The Ising model in a random field

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February 22, 2008

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The Ising model in a random field is a prominent example of a lattice spin model with quenched disorder. The question whether and when quenched disorder destroys the Ising phase transition at low temperatures was under dispute among theoretical physicists for quite some time [?, ?]. It was finally resolved by two papers in mathematical physics proving ferromagnetic ordering in 3 or more dimensions at low temperature [?], and absence of ferromagnetic ordering in 2 dimensions at any temperature [?].

The joint measures of the model on the product of spin-space and the space of the disorder variables violate the Gibbsian variational principle and provide an example of the breakdown of classical Gibbsian theory when one has only a weakly Gibbs representation [?].

Moreover the random field Ising model on the lattice and its exactly solvable mean-field counterpart play a role as an important case study of a disordered system in the study of chaotic volume dependence (metastates), metastability, and stochastic time-evolutions of Gibbsian measures and their Gibbsian properties.

The model. The model is defined in terms of the formal Hamiltonian that associates to an infinite-volume spin configuration $\sigma = (\sigma_i)_{i\in\mathbb{Z}^d}$ with $\sigma_i \in \{-1, 1\}$ the formal energy

$$H^\eta(\sigma) = -\sum_{\langle i,j\rangle} \sigma_i\sigma_j - h\sum_i \eta_i\sigma_i$$

Here the first sum is over pairs of nearest neighbors on the lattice. The minus sign provides for a ferromagnetic coupling making the spins prefer to be aligned in the same direction to lower the energy, like in the usual Ising model where the second term is absent. The parameter h denotes the strength of the coupling to the random fields $\eta = (\eta_i)_{i\in\mathbb{Z}^d}$. A configuration of such random fields is drawn according to a symmetric, sitewise independent probability distribution $\mathbb{P}$ and is kept fixed ("quenched") in the course of the analysis. Main examples of this are the standard Gaussian distribution and the symmetric Bernoulli distribution where the values plus and minus one are taken with the same probability one half.

As usual in quenched systems one is interested in properties that hold for *typical* realizations η, that is for properties that hold for a set of realizations of $\mathbb{P}$-measure one.

Following the standard definition in mathematical equilibrium statistical mechanics, the *quenched Gibbs measures* $\mu[\eta]$ are those measures on infinite-volume spin configurations σ that, for a fixed random field configuration η, at inverse temperature $\beta \geq 0$ are solutions to the Dobrushin-Lanford-Ruelle (DLR-)equations.

This means that they are those measures $\mu[\eta]$ whose conditional measures (obtained by conditioning to a configuration $\bar{\sigma}$ outside of a finite volume $\Lambda \subset \mathbb{Z}^d$) are given by the Boltzmann weights computed from the restriction of the Hamiltonian to Λ, including the couplings to the boundary condition $\bar{\sigma}$.

The fundamental question is to understand the nature of these measures, for a typical realization η, and whether there is more than one of them, in which case there is a phase transition.

Ordering at low temperatures, small disorder, in $d \geq 3$. [**?**] showed that, in the three- or more-dimensional random field Ising model, for a symmetric random field distribution, small coupling h, and large inverse temperature, disorder does not destroy the ferromagnetic ordering of the regular Ising model. There exist distinguished Gibbs measures $\mu^+[\eta]$ (and $\mu^-[\eta]$) obtained as weak limits with + (and −)-spin boundary conditions. The authors of [**?**] showed that the following picture holds: $\mu^+[\eta]$, for a fixed typical magnetic field configuration η, has typical spin configurations σ that look like small perturbations around a plus-like (respectively a minus-like) infinite-volume ground state. At fixed η, a plus-like ground state itself looks like a sea of pluses with rare islands of minuses in those regions of space where the realizations of the magnetic fields happen to be mostly oriented to favor the minus spins.

The contour renormalization group. Here is the idea of the proof of [**?**], for a pedagogic exposition see also [**?**, **?**]. Given a spin configuration, contours are the boundaries of regions on the lattice where the spins have the same sign. A spin configuration of the model can be translated into a contour description. A set of such contours has a probability weight that is directly inherited from the random field Hamiltonian. The situation would be simple and a phase transition could be proved at low enough temperatures if we had a model where there is an energetic cost for the formation of an additional single contour that is of the order of the length of the contour (Peierls estimate). The situation in the random model is difficult precisely because such an estimate is not uniformly true; in fact for some configurations of random fields the introduction of a contour will lower the energy in comparison to the (say) all plus state. This is to say that already the ground state will contain contours. The situation is hopeful nevertheless, because in $d \geq 3$ a failure of a Peierls estimate in a given contour is unlikely (Imry-Ma argument). It is still very non-trivial because there are too many contours to add up corresponding probabilities (entropy of contours).

The method used to overcome this difficulty is a renormalization group in contour space. It is a suitably defined map between an initial space of contours to an image space of coarser contours (containing less information about the initial spin configuration). A spatial rescaling takes place to be able to interpret the map as acting between the same initial and final contour space. This map now will be iterated and gives rise to a flow on probability distributions on these contour spaces, with discrete "time"-variable given by the level of the iteration. The control of the flow of these probability distributions is technically difficult. It turns out however that the relevant features of these probability distributions can be described by an effective (or renormalized) temperature and an effective (or renormalized) field. Both of these variables converge to zero (in distribution) under the iteration of the renormalization map in spatial dimensions $d \geq 3$. This shows that they become irrelevant on large length scales. Hence the model shows ferromagnetic ordering.

Non-rigorous manipulations of perturbative expansions (the dimensional reduction of Parisi and Sourlas) in the theoretical physics literature had previously led to a wrong prediction of the low-temperature behavior. Opposed to that, the contour renormalization group method is a rigorous solution to the entropy of contours-problem based on the ideology to construct self-maps. This ideology, however, is well-inspired by non-rigorous methods of field theory as applied to the study of criticial phenomena. Extensions of the contour-renormalization group method for the random field Ising model have been used in the study of random interfaces in a random environment [?], the long-range random field Ising model with Kac-potential, and for the $\mathbb{R}$-valued spin model (φ^4-theory) in the presence of external random fields [?].

Non-existence of ferromagnetic order in $d = 2$. It was proved in [?] that there is uniqueness of the Gibbs measure in 2 dimensions, at any fixed temperature, for P-a.e. η. It shows that randomness can potentially alter the behavior of the system in a fundamental way, and cannot always be treated as a small perturbation. The method of [?] is based on getting lower estimates on the fluctuations w.r.t. the distribution $\mathbb{P}$ of differences of free energies in finite volume taken with different boundary conditions. This method uses a martingale convergence result from probability theory to show the persistence of fluctuations w.r.t. the underlying randomness of free energies obtained with different boundary conditions, under the assumption that the state $\mu^+(\eta)$ is different from $\mu^-(\eta)$. This leads to a contradiction to the a priori bounds on such fluctuations which one obtains from varying the boundary condition. So it implies the impossibility of having different phases.

Joint measures and failure of variational principle. The joint measures of the random field Ising model (whose study is advocated by the so-called Morita-approach in theoretical physics) provide an example of the subtleties that may occur in infinite-volume lattice spin measures. While the joint measures are built of Gibbsian measures they are not Gibbsian themselves, and very strongly not so. To define them, we define first the joint variable ξ_i at lattice site i to be the pair of spin variable σ_i and random field variable η_i. In the case of the plus-minus random field this leaves us with 4 possible values. Call the measure on the infinite-volume space of joint variables $K^+(d\xi) = \mathbb{P}(d\eta)\mu^+(\eta)(d\sigma)$. We put ourselves again at low temperature in 3 dimensions. Then the following properties hold.

1) The conditional probability under K^+ to see a configuration ξ_0 at the origin given a joint configuration outside of the origin is different from that for K^-.

2) The relative entropy density (or information gain per site) between K^+ and the measure K^- vanishes (where K^- is correspondingly defined with minus boundary conditions.)

These two properties are a violation to the classical variational principle which states that the relative entropy density between two proper Gibbs measures (with uniformly absolutely summable Hamiltonian) only vanishes if they have the same conditional probabilities. More can be said about the conditional probabilities of the joint measures.

3) They have a representation in terms of an exponential of a Hamiltonian as an infinite sum of local terms indexed by subsets of $\mathbb{Z}^d$ and this sum converges exponentially fast when one restricts to a set of full joint measure (weakly Gibbsian representation). It is impossible to extend it such that it converges absolutely uniformly for *all* joint configurations.

It is important to mention this property since it shows that a weakly Gibbsian representation is in general not good enough to insure the validity of Gibbsian theory, as was conjectured before. In fact, the joint measures are non-Gibbs in a strong sense (almost surely non-Gibbs) which means that:

4) The set of continuity points (in the product topology) has joint measure zero. This happens although the set of points where the Hamiltonian converges has joint measure one.

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