega_1 = \varepsilon_1 + \varepsilon_2\ , \quad \omega_2 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\ , \quad \omega_3 = \frac{1}{2}(3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) \quad \text{and} \quad \omega_4 = \varepsilon_1\ .$$

**Type $G_2$** Consider the vector space $V \subset \mathbb{R}^3$ consisting of all vectors $\lambda = \sum_{i=1}^{3} \lambda_i \varepsilon_i$ such that the coefficients sum up to zero. The root system $G_2$ and $A_2$ The positive roots are

$$\Phi^+ = \{\alpha_1,\ \alpha_2,\ \alpha_1 + \alpha_2,\ 2\alpha_1 + \alpha_2,\ 3\alpha_1 + \alpha_2,\ 3\alpha_1 + 2\alpha_2\}$$

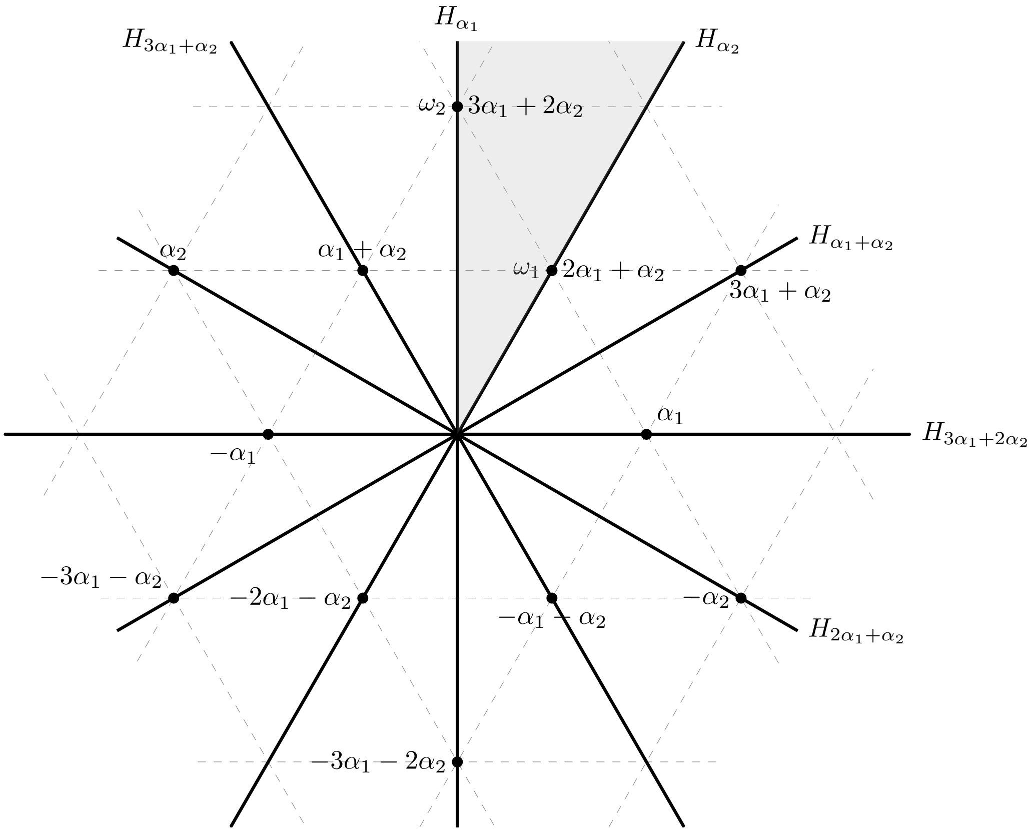
with simple roots

$$\alpha_1 := \varepsilon_1 - \varepsilon_2 \quad \text{and} \quad \alpha_2 := -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\ .$$

Thus, the highest root is $\varphi = 3\alpha_1 + 2\alpha_2 = 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2$ with height $\mathrm{ht}(\varphi) = 5$. The fundamental weights are

$$\omega_1 = -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3 \qquad \text{and} \qquad \omega_2 = -\varepsilon_2 + \varepsilon_3.$$

The weight lattice $P$ in Type $G_2$ is the same as in Type $A_2$ as you can see in the following image.




### 1.3 The affine Weyl group

Let $V$ be a real vector space and $\langle\, ,\rangle$ a non-degenerated $W$-invariant scalar product on $V$. Let $\Phi$ be a root system with positive roots $\Phi^+$ and positive simple roots $\alpha_1, \ldots, \alpha_n$ . Define (affine) hyperplanes

$$H_{\alpha,k} := \left\{\lambda \in V \mid \langle \lambda\, , \alpha^\vee\rangle = k\right\}$$

for each positive root $\alpha$ and each nonnegative integer $k$. Denote the set of those affine hyperplanes and hyperplanes by $\mathcal{H}$.

**Definition 1.5.** The connected components of

$$V \setminus \bigcup_{\substack{\alpha\in\Phi^+\\ k\in\mathbb{Z}}} H_{\alpha,k}$$

are called ***alcoves***. The ***fundamental alcove*** is

$$A = \left\{\lambda \in V \mid 0 < \langle \lambda\, , \alpha^\vee\rangle < 1 \text{ for all } \alpha \in \Phi\right\}.$$

Denote the group of affine linear functions on $V$ by $\mathrm{Aff}(V)$. Likewise to the reflections $s_\alpha$ at the hyperplanes $H_\alpha$, let $s_{\alpha,k}$ be the affine reflection at the affine hyperplane $H_{\alpha,k} \in \mathcal{H}$, that is, $s_{\alpha,k} \in \mathrm{Aff}(V)$ is defined by

$$s_{\alpha,k}(\lambda) = \lambda - (\langle \lambda\, , \alpha^\vee\rangle - k)\alpha = s_\alpha(\lambda) + k\alpha\ .$$

The Weyl group is generated by the reflections $s_{\alpha,0}$ for $\alpha$ in $\Phi^+$ and by Theorem 1.1 it is also generated by $s_i := s_{\alpha_i,0}$ for $i = 1, \ldots, n$.

**Definition 1.6.** The ***affine Weyl group*** $W_{\text{aff}}$ is the subgroup of $\text{Aff}(V)$ generated by the reflections $s_{\alpha,k}$ for any positive root $\alpha$ and integer $k$.

We see immediately (by (1.2)), that

$$ws_{\alpha,k} = s_{w\alpha,k}w \qquad \text{for any } w \in W \ ,$$

so the Weyl group $W$ acts normal in $W_{\text{aff}}$.

Analogously to Theorem 1.1, we want an alternative and smaller spanning set for the affine Weyl group. For that, determine the walls of the fundamental alcove $A$, that is, the (affine) hyperplanes $H_{\alpha,k}$ which bound $A$. Let $\varphi \in \Phi^+$ be the root such that its coroot $\varphi^\vee$ is the highest root $\varphi^\vee$ in the dual root system $\Phi^\vee$. By [3, Section 4.2], we have

$$A = C \cap \left\{\lambda \in P \mid \langle \lambda , \varphi^\vee \rangle < 1\right\} .$$

It follows directly, that $H_{\alpha_1}, \dots, H_{\alpha_n}$ are walls of $A$. The last missing wall is $H_{\varphi,1}$.
We denote $H_{\alpha_0} = H_{\varphi,1}$ and $s_0 = s_{\alpha,1}$ such that the walls of $A$ are denoted by

$$H_{\alpha_0},\ H_{\alpha_1}, \dots,\ H_{\alpha_n}$$

and such that we can describe the affine Weyl group as follows:

**Theorem 1.7.** *The affine Weyl group $W_{\text{aff}}$ is generated by the reflections $s_0, s_1, \dots, s_n$.*

We refer to [3, Section 4.3] for the proof of Theorem 1.7. The proof requires certain properties about how the Weyl group $W$ and the affine Weyl group $W_{\text{aff}}$ act on the collection of hyperplanes $\mathcal{H}$. For instance, that the affine Weyl group $W_{\text{aff}}$ permutes $\mathcal{H}$. These properties and their proof are given in [3, Sections 4.1 and 4.2].

Unless defined otherwise, we consider every element of $W_{\text{aff}}$ as word in the simple reflections $s_0, s_1, \dots, s_n$ as generators. The ***length*** of an element $w \in W_{\text{aff}}$, denoted by $\ell(w)$, is the number of these generators in a reduced word. This coincides with our previous definition for the length of an Weyl element $w \in W$.
Note, that the identity element in $W_{\text{aff}}$, denoted by 1, still has length 0.

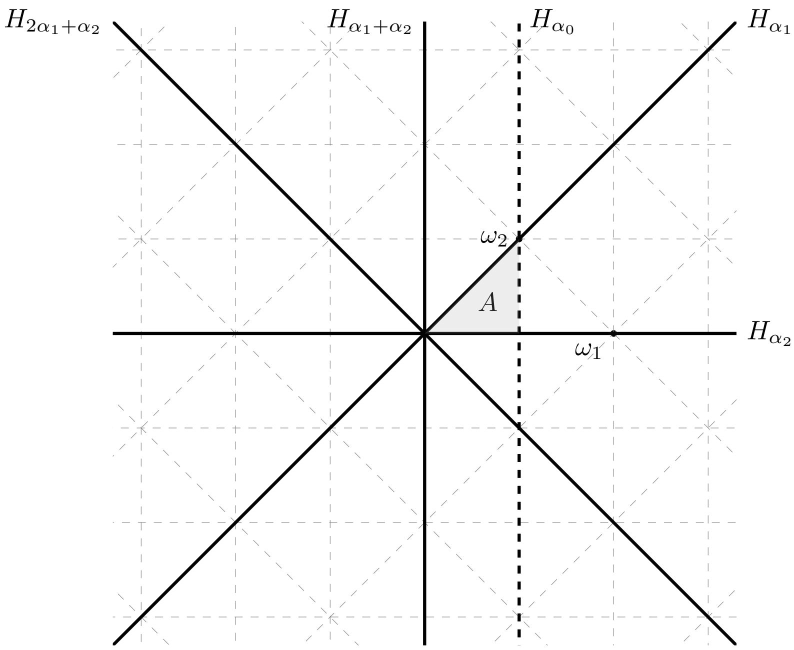
**Example.** The highest coroot in Type $A$, $B$, $C$, $D$ and $G$, respectively, are given in the following tabular:

| $\Phi$ of Type | highest root $\varphi^\vee$ in $\Phi^\vee$ | $\varphi$ in $\Phi$ |
|---|---|---|
| $A_n$ | $\varepsilon_1 - \varepsilon_n$ | $\varepsilon_1 - \varepsilon_n$ |
| $B_n$ | $2\varepsilon_1$ | $\varepsilon_1 = \alpha_1 + \dots + \alpha_n$ |
| $C_n$ | $\varepsilon_1 + \varepsilon_2$ | $\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$ |
| $D_n$ | $\varepsilon_1 + \varepsilon_2$ | $\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ |
| $G_2$ | $\varepsilon_3 - \varepsilon_2$ | $2\alpha_1 + \alpha_2$ |

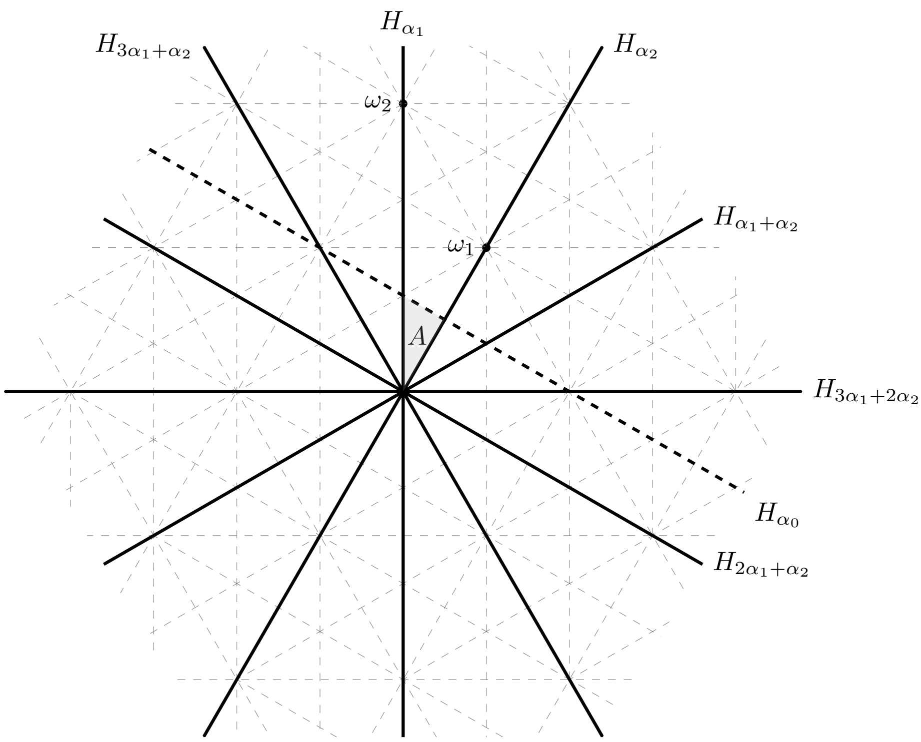
Note that the root $\varphi$ in general is not the highest root in $\Phi$. They coincide just in Type $A$, since the dual of $A_n$ is again of Type $A$ and the roots are all of the same length. Although Type $G_2$ is also self-dual, the root $\phi = 2\alpha_1 + \alpha_2$ is not the highest root. The roots in Type $G_2$ have two different length, thus they can be divided into short and long roots. Taking the dual of a root system of Type $G_2$ swaps the length of the roots.

We draw the affine hyperplanes $H_{\alpha,k}$ in Type $B_2$ and $G_2$. The fundamental alcove $A$ is filled in gray, the hyperplanes are the black lines (as in Section 1.2) and the affine hyperplanes are the dashed lines. The hyperplane $H_{\alpha_0} = H_{\varphi,1}$ is drawn in black dashed lines in contrast to the grey dashed lines of the other affine hyperplanes $H_{\alpha,k}$.

For Type $B_2$, we have




and for Type $G_2$, we have




*Remark.* The affine Weyl group is itself a Weyl group for an affine root system on a real vector space. The affine root system is related to a root system with simple roots $\alpha_1, \ldots, \alpha_n$ but has an additional simple root $\alpha_0$.

**Theorem 1.8.** *The fundamental alcove $A$ is a fundamental domain for the action of $W_{\text{aff}}$.*

We refer to [3, Sections 4.3 and 4.8] for the proof.

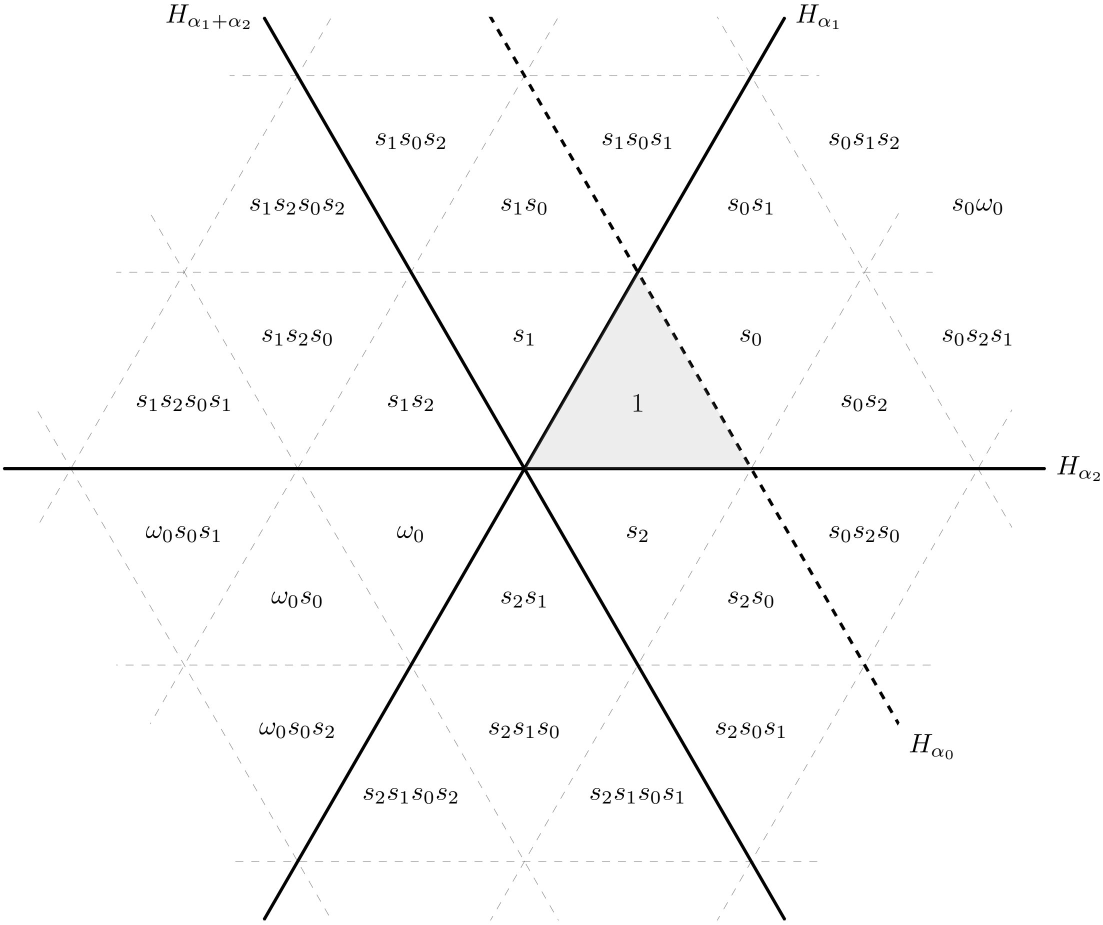
**Corollary 1.9.** *The mapping*

$$\begin{array}{ccc} W_{\text{aff}} & \to & \{\textit{alcoves in } V\} \\ w & \mapsto & wA \end{array}$$

*is a bijection.*

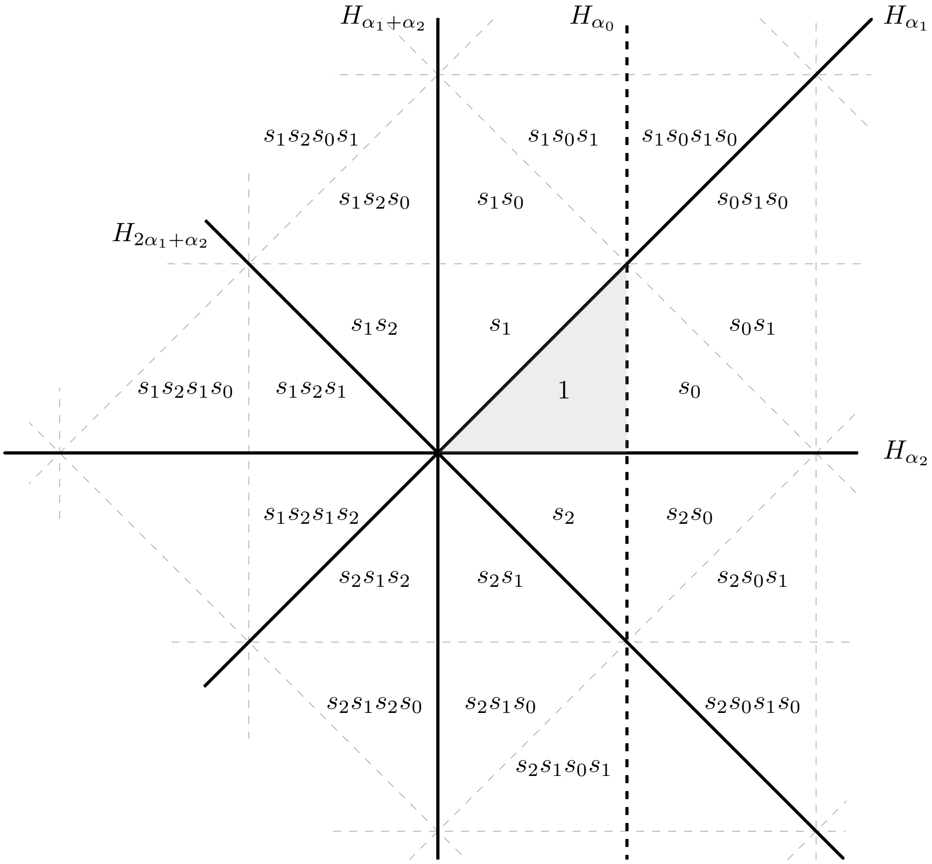
*Proof.* This follows directly from Theorem 1.8. □

**Example.** We draw the alcoves of the rank 2 root systems $A_2$ and $B_2$. In each picture the gray colored alcove is the fundamental alcove $A$. The other alcoves are labeled by their corresponding element of $W_{\text{aff}}$, given by the bijection in Theorem 1.8. Moreover, each element is given as a reduced word such that we also get their length.
For Type $A$, we get




Note that some words have $\omega_0 = s_1s_2s_1 = s_2s_1s_2$ as factor. This is for simplicity, and these words are still reduced if we write $w_0$ itself as one of its two reduced words in our used generators.

For Type $B_2$, we just label the alcoves corresponding to an element in $W_{\text{aff}}$ of length 4 or less.




## 1.4 The extended affine Weyl group

Let $\lambda \in V$ be a vector and $t_\lambda$ be the translation $t_\lambda(x) = \lambda + x$ on $V$. Denote the image of a set $X$ by $t_\lambda X = \lambda + X$.
The translations $t_\lambda$ for $\lambda \in P$ a weight form a subgroup of $\text{Aff}(V)$ and the affine Weyl group $W_{\text{aff}}$ acts normal on $t_\lambda$:

$$wt_\lambda = t_{w\lambda}w \text{ for any } \lambda \in P \text{ and } w \in W_{\text{aff}}. \tag{1.9}$$

For an affine reflection, we have

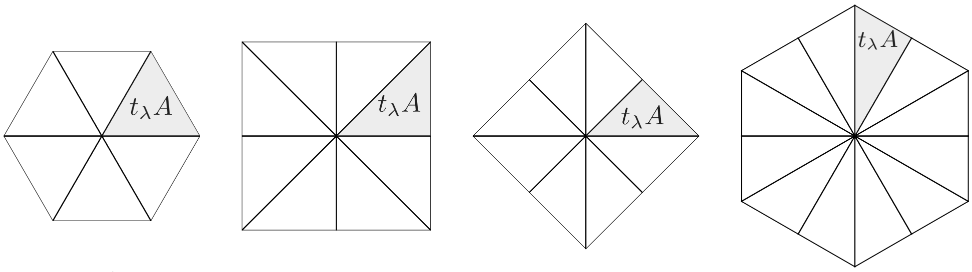
$$s_{\alpha,k}t_\lambda = t_{s_{\alpha,k}(\lambda)}s_\alpha, \tag{1.10}$$

so

$$t_\lambda s_{\alpha,k} = (s_{\alpha,k}t_{(-\lambda)})^{-1} = (t_{s_{\alpha,k}(-\lambda)}s_\alpha)^{-1} = s_\alpha t_{s_{\alpha,k}(\lambda)}.$$

Thus, $t_\lambda$ acts on the set of hyperplanes $\mathcal{H}$ if and only if $\lambda \in P$ is a weight. In particular, the translation of the fundamental alcove $t_\lambda A$ for $\lambda \in P$ is again an alcove and we define the $\lambda$-polygon as $\lambda + WA$, i.e., the translation of the $W$-orbit of $A$.

**Example.** Consider the rank 2 root systems and let $\lambda$ be a weight. The $\lambda$-polygon is




in $A_2$ in $B_2$ in $C_2$ in $G_2$.

As we have seen in Corollary 1.9, the alcoves in $V$ are in bijection with the elements of $W_{\text{aff}}$. The alcoves in the 0-polygon are the Weyl group $W$ under this bijection.

**Corollary 1.10.** *The following mapping is a bijection:*

$$\begin{array}{rcc} W & \longrightarrow & \{\textit{alcoves in } 0\textit{-polygon}\} \\ w & \longmapsto & wA \end{array}$$

*Proof.* Like Corollary 1.9, this follows directly from Theorem 1.8. □

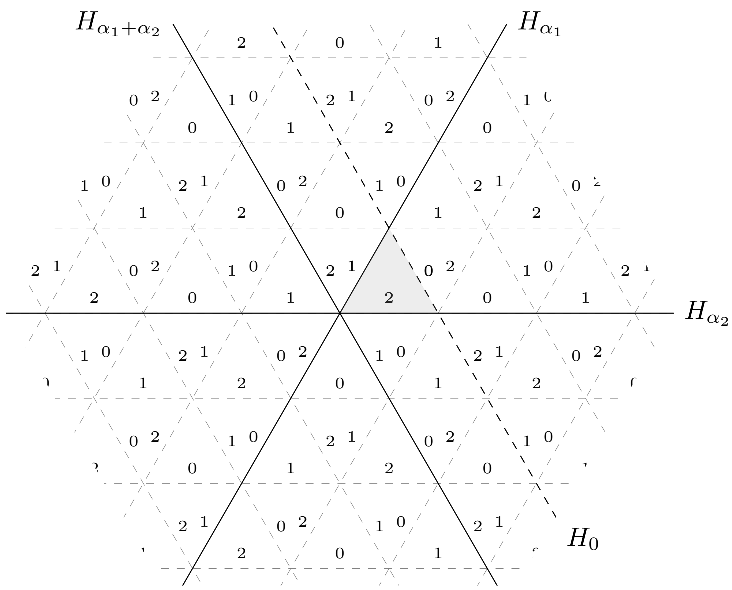
**Definition 1.7.** The subgroup of $\mathrm{Aff}(V)$

$$\widetilde{W} = \{t_\lambda w \mid \lambda \in P,\ w \in W\}$$

is called the ***extended affine Weyl group***.

The Weyl group $W$ is a normal subgroup of $\widetilde{W}$ by (1.9). Moreover, the extended affine Weyl group $\widetilde{W}$ acts on the set of alcoves, since $W_{\text{aff}}$ does. But note, that this action differs from the action of $W_{\text{aff}}$: Recall from Section 1.3, that the fundamental alcove $A$ has walls $H_{\alpha_0}, \dots, H_{\alpha_n}$ and that $W_{\text{aff}}$ is generated by the simple reflections $s_0, s_1, \dots, s_n$. We numerate the codimension one faces of $A$ by integers 0 to $n$ such that face $j$ is on $H_{\alpha_j}$. Because of Theorem 1.8 we can extend this numeration to all alcoves by acting on $A$ with the affine Weyl group $W_{\text{aff}}$.

For the root system $A_2$, this is



This numeration is not invariant under the action of $\widetilde{W}$. More precisely, in contrast to the affine Weyl group, $\widetilde{W}$ is not in bijection to the alcoves, since there can be two different elements $t_\lambda w,\ t_\mu w' \in \widetilde{W}$ such that $t_\lambda w(A) = t_\mu w'(A)$. Since $W \in O(V)$ and the set of hyperplanes is invariant under the action of $\widetilde{W}$, the numeration of the facets of $t_\lambda w(A)$ must be different from that of $t_\mu w'(A)$.

Define the group

$$\Omega = \widetilde{W}/W_{\text{aff}} = \{t_\lambda w W_{\text{aff}} \mid \lambda \in P,\ w \in W\}\,.$$

Let $g$ be a representative of an element $[g]$ in $\Omega$ with $g(A) = A$. Indeed, such an element always exists. Thus, we can identify $\Omega$ as the group of the elements of $\widetilde{W}$ with length 0, where the length of $w \in \widetilde{W}$ is defined as

$$\ell(w) = \#\{H_{\alpha,k} \mid H_{\alpha,k} \text{ seperates } A \text{ from } wA\}\,.$$

By [3, Section 4.5], this definition is equivalent to the previous definitions of the length $\ell(w)$ for a $w \in W_{\text{aff}}$.

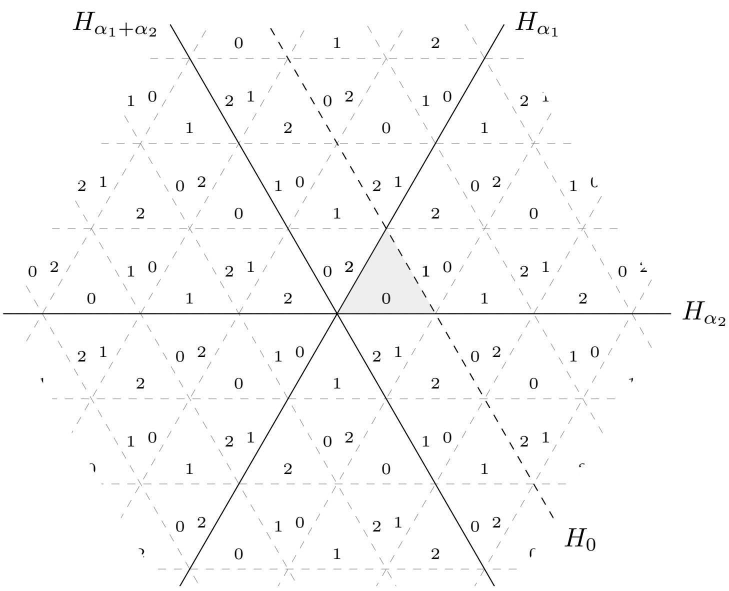
We can interpret $\Omega$ as follows: An element $g \in \Omega$ is a permutation of the walls of $A$. Any $g$ yields a different "sheet" in $\Omega \times V$ by acting on $g(\overline{A})$ with the affine Weyl group $W_{\text{aff}}$, so we call a $g \in \Omega$ a ***change of sheet***.

**Corollary 1.11.** *The extended affine Weyl group can be identified with the alcoves in $\Omega \times V$ by the bijection*

$$\begin{array}{ccc} \widetilde{W} & \to & \{\text{alcoves in } \Omega \times V\} \\ w & \to & wA \end{array}\,.$$

*Proof.* In fact, $w \in \widetilde{W}$ has a unique presentation $w = t_\lambda w'$ for a weight $\lambda$ and $w \in W$. Suppose $w = t_\mu w''$, then $t_\lambda = t_\mu w' w''$ holds and the $\lambda$-polygon is $t_\lambda WA = t_\mu WA$ the $\mu$-polygon, so $\lambda = \mu$. But then, $w' = w''$ must hold as well. □

**Example.** The root system $A_2$ has three sheets. Three representatives of distinct elements in $\Omega$ are id, $g$ and $g^2 = gg$, where $g = t_{\omega_1} s_1 s_2$. The id-sheet is the one given above. The $g$-sheet is




If we consider the representatives as permutations, we have $g = (012)$ and $g^2 = (021)$.

# 2 Macdonald polynomials

We use the notations from the previous section. Fix a root system $\Phi$ with its associated Weyl group $W$. Let $F = \mathbb{Q}(q,t)$ be the field of rational functions in variables $q, t$ over $\mathbb{Q}$ and let $A = F[P]$ be the group algebra over $F$ of the weight lattice $P$. The group operation on $P$ is addition. Therefore, we denote an element of $A$ corresponding to a weight $\lambda$ by $x^\lambda$, such that $x^\lambda x^\mu = x^{\lambda+\mu}$, $(x^\lambda)^{-1} = x^{-\lambda}$ and $x^0 = 1$. The $x^\lambda$ form a basis on $A$, therefore each $f \in A$ can be written uniquely as $\sum_{\lambda\in P} f_\lambda x^\lambda$ where $f_\lambda \in F$. This is well-defined, since $f$ has finite support.

*Remark.* Consider a group $G$ and a ring $R$. Define $\delta_g \in R[G]$ by $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for any $h \in G \setminus \{g\}$. Since $\{\delta_g \mid g \in G\}$ is a basis of $R[G]$ and $f = \sum_{g\in G} f(g)\delta_g$, one can identify $g \in G$ by $\delta_g \in R[G]$ and write $f_g = f(g)$. That is why we write $f \in R[G]$ as a formal linear combination of elements in $G$ with coefficients in $R$, that is,

$$\sum_{g\in G} f_g g.$$

In fact, the product $*$ on $R[G]$ satisfies $\delta_g * \delta_h = \delta_{g\cdot h}$. If the group operation of $G$ is addition, it is useful to denote $\delta_g$ as formal exponential $e^g$, such that we have

$$e^g e^h = e^{g+h}, \quad (e^g)^{-1} = e^{-g} \quad \text{and} \quad e^0 = 1,$$

where $+$ is the group operation, $-g$ the inverse of $g$ and $0$ the identity element of $G$.

The Weyl group $W$ acts on $P$ and we extend this action linearly to get a well-defined action of $W$ on $A$, that is,

$$w(f) = \sum_{\lambda\in P} f_\lambda x^{w\lambda}$$

for every $w \in W$. Let $A^W$ denote the subalgebra of the $W$-invariant functions in $A$.

**Definition 2.1.** Let $\lambda$ be a dominant weight. The ***orbit sum*** corresponding to $\lambda$ is

$$m_\lambda = \sum_{\mu\in W\lambda} x^\mu,$$

where $W\lambda$ denotes the $W$-orbit of $\lambda$.

Clearly, $m_\lambda$ is an element in $A^W$ for every $\lambda \in P$. But $m_\lambda = m_{w\lambda}$ for every $w \in W$, and – as proven in Corollary 1.6 – the set of dominant weights $P^+$ is a fundamental region for the action of the Weyl group $W$. Thus, $\{m_\lambda \mid \lambda \in P^+\}$ is an $F$-basis of $A^W$. Another $F$-basis is given in [8] by the ***Weyl characters*** $\chi_\lambda$, $\lambda \in P^+$, which are defined by

$$\chi_\lambda = \delta^{-1} \sum_{w\in W} \varepsilon(w) x^{w(\lambda+\rho)}$$

where

$$\delta = \prod_{\alpha\in\Phi^+} (x^{\alpha/2} - x^{-\alpha/2}) = \sum_{w\in W} \varepsilon(w) x^{w(\rho)} \in A,$$

with $\varepsilon(w) = \det(w) \in \{\pm 1\}$ and

$$\rho = \frac{1}{2} \sum_{\alpha\in\Phi^+} \alpha \in P^{++} \quad \text{is the Weyl vector.}$$

Writing this basis in terms of the orbit sums yields

$$\chi_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu \ , \qquad \text{where } K_{\lambda\mu} \in \mathbb{Z} \ . \tag{2.1}$$

The sum is over lower weights than $\lambda$ relative to the partial order on weight defined in (1.4).

When $\Phi$ is of Type $A$, we will see in Section 4.2 that the orbit sums and the Weyl characters are the well known monomial symmetric functions and Schur functions, respectively. The coefficients $K_{\lambda\mu}$ in (2.1) – known as *Kostka numbers* – can then be described by the number of certain *semistandard Young tableaux* relative to $\lambda$ and $\mu$. We will follow up on this in Section 4.2.

The Macdonald polynomials will be another basis of $A^W$ – an orthogonal basis relative to the scalar product on $A$ we define next.

Denote

$$(a;q)_\infty = \prod_{\alpha\in\Phi}\prod_{r=0}^{\infty}(1-aq^r)$$

and define

$$\Delta_{q,t} = \prod_{\alpha\in\Phi} \frac{(x^\alpha;q)_\infty}{(tx^\alpha;q)_\infty} \ .$$

We will only consider the case $t = q^k$ for any nonnegative integer $k$. Otherwise, we refer to [8]. The defined $\Delta_{q,t}$ reduces to

$$\Delta_{q,t} = \prod_{\alpha\in\Phi}\prod_{r=0}^{\infty} \frac{1-q^r x^\alpha}{1-q^{r+k}x^\alpha} = \prod_{\alpha\in\Phi}\prod_{r=0}^{k-1}(1-q^r x^\alpha). \tag{2.2}$$

Since the product in $\Delta_{q,t}$ is over all roots and $w\Phi = \Phi$ for every $w \in W$, $\Delta_{q,t}$ is an element of $A^W$. Now, let $f \in A$ and write $f = \sum_{\lambda\in P} f_\lambda x^\lambda$. Define

$$\overline{f} := \sum_{\lambda\in P} f_\lambda x^{-\lambda}$$

and denote the constant term $f_0$ of $f$ by $[f]_1$. The bilinear form $\langle\, , \rangle_{q,t}$ on $A$, defined by

$$\langle f\,, g\rangle_{q,t} := \frac{1}{|W|}\,[f\overline{g}\Delta_{q,t}]_1\,, \tag{2.3}$$

is symmetric and nondegenerate.

**Theorem 2.1** (Macdonald)**.** *There is a unique $F$-basis $(P_\lambda)_{\lambda\in P}$ of $A^W$, such that*

*(i)* $P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu$ *where* $u_{\lambda\mu} \in F$,

*(ii)* $\langle P_\lambda\,, P_\mu\rangle_{q,t} = 0$ *for* $\lambda \neq \mu$.

*The polynomials $P_\lambda$ are called the* **Macdonald polynomials**.

The complete proof is given in [8, Section 2.6 and 2.7]. But we shall sketch the proof given there very briefly. To prove the existence of the Macdonald polynomials, construct a linear operator $D : A^W \to A^W$, which satisfies

1. $\langle Df\,, g\rangle = \langle f\,, Dg\rangle$ for all $f, g \in A^W$,

2. $Dm_\lambda = \sum_{\mu \leq \lambda} c_{\lambda\mu} m_\mu$ for each $\lambda \in P^{++}$ and

3. $c_{\lambda\lambda} \neq c_{\mu\mu}$ for all different $\lambda, \mu \in P^{++}$.

The Macdonald polynomials will appear as eigenfunctions of $D$. To see this, denote for each dominant regular weight $\lambda$ the eigenfunction of $D$ with eigenvalue $c_{\lambda\lambda}$ by $P_\lambda$. We can assume that the coefficient $c_{\lambda\lambda}$ of $m_\lambda$ is 1 (otherwise we normalize $P_\lambda$). Then, $P_\lambda$ already satisfy the first condition of Theorem 2.1. The second condition of Theorem 2.1 holds, since

$$c_{\lambda\lambda}\langle P_\lambda \, , P_\mu \rangle = \langle DP_\lambda \, , P_\mu \rangle = \langle P_\lambda \, , DP_\mu \rangle = c_{\mu\mu}\langle P_\lambda \, , P_\mu \rangle,$$

what implies that $\langle P_\lambda \, , P_\mu \rangle = 0$ if $\lambda \neq \mu$ because of property (3) of $D$.

*Remark.* The definition of Macdonald polynomials in Theorem 2.1 is the very first Macdonald gave. He later generalized this definition replacing the variable $t$ in $F = \mathbb{Q}(q,t)$ by pairwise different $t_1, \dots, t_m$, where $m$ is the number of $W$-orbits of $\Phi$. More detailed, fix parameters $c_1, \dots, c_m$ (one for each $W$-orbit) and let $t_i = q^{c_i}$. When $\Phi = A_n$, we only have one $W$-orbit, so $t_1 = t = q^k$ always holds. For simplicity, we assume $c_i = k \in \mathbb{N}$ for every root system in this thesis. The more generalized definition is given in [7].

## 2.1 Special cases for Macdonald polynomials

The Macdonald polynomial specializes to other basis of $A^W$ if $q$ or $t$ are concrete values in $F$. We will discuss some of these special cases for any root system $\Phi$. In Section 4.2, we will see further special cases for Type $A$.

**Orbit sums** Let $t = 1$, then

$$\Delta_{q,1} = \prod_{\alpha \in \Phi} \frac{(x^\alpha; q)_\infty}{(x^\alpha; q)_\infty} = 1.$$

The orbit sums $m_\lambda$ satisfy condition $(i)$ of Theorem 2.1. Let $\lambda, \mu \in P^+$ be dominant weights and write $m_\lambda = \sum_{\tau \in W\lambda} x^\tau$ and $m_\mu = \sum_{\sigma \in W\mu} x^\sigma$ for the orbit sums corresponding to $\lambda$ and $\mu$, respectively. Then,

$$\begin{aligned} \langle m_\lambda \, , m_\mu \rangle_{q,1} &= \sum_{\tau \in W\lambda} \sum_{\sigma \in W\mu} \langle x^\tau \, , x^\sigma \rangle \\ &= \sum_{\tau \in W\lambda} \sum_{\sigma \in W\mu} \frac{1}{|W|} \left[ x^{\tau - \sigma} \right]_1 \\ &= \sum_{\tau \in W\lambda} \sum_{\sigma \in W\mu} \frac{1}{|W|} \delta_{\tau\sigma} \\ &= \begin{cases} 0 & \text{if } W\lambda \cap W\mu = \emptyset, \\ \frac{2|W\lambda|}{|W|} & \text{else,} \end{cases} \end{aligned}$$

since each $W$-orbit contains exactly one dominant weight. Because of the uniqueness in Theorem 2.1, the Macdonald polynomials are the orbit sums $P_\lambda = m_\lambda$ if $t = 1$.

**Weyl characters** Let $t = q$. We will show that $P_\lambda = \chi_\lambda$ the Weyl character for any dominant weight $\lambda$. We can write the Weyl character as sum of orbit sums like in (2.1). Thus, we only need to compute the inner product of two Weyl characters.
We claim, that $\Delta_{q,q} = \delta\overline{\delta}$. First, note that

$$\Delta_{q,q} = \prod_{\alpha\in\Phi}\prod_{r=0}^{\infty}\frac{(1-x^\alpha q^r)}{(1-x^\alpha q^{r+1})} = \prod_{\alpha\in\Phi}(1-x^\alpha).$$

We have

$$\begin{aligned}
\delta\overline{\delta} &= \prod_{\alpha\in\Phi^+}(x^{\alpha/2}-x^{-\alpha/2})(x^{-\alpha/2}-x^{\alpha/2}) \\
&= \prod_{\alpha\in\Phi^+}(1-x^\alpha-x^{-\alpha}+x^0) \\
&= \prod_{\alpha\in\Phi^+}(1-x^\alpha)(1-x^{-\alpha}) \\
&= \prod_{\alpha\in\Phi}(1-x^\alpha)\ ,
\end{aligned}$$

which proves $\Delta_{q,q} = \delta\overline{\delta}$. Since $t$ only effects $\Delta_{q,t}$ and since $\Delta_{q,1} = 1$, we can write

$$\langle \chi_\lambda\, , \chi_\mu\rangle_{q,q} = \langle \chi_\lambda\delta\, , \chi_\mu\delta\rangle_{1,1} = \sum_{w,s\in W}\varepsilon(w)\varepsilon(s)\langle x^{w(\lambda+\rho)}\, , x^{w(\mu+\rho)}\rangle_{1,1}\ .$$

The weights $\lambda+\rho,\ \mu+\rho$ are dominant regular, so by Corollary 1.6 they are in distinct $W$-orbits if and only if $\lambda \neq \mu$. Therefore,

$$\langle \chi_\lambda\, , \chi_\mu\rangle_{q,q} = 0$$

if $\lambda \neq \mu$, as desired.

**Macdonald spherical functions** Macdonald spherical functions are defined as the Macdonald polynomials for $q = 0$. As we will see in Section 4.4, the Macdonald polynomials in Type $A$ reduce to the Hall-Littlewood polynomials by setting $q = 0$. Macdonald spherical functions are their generalization in any root system. If we are in the case $t = q^k$, we are again in the case $q = t = 0$ and get the Weyl characters. Since we did not introduce the scalar product in case $t \neq q^k$, we just mention the existence of the Macdonald spherical functions.

# 3 A combinatorial formula for Macdonald polynomials

In this section, we will give two equivalent combinatorial formulas for the Macdonald polynomials, which will be a sum over folded *alcove walks*. More accurately, we will construct a certain walk – that is a sequence of adjacent alcoves – from the fundamental alcove $A$ to an alcove nearby a given dominant weight $\lambda$ and we will just modify this walk by using elements of the affine Weyl group. The first formula we present in Section 3.2 is Ram and Yip's formula from [10]. We will then get an equivalent formula by using an equivalent definition for alcove walks. This second perspective on alcove walks will be needed in Section 5.

### 3.1 Alcove walks

Let $F = \mathbb{Q}(q,t)$ be the same field as in Section 2 and let $\Phi$ be a root system in $V$ of rank $n$. Recall the (affine) hyperplanes $H_{\alpha,k} \in \mathcal{H}$ defined in Section 1, which determine (affine) reflections $s_{\alpha,k}$. Each hyperplane $H_{\alpha,0} = H_\alpha$ tiles $V \setminus H_\alpha$ into two half spaces, called the ***positive*** and the ***negative side***, relative to the sign of the inner product $\langle \cdot\,, \alpha^\vee \rangle$. That is, an element $\lambda \in V$ belongs to the positive side of $H_\alpha$ if and only if $\langle \lambda\,, \alpha^\vee \rangle > 0$.
Equivalently, the positive sides are determined by forcing $C$ to be on the positive side of each $H_\alpha$. We extend this orientation to the affine hyperplanes:

$$\lambda \in V \text{ is on the positive side of } H_{\alpha,k} \text{ if and only if } \langle \lambda\,, \alpha^\vee \rangle - k > 0.$$

We want to construct ***alcove walks*** as sequences of finitely many steps between alcoves, where a step from alcove $B_1$ to $B_2$ means either

- crossing their common wall if $B_1 \neq B_2$, or (3.1)
- walking towards a wall, turning around and staying in the alcove $B_1 = B_2$, or (3.2)
- changing the sheet if $g(B_1) = B_2$ for any $g \in \Omega$. (3.3)

We use the same notation and construction given in [9]. A step satisfying (3.1) is a ***crossing***, a step satisfying (3.2) is a ***fold***. Including the orientation of a hyperplane and the numeration of the facets, we define four types of crossings and folds:


positive $j$-crossing $c_j^+$ | negative $j$-crossing $c_j^-$ | positive $j$-fold $f_j^+$ | negative $j$-fold $f_j^-$

The ***alcove walk algebra*** is defined in [9] as the algebra over $F$ generated by $\Omega$ and the steps $c_j^\pm$, $f_j^\pm$ for $1 \leq j \leq n$ subject to the relations

$$c_i^+ = c_i^- + f_i^+ \quad \text{and} \quad c_i^- = c_i^+ + f_i^- \tag{3.4}$$
$$g c_i^\pm = c_{g(i)}^\pm g \quad \text{and} \quad g f_i^\pm = f_{g(i)}^\pm g. \tag{3.5}$$

A sequence of steps satisfying (3.1), (3.2) and (3.3) can be obtained by reading a product of the generators as concatenation. Then, a word in the generators is a concatenations of arrows and changes of sheets.

Because of the relation in (3.5), we can always omit that a change of sheet (a $g \in \Omega$) can only be the last step of a walk, that is, the last entry of a word written in the generators. A step starts or ends at the alcove, where the drawn arrow starts or ends, respectively. The last step $g \in \Omega$ just permutes the numeration of the walls. Thus, we identify an alcove walk by the start alcove and the sequence of crossings and folds, which leads to the following definition.

**Definition 3.1.** Let $v \in \widetilde{W}$. An ***alcove walk*** of type $\vec{w} = (i_1, \ldots, i_\ell)$ beginning at $v$ is a concatenation of $\ell$ steps and a $g \in \Omega$, satisfying

(i) the first step starts in the alcove corresponding to $v$,

(ii) the $k$th step starts at the alcove where the $(k-1)$th step ends,

(iii) the $k$th step is either an $i_k$-crossing $c_{i_k}^{\pm}$ or an $i_k$-fold $f_{i_k}^{\pm}$,

(iv) if the last step ends in $\tilde{v}$, the alcove walk ends in $g\tilde{v} \in \widetilde{W}$.

By [9], the set of alcove walks beginning at the fundamental alcove $A$ and ending in the same sheet is a basis for the alcove walk algebra.

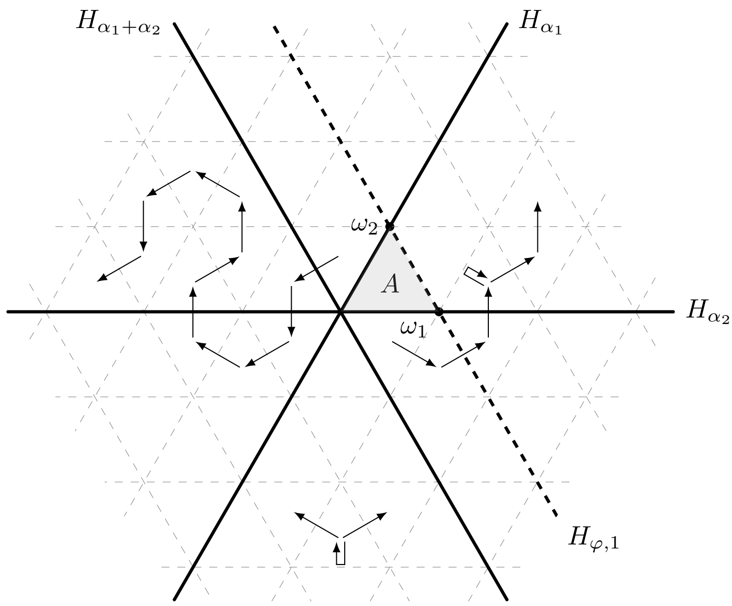
*Remark.* Definition 3.1 is slightly different from the definition of alcove walks in [9]. There, the initial alcove is always the fundamental alcove $A$ and a walk stays in one sheet. An alcove walk is then a word only in the generators $c_j^{\pm}$ and $f_j^{\pm}$ of the alcove walk algebra. In our definition, there are several alcove walks corresponding to the same word, since the initial alcove is not fixed and the type does not include the change of sheets $g \in \Omega$.

Write $\mathcal{B}(v, \vec{w}, g)$ for the set of alcove walks of type $\vec{w} = (i_1, \ldots, i_\ell)$ beginning at $v$ and ending on the $g$-sheet of $\Omega \times V$. We call $v = \iota(p)$ the ***initial alcove*** of $p \in \mathcal{B}(v, \vec{w}, g)$ and define the ***endpoint*** $\mathrm{end}(p) \in \widetilde{W}$ of $p$ as the corresponding alcove where $p$ ends. The set $\mathcal{B}(v, \vec{w}, g)$ includes $2^\ell$ alcove walks since the type of $\vec{w}$ has $\ell$ entries. To differ between them it suffices to know which step is a fold. For a walk $p \in \mathcal{B}(v, \vec{w}, g)$ let

$$f^+(p) = \{j \mid \text{step } j \text{ of } p \text{ is a positive fold}\},$$
$$f^-(p) = \{j \mid \text{step } j \text{ of } p \text{ is a negative fold}\},$$

and let $f(p) = f^+(p) \cup f^-(p)$ be the set of fold positions. Note, that $f(p)$ determines $f^+(p)$ and $f^-(p)$ (for a fixed type of the walk), so it determines $p$.

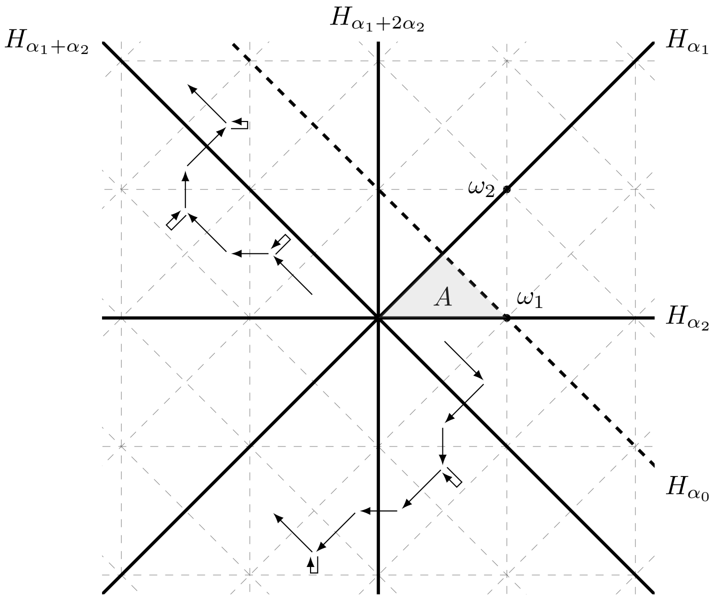
**Example.** Consider the rank 2 root systems $A_2$ and $C_2$ with multiple alcove walks. We do not label the walls since the image of the walks does not depend on the change of sheets $g$ at the end of a walk.
First, have a look at $A_2$. The image below shows some alcove walks.




Let $p_1$ be the alcove walk, which lies on the negative side of $H_{\alpha_1}$, and $p_2$ the alcove walk, which lies on the positive side of $H_{\alpha_1+\alpha_2}$.
The walk $p_1$ is of type $(2, 1, 0, 1, 0, 1, 2, 1, 2, 1, 0)$ beginning at $s_1 \in W_{\mathrm{aff}}$ and with no folds. The

alcove walk $p_2$ is of type $(0, 2, 0, 2, 1, 2)$ beginning at $s_2 \in W_{\text{aff}}$ with folds $f(p_2) = \{4\} = f^+(p_2)$. There are three more arrows drawn in the chamber $s_2 s_1 C$. These do not belong to one and the same alcove walk because any alcove walk concerning just these three steps would breach condition $(ii)$ of Definition 3.1. But we can define two alcove walks, each with one of the crossings, and the fold could belong to both of these walks.

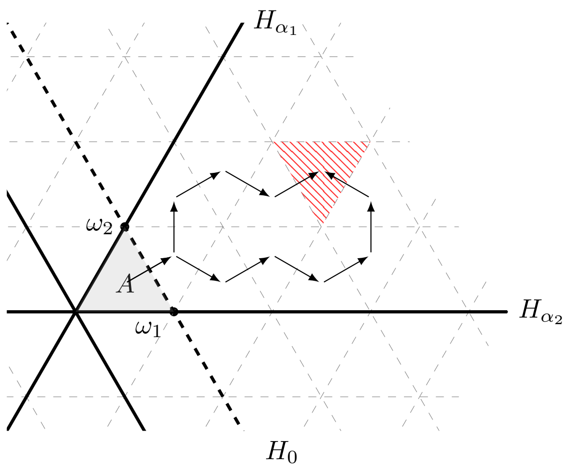
Now, consider Type $C_2$. The next image shows two alcove walks – say $q_1$ and $q_2$ – of the same type $\vec{w} = (0, 1, 2, 1, 0, 2, 1, 2, 0)$ but beginning at different alcoves and having different sets of fold positions. Let $q_1$ be the walk beginning at $s_1 s_2 s_1 \in W_{\text{aff}}$. This walk has positive folds $f^+(q_1) = \{5\}$ and negative folds $f^-(q_1) = \{2, 8\}$.




Let $q_2$ be the other walk, beginning at $s_1 \in W_{\text{aff}}$ and with one positive fold $f^+(q_2) = \{4\}$ and one negative fold $f^-(q_1) = \{8\}$.

The ***length*** $\ell(p)$ of an alcove walk $p \in \mathcal{B}(v, \vec{w}, g)$ is the number of entries in its type $\vec{w}$, and we call $p$ a minimal alcove walk if $\ell(p)$ is the minimal length for a walk from $\iota(p)$ to $\text{end}(p)$.

**Example.** Consider the following two walks from 1 (fundamental alcove $A$) to $s_0 s_1 s_2 s_1 s_0$ (red striped alcove):



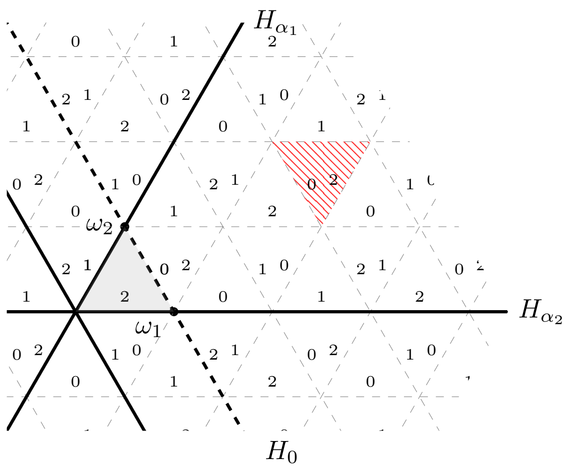
Of course, a minimal walk has no folds and its first step is a positive 0-crossing. Both walks satisfy these conditions, but only the northern walk is a minimal walk, since it has the same length as the ending alcove. The type of the minimal walk is $\vec{w} = (0, 1, 2, 1, 0)$.

The alcove $\mathrm{end}(p)$ determines a weight $\mathrm{wt}(p) \in P$ and a $\varphi(p) \in W$ by

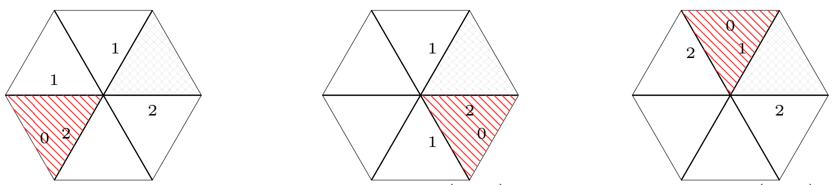
$$\mathrm{end}(p) = t_{\mathrm{wt}(p)}\varphi(p) \in \widetilde{W}.$$

We call $\varphi(p)$ the ***final direction*** of $p$ and $\mathrm{wt}(p)$ the ***weight*** of $p$.

**Example.** Continue the previous example, let $p \in \mathcal{B}(1, \vec{w}, g)$ be the minimal walk given there. The definition of minimal walk does not depend on the change of sheets $g \in \Omega$, but $\mathrm{end}(p)$ does. In Type $A_2$, we have $\Omega = \{\mathrm{id}, (012), (021)\}$, where an element $g \in \Omega$ is a permutation of the labels on the walls (in cycle notation).
The labeling for $g = \mathrm{id}$ is




Depending on $g \in \Omega$, we must consider the following $\lambda$-polygons, such that $\mathrm{end}(p)$ is an alcove in $t_\lambda(WA)$:



| $g = \mathrm{id}$ | $g = (012)$ | $g = (021)$ |
|---|---|---|
| $\mathrm{wt}(p) = 2\omega_1 + 2\omega_2$ | $\mathrm{wt}(p) = \omega_1 + 2\omega_2$ | $\mathrm{wt}(p) = 2\omega_1 + \omega_2$ |
| $\varphi(p) = \omega_0$ | $\varphi(p) = s_2$ | $\varphi(p) = s_1$ |

where the alcove with the squared gray pattern is $t_\lambda A$ and the weight of $p$ is the center $\lambda$.

From a different point of view, given a dominant weight $\mu \in P^+$ there is a unique element of minimal length in the $\mu$-polygon, say $t_\mu v_\mu$ for $v_\mu \in W$, and we can construct a minimal alcove walk $p$ of weight $\mu$ by forcing $\iota(p) = 1$ and $\mathrm{end}(p) = t_\mu v_\mu$. We call $p$ a ***$\mu$-path***. Note, that there is a unique $g \in \Omega$ such that $g^{-1}(t_\mu v_\mu) \in W_{\mathrm{aff}}$. The $\mu$-path has type $\vec{\mu} = (i_1, \ldots, i_\ell)$, where $g^{-1}(t_\mu v_\mu) = s_{i_1} \cdots s_{i_\ell}$ is a reduced word. Let $\mathcal{P}(\vec{\mu})$ be the set of alcove walks of type $\vec{\mu}$ beginning at $v \in W$ and ending at the same sheet as the $\mu$-path.

Given a minimal alcove walk $p$ in the fundamental chamber $C$ of type $(i_1, \ldots, i_\ell)$, denote sequences $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ and $j = (j_1, \ldots, j_\ell)$ such that the $k$th step of $p$ crosses the affine hyperplane $H_{\gamma_k, j_k}$ for a positive root $\gamma_k$ and a nonnegative integer $j_k$. That is,

$$\gamma_k = \pm r_{i_1} r_{i_2} \cdots r_{i_{k-1}} \beta_{i_k} \qquad \text{a positive root,} \tag{3.6}$$

where $\beta_{i_k} = \alpha_{i_k}$, $r_k = s_k$ for $1 \leq i_k \leq n$, $\beta_0$ is the coroot of the highest root in $\Phi^\vee$ (see Section 1.3) and $r_0 = s_{\beta_0}$. By Section 1, exactly one sign yields a positive root.
Since $p$ has minimal length and is in $C$, we have

$$j_k = |\,\{i \leq k \mid \beta_i = \beta_{j_k}\}\,|. \tag{3.7}$$

*Remark.* The sequence $\gamma$ is a $\lambda$-*chain* for a dominant weight $\lambda$. We will discuss this in the next section.

## 3.2 A combinatorial formula by Ram and Yip

In 2008, Ram and Yip provided in [10] the following combinatorial formula for Macdonald polynomials:

**Theorem 3.1** (Ram and Yip, 2008)**.** *Let $\mu \in P^+$ be a dominant weight. For a fixed $\mu$-path, define the sequences $(\gamma_1, \ldots, \gamma_\ell)$ and $(j_1, \ldots, j_\ell)$ as in* (3.6) *and* (3.7)*. Then*

$$P_\mu = \sum_{p \in \mathcal{P}(\vec{\mu})} x^{\mathrm{wt}(p)} t^{\frac{1}{2}(\ell(\varphi(p)) - \ell(\omega_0 \iota(p)))} \left( \prod_{k \in f^+(p)} \frac{t^{-\frac{1}{2}}(1-t)}{1 - q^{j_k} t^{\langle \gamma_k^\vee, \rho \rangle}} \right) \left( \prod_{k \in f^-(p)} \frac{t^{-\frac{1}{2}}(1-t) q^{j_k} t^{\langle \gamma_k^\vee, \rho \rangle}}{1 - q^{j_k} t^{\langle \gamma_k^\vee, \rho \rangle}} \right).$$

*Remark.* The original formula in [10, Theorem 3.4] is for the generalized definition of Macdonald polynomials, which uses several variables $t_i$ (we mentioned this in Section 2). But when $t_i = t = q^k$ for each $t_i$ – as we assume – the original formula reduces to the one in Theorem 3.1.

If $\mu$ is not regular, the formula computes a scalar multiple of the Macdonald polynomial $P_\mu$. In this case, we just have to normalize the polynomial, i.e., we have to divide it by the leading coefficient (the coefficient of $m_\mu$).

*Remark.* We just have to normalize the polynomial, because we did not introduce the theory and methods, which provide the Ram and Yip's formula. In their proof of [10, Theorem 3.4], some terms vanish or are equal to smaller terms under a certain relation. The details of the applied methods are given in [10, Section 2]. In fact, Ram and Yip state an alternate formula for the Macdonald polynomials $P_\mu$ when $\mu$ is not regular. This formula follows from the property of a $\mu$ that it is the same as $w\mu$ for at least one $w \in W$. That is, the stabilizer $W_\mu$ of $\mu$ is not trivial. The formula in Theorem 3.1 can then be compressed as a sum over walks in $\mathcal{P}(\vec{\mu})$, which start in certain representatives of $W/W_\mu$, multiplied by the *Poincaré polynomial* of $W_\mu$. The concrete formula is given in [10, Section 3].

Indeed, the formula in Theorem 3.1 can be written as

$$P_\mu = \sum_{p \in \mathcal{P}(\vec{\mu})} x^{\mathrm{wt}(p)} t^{\frac{1}{2}(\ell(\varphi(p)) - \ell(\omega_0 \iota(p)) - |f(p)|)} (1-t)^{|f(p)|} \left( \prod_{k \in f(p)} \frac{1}{1 - q^{j_k} t^{\mathrm{ht}(\gamma_k^\vee)}} \right) \left( \prod_{k \in f^-(p)} q^{j_k} t^{\mathrm{ht}(\gamma_k^\vee)} \right)$$

which follows directly by Lemma 1.2 and (1.3).

The formula in Theorem 3.1 just needs one alcove walk, namely a $\mu$-path. The sum is then over alcove walks similar to this $\mu$-path, but transformed by starting in another alcove than the fundamental alcove $A$, or changing crossings into folds. The following lemma shows, how this affects the weights and final directions of walks in $\mathcal{P}(\vec{\mu})$.

**Lemma 3.2.** *Let $p, q, t$ be alcove walks of the same type $\vec{w} = (i_1, \ldots, i_\ell)$, beginning in the 0-polygon and ending with the same change of sheet $g \in \Omega$. Let $f(p) = f(q) \subset f(t)$ be the sets of folds such that $f(t) \setminus f(p) = \{j\}$ and let $H_{\alpha,k}$ be the hyperplane which is crossed by the $j$th step of $p$. Then*

*(i)* $\iota(p)^{-1} \operatorname{end}(p) = \iota(q)^{-1} \operatorname{end}(q)$,

*(ii)* $\iota(p)^{-1} \operatorname{wt}(p) = \iota(q)^{-1} \operatorname{wt}(q)$ *and* $\iota(p)\varphi(p) = \iota(q)\varphi(q)$*, and*

*(iii)* $\operatorname{wt}(t) = s_{\alpha,k} \operatorname{wt}(p)$ *and* $\varphi(t) = s_\alpha \varphi(p)$.

*Proof.* Without loss of generality, suppose $\iota(p) = 1$ and - because of (3.5) - suppose $g = 1$.

(i) The alcove $\operatorname{end}(p)$ is then defined as the alcove, where the last step ends. The directions of the steps are determined by the numeration of the walls, which is invariant under the action of the affine Weyl group. Thus, if the $k$th step of $p$ ends at an alcove $B$, the $k$th step of $q$ ends at $\iota(q)B$. It follows directly, that $\operatorname{end}(q) = \iota(q) \operatorname{end}(p)$.

(ii) The weights $\operatorname{wt}(p), \operatorname{wt}(q)$ and the final directions $\varphi(p), \varphi(q) \in W$ are determined by

$$\operatorname{end}(p) = t_{\operatorname{wt}(p)}\varphi(p) \qquad \text{and} \qquad \operatorname{end}(q) = t_{\operatorname{wt}(q)}\varphi(q)$$

By $(i)$ and (1.9), we have

$$\operatorname{end}(q) = t_{\operatorname{wt}(q)}\varphi(q) = \iota(q)t_{\operatorname{wt}(p)}\varphi(p) = t_{\iota(q)\operatorname{wt}(p)}\iota(q)^{-1}\varphi(p).$$

The action of the Weyl group does not change sheets, therefore

$$\operatorname{wt}(q) = \iota(q)\operatorname{wt}(p) \qquad \text{and} \qquad \varphi(q) = \iota(q)^{-1}\varphi(p)$$

as desired.

(iii) The first $j-1$ steps of $p$ and $t$ are the same. Let $H_{\alpha,k}$ be the hyperplane which is crossed by the $j$th step of $p$. Recall that the labeling on the facetts of the alcoves is given by acting on the labeling of the fundamental alcove (on any sheet) with $W_{\text{aff}}$. Thus, if a step $j+m$ of $p$, $m \geq 0$, ends at an alcove $\tilde{v}$, step $j+m$ of $t$ ends at $s_{\alpha,k}\tilde{v}$. By (1.10), the walk $t$ ends at

$$\operatorname{end}(t) = s_{\alpha,k}\operatorname{end}(p) = s_{\alpha,k}t_{\operatorname{wt}(p)}\varphi(p) = t_{s_{\alpha,k}(\operatorname{wt}(p))}s_\alpha\varphi(p),$$

which is on the same sheet as $\operatorname{end}(p)$. It follows, that

$$\operatorname{wt}(t) = s_{\alpha,k}(\operatorname{wt}(p)) \quad \text{and} \quad \varphi(t) = s_\alpha\varphi(p). \qquad \square$$

Let $p$ be a minimal alcove walk with sequences $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ and $j = (j_1, \ldots, j_\ell)$ like defined in (3.6) and (3.7). For a subset $T = \{k_1, \ldots, k_s\} \in [\ell]$, define

$$\psi(T) = s_{\gamma_1,j_1}\cdots s_{\gamma_\ell,j_\ell} \qquad \text{and} \qquad \Psi(T) = s_{\gamma_1}\cdots s_{\gamma_\ell}. \tag{3.8}$$

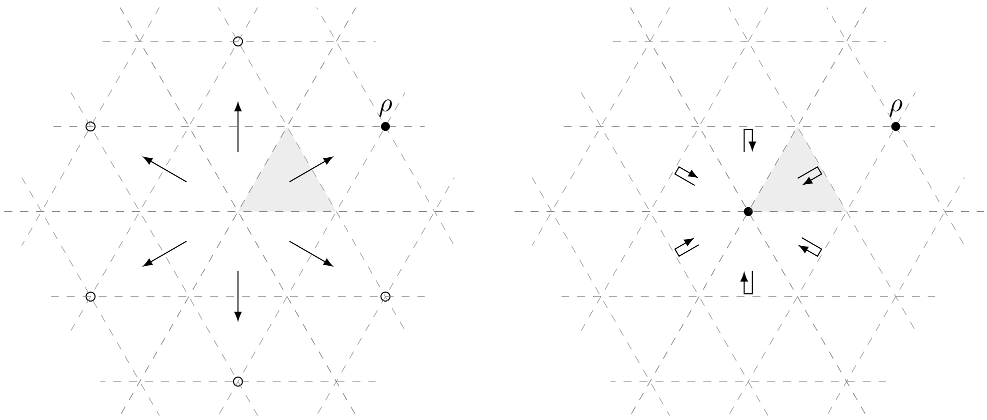
**Corollary 3.3.** *Let $\mu \in P^+$ be a dominant weight and define the sequences $\gamma$ and $j$ as in (3.6) and (3.7) for the $\mu$-path. Let $v_\mu$ be its final direction. Then,*

$$P_\mu = \sum_{\substack{w\in W\\ T\subseteq[l]}} X^{w(\psi(T)\mu)}t^{\frac{1}{2}(\ell(w^{-1}\Psi(T)(v_\mu))-\ell(\omega_0 w)-|T|)}(1-t)^{|T|}\left(\prod_{k\in T}\frac{1}{1-q^{j_k}t^{\operatorname{ht}(\gamma_k^\vee)}}\right)\left(\prod_{k\in T_w^-}q^{j_k}t^{\operatorname{ht}(\gamma_k^\vee)}\right)$$

*where $T_w^- \subseteq T$ is the set of negative folds of a walk $q \in \mathcal{P}(\vec{\mu})$ with folds $f(q) = T$ and initial alcove $\iota(p) = w$.*

**Example in type $A_2$** Recall the construction of the root system $A_2$ in Section 1.2. We want to compute the Macdonald polynomial $P_\rho$ for the Weyl vector $\rho = \omega_1 + \omega_2$.

The minimal length representative of the $\rho$-polygon is $s_0$. We have 12 walks of type (0):




First, consider the 6 unfolded walks on the left. The corresponding weight to a walk is marked with an unfilled circle. For each unfolded walk $p$, we have $\mathrm{wt}(p) = \iota(p)\rho$ and $\varphi(p) = w_0\iota(p)$ such that these 6 walks produce the sum

$$\sum_{w \in W} X^{w\rho} t^{1/2(\ell(w_0 w) - \ell(w_0 w))} = \sum_{w \in W} X^{w\rho} = m_\rho \tag{3.9}$$

in the formula. In fact, the coefficient of $m_\rho$ in $P_\rho$ is one by definition. Since $\rho$ is a regular weight, any fold in the walk would lead to a smaller weight than $\rho$. Thus, the unfolded walks are exactly the ones corresponding to the term $m_\rho$, like we have also computed above.

The sequences by (3.6) and (3.7) are just

$$\gamma_1 = \alpha_1 + \alpha_2 \qquad \text{and} \qquad j_1 = 1$$

so $\mathrm{ht}(\gamma_1^\vee) = \mathrm{ht}(\alpha_1^\vee + \alpha_2^\vee) = 2$. Each walk $p$ of the 6 alcove walks with fold satisfy

$$\mathrm{end}(p) = \iota(p) = \varphi(p) \quad \text{and} \quad \mathrm{wt}(p) = 0.$$

The fold of a walk with initial alcove $1, s_1$ or $s_2$ is negative, otherwise positive. Thus, the remaining sum of the formula is

$$\begin{aligned}\sum_{w \in W} t^{1/2(\ell(w) - \ell(w_0 w) - 1)} \frac{(1-t)}{1-qt^2} \prod_{w \in \{1, s_1, s_2\}} qt^2 =& \frac{1-t}{1-qt^2}(t + 1 + 1 + qt^2(t^{-2} + t^{-1} + t^{-1})) \\ =& \frac{1-t}{1-qt^2}(t + 2 + q + 2qt)\end{aligned}$$

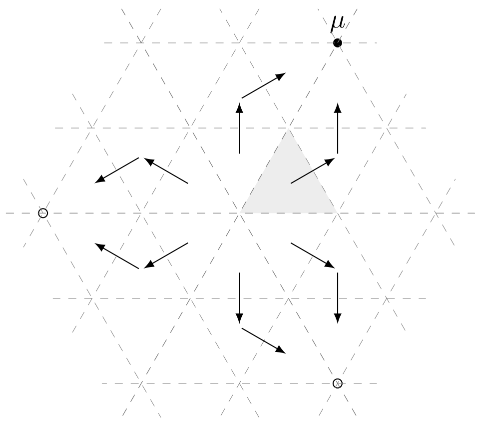
and together with (3.9), we have

$$P_\rho(q,t) = m_\rho + \frac{1-t}{1-qt^2}(t + 2 + q + 2qt). \tag{3.10}$$

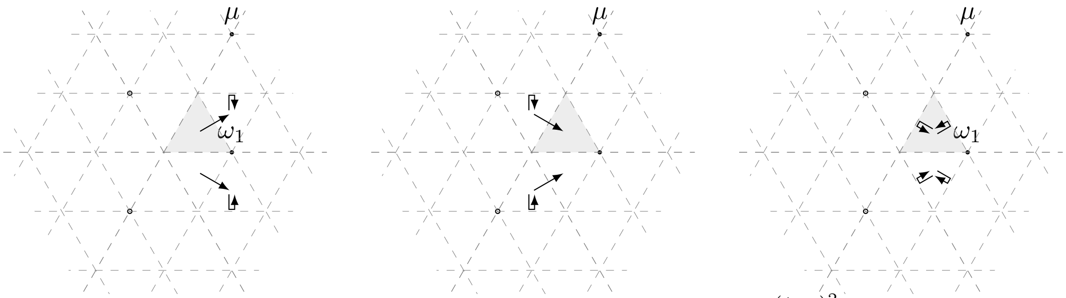
Now, let us compute the Macdonald polynomial for a not regular weight. Let $\mu = 2\omega_2$ be this weight, then we already know which terms appear in the corresponding Macdonald polynomial:

$$P_\mu = m_\mu + u_{\mu\omega_1} m_{\omega_1},$$

since $\omega_1$ is the only weight such that $\mu - \omega_1 \in Q^+$.
There are two walks per each weight in $W\mu$:




Using the formula of Theorem 3.1, the leading coefficient is $(t^{-\frac{1}{2}} + t^{\frac{1}{2}}) = t^{-\frac{1}{2}}(1+t)$. Thus, the formula must be divided by this scalar to normalize the polynomial. Since $P_\mu$ is symmetric, we only need to consider the alcove walks in $\mathcal{P}(\vec{\mu})$ with weight $\omega_1$ to determine the coefficient $u_{\mu\omega_1}$ of $m_{\omega_1}$. These are the following six:




$\frac{1-t}{1-qt}(qt^{-1/2} + t^{-1/2})$ $\quad$ $\frac{1-t}{1-qt^2}(qt^{3/2} + t^{1/2})$ $\quad$ $\frac{(1-t)^2}{(1-qt)(1-qt^2)}(qt^{1/2} + qt^{3/2})$

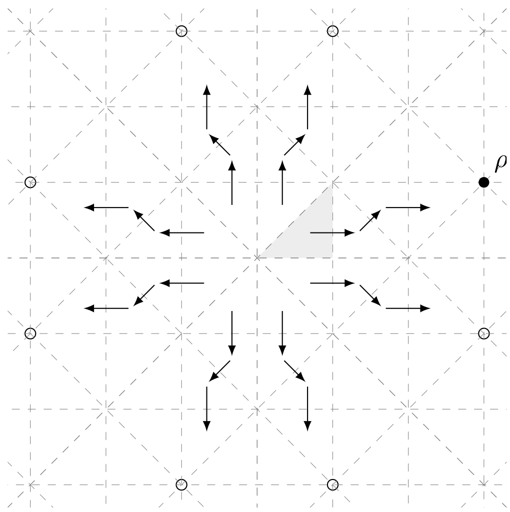
where the given terms are determined by the appearing folds $T^- \subset T \in \{1,2\}$ and the sequences $\gamma = (\alpha_1 + \alpha_2, \alpha_2)$ and $j = (1,1)$. Summarize these terms to get the coefficient of $m_{\omega_1}$ in $P_\mu$, which is

$$\begin{aligned} u_{\mu\omega_1} &= \tfrac{t^{1/2}}{1+t}\left(\tfrac{1-t}{1-qt}(qt^{-1/2} + t^{-1/2}) + \tfrac{1-t}{1-qt^2}(qt^{3/2} + t^{1/2}) + \tfrac{(1-t)^2}{(1-qt)(1-qt^2)}(qt^{1/2} + qt^{3/2})\right) \\ &= \tfrac{q-qt-t+1}{1-qt} \end{aligned}$$

**Example in type $B_2$** Recall the construction of the root system $B_2$ in Section 1.2. Like in the previous example we want to compute the Macdonald polynomial $P_\rho$ for the Weyl vector $\rho = \omega_1 + \omega_2$. Now, the minimal length representative of the $\rho$-polygon is $s_1 s_2 s_0$, so the sum in formula of Theorem 3.1 is over $2^3 = 8$ walks per element of the Weyl group, i.e., 64 walks. Moreover, we already know, which weights of the walks will appear: each weight in the $W$-orbit $W\mu$ for $\mu \leq \rho$, that is $\mu \in \{\rho, \omega_2\}$. Thus, the Macdonald polynomial $P_\rho$ is

$$P_\rho = m_\rho + u_{\rho\omega_2} m_{\omega_2}$$

for a coefficient $u_{\rho\omega_2} \in F$.
We first have to figure out, which walk belongs to which weight. Lemma 3.2 tells us, what the weights (and thus the final directions) of a walk are. Like in the previous example, we mark a weight by an unfilled circle and by a filled circle if it is the weight corresponding to a walk with initial alcove 1. Let us start with the 8 unfolded walks:




Any walk $p$ of type $(0,1,2)$ without folds has weight $\mathrm{wt}(p) = \iota(p)\rho$ and final direction $\varphi(p) = w_0\iota(p)$. It is not surprising, that this yields the term

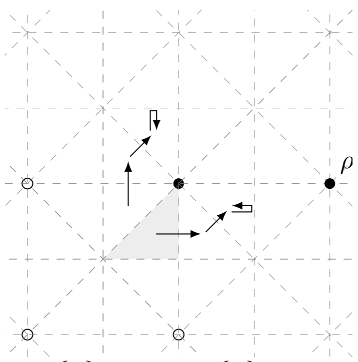
$$m_\rho = \sum_{w\in W} X^{w\rho}.$$

For every other walk, we need the sequences defined by (3.6) and (3.7), which are

$$\gamma = (\alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_1+\alpha_2) \qquad \text{and} \qquad j=(1,1,2)$$

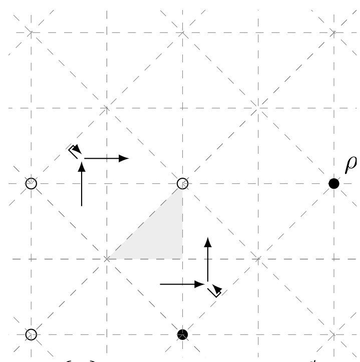
so $\mathrm{ht}(\gamma_1^\vee) = \mathrm{ht}(\gamma_3^\vee) = \mathrm{ht}(2\alpha_1^\vee + \alpha_2^\vee) = 3$ and $\mathrm{ht}(\gamma_2^\vee) = \mathrm{ht}(\alpha_1^\vee + \alpha_2^\vee) = 2$.

In fact, we do not have to consider every walk: the Macdonald polynomial is symmetric by its definition in Theorem 2.1, so it suffices to consider the walks with weight $\omega_2$ to determine the coefficient $u_{\rho\omega_2}$. We draw these walks and give their negative folds and the term they produce:



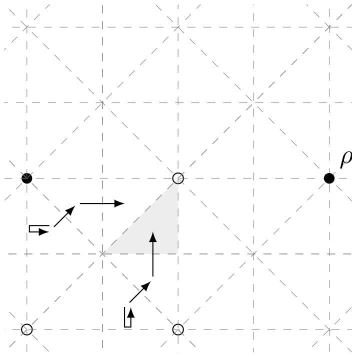

$T = \{3\} = T_1^- = \{3\} = T_{s_1}^-$

$$\frac{(1-t)}{1-q^2t^2}(q^2+q^2t)$$



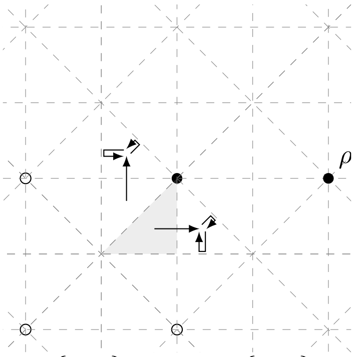

$T = \{2\} = T_{s_2}^-,\ T_{s_1s_2}^- = \emptyset$

$$\frac{(1-t)}{1-qt^3}(qt^2+1)$$



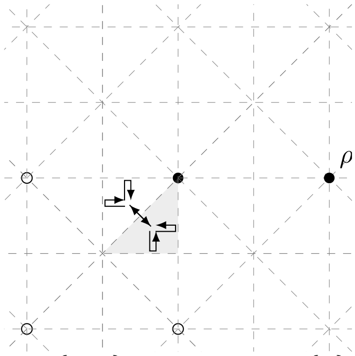

$T = \{1\},\ T^-_{s_2 s_1} = T^-_{s_1 s_2 s_1} = \emptyset$

$\frac{(1-t)}{1-qt^2}(1+t)$



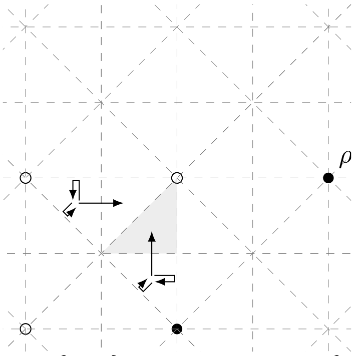

$T = \{2,3\} = T^-_{s_2} = \{2,3\},\ T^-_{s_1 s_2}$

$\frac{(1-t)^2}{(1-qt^3)(1-q^2t^2)}(qt+qt^2)$



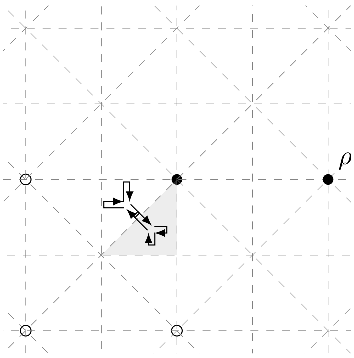

$T = \{1,3\},\ T^-_{s_1} = T^-_{s_1} = \{1\}$

$\frac{(1-t)^2}{(1-qt^2)(1-q^2t^2)}(qt+qt)$




$T = \{1,2\},\ T^-_{s_2} = T^-_{s_1 s_2} = \{1\}$

$\frac{(1-t)^2}{(1-qt^2)(1-qt^3)}(qt+qt^2)$




$T = \{1,2,3\},\ T^-_1 = \{1\},\ T^-_{s_1} = \{1,2\}$

$\frac{(1-t)^3}{(1-qt^2)(1-qt^3)(1-q^2t^2)}(q+q^2t^4)$

The coefficient $u_{\rho\omega_2}$ is the sum of the given terms:

$$\begin{aligned} u_{\rho\omega_2} =& \frac{(1-t)}{1-q^2t^2}(q^2+q^2t) + \frac{(1-t)}{1-qt^3}(qt^2+1) + \frac{(1-t)}{1-qt^2}(1+t) \\ &+ \frac{(1-t)^2}{(1-qt^3)(1-q^2t^2)}(qt+qt^2) + \frac{(1-t)^2}{(1-qt^2)(1-q^2t^2)}(qt+qt) \\ &+ \frac{(1-t)^2}{(1-qt^2)(1-qt^3)}(qt+qt^2) + \frac{(1-t)^3}{(1-qt^2)(1-qt^3)(1-q^2t^2)}(q+q^2t^4) \end{aligned}$$

## 3.3 Combinatorial formulas for special cases

Recall the special cases for the Macdonald polynomials defined in Section 2.1: the Macdonald spherical functions, the Weyl characters and the orbit sums. We obtain formulas from Theorem 3.1 for these polynomials by setting $q = 0$, $q = t$ and $t = 1$, respectively.

Given a dominant weight $\mu$, let $\mathcal{P}^+(\vec{\mu}) \subset \mathcal{P}(\vec{\mu})$ be the subset of all non-negatively folded walks in $\mathcal{P}(\vec{\mu})$.

**Corollary 3.4.** *The Macdonald spherical function for a dominant weight $\mu$ is*

$$P_\mu = \sum_{p\in\mathcal{P}^+(\vec{\mu})} x^{\mathrm{wt}(p)} t^{\frac{1}{2}(\ell(\varphi(p))-\ell(\omega_0\iota(p))-|f(p)|)}(1-t)^{|f(p)|}.$$

**Corollary 3.5.** *The Weyl character for a dominant weight $\mu$ is*

$$\chi_\mu = \sum_{\substack{p\in\mathcal{P}^+(\vec{\mu})\\ |f(p)|=\ell(\varphi(p))-\ell(\omega_0\iota(p))}} x^{\mathrm{wt}(p)}.$$

Setting $t = 1$ in theorem 3.1 yields the same formula for orbit sums as in their Definition 2.1.

**Corollary 3.6.** *The orbit sum for a dominant weight $\mu$ is*

$$m_\mu = \sum_{\substack{p\in\mathcal{P}(\vec{\mu})\\ |f(p)|=0}} x^{\mathrm{wt}(p)} = \sum_{w\in W} x^{w\mu}.$$

If $\mu$ is a dominant regular weight, the final direction of a $\mu$-path is $\omega_0$ (otherwise, the leading term of the $P_\mu$ in Theorem 3.1 would not be 1). Thus, for a dominant regular weight $\mu \in P^{++}$, the formula in Corollary 3.4 reduces to

$$P_\mu = \sum_{p\in\mathcal{P}^+(\vec{\mu})} x^{\mathrm{wt}(p)} t^{\frac{1}{2}(\ell(\iota(p))-|f(p)|)}(1-t)^{|f(p)|}$$

and Corollary 3.5 to

$$\chi_\mu = \sum_{\substack{p\in\mathcal{P}^+(\vec{\mu})\\ |f(p)|=\ell(\iota(p))}} x^{\mathrm{wt}(p)}.$$

In addition, using Corollary 3.3 we can cancel some exponents and get the next corollary.

**Corollary 3.7.** *The Macdonald polynomial for a regular weight $\mu$ is*

$$P_\mu = \sum_{\substack{w \in W \\ T \subseteq [l]}} X^{w(\psi(T)\mu)} t^{\frac{1}{2}(\ell(w)-\ell(w\Psi(T))-|T|)} (1-t)^{|T|} \left( \prod_{k \in T} \frac{1}{1-q^{j_k} t^{\mathrm{ht}(\gamma_k^\vee)}} \right) \left( \prod_{k \in T_w^-} q^{j_k} t^{\mathrm{ht}(\gamma_k^\vee)} \right)$$

*where $T_w^- \subseteq T$ is the set of negative folds of a walk $q \in \mathcal{P}(\vec{\mu})$ with folds $f(q) = T$ and initial alcove $\iota(p) = w$.*

*Remark.* In fact, the formulas for Macdonald spherical functions and Weyl characters were stated independent of Ram and Yip's combinatorial formula for Macdonald polynomials and Theorem 3.1 reduces to them. For more details, we refer to [12] for the Macdonald spherical functions in terms of positively folded alcove walks. An equivalent formula for the Weyl characters in terms of certain positively folded alcove walks is given in [1] using the *Littelmann path model.*

### 3.4 An equivalent combinatorial formula

Recall Definition 3.1 of alcove walks. As discussed in section 3.1, any alcove walk $p$ is uniquely determined by

its type $\vec{w}$, its initial alcove $\iota(p)$, a $g \in \Omega$ and the set of fold positions $f(p)$.

Let us call them for a moment the defining information. The formula in Theorem 3.1 needs further statistics about walks – like the weight, final direction and certain sequences of roots – which are determined by the defining information. Therefore, it might be useful to give an equivalent definition of alcove walks using those statistics. This subsection will give such an definition and we will transfer further definitions out of Section 3.1 to this new point of view. We will finally get an equivalent formula for the Macdonald polynomials $P_\mu$, with which $P_\mu$ can be computed without drawing alcove walks. In fact, we get the same formula as stated as Ram and Yip's formula in [5]. This formula will be used in Section 5 to obtain a combinatorial formula for the Macdonald symmetric functions. Note that the following definitions are from [5], but not the explanations why they are equivalent to the previous ones.

Given the defining information but $g \in \Omega$, the alcove $\tilde{v}$ – the alcove where the $\ell$th step ends – is already uniquely determined. Then, $\mathrm{end}(p) \in \{g\tilde{v} \mid g \in \Omega\}$ is uniquely determined either by $g \in \Omega$, the weight $\mathrm{wt}(p)$ or the final direction $\varphi(p)$.
Instead of giving the type $\vec{w} = (i_1, \dots, i_\ell)$ and the set of folds, we can give the sequence of alcoves and walls like in the following definition from [5]:

**Definition 3.2.** An ***alcove walk of weight*** $\mu$ is a sequence $\Omega = (A_0, F_1, A_1, F_2, \dots F_{\ell-1}, A_\ell, \mu)$ where $A_0, \dots A_\ell$ are alcoves, $\lambda \in P$ is a weight, and for each $i = 1, \dots, \ell$, $F_i$ is a codimensional one common face of the alcoves $A_{i-1}$ and $A_i$. We call $\Omega$ a ***minimal alcove walk*** if it is a minimal length alcove walk from $A_0$ to $A_\ell$. The weight of the alcove walk is denoted by $\mathrm{wt}(\Omega)$.

Let $\Omega = (A_0, F_1, A_1, F_2, \dots F_{\ell-1}, A_\ell, \mu)$ be a sequence in the sense of Definition 3.2. We can identify $\Omega$ as alcove walk by Definition 3.1: The walk begins at the alcove $A_0$, which belong to a unique corresponding element $v \in W_{\mathrm{aff}}$ by Corollary 1.9. Let $H_{\alpha_j,k}$ be the wall which includes $F_i$ for any $1 < i < n$. Since $F_i$ is defined as part of the common wall $H_{\alpha_j,k}$, the alcoves $A_{i-1}$ and $A_i$ must be either adjacent or equal. Thus, a consecutive triple $(A_{i-1}, F_i, A_i)$ is either a $j$-crossing or a $j$-fold, respectively.

**Definition 3.3.** Let $\mu$ be a dominant weight. Analogously to the definition of a $\mu$-path using Definition 3.1, we define a ***$\mu$-path*** as minimal alcove walk

$$(A, F_1, A_1, F_2, \ldots F_{\ell-1}, A_\ell, \mu)$$

which has weight $\mu$, starts at the fundamental alcove $A$ and ends at the alcove $A_\ell$, which has the minimal length representative of all alcoves in the $\mu$-polygon. The ***$\mu$-chain*** is a sequence of positive roots $\Gamma = (\gamma_1, \ldots, \gamma_\ell)$, such that $F_k$ is part of a hyperplane $H_{\gamma_k, j_k} \in \mathcal{H}$ for every $k \in \{1, \ldots, \ell\}$.

A $\mu$-chain $\Gamma$ is the sequence we already defined in (3.6). The sequence $j = (j_1, \ldots, j_\ell)$, defined in (3.7), can be obtained by $\Gamma$. Thus, the combinatorial formula in Theorem 3.1 just needs the $\mu$-chain $\Gamma = (\gamma_1, \ldots, \gamma_\ell)$ of a dominant weight $\mu$.

Because of Lemma 3.2, we define the action of $W$ on alcove walks by

$$w\Omega = (wA_0, wF_1, wA_1 \ldots, wF_{\ell-1}, wA_\ell, w\mu)$$

to obtain alcove walks of the same type (relative to Definition 3.1) but with another initial alcove. Analogously, if we change a crossing into a fold at the $k$th step of $\Omega$, we can describe this by an operator $F_k$, which acts on $\Omega$ by

$$f_k(\Omega) = (A_0, F_0, \ldots, A_k, F_k, s_{\gamma_k, j_k} A_{k+1}, s_{\gamma_k, j_k} F_{k+1}, \ldots, s_{\gamma_k, j_k} F_{l-1}, s_{\gamma_k, j_k} A_l, s_{\gamma_k, j_k} \mu)$$

Fix a $\mu$-chain to a dominant weight $\mu$. Let $\mathcal{F}(\Gamma) := \{(w, T) \mid w \in W,\ T \subset [\ell]\}$ and encode an element $(w, T) \in \mathcal{F}(\Gamma)$ as alcove walk $w f_T(\Omega)$, where $\Omega$ is the $\mu$-path corresponding to $\Gamma$, and $f_T$ is the concatenation of operators $f_{k_1} \cdots f_{k_m}$ for $T = \{k_1, \ldots, k_m\}$. Lemma 3.2 holds and we define $\Psi(T)$ and $\psi(T)$ like in (3.8).

Now, we can rewrite Theorem 3.1 using the notations in this subsection:

**Corollary 3.8.** *Let $\mu \in P^+$ be a dominant weight and $\Gamma$ a $\mu$-chain. The Macdonald polynomial is*

$$P_\mu = \sum_{(w,T) \in \mathcal{F}(\Gamma)} x^{w\psi(T)\mu} t^{\frac{1}{2}(\ell(w^{-1}\Psi(T)v_\mu) - \ell(\omega_0 w) - |T|)} \left( \prod_{k \in T} \frac{1}{1 - q^{j_k} t^{\mathrm{ht}(\gamma_k^\vee)}} \right) \left( \prod_{k \in T_w^-} q^{j_k} t^{\mathrm{ht}(\gamma_k^\vee)} \right)$$

*where $v_\mu$ is the final direction of the $\mu$-path, and $T_w^-$ denotes the set of negative folds of the walk $(w, T)$.*

**Corollary 3.9.** *Let $\mu \in P^{++}$ be a dominant regular weight and $\Gamma$ a $\mu$-chain. The Macdonald polynomial is*

$$P_\mu = \sum_{(w,T) \in \mathcal{F}(\Gamma)} x^{w\psi(T)\mu} t^{\frac{1}{2}(\ell(w) - \ell(w\Psi(T)) - |T|)} \left( \prod_{k \in T} \frac{1}{1 - q^{j_k} t^{\mathrm{ht}(\gamma_k^\vee)}} \right) \left( \prod_{k \in T_w^-} q^{j_k} t^{\mathrm{ht}(\gamma_k^\vee)} \right)$$

*where $v_\mu$ is the final direction of the $\mu$-path, and $T_w^-$ denotes the set of negative folds of the walk $(w, T)$.*

This formula gets by so far almost without drawing alcove walks. The remaining question is, how the negative folds $T_w^-$ can be obtained from $T$ without drawing alcove walks. To answer this, define the most known partial order on the Weyl group – the Bruhat order.

**Definition 3.4.** Write $w \geq u$ for two elements $w, u \in W$ if the following two conditions hold:

(i) $w = us_\beta$ for a $\beta \in \Phi$,

(ii) $\ell(w) \geq \ell(u)$.

The transitive closure of this relation defines a partial order on $W$, called the ***Bruhat order***.

*Remark.* For consistency, we use the same definition for the Bruhat order as Lenart in [5]. This definition can also be found in [3, Section 5.9], however, there are equivalent definitions mentioned. Let $w = us_\beta$ for $w, u \in W$ and a reflection $s_\beta \in W$. Then, by (1.2),

$$w = us_\beta = u(u^{-1}s_{u(\beta)}u) = s_{u(\beta)}u.$$

Since $u(\beta)$ is a root, we can exchange condition (i) of Definition 3.4 by $w = s_\beta u$ or $u = s_\beta w$. The same argument shows that $w \geq u$ if and only if $w^{-1} \geq u^{-1}$. Let $w \geq u$, so $w = us_\beta$ for a root $\beta$. Then $u^{-1} = s_\beta w^{-1}$, and hence $u^{-1} = w^{-1}s_{u^{-1}(\beta)}$. Condition (ii) of Definition 3.4 holds, since $\ell(v) = \ell(v^{-1})$ for any $v \in W$.

By [5], we finally get a tool to determine a negative fold of a walk.

**Lemma 3.10.** *Let $\Gamma = (\gamma_1, \dots, \gamma_\ell)$ be a $\lambda$-chain and let $(w, T) \in \mathcal{F}(\Gamma)$ be an alcove walk with set of folds $T = \{t_1 < \cdots < t_m\}$. A fold $t_k$ in $(w, T)$ is positive if and only if*

$$ws_{\beta_1} \dots s_{\beta_{k-1}} > ws_{\beta_1} \dots s_{\beta_{k-1}} s_{\beta_k},$$

*where $\beta_i = \gamma_{t_i}$ for $i = 1, \dots, m$.*

Likewise to [5], use the ***Bruhat chain***

$$w,\ ws_{\gamma_{t_1}},\ ws_{\gamma_{t_1}}s_{\gamma_{t_2}},\ \dots,\ ws_{\gamma_{t_1}} \cdots s_{\gamma_{t_m}}$$

to determine all the positive and negative folds in $T = \{t_1 < \cdots < t_m\}$.

**Example.** Recall the example in Section 3.2 about $P_\rho$ in Type $B_2$ with $\rho$-chain

$$\gamma = (\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2).$$

Consider the walk in $\mathcal{P}(\rho)$ with initial alcove $w = s_1$ and folds $T = \{2, 3\}$, so let $\beta_1 = \alpha_1 + 2\alpha_2$ and $\beta_2 = \alpha_1 + \alpha_2$ such that $s_{\beta_1} = s_1 s_2 s_1$ and $s_{\beta_2} = s_2 s_1 s_2$. Pictorially, the set of negative folds is $T_w^- = \{2\}$. The same set of negative folds follows by the Bruhat chain, which is

$$s_1 < \omega_0 > s_2.$$

# 4 Macdonald polynomials in Type $A$

Historically, mathematicians started studying symmetric functions, which had a partition as input. It turned out, partitions are in bijection to dominant weights in root systems of Type $A$. This leaded to generalizations of symmetric functions by changing the input to weights of any root system. In this section we will explain the relation between partitions and weights in Type $A$, and how this effects the group algebra $A = F[P]$.

## 4.1 Symmetric polynomials

Macdonald polynomials are defined in the group algebra $F[P]$. If the root system $\Phi$ is of Type $A_{n-1}$, the group algebra becomes the polynomial ring of symmetric functions. We introduce them similar to [8] and [11].

Consider the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$, where $x_1, \dots, x_n$ are independent indeterminates. The symmetric group $S_n$ acts naturally on $\mathbb{Z}[x_1, \dots, x_n]$ by permuting the $x_i$'s and the polynomials which are invariant under this action form a subring $\Lambda_n$ – the subring of symmetric polynomials in $x_1, \dots, x_n$. In fact, $\Lambda_n$ is a graded ring, that is

$$\Lambda_n = \bigoplus_{r \geq 0} \Lambda_n^r \qquad \text{and} \qquad \Lambda_n \Lambda_m = \Lambda_{n+m},$$

where $\Lambda_n^r$ is the additive group of homogeneous symmetric polynomials of degree $r$. There is a surjective homomorphism $\Lambda_{n+1} \to \Lambda_n$ defined by setting $x_{n+1} = 0$. Thus, we consider a polynomial in $\Lambda_n$ in

$$\Lambda = \bigoplus_{r \geq 0} \Lambda^r$$

where $\Lambda^r = \varprojlim_n \Lambda_n^r$ is the projective limit of $\Lambda_n^r$ for $r \geq 0$.

The property of a polynomial to be symmetric does not depend on the ring $\mathbb{Z}$. We can easily change the ring $\mathbb{Z}$ to any ring $R$ and shall write

$$\Lambda_R = \Lambda \oplus_{\mathbb{Z}} R \qquad \text{and} \qquad \Lambda_{R,n} = \Lambda_n \oplus_{\mathbb{Z}} R.$$

*Remark.* Let $f$ be a symmetric function in $\Lambda_R$. We write $f(x_1, \dots, x_n)$ for the image of $f$ in $\Lambda_R$, obtaining that $x_k = 0$ for all indices $k > n$. Thus, the image $f(x_1, \dots, x_n)$ is in $\Lambda_{R,n}$. Instead of permuting the $x_i$'s, let the symmetric group act on the exponents of the monomials $x_1^{\mu_1} x_2^{\mu_2} \cdots$ in $f$. This provides the same symmetric polynomials as in $\Lambda_n$.

The exponents of a symmetric function in are determined by a sequence of nonnegative integers $\mu = (\mu_1, \mu_2, \dots)$, the symmetric group $S_n$ acts on them naturally by permuting the first $n$ entries:

$$w\mu = (\mu_{w(1)}, \mu_{w(2)}, \dots, \mu_{w(n)}, \mu_{n+1}, \mu_{n+2}, \dots).$$

When the sequence has length $n$, it suffices to regard it ordered by size – as a partition. Therefore, many symmetric polynomials of interest are defined corresponding to a partition.

**Definition 4.1.** A ***partition*** of a nonnegative integer $m$ is a sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of weakly decreasing nonnegative integers which sum up to $m$. We then write $|\lambda| = m$. The ***length*** $\ell(\lambda)$ of $\lambda$ is the number of parts $n$ in the sequence. If the sequence is strictly decreasing and ends with $\lambda_n = 0$, we call $\lambda$ a ***regular partition***.

To simplify the notation of a concrete partition, write $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ if $\lambda$ has $m_r$ parts of size $r$.

It is convenient to ignore parts of size zero since adding or subtracting such parts only changes the length. That means especially, we can always assume that a partition of $m$ has length $m$ by adding or subtracting parts of size zero. To avoid confusion, denote the length of a partition $\lambda$ after subtracting all parts of size zero by $\ell_{\min}$. Hence, the length of a regular partition is $\ell(\lambda) = \ell_{\min}(\lambda) + 1$.

Given two partitions $\lambda$ and $\mu$, without loss of generality of the same length $n$, define the sum $\nu = \lambda + \mu$ as partition with

$$\nu_i = \lambda_i + \mu_i \quad \text{for every } 1 \leq i \leq n.$$

This defines an associative operation on the set of partitions of any length.

**Definition 4.2.** Let $\lambda, \mu$ be partitions of $m$, without loss of generality of the same length $n$. Write $\lambda \trianglerighteq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

for every $i = 1, \ldots, n$. This defines a partial order, called the ***dominance order***, on the partitions of $m$.

**Example.** The dominance order is a total order for $m = 1, 2, 3, 4$:

$$\begin{aligned}&(1)\\ &(2) \triangleright (1,1)\\ &(3) \triangleright (2,1) \triangleright (1,1,1)\\ &(4) \triangleright (3,1) \triangleright (2,2) \triangleright (2,1,1) \triangleright (1,1,1,1).\end{aligned}$$

The two partitions $(3, 1, 1)$ and $(4, 2)$ of 6 are not related to each other, so the dominance order in indeed just a partial order in general.

The dominance order can be visualized using *Young diagrams.* We will follow up on this in Section 5.1.

**Theorem 4.1.** *There is a bijection between the set of partitions of length $n-1$ and the set of dominant weights in Type $A_{n-1}$, which is compatible with the additions defined on both sides. The partial order on weights is then identified with the dominance order on partitions under this bijection.*

*Proof.* Recall from Section 1.2 the realization for Type $A_n$, so let $V \subset \mathbb{R}^n$ be the vector space consisting of all vectors whose coefficients in the linear combination in the standard basis sum up to zero, and let $\omega_i = v_i - \frac{i}{n} v_n$ for $1 \leq i \leq n-1$ be the fundamental weights where $v_j = \varepsilon_1 + \cdots + \varepsilon_j$ for $1 \leq j \leq n$.
Let $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, 0)$ be a partition. We can think of $\lambda$ as vector $\sum_{i=1}^n \lambda_i \varepsilon_i$ in $\mathbb{R}^n$, and we can transform $\lambda$ to a vector in $V$ by subtracting $\frac{|\lambda|}{n} v_n$. We prove that

$$\mu := \lambda - \frac{|\lambda|}{n} v_n \in V$$

is a dominant weight. Indeed, we can write $\mu = \sum_{i=1}^{n-1} a_i \omega_i$ for unique coefficients $a_i$, and the definition of the fundamental weights yields

$$\mu_i - \mu_{i+1} = a_i.$$

On the other hand, we have

$$\mu_i - \mu_{i+1} = \left(\lambda_i - \frac{|\lambda|}{n}\right) - \left(\lambda_{i+1} - \frac{|\lambda|}{n}\right) = \lambda_i - \lambda_{i+1}$$

which is a nonnegative integer since $\lambda$ is a partition. Thus, $\mu$ can be written as a nonnegative integer linear combination of the fundamental weights, as desired.
Vice versa, a dominant weight $\mu = \sum_{i=1}^{n-1} a_i \omega_i$ determines a unique partition $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ by the rule

$$\lambda_k = \sum_{i=k}^{n-1} a_i \quad \text{for } 1 \leq k \leq n-1.$$

The described mappings are inverse to each other, so they are bijections. It should be clear by construction, that this bijection preserves the additions defined on partitions and weights, respectively. □

*Remark.* The described bijection in the proof of Theorem 4.1 is a change of bases. A dominant weight $\mu$ corresponds to a partition $\lambda = (\lambda_1, \dots, \lambda_{n-1}, \lambda_n = 0)$ under the bijection if and only if

$$\mu = \sum_{i=1}^{n-1} \mu_i \omega_i = \sum_{i=1}^{n} \left( \lambda_i - \frac{i}{n} \right) \varepsilon_i.$$

Analogously, there is a bijection between dominant regular weights and regular partitions what explains the similar name.

**Corollary 4.2.** *There is an bijection between regular partitions of length $n$ and dominant weights in Type $A_{n-1}$, which is compatible with the additions defined on both sides.*

**Corollary 4.3.** *The group algebra $F[P]$ in Type $A$ is isomorphic to the ring of symmetric functions $\Lambda_F$.*

We next describe more relations between symmetric functions and the root system of Type $A$ by considering some well known bases for the ring of symmetric functions $\Lambda_R$. Their definitions and properties – like to be a basis for $\Lambda_R$ – are from and given in [8, Section 1].

**Monomial symmetric function** Given a sequence $\mu$ of $n$ nonnegative integers, consider the monomial $x^\mu = \prod_{i=1}^n x^{\mu_i}$. If $\lambda$ is a partition of length $n$, we define the ***monomial symmetric function*** $m_\lambda$ as sum of all distinct monomials $x^\mu$ such that $\mu = \sigma(\lambda)$ for any permutation $\sigma \in S_n$. That is,

$$m_\lambda = \sum_{\mu \in S_n \lambda} x^\mu,$$

where $S_n\lambda$ is the $S_n$-orbit of the partition $\lambda$.
The monomial symmetric functions $m_\lambda$ form a $\mathbb{Z}$-basis of the ring $\Lambda_\mathbb{Z}$.

**Example.** We determine some monomial symmetric functions $m_\lambda$. The simplest ones are

$$m_{(m)} = \sum_i x_i^m \quad \text{and}$$

$$m_{(1^m)} = \sum_{i_1 < \cdots < i_m} x_{i_1} \cdots x_{i_m}$$

where the sum is over numbers between 1 and $n = \ell(\lambda)$. To shorten the index $\lambda$ of $m_\lambda$, we write the partition often of minimal length, and the sum is then from 1 to $n$ for an $n \geq \ell_{\min}(\lambda)$. In the next sections, in particular in Section 5.3, we will express symmetric functions as linear combinations of the monomial symmetric functions corresponding to $\lambda$ of $m = 1, 2, 3, 4$. These are the following (except $m_{(m)}$ and $m_{(1^m)}$):

$$m_{(2,1)} = \sum_{i \neq j} x_i^2 x_j \ ,$$

$$m_{(3,1)} = \sum_{i \neq j} x_i^2 x_j \ ,$$

$$m_{(2,1,1)} = \sum_{\substack{i \neq j,k \\ j<k}} x_i^2 x_j \ .$$

**Theorem 4.4.** *The orbit sums in Type $A_{n-1}$ are the monomial symmetric functions subject to $x_1 \cdots x_n = 1$.*

*Proof.* Let $\mu$ be a dominant weight in Type $A_{n-1}$ and let $\lambda = (\lambda_1, \dots, \lambda_{n-1}, 0)$ be the corresponding partition by the bijection in Theorem 4.1, that is

$$\mu = \sum_{i=1}^{n-1} \mu_i \omega_i = \sum_{i=1}^{n} \left(\lambda_i - \frac{i}{n}\right) \varepsilon_i.$$

Likewise to [8, Section 2.3], let $x_i = x^{\varepsilon_i - \frac{1}{n} v_n}$ where $v_n = \varepsilon_1 + \cdots + \varepsilon_n$. Then, we have

$$x_1 \cdots x_n = x^0 = 1.$$

Moreover, the orbit sum corresponding to $\mu$ is then a sum over distinct sequences which can be obtained by acting on $\lambda$ with the symmetric group $S_n$, and each such sequence $w\lambda$, $w \in S_n$, determines a distinct monomial $x^{w\lambda}$. In other words, the orbit sum $m_\mu$ becomes the monomial symmetric function $m_\lambda$. □

**Power sums** Let $r$ be a nonnegative integer. The ***rth power sum*** is

$$p_r = m_{(r)} = \sum_i x_i^r.$$

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, define its ***power sum*** $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}$. These products form a $\mathbb{Q}$-basis of $\Lambda_{\mathbb{Q}}$.

**Completly symmetric functions** Let $r$ be a nonnegative integer. The ***rth complete symmetric function*** $h_r$ defined as

$$h_r = \sum_{|\mu|=r} m_\mu.$$

For each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ let $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$. These products $h_\lambda$ form a $\mathbb{Z}$-basis of $\Lambda_{\mathbb{Z}}$.

The complete symmetric function in terms of the power sums is given by

$$h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda,$$

where

$$z_\lambda = \prod_{r \geq 1} (r^{m_r} m_r!)$$

for $\lambda = (1^{m_1}, 2^{m_2}, \dots)$.

**Definition 4.3.** Define a scalar product on $\Lambda$ by the condition

$$\langle h_\lambda \, , m_\mu \rangle = \delta_{\lambda\mu}$$

for all partitions $\lambda$ and $\mu$.

Indeed, the scalar product is defined on $\Lambda_R$ for any $\mathbb{Z} \subset R$.

**Lemma 4.5.** *For any two partitions $\lambda$ and $\mu$, we have*

$$\langle p_\lambda \, , p_\mu \rangle = \delta_{\lambda\mu} z_\lambda,$$

*where $z_\lambda$ is defined in section 4.1.*

## 4.2 Macdonald symmetric functions

In the following subsections, we will introduce further basis for the ring of symmetric functions $\Lambda_R$, written as a linear combination of the monomials. The ring $R = F = \mathbb{Q}(q,t)$ is the ring we already considered in Section 2. To shorten the notation of a function indexed by a partition $\lambda$, we write "***lower terms***" for a linear combination of monomials $m_\mu$ with $\mu \lhd \lambda$. In this subsection, we will introduce the equivalent to the Macdonald polynomials:

**Theorem 4.6.** *The Macdonald polynomials $P_\mu$ in Type $A_{n-1}$ are symmetric functions subject to the relation $x_1 \cdots x_n = 1$, and defined by*

*(i) $P_\lambda = P_\lambda(q,t) = m_\lambda +$ lower terms,*

*(ii) $\langle P_\lambda , P_\mu \rangle_{q,t} = 0$ if $\lambda \neq \mu$,*

*where the scalar product is defined by*

$$\langle p_\lambda , p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}. \tag{4.1}$$

*We call them* **Macdonald symmetric functions**.

*Proof.* The two defining conditions follow directly from the definition of Macdonald polynomials in Theorem 2.1 and Theorem 4.4. More interesting is the equivalent definition for the scalar product $\langle\, , \rangle_{q,t}$ on $\Lambda_F$. Laurent polynomials in $x_1, \ldots, x_n$ are polynomials in $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$. Denote the $F$-algebra of those Laurent polynomials by

$$L_n = F\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right].$$

Following the construction in the proof of Theorem 4.4, we set $x_i = x^{\varepsilon_i - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n)}$. Thus, for any root $\alpha = \varepsilon_i - \varepsilon_j$,

$$x^\alpha = x^{\varepsilon_i - \varepsilon_j} = x_i x_j^{-1}$$

such that $\Delta_{q,t}$ of (2.2) becomes

$$\Delta_{q,t} = \prod_{i \neq j} \frac{(x_i x_j^{-1}; q)_\infty}{(t x_i x_j^{-1}; q)_\infty} = \prod_{i \neq j} \prod_{r=0}^{k-1} (1 - q^r x_i x_j^{-1})$$

an element in $L_n$. Let $f, g \in \Lambda_{n,F}$, then the scalar product defined in (2.3) reduces to

$$\langle f , g \rangle_{q,t} = \frac{1}{n!} \left[f \overline{g} \Delta_{q,t}\right]_1,$$

where $\overline{g} = g(x_1^{-1}, \ldots, x_n^{-1})$ in $L_n$. By [8, Section 1.12], the $P_\lambda$ are pairwise orthogonal relative to this inner product. More precisely, this defines an equivalent scalar product to the one defined in (4.1). □

Note that the scalar product defined in (4.1) is just equivalent to the scalar product $\langle\, , \rangle_{q,t}$ defined on $F[P]$ and restricted to the case, where the root system $\Phi$ is of Type $A$.

When $q = t$, the scalar product (4.1) becomes

$$\langle p_\lambda , p_\mu \rangle_{q,q} = \delta_{\lambda\mu} z_\lambda$$

the scalar product in Lemma 4.5.

*Remark.* A breakthrough in the theory of symmetric polynomials was the proof of the *positivity conjecture.* It concerns a modification of the Macdonald symmetric functions $H(q,t)$ via *plethystic substitution.* These modified Macdonald polynomials are again a basis for the ring of symmetric functions. Written as a linear combination in the Schur functions, the coefficients $K_{\lambda\mu}$ are called *Kostka-Macdonald coefficients* and Macdonald conjectured, that they are polynomials in $q$ and $t$ with nonnegative integer coefficients. In 2001, this conjecture was proven by Haiman in [2] for the first time.

The following corollary follows directly by Theorem 4.4 and Section 2.1.

**Corollary 4.7.** *The monomial symmetric functions $m_\lambda$ are the Macdonald symmetric functions $P_\lambda$ at $t = 1$.*

The Macdonald symmetric functions generalize further polynomials of interest, like the Schur polynomials we will define in the next section.

## 4.3 Schur functions

The Schur functions are a well-studied basis for the ring of symmetric functions. There are several combinatorial formulas using the strong relation between them and partitions. Of course, we could define the Schur functions by these combinatorial considerations, but we will choose an algebraic definition (see [8, Section 1.7]).
First, let us work in $\mathbb{Z}[x_1, \dots, x_n]$ again. Given an partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of length $m \leq n$ we can extend it to a partition of length $n$ by adding parts of size zero. Define

$$a_\lambda = \sum_{w \in S_n} \varepsilon(w) x^{w(\lambda)},$$

where $x^{w\lambda} = \prod_{i=1}^n x_i^{\lambda_{w(i)}}$.
Considering $\delta_n = (n-1, n-2, \dots, 1, 0)$, the minimal regular partition of length $n$, it yields

$$a_{\delta_n} = \prod_{i<j}(x_i - x_j) \qquad \text{the Vandermonde determinant.}$$

**Definition 4.4.** Let $\lambda$ be a partition of length $n$. Define the ***Schur function*** corresponding to $\lambda$ as

$$s_\lambda = \frac{a_{\lambda+\delta_n}}{a_{\delta_n}}.$$

The Schur functions form a $\mathbb{Z}$-basis of $\Lambda$ like the following theorem shows.

**Theorem 4.8.** *The Schur function $s_\lambda$ can be written as linear combination of the monomials*

$$s_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu$$

*for nonnegative integers $K_{\lambda\mu}$, called the* **Kostka numbers**.

We refer to [8, Section 1.7] for the proof of Theorem 4.8.
The Kostka number $K_{\lambda\mu}$ has a combinatorial interpretation as number of certain Young tableaux related to $\lambda$ and $\mu$. We come back to this in Section 5.2.
The Schur functions satisfy condition $(i)$ of Theorem 4.6. By [8, Section 1.6] it also applies $\langle s_\lambda\, , s_\mu \rangle = \delta_{\lambda\mu}$. Thus, the Schur functions are Macdonald symmetric functions:

**Corollary 4.9.** *The Schur functions are the Macdonald symmetric functions for $q = t$, so they are the Weyl characters in Type A.*

## 4.4 More special cases of Macdonald symmetric functions

Although the Macdonald symmetric functions are a special case for the Macdonald polynomials, they are themselves a family of some well-studied symmetric functions. Conversely to the historical order, we define some of them as special cases of the Macdonald symmetric functions, and follow the notations of [8].

**Hall-Littlewood symmetric functions** We get the ***Hall-Littlewood symmetric functions*** $P_\lambda(t)$ by setting $q = 0$ for the Macdonald symmetric functions. Thus, they are defined by

(i) $P_\lambda(t) = m_\lambda +$ lower terms,

(ii) $\langle P_\lambda(t)\,, P_\mu(t)\rangle_t = 0$ if $\lambda \neq \mu$,

where the scalar product is given by

$$\langle p_\lambda\,, p_\mu\rangle_t = \langle p_\lambda\,, p_\mu\rangle_{0,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} (1 - t^{\lambda_i})^{-1}.$$

In fact, a closed formula is known for $P_\lambda(t)$ and they form a basis for $\Lambda_{\mathbb{Z}[t]}$ (see [8, Section 1.10]).

**Jack polynomials** Let $q = t^\alpha$ for $\alpha > 0$. The limits

$$P_\lambda^{(\alpha)} = \lim_{t \to 1} P_\lambda$$

are called the ***Jack polynomials***. They form a unique $\mathbb{Q}(\alpha)$-basis for the ring $\Lambda_{\mathbb{Q}(\alpha)}$ satisfying

(i) $P_\lambda^{(\alpha)} = m_\lambda +$ lower terms,

(ii) $\langle P_\lambda^{(\alpha)}\,, P_\mu^{(\alpha)}\rangle_\alpha = 0$ if $\lambda \neq \mu$,

where the scalar product is defined by

$$\langle p_\lambda\,, p_\mu\rangle = \delta_{\lambda\mu} \alpha^{\ell(\lambda)} z_\lambda.$$

**Zonal polynomials** Define the zonal polynomial $Z_\lambda$ corresponding to a partition $\lambda$ the Jack polynomials at $\alpha = 2$, that is, as the unique symmetric polynomial in $\Lambda$ satisfying

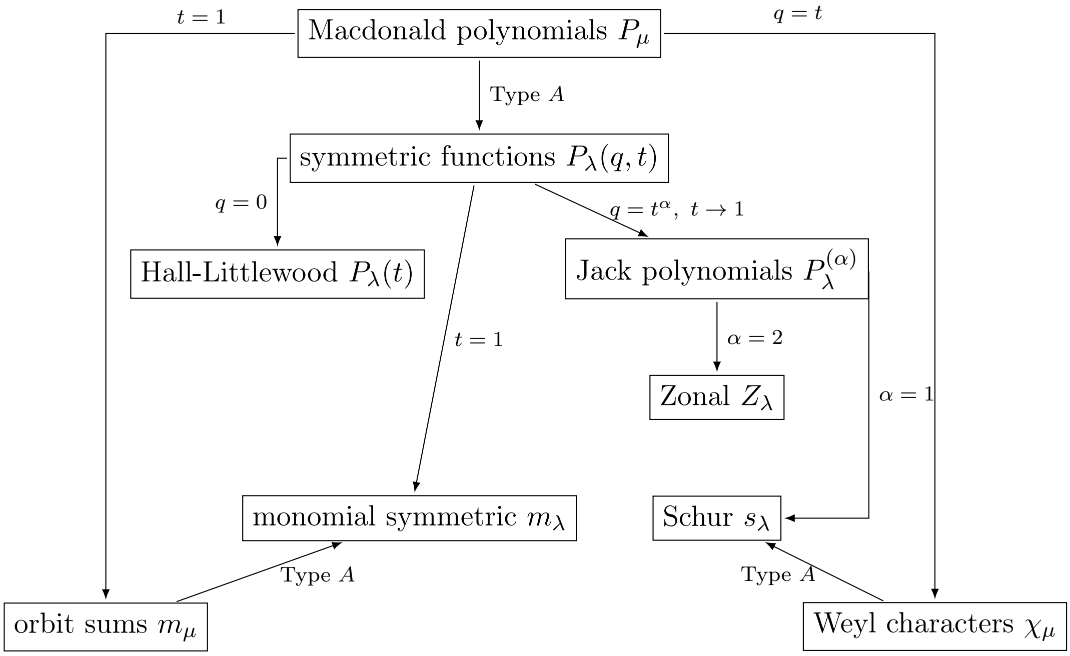
(i) $Z_\lambda = m_\lambda +$ lower terms,

(ii) $\langle Z_\lambda\,, Z_\mu\rangle_2 = 0$ if $\lambda \neq \mu$,

where the definition of the scalar product reduces to

$$\langle p_\lambda\,, p_\mu\rangle_2 = \delta_{\lambda\mu} 2^{\ell(\lambda)} z_\lambda.$$

We summarize the results and definitions of this section and of Section 2.1 by the following diagram:




# 5 Combinatorial formulas for Macdonald symmetric functions

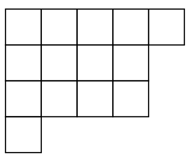
In Section 3, we have already seen a combinatorial formula for Macdonald polynomials, so in particular for those in Type $A$ - the Macdonald symmetric functions. This formula (see Theorem 3.1) uses combinatorics based on alcove walks in the affine Weyl arrangement. Since there is a bijection between partitions and weights in Type $A$, we ask for a formula using combinatorics on partitions. Such a formula is given by Lenart in [5]. Before we state Lenart's formula in Theorem 5.1 we will give some definitions on partitions.

## 5.1 A combinatorial formula using partitions

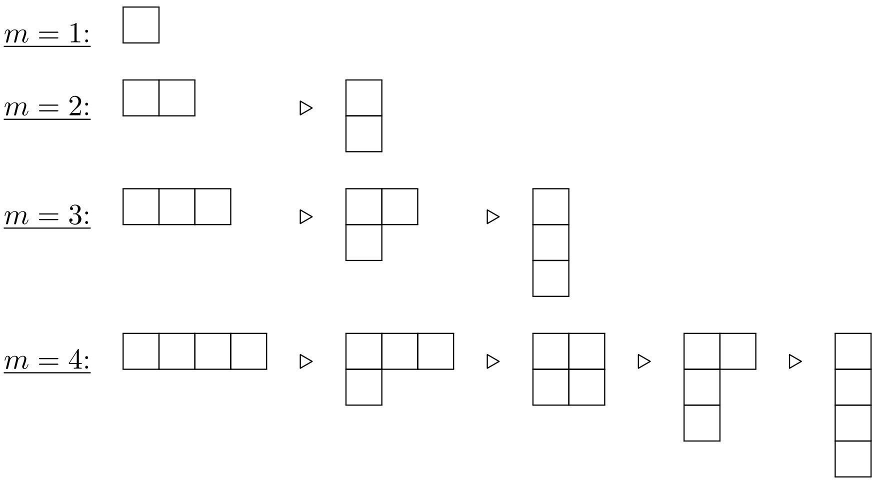
A partition $\lambda$ is identified with the ***Young diagram*** of shape $\lambda$, which is defined as

$$\{(i,j)\in\mathbb{N}\times\mathbb{N}\mid j\leq\lambda_i\}\,.$$

We draw the elements, called ***cells***, as connected boxes. We use the english notation, where the cells are ordered like the numeration in a matrix. For example, the Young diagram of shape $\lambda=(5,4,4,1)$ is

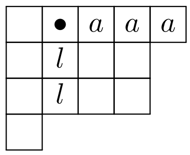


The Young diagram visualize many definitions corresponding to the partition. For example, we can view the dominance order on partitions as bumping down boxes of a Young diagram into a lower row. As in the corresponding example in Section 4.1, we have




A second example is the ***conjugate*** $\lambda' = (\lambda'_1, \ldots, \lambda'_{\lambda_1})$ of a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. It is defined by $\lambda'_j = \#\{i \mid \lambda_i \geq j\}$, or equivalently, $\lambda'_j$ is the number of cells in the $jth$ column of the Young diagram of shape $\lambda$.

We define some combinatorial statistics we need in the combinatorial formula in Theorem 5.1, using Young diagrams. Still considering the english notation for the Young diagram of shape $\lambda$, the ***arm*** and ***leg*** of a cell $u = (i, j)$ of $\lambda$ are defined as the number of cells strictly to the left of $u$ or below $u$, respectively. Take again the partition $\lambda = (5, 4, 4, 1)$ and denote the cell $\bullet = (1, 2)$:




$\text{arm}(\bullet) = 3$ and $\text{leg}(\bullet) = 2$.

Two cells $u, v$ ***attack each other*** if one of the following holds:

(1) $u$ and $v$ are in the same column.

(2) $u$ is in the column to the right of $v$ and in a strictly higher row than $v$.

We can define functions $\sigma : \lambda \to [n]$ for any positive integer $n$ and identify them with Young diagrams of shape $\lambda$, where the image of each cell is written in the corresponding box. Therefore, call such a function a ***filling***. The ***content*** of a filling $\sigma$ is

$$\text{content}(\sigma) := (c_1, \ldots, c_n), \text{ where } c_i := |\sigma^{-1}(i)|.$$

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, the statistic

$$n(\lambda) = \sum_{i=1}^{\ell} (i-1)\lambda_i$$

can be seen as the sum of cells in the filling $\tau : (i, j) \to (i - 1)$, which has content $\lambda$. We will use this point of view in Section 5.2.

The formula in Theorem 5.1 for Macdonald polynomials in Type $A$ is a sum over certain fillings:

**Definition 5.1.** A filling is called ***nonattacking*** if any two attacking cells $u, v$ have different images $\sigma(u) \neq \sigma(v)$. Denote the set of nonattacking fillings $\sigma : \lambda \to [n]$ by $\mathcal{T}(\lambda, n)$.

**Example.** Let us consider two fillings $\pi, \sigma : \lambda \to [7]$:

$\pi(\lambda) =$

| | | | | |
|---|---|---|---|---|
| 1 | 2 | **4** | 5 | 1 |
| 7 | 7 | 1 | 1 | |
| 4 | **4** | 2 | 4 | |
| 5 | | | | |

and $\sigma(\lambda) =$

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 4 | 6 | 5 |
| 1 | 5 | 5 | 4 | |
| 4 | 7 | 1 | 6 | |
| 6 | | | | |

The colored cells in $\pi$ attack each other and have the same filling, thus, $\pi$ does not satisfy Definition 5.1. It has content $\text{content}(\pi) = (4, 2, 0, 4, 2, 0, 2)$. The filling $\sigma$ is a nonattacking filling with $\text{content}(\sigma) = (2, 1, 1, 3, 3, 3, 1)$.

We define the ***column reading order*** on the cells of $\lambda$ as total order, for which a cell $(i, j)$ precedes a cell $(k, \ell)$ if $(j, i)$ precedes $(\ell, k)$ in lexicographic order, that is, if either $j < \ell$ or if $j = \ell$ and $i < k$.

For the following definitions, let $\sigma$ be any filling of $\lambda$.

**Definition 5.2.** A pair of cells $(u, v)$ with $\sigma(u) < \sigma(v)$ is said to be an ***inversion*** if $u, v$ attack each other and $u$ precedes $v$ in column reading order. The set of inversions is $\text{Inv}(\sigma)$.

**Example.** The nonattacking filling $\sigma$ of the previous example has 8 inversions:

| | | | | |
|---|---|---|---|---|
| **3** | 2 | 4 | 6 | 5 |
| **1** | 5 | 5 | 4 | |
| 4 | 7 | 1 | 6 | |
| 6 | | | | |

| | | | | |
|---|---|---|---|---|
| 3 | **2** | 4 | 6 | 5 |
| 1 | 5 | 5 | 4 | |
| **4** | 7 | 1 | 6 | |
| 6 | | | | |

| | | | | |
|---|---|---|---|---|
| 3 | **2** | 4 | 6 | 5 |
| 1 | 5 | 5 | 4 | |
| 4 | 7 | 1 | 6 | |
| **6** | | | | |

| | | | | |
|---|---|---|---|---|
| 3 | **2** | 4 | 6 | 5 |
| 1 | 5 | 5 | 4 | |
| 4 | 7 | 1 | 6 | |
| **6** | | | | |

| | | | | |
|---|---|---|---|---|
| 3 | 2 | **4** | 6 | 5 |
| 1 | **5** | 5 | 4 | |
| 4 | 7 | 1 | 6 | |
| 6 | | | | |

| | | | | |
|---|---|---|---|---|
| 3 | 2 | **4** | 6 | 5 |
| 1 | 5 | 5 | 4 | |
| 4 | **7** | 1 | 6 | |
| 6 | | | | |

| | | | | |
|---|---|---|---|---|
| 3 | 2 | **4** | 6 | 5 |
| 1 | 5 | 5 | 4 | |
| 4 | 7 | **1** | 6 | |
| 6 | | | | |

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 4 | 6 | 5 |
| 1 | 5 | **5** | 4 | |
| 4 | 7 | **1** | 6 | |
| 6 | | | | |

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 4 | **6** | 5 |
| 1 | 5 | 5 | **4** | |
| 4 | 7 | 1 | 6 | |
| 6 | | | | |

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 4 | 6 | **5** |
| 1 | 5 | 5 | 4 | |
| 4 | 7 | 1 | **6** | |
| 6 | | | | |

**Definition 5.3.** A cell $u = (i, j)$ is called a ***descent*** if the cell $v = (i, j + 1)$ exist in $\lambda$ and $\sigma(u) > \sigma(v)$. The set of descents is denoted by $\text{Des}(\sigma)$. Extend this set to $\text{Diff}(\sigma)$ by requiring different images in the definition of descents.

**Definition 5.4.** Given a filling $\sigma$, define two combinatorial statistics:

$$\text{maj}(\sigma) := \sum_{u \in \text{Des}(\sigma)} \text{arm}(u)$$

$$\text{inv}(\sigma) := |\,\text{Inv}(\sigma)| - \sum_{u \in \text{Des}(\sigma)} \text{leg}(u)$$

**Example.** Our running example has 4 descents, which are colored in the left tableaux. The cells in $\text{Diff}(\sigma)$ are colored in the right tableaux.

$\text{Des}(\sigma)$ :

| | | | | |
|---|---|---|---|---|
| **3** | 2 | 4 | **6** | 5 |
| 1 | 5 | **5** | 4 | |
| 4 | **7** | 1 | 6 | |
| 6 | | | | |

$\text{Diff}(\sigma)$ :

| | | | | |
|---|---|---|---|---|
| **3** | **2** | **4** | **6** | 5 |
| **1** | 5 | **5** | 4 | |
| **4** | **7** | **1** | 6 | |
| 6 | | | | |

We obtain $\text{maj}(\sigma) = 4 + 1 + 1 + 2 = 8$ and $\text{inv}(\sigma) = 8 - (3 + 0 + 1 + 2) = 2$.

Now that we have fixed some definitions on partitions, we can state Lenart's combinatorial formula for Macdonald symmetric functions from [5, Thm. 2.12].

**Theorem 5.1** (Lenart, 2008)**.** *Let $\lambda$ be a regular partition of length $n$. Then, we have*

$$P_\lambda(q,t) = \sum_{\sigma \in \mathcal{T}(\lambda,n)} t^{n(\lambda)-\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} \left( \prod_{u \in \mathrm{Diff}(\sigma)} \frac{1-t}{1-q^{\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}} \right) x^{\mathrm{content}(\sigma)}.$$

In [5], Lenart shows how to obtain this formula, which is related to partitions, from the formula of Ram and Yip, which uses combinatorics based on alcove walks we discussed in Section 3. His proof can be split in two parts. In the first part, he constructs nonattacking fillings from alcove walks and expresses Ram and Yip's formula as sum over those fillings. The second part compresses the result to the formula in Theorem 5.1. The details of the proof are given in [5]. Although, we will give the proof of the first part in Section 5.4 to explain the relation between nonattacking fillings and alcove walks. But first, we provide a formula for Schur functions and examples of Macdonald symmetric functions using Lenart's formula.

## 5.2 Combinatorial formulas for the Schur functions

By Theorem 4.8 we already know, that

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu$$

where $K_{\lambda\mu}$ are the ***Kostka numbers***, which have the following combinatorial interpretation: A Young tableau of shape $\lambda$ is called semistandard if the numbers in the cells are strictly increasing in the columns and weakly increasing in the rows. To be exact, a semistandard Young tableau of shape $\lambda$ is a filling $\sigma$ of $\lambda$, such that the conditions above hold. The Kostka number $K_{\lambda\mu}$ is then the number of semistandard Young tableaux of shape $\lambda$ with content $\mu$. Thus, we have a second formula for the Schur functions as sum of semistandard Young tableaux

$$s_\lambda = \sum_{\sigma \in \mathrm{SSYT}(\lambda)} x^{\mathrm{content}(\sigma)}, \tag{5.1}$$

where $\mathrm{SSYT}(\lambda)$ is the set of all semistandard fillings of $\lambda$.
Like we have seen in Section 2.1 and Section 3.3, we can use combinatorial formulas of Macdonald polynomials to obtain Schur functions by setting $q = t = 0$. In contrast to the previous given formulas for special cases of Macdonald polynomials, we get a new one for the Schur functions:

**Corollary 5.2.** *The Schur function for a partition $\lambda$ is*

$$s_\lambda = \sum_{\substack{\sigma \in \mathcal{T}(\lambda,n) \\ \mathrm{Des}(\sigma)=\emptyset \\ n(\lambda)=\mathrm{Inv}(\sigma)}} x^{\mathrm{content}(\sigma)} \tag{5.2}$$

*a sum over nonattacking fillings without descents and with the maximal number of inversions* $\mathrm{Inv}(\sigma) = n(\lambda)$.

*Proof.* Consider the formula of Theorem 5.1. Let $\lambda$ be a regular partition. The used combinatorial statistics $n(\lambda), \mathrm{inv}(\lambda), \mathrm{maj}(\lambda), \mathrm{arm}(\lambda)$ and $\mathrm{leg}(\lambda)$ are nonnegative. We have to determine when an exponent of $q$ or $t$ is zero. Remark that a cell with arm length 0 cannot be in $\mathrm{Diff}(\sigma)$. Now, we write

$$\begin{aligned}
P_\lambda(q,t) &= \sum_{\sigma \in \mathcal{T}(\lambda,n)} t^{n(\lambda)-\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} \left( \prod_{u \in \mathrm{Diff}(\sigma)} \frac{1-t}{1-q^{\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}} \right) x^{\mathrm{content}(\sigma)} \\
&= \sum_{\substack{\sigma \in \mathcal{T}(\lambda,n) \\ \mathrm{Des}(\sigma)=\emptyset}} t^{n(\lambda)-\mathrm{Inv}(\sigma)} \left( \prod_{u \in \mathrm{Diff}(\sigma)} \frac{1-t}{1-q^{\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}} \right) x^{\mathrm{content}(\sigma)} \\
&\quad + \sum_{\substack{\sigma \in \mathcal{T}(\lambda,n) \\ \mathrm{Des}(\sigma)\neq\emptyset}} t^{n(\lambda)-\mathrm{Inv}(\sigma)} q^{\mathrm{maj}(\sigma)} \left( \prod_{u \in \mathrm{Diff}(\sigma)} \frac{1-t}{1-q^{\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}} \right) x^{\mathrm{content}(\sigma)} \\
&\overset{q=0}{=} \sum_{\substack{\sigma \in \mathcal{T}(\lambda,n) \\ \mathrm{Des}(\sigma)=\emptyset}} t^{n(\lambda)-\mathrm{Inv}(\sigma)} \left( \prod_{u \in \mathrm{Diff}(\sigma)} \frac{1-t}{1-t^{\mathrm{leg}(u)+1}} \right) x^{\mathrm{content}(\sigma)} \\
&= \sum_{\substack{\sigma \in \mathcal{T}(\lambda,n) \\ \mathrm{Des}(\sigma)=\emptyset}} t^{n(\lambda)-\mathrm{Inv}(\sigma)} \left( \prod_{u \in \mathrm{Diff}(\sigma)} \frac{1-t}{1-t^{\mathrm{leg}(u)+1}} \right) x^{\mathrm{content}(\sigma)}.
\end{aligned}$$

This formula reduces to a sum over fillings $\sigma$ (nonattacking and without descents) such that $n(\lambda) - \mathrm{Inv}(\sigma) = 0$. Recall that a pair of cells $(u,v)$ is an inversion if $u$ precedes $v$ in column reading order, $u$ and $v$ attack each other and $\sigma(u) \geq \sigma(v)$. Construct a nonattacking filling $\sigma$ without descents, which has a maximal number of inversions. This is (for example):

$$\begin{array}{cccc} \sigma: & \lambda & \to & [\ell] \\ & (i,j) & \mapsto & k_i \end{array}$$

for pairwise different $k_1, \ldots, k_\ell$, where $\ell = \ell(\lambda)$, that is

$$\begin{array}{llll}
\boxed{k_1}\boxed{k_1}\boxed{k_1} & \cdots\cdots & \boxed{k_1} \\
\boxed{k_2}\boxed{k_2} & \cdots & \\
\boxed{k_3} & \quad\iddots & \\
\vdots & & \\
\boxed{k_\ell} & \cdots &
\end{array}$$

Note that there can be other types of fillings with the same number of inversions, but no filling with more. Construct a second filling

$$\tau_\sigma: \quad u \quad \mapsto \quad \#\{v \mid (u,v) \in \mathrm{Inv}(\sigma)\},$$

which counts how many inversions $(u,v)$ a cell $u$ makes up. Let $u = (i,j)$ be a cell in the $j$th column and let $\lambda'_k$ be the number of cells in the $k$th column for any $k = 1, \ldots, \lambda_1$. The image $\tau_\sigma(u)$ is an integer between 0 and $\lambda'_j$. Since $\lambda$ is a regular partition, there are at least $\lambda'_j - 1$ cells in the $(j+1)$-th column. In addition, the entries in $\sigma$ are the same for each row. Thus, $\tau_\sigma(u) = \#\{v \mid v \text{ in } j\text{th column}, \sigma(u) > \sigma(v)\}$ counts the number of smaller entries in the column of $u$, so each number between 0 and $\lambda'_j$ is exactly once in the $j$th column of $\lambda$. Ordering each column by size yields a third filling

$$\tau : \quad (i,j) \quad \mapsto \quad i-1 \ .$$

For example, if $\lambda = (4,2,1)$, $k_1 = 3$, $k_2 = 1$ and $k_3 = 2$, the three described fillings are

$$\sigma(\lambda) = \begin{array}{|c|c|c|c|}\hline 3&3&3&3\\\hline 1&1 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array} \qquad \tau_\sigma = \begin{array}{|c|c|c|c|}\hline 2&1&0&0\\\hline 0&0 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \qquad \tau = \begin{array}{|c|c|c|c|}\hline 0&0&0&0\\\hline 1&1 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array} \ .$$

The maximal number of inversions of a nonattacking filling without descents is

$$|\operatorname{Inv}(\sigma)| = \sum_{u\in\lambda} \tau_\sigma(u) = \sum_{u\in\lambda} \tau(u) = \sum_{i=1}^{\ell_{\min}(\lambda)} (i-1)\lambda_i = n(\lambda). \qquad \square$$

Indeed, if $\operatorname{Des}(\sigma)$ is not empty, there are fillings with more than $n(\lambda)$ inversions. But the proof shows, that a nonattacking filling without descents has at most $n(\lambda)$ inversions.

*Remark.* Lenart calls the difference $n(\lambda) - \operatorname{inv}(\sigma)$ the ***complementary inversion statistic*** (see [4, Definition 2.11 and Proposition 2.12]) and denotes it by $\operatorname{cinv}(\sigma)$. In addition, he proves in [4, Proposition 2.12] that $\operatorname{cinv}(\sigma)$ counts pairs of cells $(u, v)$, such that the following two conditions hold:

(1) $u$ is below $v$ and $\sigma(u) < \sigma(v)$ and

(2) if the cell $w$ directly to the right of $u$ exist, then $\sigma(v) < \sigma(w)$.

The stated formula (5.1) for the Schur function looks very similar to the well known formula (5.2). But in general, they are not the same. That is, the sum is over different fillings but with the same content as the next example shows.

**Example.** Compare (5.1) with (5.2) for the partition $\lambda = (2,1)$ and $n = 3$. The semistandard Young tableaux which appear in (5.1) are

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \ ,$$

while in (5.2) appear the tableaux

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline 1\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3&3\\\hline 1\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array} \ .$$

The monomials $x_1x_2^2$, $x_1x_3^2$ and $x_2x_3^2$ correspond to different tableaux in both equations. The monomial $x_1x_2x_3$ corresponds to the last two tableaux, but just one of them appears in both cases.

### 5.3 Examples of Lenart's formula

In this subsection, we want to compute the Macdonald polynomial for every regular partition of $m \in \{1,2,3,4\}$ via Lenart's formula in Theorem 5.1. Since $P_\lambda(q,t) = m_\lambda \sum_{\mu\leq\lambda} u_{\lambda\mu} m_\mu$ for coefficients $u_{\lambda\mu} \in \mathbb{Q}(q,t)$, we will consider each part $u_{\lambda\mu}m_\mu$ of the sum individually. Moreover, we will consider $m_{\lambda\mu} \in \Lambda_{F,n}$ for a $n \geq m$ (see the paragraph about monomial symmetric functions in Section 4.1).

In fact, given a regular partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell-1}, 0)$, the part of the formula corresponding to the monomials $x_{k_1}^{\lambda_1} \cdots x_{k_{\ell-1}}^{\lambda_{\ell-1}}$ is the monomial symmetric function $m_\lambda$. This is clear by the definition of the Macdonald symmetric functions in Theorem 4.6, but it is also clear by Theorem 5.1: A nonattacking filling $\sigma$ of $\lambda$ with content $(c_1, \ldots, c_n)$, $c_{k_i} = \lambda_i$, must be

$$\begin{array}{|c|c|c|cc|c|}\hline k_1 & k_1 & k_1 & \cdots & \cdots & k_1 \\ \hline k_2 & k_2 & \cdots & & & \\ \cline{1-2} \vdots & & & & & \end{array}$$

so $\operatorname{maj}(\sigma) = 0$ and $\operatorname{inv}(\sigma) = n(\lambda)$.
A regular partition $\lambda$ of $m$ with length $\ell$ exists if and only if

$$m > \sum_{i=1}^{\ell-1} i = \frac{(\ell-1)\ell}{2}.$$

Hence, we now only consider partitions of minimal length 1 or 2.

$\boldsymbol{\lambda = (m, 0)}$ Partitions of minimal length 1, that is, $\lambda = (m, 0)$, satisfy $n(\lambda) = 0$, $\operatorname{leg}(u) = 0$ for any cell $u$ and $\operatorname{Inv}(\sigma)$ is empty for every filling $\sigma$. The formula in Theorem 5.1 reduces to

$$P_\lambda(q,t) = \sum_{\sigma \in \mathcal{T}(\lambda,n)} q^{\operatorname{maj}(\sigma)} \left( \prod_{i \in \operatorname{Diff}(\sigma)} \frac{1-t}{1-q^{m-i}t} \right) x^{\operatorname{content}(\sigma)} \tag{5.3}$$

where $i \in \operatorname{Diff}(\sigma)$ means $(1, i) \in \operatorname{Diff}(\sigma)$. Since no two cells attack each other, every filling is nonattacking. A filling $\sigma_k$, which has just one integer $k$ as image, generates those parts of the sum corresponding to the monomial $x_k^m$. Such a filling has empty sets $\operatorname{Des}(\sigma)$ and $\operatorname{Diff}(\sigma)$, so all those fillings $\sigma_k$, $k \in [n]$, generate

$$m_{(m,0)} = \sum_{k=1}^{n} x_k^m$$

as term of (5.3).
A filling $\tilde{\sigma}$ which has pairwise different images – say $\tilde{\sigma}(1, i) = k_i$ – corresponds to the monomial $x_{k_1} \cdots x_{k_m}$. For a fixed choice of integers $k_1 < \cdots < k_m$, we can get each filling $\sigma$ with $k_1, \ldots, k_m$ as image, if we act on the index set with the symmetric group $S_m$. That is, each filling $\sigma$ with $k_1, \ldots, k_m$ as image can be written as

$$\sigma = w\tilde{\sigma} \quad \text{where } \sigma(1, i) = k_{w(i)}.$$

A descent of the filling $\sigma$ then becomes a descent of the permutation $w$, while the set $\operatorname{Diff}(\sigma)$ includes all cells except the last $(1, m)$.
The major statistic of a filling becomes the comajor index

$$\operatorname{comaj}(\pi) := \sum_{\pi(i) < \pi(i+1)} m - i$$

of the corresponding permutation $\pi$. Thus, fillings whose image is a set of pairwise different integers generate

$$u_{\lambda(1^m)} m_{(1^m)} = \sum_{w \in S_m} q^{\operatorname{comaj}(w)} \left( \prod_{i=1}^{m-1} \frac{1-t}{1-q^i t} \right) \left( \sum_{k_1 < \cdots < k_m} x_{k_1} \cdots x_{k_m} \right)$$

as term of (5.3). Note that in smaller examples it might be easier to stay with the major statistic for fillings than the comajor index.

**$\boldsymbol{\lambda = (1,0)}$** Consider $\lambda = (1,0)$, we have

$$P_{(1,0)}(q,t) = m_{(1,0)} = \sum_{k=1}^{n} x_k.$$

**$\boldsymbol{\lambda = (2,0)}$** We differ between two types of fillings $\sigma$

$$\boxed{k \mid k} \qquad \text{and} \qquad \boxed{k \mid \ell}$$

for different $k, \ell \in \{1, \dots, n\}$. Depending on whether $k > \ell$ or $k < \ell$, the statistic $\mathrm{maj}(\sigma)$ is equal to 1 or 0, respectively. Thus, we have

$$P_{(2,0)}(q,t) = m_{(2,0)} + (1+q)\frac{1-t}{1-qt} m_{(1,1)}.$$

**$\boldsymbol{\lambda = (3,0)}$** We differ between fillings relative to the number of equal entries: The monomials $x_k^3$ and $x_k x_\ell x_r$, $1 \le k < \ell < r \le n$, correspond to the fillings

$$\boxed{k \mid k \mid k} \qquad \text{and} \qquad \boxed{k' \mid \ell' \mid r'} \quad \text{for } \{k', \ell', r'\} = \{k, \ell, r\},$$

respectively. The monomial $x_k^2 x_\ell$ corresponds to three types of fillings $\sigma$ and since $P_\lambda$ is symmetric, we only have to consider $k < \ell$:

| $\sigma$ | $\boxed{k \mid k \mid \ell}$ | $\boxed{k \mid \ell \mid k}$ | $\boxed{\ell \mid k \mid k}$ |
|---|---|---|---|
| $i \in \mathrm{Diff}(\sigma)$ | 2 | 1, 2 | 1 |
| $\mathrm{maj}(\sigma)$ | 0 | 1 | 2 |

Thus, we have

$$\begin{aligned} P_{(3,0)}(q,t) =& m_{(3,0)} + (q^3 + 2q^2 + 2q + q^0)\frac{1-t}{1-qt}\frac{1-t}{1-q^2t} m_{(1^3)} \\ &+ \left(\frac{1-t}{1-qt} + q\frac{1-t}{1-qt}\frac{1-t}{1-q^2t} + q^2\frac{1-t}{1-q^2t}\right) m_{(2,1)} \\ =& m_{(3,0)} + \frac{(q^2+q+1)(1-t)}{1-q^2t} m_{(2,1)} + \frac{(q^3+2q^2+2q+1)(1-t)^2}{(1-qt)(1-q^2t)} m_{(1^3)} \\ =& m_{(3,0)} + \frac{(q^2+q+1)(1-t)}{1-q^2t} m_{(2,1)} + \frac{(q+1)(q^2+q+1)(1-t)^2}{(1-qt)(1-q^2t)} m_{(1^3)}. \end{aligned}$$

**$\boldsymbol{\lambda = (4,0)}$** Let $1 \le k, \ell, r, s \le n$ be pairwise different. We have five types of monomials. Analogously to the previous examples, the monomials $x_k^4$ and $x_k x_\ell x_r x_s$, $k, \ell, r, s \le n$, correspond to the fillings

$$\boxed{k \mid k \mid k \mid k} \qquad \text{and} \qquad \boxed{k' \mid \ell' \mid r' \mid s'} \quad \text{for } \{k', \ell', r', s'\} = \{k, \ell, r, s\},$$

respectively. The monomial $x_k^3 x_\ell$ corresponds to four types of fillings and since $P_\lambda$ is symmetric, we only have to consider $k < \ell$:

| $\sigma$ | $\boxed{k \mid k \mid k \mid \ell}$ | $\boxed{k \mid k \mid \ell \mid k}$ | $\boxed{k \mid \ell \mid k \mid k}$ | $\boxed{\ell \mid k \mid k \mid k}$ |
|---|---|---|---|---|
| $i \in \mathrm{Diff}(\sigma)$ | 3 | 2, 3 | 1, 2 | 1 |
| $\mathrm{maj}(\sigma)$ | 0 | 1 | 2 | 3 |

The monomial $x_k^2 x_\ell^2$ corresponds to three types of fillings (or 6 if we differ between $k < \ell$ and $k > \ell$):

| $\sigma$ | $\boxed{k\,\vert\,k\,\vert\,\ell\,\vert\,\ell}$ | $\boxed{k\,\vert\,\ell\,\vert\,\ell\,\vert\,k}$ | $\boxed{k\,\vert\,\ell\,\vert\,k\,\vert\,k}$ |
|---|---|---|---|
| $i \in \mathrm{Diff}(\sigma)$ | 2 | 1, 3 | 1, 2, 3 |
| $\mathrm{maj}(\sigma),\ k < \ell$ | 0 | 1 | 2 |
| $\mathrm{maj}(\sigma),\ k > \ell$ | 2 | 3 | 4 |

The fifth type of monomials is $x_k^2 x_\ell x_r$ and again, we only have to consider $k < \ell, r$ for the following six types of corresponding fillings:

| $\sigma$ | $\boxed{k\,\vert\,k\,\vert\,\ell\,\vert\,r}$ | $\boxed{k\,\vert\,\ell\,\vert\,k\,\vert\,r}$ | $\boxed{k\,\vert\,\ell\,\vert\,r\,\vert\,k}$ | $\boxed{\ell\,\vert\,k\,\vert\,k\,\vert\,r}$ | $\boxed{\ell\,\vert\,k\,\vert\,r\,\vert\,k}$ | $\boxed{\ell\,\vert\,r\,\vert\,k\,\vert\,k}$ |
|---|---|---|---|---|---|---|
| $i \in \mathrm{Diff}(\sigma)$ | 2, 3 | 1, 2, 3 | 1, 2, 3 | 1, 3 | 1, 2, 3 | 1, 2 |
| $\mathrm{maj}(\sigma),\ \ell < r$ | 0 | 2 | 1 | 3 | 4 | 2 |
| $\mathrm{maj}(\sigma),\ \ell > r$ | 1 | 2 | 3 | 3 | 4 | 5 |

Thus, we have

$$\begin{aligned}
P_{(4,0)}(q,t) =& m_{(4,0)} + \frac{(q+1)(q^2+1)(1-t)}{1-q^3t} m_{(3,1)} + \frac{(1+q^2)(q^2+q+1)(1-t)(1-qt)}{(1-q^2t)(1-q^3t)} m_{(2,2)} \\
&+ \frac{(q^5+2q^4+3q^3+3q^2+2q+1)(1-t)^2}{(1-q^2t)(1-q^3t)} m_{(2,1,1)} \\
&+ \frac{(q+1)^2(q^4+q^3+2q^2+q+1)(1-t)^3}{(1-qt)(1-q^2t)(1-q^3t)} m_{(1^4)} \\
=& m_{(4,0)} + \frac{(q+1)(q^2+1)(1-t)}{1-q^3t} m_{(3,1)} + \frac{(q^2+1)(q^2+q+1)(1-t)(1-qt)}{(1-q^2t)(1-q^3t)} m_{(2,2)} \\
&+ \frac{(q+1)(q^2+1)(q^2+q+1)(1-t)^2}{(1-q^2t)(1-q^3t)} m_{(2,1,1)} \\
&+ \frac{(q+1)^2(q^2+1)(q^2+q+1)(1-t)^3}{(1-qt)(1-q^2t)(1-q^3t)} m_{(1^4)}.
\end{aligned}$$

Now, let us compute the Macdonald symmetric functions for the partitions of minimal length 2: $(2,1,0)$ and $(3,1,0)$. For these partitions, not every filling is nonattacking.

$\boldsymbol{\lambda = (2,1,0)}$ By Theorem 4.1, the partition $(2,1) = (2,1,0)$ is identified with the dominant weight $\omega_1 + \omega_2$ in Type $A_2$. We already computed the Macdonald polynomial of this weight in Section 3.2 by using Theorem 3.1. However, we will compute it again, but this time with Theorem 5.1. We differ between two types of fillings:

$$\begin{array}{|c|c|}\hline k & k \\ \hline \ell & \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline k & r \\ \hline \ell & \\ \cline{1-1}\end{array}\,, \tag{5.4}$$

for pairwise different $k, \ell, r \in [n]$, where the left filling corresponds to the monomial $x_k^2 x_{k_2}$ and the right filling to $x_{k_1} x_{k_2} x_{k_3}$. Denote $c_1 = (1,1)$, $c_2 = (2,1)$ and $c_3 = (1,2)$ for the cells of $\lambda$. A filling $\sigma$ of the left type corresponds to the monomial $x_k^2 x_\ell$, has no descent and one inversion: either $(c_1, c_2)$ if $k > \ell$ or $(c_2, c_3)$ if $k < \ell$.
now, consider a filling $\sigma$ of the right type in (5.4). It corresponds to the monomial $x_k x_\ell x_r$, and provides $\mathrm{Diff}(\sigma) = \{c_1\}$. Fix $k < \ell < r$, then we have the following six cases:

| $\sigma$ | $\begin{array}{\|c\|c\|}\hline k & r \\ \hline \ell \\ \cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|}\hline \ell & r \\ \hline k \\ \cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|}\hline r & k \\ \hline \ell \\ \cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|}\hline k & \ell \\ \hline r \\ \cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|}\hline r & \ell \\ \hline k \\ \cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|}\hline \ell & k \\ \hline r \\ \cline{1-1}\end{array}$ |
|---|---|---|---|---|---|---|
| $\mathrm{maj}(\sigma)$ | 0 | 0 | 1 | 0 | 1 | 1 |
| $\mathrm{inv}(\sigma)$ | 0 | 1 | 1 | 1 | 0 | 0 |
| $n(\lambda) - \mathrm{inv}(\sigma)$ | 1 | 0 | 0 | 0 | 1 | 1 |

Thus, the Macdonald polynomial is

$$\begin{aligned} P_{(2,1,0)} &= \sum_{k \neq \ell} x_k^2 x_\ell + \sum_{k<\ell<r} \frac{1-t}{1-q^1t^2}(q^0t + 2q^0t^0 + q^1t^0 + 2q^1t^1)x_k x_\ell x_r \\ &= m_{(2,1,0)} + \frac{1-t}{1-qt^2}(t+2+q+2qt)m_{(1^3)}. \end{aligned}$$

Recall our result (3.10) by the formula in Theorem 3.1:

$$\begin{aligned} P_\rho(q,t) &= \frac{1-t}{1-qt^2}(t+2+q+2qt) + \sum_{w \in W} x^{w\rho} \\ &= \frac{1-t}{1-qt^2}(t+2+q+2qt) + \sum_{i \neq j} x^{\varepsilon_i - \varepsilon_j} \\ &= \frac{1-t}{1-qt^2}(t+2+q+2qt) + \sum_{i \neq j} (x^{\varepsilon_i}(x^{\varepsilon_j})^{-1}). \end{aligned}$$

Using the bijection given in Theorem 4.1 (or Theorem 4.4), this polynomial is isomorphic to

$$P_{(2,1,0)} = \frac{1-t}{1-qt^2}(t+2+q+2qt) + \sum_{i \neq j} \frac{x_i}{x_j}$$

subject to the relation $x_1x_2x_3 = 1$. Thus, multiplying with $1 = x_1x_2x_3$ yields the better known representative

$$P_{(2,1,0)} = \frac{1-t}{1-qt^2}(t+2+q+2qt) + \sum_{i \neq j} x_i x_j.$$

$\boldsymbol{\lambda = (3,1,0)}$ Let $1 \leq k, \ell, r, s \leq n$ be pairwise different. By Theorem 4.6, the Macdonald polynomial is

$$P_\lambda = m_\lambda + u_{\lambda(2,2)}m_{(2,2)} + u_{\lambda(2,1,1)}m_{(2,1,1)} + u_{\lambda(1^4)}m_{(1^4)},$$

so it has four types of monomials: $x_k^3x_\ell$, $x_k^2x_\ell^2$, $x_k^2x_\ell x_r$ and $x_kx_\ell x_r x_s$. The corresponding nonattacking fillings are

$$\begin{array}{|c|c|c|}\hline k & k & k \\ \hline \ell \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline k & k & l \\ \hline \ell \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline k & k & r \\ \hline \ell \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline k & r & k \\ \hline \ell \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline r & k & k \\ \hline \ell \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline \ell & r & k \\ \hline k \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline k & r & s \\ \hline \ell \\ \cline{1-1}\end{array} \tag{5.5}$$

where four of them correspond to $x_k^2x_\ell x_r$. A filling of the first type in (5.5) has no descents and one inversion. Thus, this type produces the term

$$m_\lambda = \sum_{k \neq \ell} x_k^3 x_\ell.$$

A filling $\sigma$ of the second type (corresponding to $x_k^2 x_\ell^2$) has a descent if and only if $k > \ell$. But it always has exactly one inversion. Thus, we get the next summand

$$u_{\lambda(2,2)} m_{(2,2)} = \frac{1-t}{1-qt}(q^0 t^0 + q^1 t^0) m_{(2,2)} = \frac{1-t}{1-qt}(1+q) m_{(2,2)}.$$

Now, let us compute $u_{\lambda\mu} m_\mu$ for $\mu = (2,1,1)$, that is, the part of the formula corresponding to $x_k^2 x_\ell x_r$. Denote $c_1 = (1,1)$, $c_2 = (2,1)$, $c_3 = (1,2)$ and $c_3 = (1,3)$ for the cells of $\lambda$. The set of inversion $\operatorname{Inv}(\sigma)$ is a subset of $\{(c_1, c_2), (c_2, c_3)\}$ and $\operatorname{inv}(\sigma) = |\operatorname{Inv}(\sigma)| - 1$ if $k > r$; otherwise it is $|\operatorname{Inv}(\sigma)|$. Likewise to the previous examples, we only need to consider a fixed choice $\{k < \ell, r\}$. We have

| $\sigma$ | $\operatorname{Diff}(\sigma)$ | $\ell < r$ | | $\ell > r$ | |
|---|---|---|---|---|---|
| | | $\operatorname{maj}(\sigma)$ | $\operatorname{inv}(\sigma)$ | $\operatorname{maj}(\sigma)$ | $\operatorname{inv}(\sigma)$ |
| $\begin{array}{\|c\|c\|c\|}\hline k & k & r \\ \hline \ell \\ \cline{1-1}\end{array}$ | $c_3$ | 0 | 1 | 0 | 1 |
| $\begin{array}{\|c\|c\|c\|}\hline k & r & k \\ \hline \ell \\ \cline{1-1}\end{array}$ | $c_1, c_3$ | 1 | 0 | 1 | 1 |
| $\begin{array}{\|c\|c\|c\|}\hline r & k & k \\ \hline \ell \\ \cline{1-1}\end{array}$ | $c_1$ | 2 | 1 | 2 | 0 |
| $\begin{array}{\|c\|c\|c\|}\hline \ell & r & k \\ \hline k \\ \cline{1-1}\end{array}$ | $c_1, c_3$ | 1 | 1 | 3 | 0 |

so the coefficient of $x_k^2 x_\ell x_r$ in $P_{(3,1,0)}$ is

$$\begin{aligned} u_{\lambda\mu} &= (1-t)\left(\frac{2}{1-qt} + \frac{(1-t)(qt+2q+q^3t)}{(1-q^2t^2)(1-qt)} + \frac{q^2+q^2t}{1-q^2t^2}\right) \\ &= \frac{2q^3t^3 - 2q^3t^2 + 2q^2t^3 - 3q^2t^2 + q^2 + qt^3 - 3qt + 2q - 2t + 2}{(1-qt)^2(1+qt)} \end{aligned}$$

for $\lambda = (3,1,0)$ and $\mu = (2,1,1)$.
We have just one type of filling left which corresponds to the monomial $x_k x_\ell x_r x_s$, that is

$$\sigma(\lambda) = \begin{array}{|c|c|c|}\hline k & r & s \\ \hline \ell \\ \cline{1-1}\end{array}\,,$$

and $\operatorname{Diff}(\sigma) = \{c_2, c_3\}$ always holds. So the factor

$$\prod_{u \in \operatorname{Diff}(\sigma)} \frac{1-t}{1-q^{\operatorname{arm}(\sigma)} t^{\operatorname{leg}(\sigma)+1}} = \frac{(1-t)^2}{(1-q^2t^2)(1-qt)} \tag{5.6}$$

does not depend on $\sigma$. Nevertheless, we have to differ $4! = 24$ cases to determine the statistics $\operatorname{maj}(\sigma)$ and $\operatorname{inv}(\sigma)$:

| $\sigma$ | $\operatorname{maj}(\sigma)$ | $\lvert\operatorname{Inv}(\sigma)\rvert$ | $\operatorname{inv}(\sigma)$ | $q^{\operatorname{maj}(\sigma)}t^{n(\lambda)-\operatorname{inv}(\sigma)}$ |
|---|---|---|---|---|
| $\ell > s > r > k$ | 0 | 1 | 1 | 1 |
| $\ell > r > k, s$ | 1 | 1 | 1 | $2q$ |
| $k > \ell > r > s$ | 3 | 2 | 1 | $q^3$ |
| $k > \ell > s > r$ | 2 | 2 | 1 | $q^2$ |
| $r > k > \ell > s$ | 1 | 1 | 1 | $q$ |
| $k, s > \ell > r$ | 2 | 2 | 1 | $2q^2$ |
| $r > s, k > \ell$ | 1 | 1 | 1 | $2q$ |
| $s > r > k > \ell$ | 0 | 1 | 1 | 1 |
| $s > \ell > r > k$ | 0 | 1 | 1 | 1 |
| $k > r > \ell > s$ | 3 | 1 | 0 | $q^3t$ |
| $\ell > k > r > s$ | 3 | 1 | 0 | $q^3t$ |
| $\ell > s, k > r$ | 2 | 1 | 0 | $2q^2t$ |
| $s > \ell > k > r$ | 2 | 1 | 0 | $q^2t$ |
| $r > \ell > k, s$ | 1 | 0 | 0 | $qt$ |
| $k, s > r > \ell$ | 2 | 1 | 0 | $2q^2t$ |
| $k > r > s > \ell$ | 3 | 1 | 0 | $q^3t$ |
| $s > r > \ell > k$ | 0 | 0 | 0 | $t$ |
| $r > s > \ell > k$ | 1 | 0 | 0 | $qt$ |

Note, that some cases are combined in the first column. The last column gives the term of all the fillings mentioned in the corresponding row, so that we get $kq^ut^v$ for $u, v \in \mathbb{Z}$ if $k$ cases are mentioned. The coefficient $u_{\lambda\nu}$ for $\nu = (1^4)$ is the sum of terms in the last column, multiplied with the factor in (5.6), that is

$$\begin{aligned} u_{\lambda\nu} &= \frac{(1-t)^2}{(1-q^2t^2)(1-qt)}(3q^3t + q^3 + 5q^2t + 3q^2 + 3qt + 5q + t + 3) \\ &= \frac{(q+1)(3q^2t + q^2 + 2qt + 2q + t + 3)(1-t)^2}{(1-q^2t^2)(1-qt)}. \end{aligned}$$

The Macdonald polynomial can know be expressed as

$$\begin{aligned} P_{(3,1,0)} =& m_{(3,1,0)} + \frac{(1-t)(1+q)}{1-qt} m_{(2,2)} \\ &+ \frac{2q^3t^3 - 2q^3t^2 + 2q^2t^3 - 3q^2t^2 + q^2 + qt^3 - 3qt + 2q - 2t + 2}{(1-qt)^2(1+qt))} m_{(2,1,1)} \\ &+ \frac{(q+1)(3q^2t + q^2 + 2qt + 2q + t + 3)(1-t)^2}{(1-q^2t^2)(1-qt)} m_{(1^4)}. \end{aligned}$$

## 5.4 Proof of Theorem 5.1

In this section, we will see, how the formula in Theorem 5.1

$$P_\lambda(q,t) = \sum_{\sigma \in \mathcal{T}(\lambda,n)} t^{n(\lambda)-\operatorname{inv}(\sigma)} q^{\operatorname{maj}(\sigma)} \left( \prod_{u \in \operatorname{Diff}(\sigma)} \frac{1-t}{1 - q^{\operatorname{arm}(u)} t^{\operatorname{leg}(u)+1}} \right) x^{\operatorname{content}(\sigma)}$$

can be obtained by the formula for regular weights in Corollary 3.9

$$P_\mu = \sum_{(w,T) \in \mathcal{F}(\Gamma)} x^{w\psi(T)\mu} t^{\frac{1}{2}(\ell(w) - \ell(w\Psi(T)) - |T|)} \left( \prod_{k \in T} \frac{1}{1 - q^{j_k} t^{\operatorname{ht}(\gamma_k^\vee)}} \right) \left( \prod_{k \in T_w^-} q^{j_k} t^{\operatorname{ht}(\gamma_k^\vee)} \right).$$

To avoid confusion, we use $P_\lambda(q,t)$ if we compute the Macdonald symmetric function via combinatorics on partitions – that is, if we use Lenart's formula. Otherwise, we use the shorter symbol $P_\mu$.
The bijection between partitions and weight has already been discussed in Section 4.1. Thus, the main part of this subsection is to relate alcove walks with nonattacking fillings. We will consider alcove walks as a pair $(w,T) \in \mathcal{F}(\Gamma)$ to a fixed $\lambda$-chain $\Gamma$ (see Section 3.4). The section follows the structure, definitions and proof by [5].

First, we construct a certain $\lambda$-chain $\Gamma$ for a regular partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, 0)$. Recall the realization of Type $A_{n-1}$ in Section 1.2. Likewise to the notation of [5], identify each positive root $\alpha = \varepsilon_i - \varepsilon_j$ for $i < j$ with the tupel $(i,j)$. Define two sequences of roots

$$\begin{aligned}\Gamma(k) =&((k,n),(k,n-1),\ldots,(k,k+1),\\ &(k-1,n),(k-1,n-1),\ldots,(k-1,k+1),\\ &\ldots,\\ &(1,n),\ldots,(1,k+1))\end{aligned}$$

and

$$\begin{aligned}\Gamma'(k) =&((k,n),(k,n-1),\ldots,(k,k+2),\\ &(k-1,n),(k-1,n-1),\ldots,(k-1,k+2),\\ &\ldots,\\ &(1,n),\ldots,(1,k+2)).\end{aligned}$$

We use these defined chains to construct a $\lambda$-chain $\Gamma$ as concatenation $\Gamma = \Gamma_{\lambda_1}\Gamma_{\lambda_1-1}\ldots\Gamma_2$, where

$$\Gamma_k = \begin{cases}\Gamma'(\lambda'_k) & \text{if } \ k = \min\{i \mid \lambda'_i = \lambda'_k\},\\ \Gamma(\lambda'_k) & \text{else.}\end{cases} \tag{5.7}$$

This is indeed a $\lambda$-chain, which follows by an equivalent definition of $\lambda$-chains given in [6, Section 4]. But notice that a $\lambda$-chain there corresponds to a minimal alcove walk from $A$ to $A-\lambda$, thus, the definition needs a modification in that sense, that any $\alpha$ in the chain have to occur one time less than required.

**Example.** Let $n = 4$ and $\lambda = (4,2,1,0)$, so its conjugate is $\lambda' = (3,2,1,1)$. The $\lambda$-chain (according to the construction via (5.7)) is

$$\begin{array}{rcccc}\Gamma &=& \Gamma_4 & \Gamma_3 & \Gamma_2\\ &=& \Gamma(\lambda'_4) & \Gamma'(\lambda'_3) & \Gamma'(\lambda'_2)\\ &=& \Gamma(1) & \Gamma'(1) & \Gamma'(2)\\ &=& (\ (1,4),(1,3),(1,2), & (1,4),(1,3), & (2,4),(2,3),(1,4),(1,3)\ ).\end{array}$$

From now on, we always consider $\Gamma = \Gamma_{\lambda_1}\ldots\Gamma_2$ as the $\lambda$-chain to a regular weight $\lambda$. Identify a set of folds $T = \{t_1 < \cdots < t_s\}$ in the sense of Section 3.4 as the subsequence of $\Gamma$

$$T = (\gamma_{t_1}, \ldots, \gamma_{t_s}).$$

It will be clear in the context, whether we consider $T$ as set of folding positions or as a sequence of roots, so we use the same notation $T$.
Write $T = T_{\lambda_1}\ldots T_2$ for subsequences $T_{\lambda_1},\ldots,T_2$ of $\Gamma_{\lambda_1},\ldots,\Gamma_2$, respectively. That is, $T_k$ consists of the roots in $T$, which came from $\Gamma_k$ in $\Gamma$.

**Example.** Continue the previous example with $T = (1 < 3 < 4 < 7 < 9)$ in the sense of Section 3.4. Underline the roots in $\Gamma$ at position $t \in T$. These make up the sequence $T$.

$$\begin{array}{rcccc}
\Gamma & = & \Gamma_4 & \Gamma_3 & \Gamma_2 \\
& = & (\ \underline{(1,4)}, (1,3), \underline{(1,2)}, & \underline{(1,4)}, (1,3), & (2,4), \underline{(2,3)}, (1,4), \underline{(1,3)}\ ) \\
\\
T & = & (\ (1,4), (1,2), & (1,4), & (2,3), (1,3)\ ) \\
& = & T_4 & T_3 & T_2
\end{array}$$

To construct a filling of $\lambda$ determined by a pair $(w, T) \in \mathcal{F}(\Gamma)$ start with the ***row filling*** $\tau(i,j) \mapsto i$ for each cell in $\lambda$. An element $w \in S_n$ acts on the filling by acting on the numbers in the cells, that is, by acting on the images of the filling $\tau$. Define an action of the folds (viewed as transpositions) on certain columns of the tableau. As discussed in Section 1.2, the Weyl group of $A_{n-1}$ is $W = S_n$ and a reflection $s_\alpha$ for $\alpha = (i,j)$ is the transposition $(i,j)$. Thus, the constructed chains can be viewed as sequences or as composition of transpositions. Let a subsequence $T_k$ of $T$ act as permutation of $S_n$ on the first $k-1$ columns of the square.

**Example.** Continue the example with the partition $\lambda = (4,3,1,0)$, the row filling $\tau$ and the folds $T = T_4T_3T_2((1,4),(1,2)|(1,4)|(2,3),(1,3))$, where the vertical lines separate the subsequences $T_4, T_3, T_2$ visually. Let $w = (1243) \in S_4$ (cycle notation). The subsequence $T_4$, viewed as permutation, is the composition of the transpositions (14) and (12), where (12) acts first. Thus, $T_4$ is the permutation $(14)(12) = (124) \in S_4$ and acts on the columns $1, 2$ and 3. Likewise, $T_3 = (14)$ acts on the columns 1 and 2, and $T_2 = (23)(13) = (123)$ acts on the first column. The action $wT = wT_4T_3T_2$ on the row filling $\tau$ of $\lambda$ is

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array} \overset{T_2}{\longmapsto} \begin{array}{|c|c|c|c|}\hline 2&1&1&1\\\hline 3&2\\\cline{1-2} 1\\\cline{1-1}\end{array} \overset{T_3}{\longmapsto} \begin{array}{|c|c|c|c|}\hline 2&4&1&1\\\hline 3&2\\\cline{1-2} 4\\\cline{1-1}\end{array} \overset{T_4}{\longmapsto} \begin{array}{|c|c|c|c|}\hline 4&1&2&1\\\hline 3&4\\\cline{1-2} 1\\\cline{1-1}\end{array} \overset{w}{\longmapsto} \begin{array}{|c|c|c|c|}\hline 3&2&4&2\\\hline 1&3\\\cline{1-2} 2\\\cline{1-1}\end{array}\ .$$

Construct a filling of $\lambda$ to each pair $(w, T) \in \mathcal{F}(\Gamma)$ as follows:

**Definition 5.5.** Let $\lambda$ be a partition with $\lambda$-chain $\Gamma$ as defined before. Define the ***filling map*** $f$ from a pair $(w,T) \in \mathcal{F}(\Gamma)$ to a filling $f(w,T) = \sigma$ of $\lambda$ by

$$\sigma(i,j) := \pi_j(i),$$

where

$$\pi_j := wT_{\lambda_1}T_{\lambda_1-1}\ldots T_{j+1}$$

for a column $j = 1, \ldots, \lambda_1$ in $\lambda$.

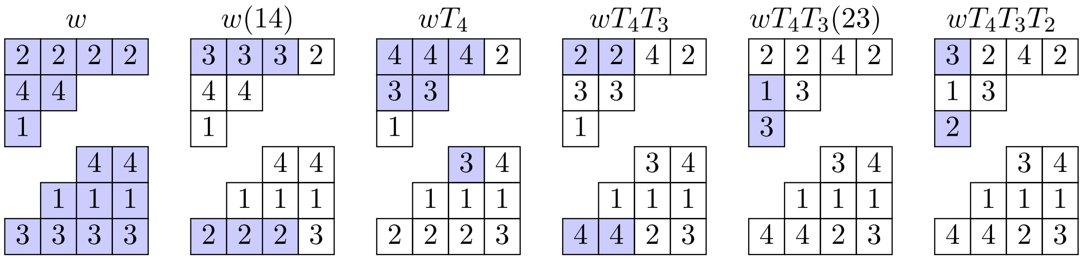
**Example.** In our running example, the filling map $f$ maps $(w,T)$ to

$$f(w,T) = \begin{array}{|c|c|c|c|}\hline 3&2&4&2\\\hline 1&3\\\cline{1-2} 2\\\cline{1-1}\end{array}\ .$$

The permutations $\pi_j$ appear in the Bruhat chain of $(w,T)$

$$\pi_{\lambda_1} = w, \ldots,\ \pi_{\lambda_1-1} = wT_{\lambda_1}, \ldots,\ \pi_2 = wT_{\lambda_1}\ldots T_3,\ \ldots,\ \pi_1 = wT_{\lambda_1}\ldots T_2 = wT.$$

This is a chain of permutations, so we can read the words as well from left to right if we make use of their inverse: Consider the partition $\lambda$ as part of a $\lambda_1 \times \lambda_1$ square and fill the boxes of this square by the row filling. The permutations then (considered from left to right) act on the rows, not on the filling. The permutation $w$ sends row $r$ to row $w^{-1}(r)$ for every row $r = 1, \ldots, n$. A transposition $(i, j)$ in $T_k$ switch the cell in row $i$ with the cell in row $j$ in every column $1, \ldots, k-1$.
In the previous example with $\lambda = (4, 2, 1)$, $w = (1243)$ and $T = ((1, 4), (1, 2)|(1, 4)|(2, 3), (1, 3))$, we get




The blue cells are the ones which have swapped.

The sum in Theorem 5.1 is over nonattacking fillings, which were defined in Definition 5.1, and each filling $\sigma$ generates a term corresponding to the monomial $x^{\text{content}(\sigma)}$. The filling map provides these conditions, as the following two corollaries show.

**Proposition 5.3.** *Let $\mu$ be the weight corresponding to a partition $\lambda$ (by the isomorphism in Theorem 4.1). For any element $w \in W$ and set of folds $T$, the content of $f(w, T)$ corresponds to the weight of the walk $(w, T) \in \mathcal{F}(\Gamma)$, that is $w\psi(T)\mu$.*

Lenart proves Proposition 5.3 by an induction on the length of $T$, so he explains, how the weight changes by adding a fold after the last one of a fixed walk. The change in the filling then corresponds to the change in the affine arrangement. To identify the concrete affine hyperplane, where the fold is at, he considers certain parts of the filling and their number of entries related to a particular root. For details, we refer to [4, Proposition 3.6 and section 5]. In addition, the proof implies for $(i, k) \in T_j$, that

$$q^{\ell} t^{\text{ht}(\gamma^\vee)} = q^{\text{arm}(i,j-1)} t^{k-i}$$

where $H_{\gamma,\ell}$ is the affine hyperplane at which the fold $(i, k)$ is.

**Proposition 5.4.** *The filling $f(w, T)$ is nonattacking. Moreover, the filling map is surjective when $\lambda$ is regular.*

*Proof.* The proof follows those to [5, Proposition 3.6 and Proposition 3.7].
Let $\sigma = f(w, T)$ be the image of a pair $(w, T) \in \mathcal{F}(\lambda)$ and let $u = (i, j-1)$ be a cell of $\lambda$. Let $v = (k, j-1)$ be a cell in the same column as $u$, but below, so $k > i$. Then,

$$\sigma(u) = \pi_{j-1}(i) \quad \text{and} \quad \sigma(v) = \pi_{j-1}(k)$$

are different, since $\pi_j$ is a permutation. Therefore, $\sigma$ fills a column with pairwise different integers.
Let $w = (i, j)$ be the cell to the right of $u$. If $\sigma(w) = \sigma(u)$, then $\sigma(u)$ differs from every entry in the column to the right, so in particular to the images of those cells, which are in an upper row.

Thus, assume $\sigma(w) \neq \sigma(u)$. Consider the subsequence $T_j = \{\tau_1, \dots, \tau_t\}$ of $T = T_{\lambda_1} \dots T_2$, and the Bruhat chain of $(w, T)$

$$w, \dots, wT_{\lambda_1}, \dots, wT_{\lambda_1}T_{\lambda_1 - 1}, \dots, wT_{\lambda_1} \dots T_2 = wT.$$

In particular, we just need the subchain from $\pi_j$ to $\pi_{j-1}$, that is

$$\pi_j = wT_{\lambda_1} \dots T_{j+1},\ wT_{\lambda_1} \dots T_{j+1}\tau_1, \dots,\ wT_{\lambda_1} \dots T_{j+1}\tau_1 \dots \tau_{t-1},\ wT_{\lambda_1} \dots T_j = \pi_{j-1}.$$

Let $s = (i, r)$, $r < i$, be a cell above $u$. A permutation $T_k$ changes the first $k - 1$ columns, so $\pi_j(u) = \pi_j(w) = \pi_{j-1}(w)$.
Thus, there must be a permutation $\pi$ in this subchain, such that $\pi(r) = \sigma(s)$, so it writes $\sigma(s)$ into row $r$.

To show, that the filling map $f$ is surjective, note, that a pair $(w, T)$ is uniquely determined by its Bruhat chain. Thus, we construct a Bruhat chain, that is, a chain of permutations, such that two consecutive permutations only differ by an transposition (so they are in relation by the Bruhat order). For a nonattacking filling of $\lambda$, construct the chain as follows:

Set $\pi_1 \in S_n$ by $\pi_1(i) = \sigma(i, 1)$ for each row $i = 1, \dots, \lambda'_1$. We construct the chain recursively and assume to have constructed $\pi_1, \dots, \pi_{j-1}$. So let $\pi_j = \pi_{j-1}$ first. Now, for each row $i$ from 1 to $\lambda'_j$, compare the consecutive cells $(i, j-1)$ and $(i, j)$. If they differ, redefine $\pi_j$ by swapping the entries $i$ and $\sigma(i, j)$ (in the image of $\pi_j$).
Proceed like this for $\pi_{j+1}$ to $\pi_{\lambda_1}$ and set $w := \pi_{\lambda_1}$. By the definition of $\pi_j$, it is $\pi_j T_j = \pi_{j-1}$, so we obtain a Bruhat chain to $(w, T)$. □

For the moment, the formula by Ram and Yip in Corollary 3.8 derives to

$$P_\lambda = \sum_{(w,T) \in \mathcal{F}(\Gamma)} x^{\mathrm{wt}(w,T)} t^{\frac{1}{2}(\ell(w) - \ell(wT) - |T|)} (1-t)^{|T|} \times$$
$$\times \left( \prod_{\substack{2 \le j \le \lambda'_1 \\ (i,k) \in T_j}} \frac{1}{1 - q^{\mathrm{arm}(i,j-1)} t^{k-i}} \right) \left( \prod_{\substack{2 \le j \le \lambda'_1 \\ (i,k) \in T_j}} q^{\mathrm{arm}(i,j-1)} t^{k-i} \right)$$

It remains to compress this formula identifying and summarizing the coefficient to the $(w, T)$, which correspond to the same nonattacking fillings in $\mathcal{T}(\lambda, n)$. That is, we have to consider the sum over walks $(w, T) \in f^{-1}$ of the coefficients. By [5, Theorem 3.7], this sum for a fixed $\sigma$ compresses to

$$\sum_{(w,T) \in f^{-1}(\sigma)}$$

so that we finally obtain

$$P_\lambda = \sum_{\sigma \in \mathcal{T}(\lambda, n)} x^{\mathrm{content}(\sigma)} t^{\frac{1}{2}(\ell(w) - \ell(wT) - |T|)} (1-t)^{|T|} \times$$
$$\times \left( \prod_{\substack{2 \le j \le \lambda'_1 \\ (i,k) \in T_j}} \frac{1}{1 - q^{\mathrm{arm}(i,j-1)} t^{k-i}} \right) \left( \prod_{\substack{2 \le j \le \lambda'_1 \\ (i,k) \in T_j}} q^{\mathrm{arm}(i,j-1)} t^{k-i} \right)$$

The proof of this last compression part requires a lot of notations, which we want to spare the reader. [5] considers consecutive columns of nonattacking fillings and associate folds to statistics on these columns, like the inversions, descents and differences. He also splits the $\lambda$-chain $\Gamma$ and the folds, to consider smaller parts of the sum, expresses them in terms of $q$, $t$ and desired statistics of the filling, and rearranges the result.

# 6 Further examples

By now, we have seen the following examples for Macdonald polynomials $P_\mu$ in Type $A$:

| $\mu$ | $\lambda$ | $A_{n-1}$ | formula | is given in |
|---|---|---|---|---|
| $\omega_1+\omega_2$ | $(2,1,0)$ | $A_2$ | Ram and Yip | Section 3.2 |
| | | | Lenart | Section 5.3 |
| $2\omega_2$ | $(2,2,0)$ | $A_2$ | Ram and Yip | Section 3.2 |
| $m\omega_1$ | $(m,0)$ for $m \geq 1$ | $A_1$ | Lenart | Section 5.3 (just a reduced form) |
| $\omega_1$ | $(1,0)$ | $A_1$ | Lenart | Section 5.3 |
| $2\omega_1$ | $(2,0)$ | $A_1$ | Lenart | Section 5.3 |
| $3\omega_1$ | $(3,0)$ | $A_1$ | Lenart | Section 5.3 |
| $4\omega_1$ | $(4,0)$ | $A_1$ | Lenart | Section 5.3 |
| $2\omega_1+\omega_2$ | $(3,1,0)$ | $A_2$ | Lenart | Section 5.3 |

where $\lambda$ denotes the corresponding partition to the weight $\mu$ by Theorem 4.1. The formula of Ram and Yip is the one in Theorem 3.1, the formula of Lenart is given in Theorem 5.1.
Let us complete the computation of $P_\lambda$ for all partitions $\lambda$ of $m = 1,2,3,4$, so that we have used both: Theorem 3.1 and Theorem 5.1. In fact, since Theorem 5.1 is just for regular partitions, we just have to make use of Theorem 3.1 for the partitions $(3,1,0)$ and $(m,0)$. In the following, we give alcove walks in $\mathcal{P}(\vec{\mu})$ to a weight $\mu$. To avoid confusion, we do not draw every alcove walk in $\mathcal{P}(\vec{\mu})$ for a fixed set of folds $T$, but we mark every weight. Denote $\mathrm{wt}(T)$ for the weight of the alcove walk in $\mathcal{P}(\vec{\mu})$, which has folds $T$ and begins at the fundamental alcove $A$. Likewise to the examples in Section 3.2, mark the weights $\mathrm{wt}(T)$ with a filled circle, and mark weights of alcove walks with a different initial alcove by an unfilled circle. For $A_2$, we use the same image as before, where the fundamental alcove $A$ is the gray colored alcove. Note, that in our examples the last arrow of each walk points at its weight, since the constructed $\mu$-walks points at $\mu$. Additionally, we give the term in the alcove walk formula in Theorem 3.1, which is generated by the walks $(w,T)$ for the fixed set $T$.

**$A_{m-1}$: partition $(1^m)$** By the definition of the Macdonald symmetric functions (see Theorem 4.6), the partition $(1^m)$ corresponds to

$$P_{(1^m)} = m_{1^m}.$$

The corresponding weight in Type $A_{m-1}$ is 0, and by the definition of Macdonald polynomials in Theorem 2.1, we have

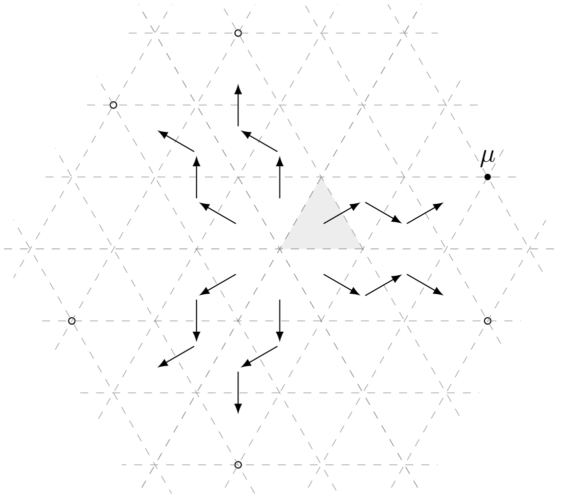
$$P_0 = m_0 = 1.$$

These two polynomials are the same under the bijection described in Theorem 4.6.

**$A_2$: partition $(3,1,0)$** The partition $(3,1,0)$ corresponds to the weight $\mu = 2\omega_1+\omega_2$. Then, the $\mu$-path is of type $(0,2,1)$. Although the walks to one weight per orbit suffice to compute the coefficients of the corresponding orbit sum in $P_\mu$, we give every walk in $\mathcal{P}(\vec{\mu})$ for a fixed $T$, unless they overlap. The Macdonald polynomial is by definition

$$P_\mu = m_\mu + u_{\mu,2\omega_2} m_{2\omega_2} + u_{\mu,\omega_1} m_{\omega_1}.$$

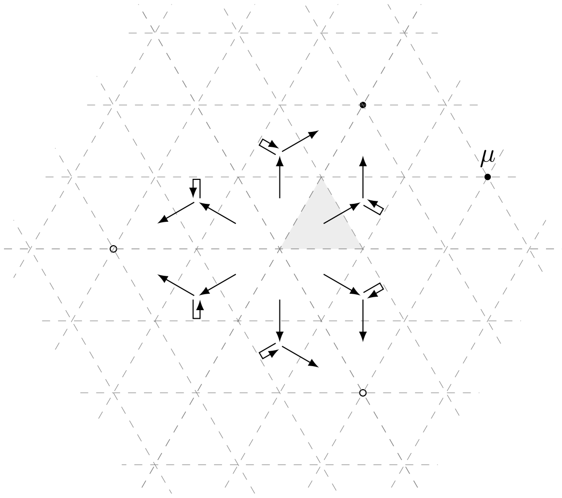
The first term $m_\mu$ corresponds to the following six walks:




$T = \emptyset, \quad \mathrm{wt}(T) = \mu = 2\omega_1 + \omega_2$

$m_{\mathrm{wt}(T)} = \sum_{w \in W} x^{w\mu}$

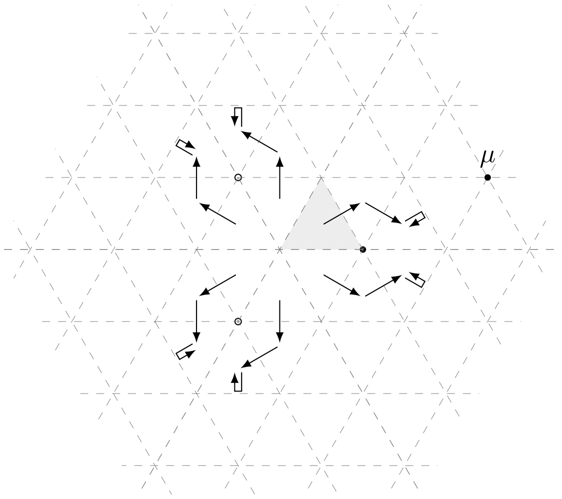
To determine the coefficient $u_{\mu,2\omega_2}$, we consider every walk in $\mathcal{P}(\vec{\mu})$ with weight $2\omega_2$. These are the walks in the following image with inital alcove 1 and $s_1$:




$T = \{2\}, \quad \mathrm{wt}(T) = 2\omega_2$

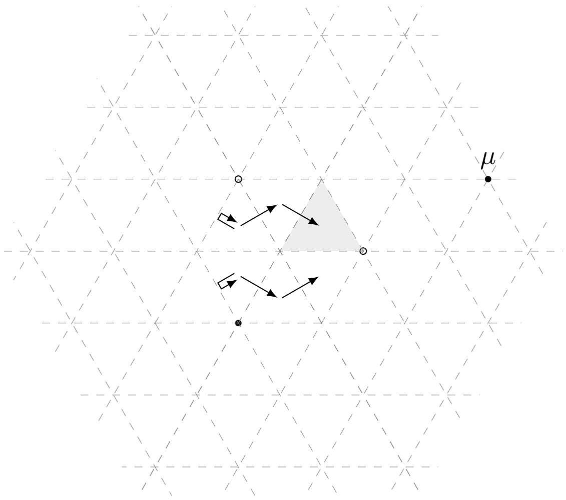
$$u_{\mu,2\omega_2} m_{2\omega_2} = \frac{(1+q)(1-t)}{1-qt} m_{2\omega_2}$$

Any walk in $\mathcal{P}(\vec{\mu})$ with $T \neq \emptyset, \{2\}$ has its weight in $W\omega_1$. Thus, consider the walks with weight $\omega_1$ (these are two per image) and summarize their terms to get $u_{\mu,\omega_1}$.



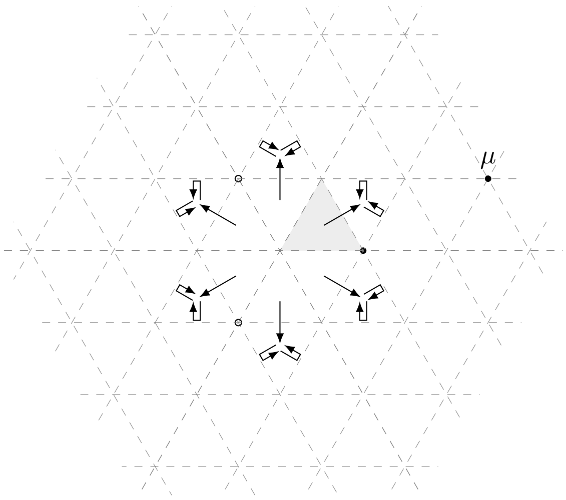

$T = \{3\} = T_1^- = T_{s_1}^- = T_{s_2}^-$,
$\mathrm{wt}(T) = \omega_1$

$$x^{\omega_1} \frac{1-t}{1-q^2t^2}(q^2 + q^2 t)$$



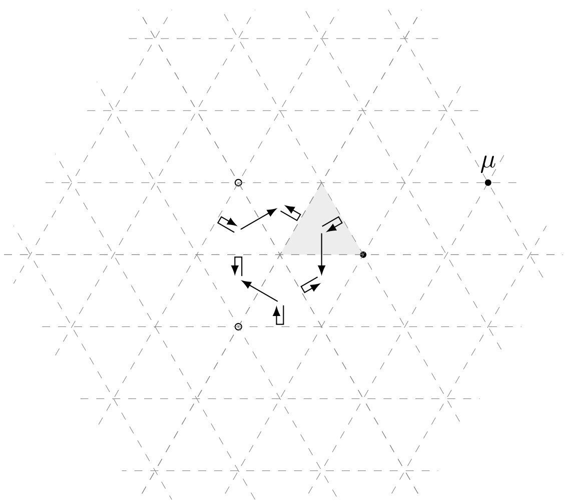

$T = \{1\}, \quad \mathrm{wt}(T) = \omega_2 - \omega_1$

$$x^{\omega_1} \frac{1-t}{(1-qt^2)}(1+t)$$



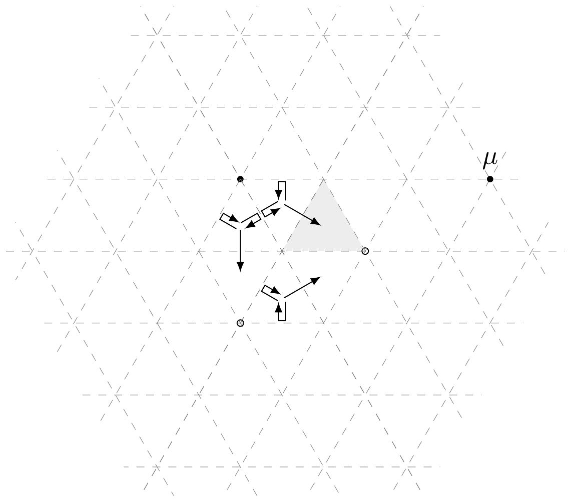

$T = \{2,3\}, \quad \mathrm{wt}(T) = \omega_2 - \omega_1$

$$x^{\omega_1} \frac{(1-t)^2}{(1-qt)(1-q^2t^2)}(q^3t + q)$$



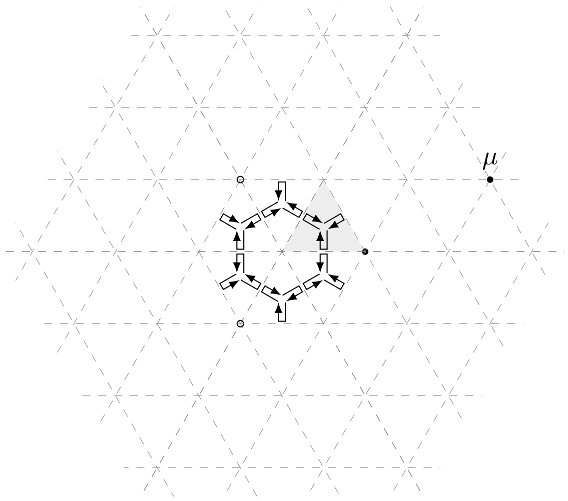

$T = \{1,3\}, \quad \mathrm{wt}(T) = \omega_2 - \omega_1$

$$x^{\omega_1} \frac{(1-t)^2}{(1-qt^2)(1-q^2t^2)}(qt + qt)$$




$T = \{1, 2\}, \quad \mathrm{wt}(T) = \omega_2 - \omega_1$

$$x^{\omega_1} \frac{(1-t)^2}{(1-qt^2)(1-qt)}(qt+1)$$




$T = \{1, 2, 3\}, \quad \mathrm{wt}(T) = \omega_1$

$$x^{\omega_1} \frac{(1-t)^3}{(1-qt^2)(1-qt)(1-q^2t^2)}(q+q^2t^2)$$

The coefficient $u_{\mu,\omega_1}$ is therefore

$$\begin{aligned} u_{\mu,\omega_1} =& \frac{1-t}{1-q^2t^2}(q^2+q^2t) + \frac{1-t}{(1-qt^2)}(1+t) \\ &+ \frac{(1-t)^2}{(1-qt)(1-q^2t^2)}(q^3t+q) + \frac{(1-t)^2}{(1-qt^2)(1-q^2t^2)}(qt+qt) \\ &+ \frac{(1-t)^2}{(1-qt^2)(1-qt)}(qt+1) + \frac{(1-t)^3}{(1-qt^2)(1-qt)(1-q^2t^2)}(q+q^2t^2) \\ =& \frac{2q^3t^3 - 2q^3t^2 + 2q^2t^3 - 3q^2t^2 + q^2 + qt^3 - 3qt + 2q - 2t + 2}{(1-qt)^2(1+qt)} \ . \end{aligned}$$

Hence, the Macdonald polynomial $P_\mu$ for the weight $\mu = 2\omega_1 + \omega_2$ is

$$\begin{aligned} P_\mu =& m_\mu + \frac{(1+q)(1-t)}{1-qt} m_{2\omega_2} \\ &+ \frac{2q^3t^3 - 2q^3t^2 + 2q^2t^3 - 3q^2t^2 + q^2 + qt^3 - 3qt + 2q - 2t + 2}{(1-qt)^2(1+qt)} m_{\omega_1} \ . \end{aligned}$$

As we have seen in Section 5.3, Lenart's formula yields

$$\begin{aligned} P_{(3,1,0)} =& m_{(3,1,0)} + \frac{(1-t)(1+q)}{1-qt} m_{(2,2)} \\ &+ \frac{2q^3t^3 - 2q^3t^2 + 2q^2t^3 - 3q^2t^2 + q^2 + qt^3 - 3qt + 2q - 2t + 2}{(1-qt)^2(1+qt))} m_{(2,1,1)} \\ &+ \frac{(q+1)(3q^2t + q^2 + 2qt + 2q + t + 3)(1-t)^2}{(1-q^2t^2)(1-qt)} m_{(1^4)} \end{aligned}$$

for the corresponding partition $(3,1,0)$ to $\mu$ by Theorem 4.1. By this theorem, or more precisely under the bijection in Theorem 4.6, these two polynomials are indeed the same: The partition $(2,1,1)$ is the same as $(1,0,0)$ in Type $A_2$, so the monomial symmetric functions $m_{(2,1,1)}$ is the orbit sum $m_{\omega_1}$. Analogously, $m_{(2,2,0)}$ is $m_{\omega_2}$. Moreover, we consider the polynomials in $\Lambda_{F,3}$, i.e., $x_k = 0$ for $k \geq 3$. Thus, the monomial symmetric function $m_{(1^4)}$ vanishes in $P_\mu$.

***A*₁: partition (m,0)** Recall the short example for a realization of Type $A_1$. The corresponding weight in Type $A_1$ to the partition $(m,0)$ is $m\omega$. The $m\omega$-path is of type $(0,1)^m$, meaning the sequence of alternating 0's and 1's, with length $2m$ and beginning at 0. Since $\alpha$ is the only root, we get the sequences

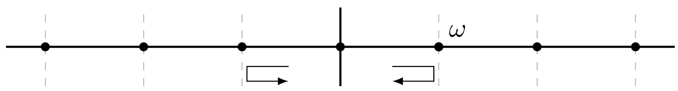
$$\gamma = (\alpha)^m \quad \text{and} \quad j = (1,2,\dots,m).$$

The weight $m\omega - \nu$ for a dominant weight $\nu = \ell\omega$ is in $Q^+$ if and only if $m - \ell$ is even. By definition, the Macdonald polynomial $P_{m\omega}$ is then

$$P_{m\omega} = m_{m\omega} + \begin{cases} \sum_{i=1}^{\frac{m}{2}} u_{m,2i} m_{2i\omega} & \text{if } m \text{ is even,} \\ \sum_{i=1}^{\frac{m-1}{2}} u_{m,m-2i+1} m_{(m-2i+1)\omega} & \text{else,} \end{cases}$$

where $u_{m,\ell} = u_{m\omega,\ell\omega}$ are the unique coefficients in $\mathbb{Q}(q,t)$. In the following paragraphs, we will compute $P_{m\omega}$ for $m = 2,3,4$.

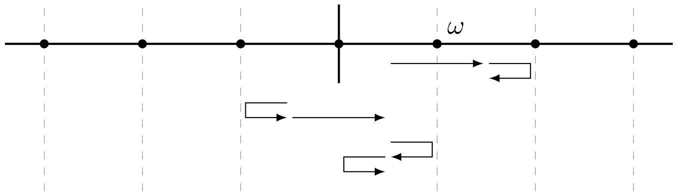
***A*₁: partition (2,0)** By the previous paragraph, we only need to compute the coefficient $u_{2,0}$ of $x^0$. We have six walks, two walks per a set of folds (drawn below the vector space $\mathbb{R}$):




The Macdonald polynomial is therefore

$$P_{2\omega} = m_{2\omega} + \frac{1-t}{1-qt}(q+1)m_0 = m_{2\omega} + \frac{(q+1)(1-t)}{1-qt} .$$

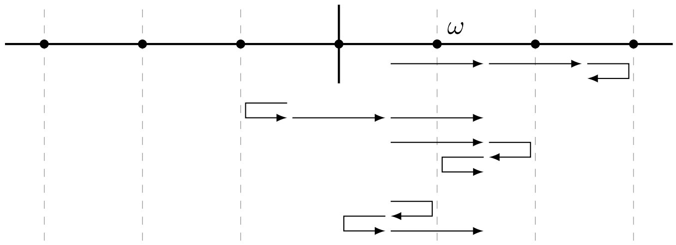
***A*₁: partition (3,0)** By the previous paragraph, we only need to compute the coefficient $u_{2,0}$ of $x^0$. We have six walks, two walks per a set of folds (drawn in the vector space $\mathbb{R}$):




The Macdonald polynomial is

$$\begin{aligned}P_{3\omega} =& m_{3\omega} + \left(\frac{1-t}{1-q^2t}q^2 + \frac{1-t}{1-qt} + \frac{(1-t)^2}{(1-qt)(1-q^2t)}q\right) m_0 \\ =& m_{3\omega} + \frac{(q^2+q+1)(1-t)}{1-q^2t} m_\omega \ .\end{aligned}$$

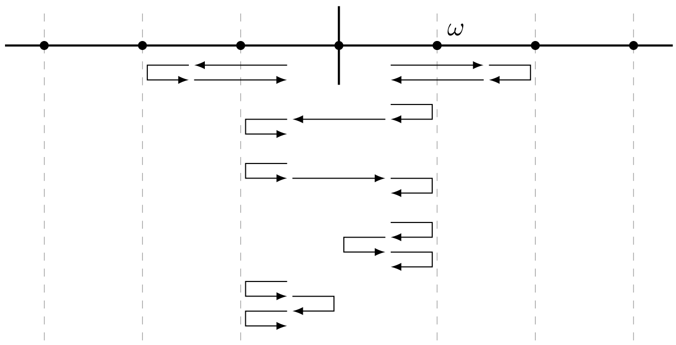
***$A_1$*: partition (4,0)** We have to consider the walks with weight $2\omega$ and 0. We have four walks with weight $2\omega$:




which determine the coefficient

$$\begin{aligned}u_{4\omega,2\omega} =& \frac{1-t}{1-q^3t}q^3 + \frac{1-t}{1-qt} + \frac{(1-t)^2}{(1-q^2t)(1-q^3t)}q^2 + \frac{(1-t)^2}{(1-qt)(1-q^2t)}q \\ =& \frac{(q+1)(q^2+1)(1-t)}{1-q^3t} \ .\end{aligned}$$

And we have six walks with weight 0:




which determine the coefficient

$$\begin{aligned}u_{4\omega,0} =& \frac{1-t}{1-q^2t}\left(q^2+1\right) + \frac{(1-t)^2}{(1-qt)(1-q^3t)}\left(q+q^3\right) + \frac{(1-t)^3}{(1-qt)(1-q^2t)(1-q^3t)}\left(q^4+q^2\right) \\ =& \frac{(q^2+1)(q^2+q+1)(1-t)(1-qt)}{(1-q^2t)(1-q^3t)} \ .\end{aligned}$$

Then, the Macdonald polynomial is

$$P_{4\omega} = m_{4\omega} + \frac{(q+1)(q^2+1)(1-t)}{1-q^3t} m_{2\omega} + \frac{(q^2+1)(q^2+q+1)(1-t)(1-qt)}{(1-q^2t)(1-q^3t)} \ .$$

**$A_2$: partition (m,0,0)** Before we compute the Macdonald polynomials for a concrete partition $(m,0,0)$, we list some general statements. By the definition, the Macdonald polynomial $P_\mu$ to the weight $\mu = m\omega_1$, which corresponds to the partition $(m,0,0)$, is

$$P_\mu = m_\mu + \sum_{\nu<\mu} u_{\mu\nu} m_\nu \ .$$

A weight $\nu$ appears in the sum, if $\nu$ is dominant and $\mu - \nu \in Q^+$, that is

$$\begin{aligned}\nu = \mu - (u_1\alpha_1 + u_2\alpha_2) &= m\omega_1 - u_1(\omega_1 - \omega_2) - u_2(\omega_2 - \omega_1)\\ &= (m - u_1 + u_2)\omega_1 + (u_1 - u_2)\omega_2\end{aligned}$$

for $u_1, u_2$ nonnegative integers such that $m - u_1 + u_2, u_1 - u_2 \geq 0$. So, $u_2 \leq u_1 \leq m + u_2$. The $\mu$-path is of type $(0,1)^m$, that is a sequence of alternating 0's and 1's, with length $2m$ and beginning at 0. Thus, we get the sequences

$$\gamma = (\alpha_1 + \alpha_2, \alpha_2)^m \quad \text{and} \quad j = (1,1,2,2,\ldots,m,m)$$

by their definitions (3.6) and (3.7), respectively. The final direction of the $\mu$-path is $v_\mu = s_1 s_2$. Consider the formula in Corollary 3.3 and reduce the following terms:

$$t^{\frac{1}{2}(\ell(w^{-1}\Psi(T)(v_\mu)) - \ell(\omega_0 w) - |T|)} = t^{\frac{1}{2}(\ell(w) - \ell(w^{-1}\Psi(T)(s_1)) - |T|)}$$

and for any $k \in [2m]$,

$$q^{j_k} t^{\mathrm{ht}(\gamma_k^\vee)} = \begin{cases} q^{\frac{k}{2}} t & \text{if } k \text{ is even,}\\ q^{\frac{k+1}{2}} t^2 & \text{else.}\end{cases}$$

We do not have to compute every coefficient $u_{\mu\nu}$ of the $m_\nu$ for $\nu < \mu$: We already computed some of them regarding the Macdonald polynomial for Type $A_1$. For example, $(k,k)$ is the same partition as $(0,0)$ under the bijection between partitions and weights. For more details, let us consider certain partitions in the next paragraphs.

**$A_2$: partition (2,0,0)** By the definition, the Macdonald polynomial is

$$P_{(2\omega_1)} = m_{2\omega_1} + u_{2\omega_1,\omega_2} m_{\omega_2}.$$

The weight $\omega_2$ corresponds to the partition $(1,1,0) = (1,1)$ which corresponds to the weight 0 in Type $A_1$. So we get the same representation as for $P_{2\omega}$ in Type $A_1$.

**$A_2$: partition (3,0,0)** By the definition, the Macdonald polynomial is

$$P_{(3\omega_1)} = m_{3\omega_1} + u_{3\omega_1,\omega_1+\omega_2} m_{\omega_1+\omega_2} + u_{3\omega_1,0} m_0.$$

The weight $\omega_1 + \omega_2$ corresponds to the partition $(2,1,0) = (2,1)$, which corresponds to the weight $\omega$ in Type $A_1$. Then, the coefficient $u_{3\omega_1,\omega_1+\omega_2}$ (Type $A_2$) is the same as $u_{3\omega,\omega}$ (Type $A_1$), so

$$u_{3\omega_1,\omega_1+\omega_2} = \frac{(q^2+q+1)(1-t)}{1-q^2 t} \ .$$

It lasts to consider 30 walks with weight 0. Since the weight is not regular, we have to normalize their terms multiplying with $t^{-1/2}(1+t)$.

Depending on the set of folds $T$, they generate the following terms:

| $T$ | term |
|---|---|
| $\{1,4\}$ | $\frac{(1-t)^2(1+t)}{(1-qt^2)(1-q^2t)}\left(q^3t^2+q^2t+qt^2+qt+q^2+1\right)$ |
| $\{2,3\}$ | $\frac{(1-t)^2(1+t)}{(1-qt)(1-q^2t^2)}\left(q^3t+q^2t^2+q^2t+qt+q+t\right)$ |
| $\{1,2,4\}$ | $\frac{(1-t)^3(1+t)}{(1-qt^2)(1-qt)(1-q^2t)}\left(2q^3t+q^2t^2+q^2+2qt\right)$ |
| $\{1,3,4\}$ | $\frac{(1-t)^3(1+t)}{(1-qt^2)(1-q^2t^2)(1-q^2t)}\left(2q^3t+q^4t^2+2q^2t^2+qt\right)$ |
| $\{1,2,3,4\}$ | $\frac{(1-t)^4(1+t)}{(1-qt^2)(1-qt)(1-q^2t^2)(1-q^2t)}\left(q^5t^3+q^4t^2+q^3t^3+q^3+q^2t+q\right)$ |

Thus, we have

$$u_{3\omega_1,0}=\frac{(q+1)(q^2+q+1)(1-t)^2}{(1-qt)(1-q^2t)}$$

and the Macdonald polynomial in Type $A_2$ to the weight $3\omega_1$ is

$$P_{3\omega}=m_{3\omega_1}\frac{(q^2+q+1)(1-t)}{1-q^2t}+m_{\omega_1+\omega_2}\frac{(q+1)(q^2+q+1)(1-t)^2}{(1-qt)(1-q^2t)}\ .$$

**$A_2$: partition (4,0,0)** By the definition, the Macdonald polynomial is

$$P_{(4\omega_1)}=m_{4\omega_1}+u_{4\omega_1,2\omega_1+\omega_2}m_{2\omega_1+\omega_2}+u_{4\omega_1,\omega_1}m_{\omega_1}+u_{4\omega_1,0}m_0.$$

The weight $2\omega_1+\omega_2$ corresponds to the partition $(3,1,0)=(3,1)$, which corresponds to the weight $2\omega$ in Type $A_1$. Then, the coefficient $u_{4\omega_1,2\omega_1+\omega_2}$ (Type $A_2$) is the same as $u_{4\omega,\omega}$ (Type $A_1$), so

$$u_{4\omega_1,2\omega_1+\omega_2}=\frac{(q+1)(q^2+1)(1-t)}{1-q^3t}.$$

The weight $2\omega_2$ corresponds to the partition $(2,2,0)=(2,2)$, which corresponds to the weight 0 in Type $A_1$. Thus, the coefficient $u_{4\omega_1,2\omega_2}$ (Type $A_2$) is the same as $u_{4\omega,0}$ (Type $A_1$), so

$$u_{4\omega_1,2\omega_2}=\frac{(q^2+1)(q^2+q+1)(1-t)(1-qt)}{(1-q^2t)(1-q^3t)}\ .$$

It lasts to consider the walks with weight $\omega_1$. They will generate the coefficient

$$u_{4\omega_1,\omega_1}=\frac{(q+1)(q^2+1)(q^2+q+1)(t-1)^2}{(1-q^2t)(1-q^3t)}\ .$$

Then, in Type $A_2$, the Macdonald polynomial to the weight $4\omega_1$ is

$$\begin{aligned}P_{4\omega_1}=&m_{4\omega_1}+\frac{(q+1)(q^2+1)(1-t)}{1-q^3t}m_{2\omega_1+\omega_2}\\&+\frac{(q^2+1)(q^2+q+1)(1-t)(1-qt)}{(1-q^2t)(1-q^3t)}m_{2\omega_2}\\&+\frac{(q+1)(q^2+1)(q^2+q+1)(t-1)^2}{(1-q^2t)(1-q^3t)}m_{\omega_1}\ .\end{aligned}$$

# References

[1] Stéphane Gaussent and Peter Littelmann. *LS-Galleries, the path model and MV-cycles.* 2003. DOI: 10.48550/ARXIV.MATH/0307122. URL: https://arxiv.org/abs/math/0307122.

[2] Mark Haiman. "Hilbert schemes, polygraphs, and the Macdonald positivity conjecture". In: (2000). DOI: 10.48550/ARXIV.MATH/0010246. URL: https://arxiv.org/abs/math/0010246.

[3] James E. Humphreys. *Reflection Groups and Coxeter Groups.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1990. DOI: 10.1017/CBO9780511623646.

[4] Cristian Lenart. *Hall-Littlewood Polynomials, Alcove Walks, and Fillings of Young Diagrams.* 2010. DOI: 10.48550/ARXIV.0804.4715v3. URL: https://arxiv.org/abs/0804.4715v3.

[5] Cristian Lenart. *On Combinatorial Formulas for Macdonald Polynomials.* 2008. DOI: 10.48550/ARXIV.0804.4716. URL: https://arxiv.org/abs/0804.4716.

[6] Cristian Lenart and Alexander Postnikov. *A Combinatorial Model for Crystals of Kac-Moody Algebras.* 2005. DOI: 10.48550/ARXIV.MATH/0502147. URL: https://arxiv.org/abs/math/0502147.

[7] Ian G. Macdonald. "Orthogonal polynomials associated with root systems". In: (2000). DOI: 10.48550/ARXIV.MATH/0011046. URL: https://arxiv.org/abs/math/0011046.

[8] Ian G. Macdonald. *Symmetric functions and orthogonal polynomials.* University lecture series, v. 12. Providence, R.I: American Mathematical Society, 1998. ISBN: 0821807706.

[9] Arun Ram. *Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux.* 2006. DOI: 10.48550/ARXIV.MATH/0601343. URL: https://arxiv.org/abs/math/0601343.

[10] Arun Ram and Martha Yip. *A combinatorial formula for Macdonald polynomials.* 2008. DOI: 10.48550/ARXIV.0803.1146. URL: https://arxiv.org/abs/0803.1146.

[11] Bruce E. Sagan. *The symmetric group - representations, combinatorial algorithms, and symmetric functions.* Wadsworth & Brooks / Cole mathematics series. Wadsworth, 1991, pp. I–XV, 1–197. ISBN: 978-0-534-15540-7.

[12] C. Schwer. "Galleries, Hall-Littlewood polynomials, and structure constants of the spherical Hecke algebra". In: *International Mathematics Research Notices* (2006). DOI: 10.1155/imrn/2006/75395. URL: https://doi.org/10.1155%2Fimrn%2F2006%2F75395.