Metastates in random spin models

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Consider a lattice spin model with a quenched random Hamiltonian, such as the Edwards-Anderson spinglass or a random field Ising model. The metastate is a concept to capture the asymptotic volume-dependence of the Gibbs states of such a system in the phase transition regime when several Gibbs states are available. It is a probability distribution on the infinite-volume Gibbs states of the system that, intuitively speaking, describes the likelihood of finding a disordered system in a particular Gibbs measure when one choses a large volume from a sequence of volumes at random. The metastate of a system may be non-degenerate for boundary conditions which do not preselect the Gibbs state that is (approximately) realized in a large volume, so that several Gibbs measures may arise as subsequence limits.

Disordered systems. To start with, suppose we are given an Ising system (for simplicity) on the lattice $\mathbb{Z}^d$ with quenched randomness. We denote the variable describing the quenched randomness by η. It will be chosen according to a probability distribution denoted by $\mathbb{P}(d\eta)$ and kept fixed (or "quenched") in the course of the analysis. A disordered spin model is usually defined by prescribing a Hamiltonian $H^\eta(\sigma)$, for each realization of η, which associates to an infinite volume spin configuration $\sigma = (\sigma_i)_{i\in\mathbb{Z}^d}$ (with σ_i taking values plus or minus one) a formal energy. (Such a Hamiltonian will typically be given in terms of an interaction potential Φ^η which collects all interaction terms in the Hamiltonian and should be considered the more basic object from a mathematical point of view.)

Fixing a boundary condition $\bar\sigma$, one may now define the *finite-volume Gibbs states* $\mu_\Lambda^{\bar\sigma}[\eta](d\sigma)$ in the finite volume $\Lambda \subset \mathbb{Z}^d$ in the usual way, namely by restricting the terms of the Hamiltonian to the volume Λ, including the couplings over the boundary to the boundary condition. These measures are conveniently interpreted as probability measures on the whole infinite-volume configurations of the system, where the configuration is fixed to be $\bar\sigma$ outside of Λ. The collection of these measures, for all volumes Λ and all boundary conditions $\bar\sigma$ forms a *(random) Gibbs specification.* Generally speaking, measures on infinite volume spin configurations σ are also referred to as the *states*.

As it is a fundamental task in statistical mechanics to describe the large-volume behavior of a system, we are interested in describing these finite-volume Gibbs measures along a sequence of cubes Λ_n, centered at the origin, of sidelength $2n+1$, as n tends to infinity. While it is common for *translation-*

invariant systems to have convergence of the finite-volume states themselves when a particular boundary condition is fixed, for *disordered systems* the situation may be more complicated and convergence on the levels of states does not hold. Indeed, when several Gibbs measures are available, it might be that a system finds itself (approximately) in one of these states for one volume, and (approximately) in another Gibbs measure for another volume, which one being dependent on the choice of the underlying randomness in the Hamiltonian H^η.

The Newman-Stein metastate. To account for disordered systems where the boundary condition does *not* preselect in an obvious way the Gibbs measure Newman and Stein proposed the following to capture the asymptotic volume dependence. In all of the following the boundary condition $\bar{\sigma}$ will be fixed and dropped in the notation.

Look at a sequence of finite-volume Gibbs measures $\mu_n[\eta]$ in the volumes Λ_n. Look at the empiral average

$$\kappa_N[\eta] := \frac{1}{N}\sum_{n=1}^{N}\delta_{\mu_n[\eta]}$$

taken along the trajectory $\mu_n[\eta]$, where δ is the Dirac measure. Intuitively it means to look at the frequency of occurrence of states along a volume sequence ("histogramm"). The limit

$$\kappa[\eta] := \lim_{N\uparrow\infty}\kappa_N[\eta]$$

is called an *Newman-Stein metastate* or *empirical metastate*, if it exists for $\mathbb{P}$-almost every realization η. It is a probability measure on the Gibbs measures of the system that depends on the particular realization of the disorder variables η.

To make sense of such a convergence, appropriate notions of convergence (topologies) have to be chosen. In accordance with the Gibbs theory for non-random systems this is done such that convergence has to be checked locally, be it on the level of spin configurations, on the level of states (measures on spin configurations), or on the level of metastates (measures on states).

This empirical mean along a trajectory of volumes is in analogy to the construction of invariant measures for dynamical systems. Now the role of the time is taken by the label n in a given sequence of volumes Λ_n, and the role of the state variable of the dynamical system is taken by the probability measures $\mu_n[\eta]$.

There are general existence results about the convergence for $\mathbb{P}$-a.e. η that follow from compactness arguments but these are only for sparse enough subsequences of n's and N's. Naturally, one would like to speak of *the* Newman-Stein metastate, but it has not been proved in general that different subsequences necessary yield the same object.

The Aizenman-Wehr (or conditional) metastate. Aizenman and Wehr [1] provided a different way to an η-dependent probability measure on the Gibbs states of a system that describes the large-volume asymptotics. They suggested to look at the probability distribution of the pair of the finite-volume Gibbs measure and the disorder variable $(\mu_n[\eta], \eta)$ under the governing measure of the disorder variable, $\mathbb{P}(d\eta)$. Suppose that a limit exists for this random pair in the sense of weak convergence. Let us call the resulting limiting distribution $K(d\mu, d\eta)$. Of this limit we may take now a conditional distribution, obtained by conditioning on the disorder variable η, and this provides us with a measure on the first variable, which we call $\kappa^{\mathrm{AW}}[\eta](d\mu)$. The resulting object is called *Aizenman-Wehr* or *conditional metastate*.

Again, the existence of the limit K is guaranteed (only) for subsequences of n's (by a compactness argument). The independence of the limit of the choice of the subsequence has not been proved in general, but it is very plausible in all examples that have been studied.

The connection between both notions. In [8] it was proved that the Aizeman-Wehr metastate coincides with the Newman-Stein metastate for sufficiently sparse subsequences. More precisely the following holds: Take sufficiently sparse sequences n_k, $k = 1, 2, \dots$ of the subsequence of n's in the construction of the Aizenman-Wehr metastate $\kappa^{\mathrm{AW}}[\eta]$. Then, for sufficiently sparse subsequences N_l, the corresponding Newman-Stein metastate converges to the Aizenman-Wehr metastate. This reads in formulas:

$$\lim_{l \uparrow \infty} \frac{1}{N_l} \sum_{k=1}^{N_l} \delta_{\mu_{n_k}[\eta]} = \kappa^{\mathrm{AW}}[\eta]$$

An example. Subsequences of n's are really necessary to have a.s. convergence, as it was observed in [5] for the *Mean-Field Random Field Ising Model.* It is probably the simplest system showing nontrivial behavior of the metastate. For more examples, see [2, 3] and the references therein. The *finite-volume Gibbs measures* $\mu_n[\eta]$ in the finite volume $\{1, \dots, n\}$ are given by

$$\mu_n[\eta]\left((\sigma_i)_{i=1,\dots,n}\right) = \frac{1}{Z_n[\eta]} \exp\left(\frac{\beta}{2n}\sum_{1\le i,j\le n}\sigma_i\sigma_j + \beta\varepsilon\sum_{1\le i\le n}\eta_i\sigma_i\right)$$

where $\sigma_i = \pm 1$ are Ising spins, η_i are taken as i.i.d. variables taking the variables plus or minus one with equal probability one half, and $Z_n[\eta]$ is the disorder-dependent partition function that makes the r.h.s. a probability measure on the σ's. Interactions between the spins take place between all pairs which makes the model a mean-field model. The phase diagram of the model is well known. In particular, at low temperatures $1/\beta$ and small ε the model is ferromagnetic, i.e. there exist two 'pure' phases, a ferromagnetic $+$ phase $\mu_\infty^+[\eta]$ and a $-$ phase $\mu_\infty^-[\eta]$. Now the metastate will be non-degenerate, since in large volumes, with large probability, the system will be approximately in one of these phases, which one, depending on the realization η through the sign of the sum of random fields on the volume. Moreover, it was shown in [5] that the empirical metastate does *not* converge for a.e. realization, if one considers the sequence of volumes $\{1,\dots,n\}$ obtained by adding one site at a time step. However it *does converge in distribution* (probability law): Looking at its expectation of a local function F on the states of the system we have

$$\lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=1}^{N}F(\mu_n[\eta]) =^{\text{law}} n_\infty F\left(\mu_\infty^+[\eta]\right) + (1-n_\infty)F\left(\mu_\infty^-[\eta]\right)$$

where n_∞ is a 'fresh' random variable, independent of η on the r.h.s., with arcsine-distribution (that is $\mathbb{P}\left[n_\infty < x\right] = \frac{2}{\pi}\arcsin\sqrt{x}$). On the other hand, if one takes a deterministic volume sequence n_k that is sufficiently sparse, convergence of the l.h.s. takes place to the limit $\frac{1}{2}F\left(\mu_\infty^+[\eta]\right) + \frac{1}{2}F\left(\mu_\infty^-[\eta]\right)$, for almost every realization of η. This expresses the picture that, choosing a very large volume at random, we see either the plus-state or the minus state with probability one half.

The EA spinglass. The metastate playes an important role in the debate between Newman and Stein [9] and the Parisi school [6] whether a (suitably interpreted) so-called Parisi-"replica-symmetry breaking" picture can hold for the Edwards-Anderson lattice-spinglass model (see also [4, 7]). Such a picture would involve the existence of many pure states that are organized in a tree-like manner. On the basis of soft (non-computational) arguments based on translation-ergodicity as a main property Newman and Stein collected strong evidence that a naively interpreted Parisi-picture cannot hold. Considering the ground states of the model they argued as a most plausible behavior for a "chaotic-pair-picture". This means that, in a given volume only one pair of groundstates should be visible, which one however, depending on the volume.

References

[1] M. Aizenman, J. Wehr, *Rounding effects of quenched randomness on first-order phase transitions*, Comm. Math. Phys. **130** no. 3, 489-528 (1990)

[2] A. Bovier, *Statistical mechanics of disordered systems. A mathematical perspective.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge (2006)

[3] A. van Enter, K. Netočný, H.G. Schaap, *Incoherent boundary conditions and metastates.* In: Dynamics & stochastics, 144–153, IMS Lecture Notes Monogr. Ser. **48** Inst. Math. Statist., Beachwood, OH (2006)

[4] D.S. Fisher, D. A. Huse, *Absence of many states in realistic spin glasses.* J. Phys. A **20**, no. 15, L1005–L1010 (1987)

[5] C. Külske, *Metastates in Disordered Mean-Field Models: Random Field and Hopfield Models*, J. Stat. Phys. **88**, 1257-1293 (1997)

[6] M. Mézard, G. Parisi, M. A. Virasoro, *Spin glass theory and beyond*, World Scientific Lecture Notes in Physics, 9. World Scientific Publishing Co., Inc., Teaneck, NJ (1987)

[7] C. M. Newman, D. L. Stein, *Are there incongruent ground states in 2D Edwards-Anderson spin glasses? Dedicated to Joel L. Lebowitz.* Comm. Math. Phys. **224** no. 1, 205–218 (2001)

[8] C. Newman, *Topics in disordered systems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel (1997)

[9] C. Newman, D. L. Stein, *The state(s) of replica symmetry breaking: mean field theories vs. short-ranged spin glasses.* J. Statist. Phys. **106** no. 1-2, 213–244 (2002)