Dependent Particle Deposition on a Graph: Concentration Properties of the Height Profile

S.R. Fleurke[1] **, M. Formentin**[2] **and C. Külske**[2]

[1] Radiocommunications Agency Netherlands, Postbus 450, 9700 AL Groningen,
  The Netherlands. E-mail: sjoert.fleurke@at-ez.nl
[2] Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstrasse 150, 44780
  Bochum, Germany. E-mail: Marco.Formentin@rub.de, Christof.Kuelske@rub.de

Abstract. We present classes of models in which particles are dropped on an arbitrary fixed finite connected graph, obeying adhesion rules with screening. We prove that there is an invariant distribution for the resulting height profile, and Gaussian concentration for functions depending on the paths of the profiles. As a corollary we obtain a law of large numbers for the maximum height. This describes the asymptotic speed with which the maximal height increases.

The results incorporate the case of independent particle droppings but extend to droppings according to a driving Markov chain, and to droppings with possible deposition below the top layer up to a fixed finite depth, obeying a non-nullness condition for the screening rule. The proof is based on an analysis of the Markov chain on height-profiles using coupling methods. We construct a finite communicating set of configurations of profiles to which the chain keeps returning.

1. Introduction

Stochastic models for particle deposition have enjoyed much interest over the years, motivated by applications ranging from car parking, physical chemistry

to frequency assignment [2, 5, 6, 10, 17, 20, 23, 24]. In a series of papers particle deposition models with a number of different deposition rules were considered and exact solutions for models in solvable geometries were given [1, 8, 11, 12, 14, 19, 22]. Natural probabilistic questions to be studied in cases where no closed solutions are available are limit laws for such processes in space [18, 21] or time. In particular one would like to have a law of large numbers for the maximum of the height variables and the behavior of the active or top region. Moreover there is a branch in probability which is interested in the investigation in concentration of measure properties for Markov chains and multidimensional stochastic processes [3, 4, 15, 16], and we also want to look at deposition models in this spirit.

In the present paper we consider models of discrete-time Markov chains describing the growth of adsorbed particles on a substrate. In our main example particles are dropped on the vertices of a finite connected graph V according to a discrete time Markov chain and obeying screening rules of adsorption. The particles pile up to integer heights according to an exclusion interaction between sites which are connected in V. Our last example softens the screening rule to allow adsorption below the top layer.

We prove a strong law of large numbers for the maximal height and show convergence of the height profile to a stationary state. As the number of deposited particles grows linearly in time when we keep the graph (and hence the volume) fixed, we will look at the heights differences relative to the maximum. This map from height configurations to relative heights is just the same as the map from interface configurations to gradient configurations considered in models of interfaces in a Gibbs state [9, 13, 25] when issues of stability of interfaces in the large volume limit are considered.

Now, in our situation we show the convergence of the height-profile as seen from the maximum to an invariant distribution using a coupling method. Our Markov chain has an unbounded state space, but the coupling turns out to be very good, namely we are able to show that the distribution of the coupling time can be controlled uniformly in the initial configurations. The physical reason for this is the following: however rough a profile is, there is always a chain of particle droppings which will make it flat and thereby erase the memory on the past. An essential ingredient for this to turn into a proof in the context of the general models we consider, is the construction of a finite set of profiles the chain communicates to in a time s which is uniform in any starting configuration. The construction of this set is slightly subtle in the case of a non-i.i.d. chain of particle droppings where it is based on irreducibility and laziness of the driving chain. In particular, from this coupling the law of large numbers for the maximal height follows as a corollary from concentration results for path observables.

2. The models and the main results

2.1. Independent particle droppings

Let $G = (V, E)$ be a finite connected graph. Write $i \sim j$ if $\{i, j\} \in E$, that is i, j are adjacent. Consider the Markov chain on the state space $\Omega := \mathbb{N}_0^V$ of height configurations $h = (h_j)_{j \in V}$ obtained by choosing a site $i \in V$ according to a probability $p(i) > 0$, where $p \in \mathcal{P}(V)$ is fixed, and adding a particle at i at height $\max\{h_j, \operatorname{dist}(j, i) \leq 1\} + 1$ where h_j is the maximum height at which a particle is already present at site j.

The formal definition is as follows. Denote by $T_i : \Omega \to \Omega$ the operator which assigns to a configuration h the configuration $T_i h$ which is the configuration obtained by adding a particle at i, i.e.

$$(T_i h)_j = \begin{cases} \max\{h_k : \operatorname{dist}(k, i) \leq 1\} + 1, & \text{if } j = i, \\ h_j & \text{else.} \end{cases} \tag{2.1}$$

Look at the discrete time Markov chain with transition matrix $(M(h, h'))_{h, h' \in \Omega}$ given by

$$M(h, h') = \begin{cases} p(i), & \text{if } h' = T_i h, \\ 0 & \text{else.} \end{cases} \tag{2.2}$$

We denote the value of the configuration at time t by $h(t) = (h_i(t))_{i \in V}$.

The model has the following property: If $h' \in \Omega$ is such that $h'_j = h_j + c$ for all $j \in V$ we have that

$$(T_i h')_j = (T_i h)_j + c$$

and hence we can define the action of T_i also on equivalence classes of height-profiles w.r.t. constant shifts c. Let us extend the local state space to $\mathbb{Z}$ and allow for arbitrary $c \in \mathbb{Z}$. We may choose then a representative of these equivalence classes in such a way that the height profile is zero at the maximum and negative elsewhere. That is, we introduce the variable $x_i = h_i - \max_{j \in V} h_j$. This is the height profile seen from the maximum.

According to the exclusion rules the process on $x = (x_i)_{i \in V}$ is a Markov chain again, now with state space $S := (-\mathbb{N}_0)^V$ and transition matrix $M(x, x') = M(h, h')$ when x is the equivalence class of h and x' is the equivalence class of h'. We will show convergence to an invariant distribution of this Markov chain. In order to do this we need to prove recurrence, and therefore we need to make use of the exclusion rules. To compare, consider the process in which particles are added without exclusion. Then the distribution of the heights becomes multinomial and the corresponding x-distribution won't stabilize but have fluctuations of the order of the square-root of the discrete time n.

2.2. Markov chain particle droppings

Now the probability where to drop the next particle depends on where the last time a particle has fallen.

Let $v(t)$ denote a Markov chain with state space V and transition matrix

$$A_{v,v'} = \mathbb{P}\big(v(t+1) = v' \mid v(t) = v\big).$$

We call this the driving Markov chain.

We assume that $A = (A(v,v'))_{v,v'\in V}$ is *irreducible* (meaning that for all $v \neq v'$ there exists a time $s(v,v')$ such that $A^{s(v,v')}(v,v') > 0$) and that it is *lazy* (meaning that $A(v,v) > 0$ for all $v \in V$).

This time look at the Markov chain $(h(t), v(t))$ with transition matrix

$$\big(M(h,v;h',v')\big)_{h,v;h',v'\in\Omega\times V}$$

given by

$$M(h,v;h',v') = \begin{cases} A(v,v'), & \text{if } h' = T_{v'}h, \\ 0 & \text{else.} \end{cases} \tag{2.3}$$

We denote the value of the configuration at time t by $(h(t), v(t))$.

2.3. Main results

Our main goal will be the following theorem which provides a concentration estimate for a specific important example of an observable. Generalizations to other observables will become clear from the proof.

Theorem 2.1. *Assume that we are given either a model of independent particle droppings or, more generally Markov chain particle droppings on a connected graph with more than two vertices. Then the following holds.*

1. *$x(t)$ converges in law to an invariant distribution, independently of the starting configuration.*

2. *Define $m_V(t) = \max_{j\in V} h_j(t)$ to be the total height of the particle profile. Then there exists a positive constant c, depending on the model, such that*

$$\mathbb{P}\big(|m_V(t) - \mathbb{E}m_V(t)| > y\big) \leq 2\exp\Big(-\frac{cy^2}{2t}\Big) \tag{2.4}$$

   where the bounds hold either if we take for $\mathbb{P} = \mathbb{P}_\pi$ the chain in equilibrium, or the chain started in any initial configuration.

3. *There exists a constant C such that*

$$\sup_t \big|\mathbb{E}_\pi m_V(t) - \mathbb{E}_0 m_V(t)\big| \leq C \tag{2.5}$$

where $\mathbb{E}_\pi$ denotes the mean in equilibrium and $\mathbb{E}_0$ is the chain started in the flat configuration $h_j = 0$ for all $j \in V$.

From the theorem follows the SLLN for the variable $m_V(t)/t$ as t tends to infinity and also the independence of the initial configuration.

3. Independent particle droppings — the proof

We will now give a self-contained presentation of the proof for the first example of independent particle droppings.

3.1. Construction of communicating set — convergence to invariant distribution

For each vertex $i \in V$ we pick an i-dependent ordering $a^{(i)} = (a_1^{(i)}, \ldots, a_{|V|-1}^{(i)})$ of the sites in the set $V \setminus \{i\}$, starting with $a_1^{(i)}$ to be a nearest neighbor of i and the additional property that $d(a_k^{(i)}, \{a_1^{(i)}, \ldots, a_{k-1}^{(i)}\}) = 1$ where d is the graph distance (see Figure 1). This means that $a^{(i)}$ describes a way how the set V can be grown starting from i by adding nearest neighbors at each step. We call $a^{(i)}$ the i-ordering.
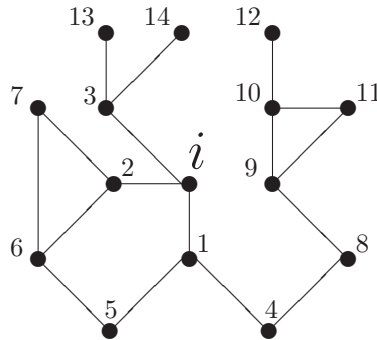

Figure 1. An example of an i-ordering.

For the given site $i \in V$ let us write $S^{(i)} = \{y \in S : y_i = 0\}$ (meaning that the maximum is realized at i). We put particles according to the corresponding i-ordering $a^{(i)} = (a_1^{(i)}, \ldots, a_{|V|-1}^{(i)})$ and look at the resulting configuration

$$T_{a_{|V|-1}^{(i)}} \ldots T_{a_1^{(i)}} y =: x^{(i)}. \tag{3.1}$$

We note that the profile on the r.h.s. is independent of the choice of $y \in S^{(i)}$ and stays bounded with $\min_{j \in V} x_j^{(i)} \geq -(|V| - 1)$ (see Figure 2).
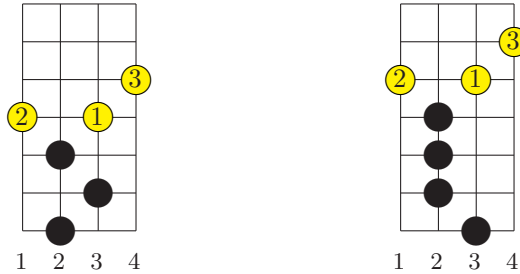

Figure 2.   In the picture we consider a graph with 4 sites and edges $\{\{1,2\}, \{2,3\}, \{3,4\}\}$. Once the $i = 2$-ordering is given, starting from different height profiles (black in the picture) having the maximum at the same vertex, using (3.1) we end up with the same configuration in S_1.

Let us put together these configurations and consider the finite subset

$$S_1 = \{x^{(i)} : i \in |V|\} \tag{3.2}$$

denoting the complement by $S_2 = S \setminus S_1$.

We note the following lemma.

Lemma 3.1. *There exists an α such that*

$$\inf_{x \in S} M^{|V|-1}(x, S_1) \geq \alpha > 0 \tag{3.3}$$

where $M^{|V|-1}$ is given by the matrix product.

This is clear since any addition of a particle has a positive probability and finitely many of those have to be considered, leading to the formula $\alpha = \min_{i \in V} \prod_{j=1}^{|V|-1} p_{a_j^{(i)}}$. Next we have the following lemma.

Lemma 3.2. *The equation $\pi M = \pi$ for the invariant distribution has a solution $\pi \in \mathcal{P}(S)$.*

Proof. We can say that there is exponential killing on the infinite part of the space S_2 and the Markov chain comes back safely to S_1. This makes it "effectively finite state". Now, to see this, let us introduce the four block-matrices $M_{ij} = (M(x,y))_{x \in S_i, y \in S_j}$, introduce the two vectors $\pi_i = (\pi(x))_{x \in S_i}$

for $\pi \in \mathcal{P}(S)$ and rewrite the equation $\pi M = \pi$ for the invariant distribution π in component form

$$(\pi_1, \pi_2) \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = (\pi_1, \pi_2). \qquad (3.4)$$

This is equivalent to the form

$$\pi_2 = \pi_1 M_{12}(1_2 - M_{22})^{-1}, \qquad (3.5)$$

$$\pi_1 (M_{12}(1_2 - M_{22})^{-1} M_{21} + M_{11}) = \pi_1$$

provided that $(1_2 - M_{22})^{-1} = \sum_{l=0}^{\infty} M_{22}^l$ exists. But to see the latter use the norm $\|M_{22}\| = \sup_{x \in S_2} \sum_{y \in S_2} M_{22}(x, y)$ and note that $\|M_{22}^{|V|-1}\| \le 1 - \alpha$, by (3.3). Hence $(M_{12}(1_2 - M_{22})^{-1} M_{21} + M_{11})$ is a well-defined positive matrix on the finite space S_1. It is even a stochastic matrix. In fact,

$$\sum_{y \in S_1} \left(M_{12}(1_2 - M_{22})^{-1} M_{21} + M_{11} \right)(x, y)$$

$$= \sum_{y \in S_1} \left(M_{12} \sum_{l=0}^{\infty} M_{22}^l M_{21} + M_{11} \right)(x, y) = \mathbb{P}_x(\tau_{S_1} < \infty)$$

$$= 1 - \lim_{R \to \infty} \mathbb{P}_x(\tau_{S_1} > R) = 1 - \lim_{R \to \infty} (1 - \alpha)^R = 1 \qquad (3.6)$$

where

$$\tau_{S_1} = \inf \big\{ t \in \mathbb{N} \mid x(t) \in S_1 \big\}.$$

So the matrix has a Perron–Frobenius eigenvector to the eigenvalue 1, which we call π_1^* (up to a positive multiple). This is (up to this multiple) the invariant distribution restricted to S_1. From this we get the invariant distribution π_2^* on the infinite part of the system by looking at the first equation of (3.5) and normalizing. $\qquad \square$

Remark. If $S_1 = \{x\}$ is a single point then define the return time $\tau_x = \inf\{t \ge 1 : W(t) = x\}$ where $W(t)$ is a random walk started at x. For a state $y \ne x$ we have that the non-normalized distribution at y is given by the expected number of visits from x to y before returning to y, i.e. $[M_{12}(1_2 - M_{22})^{-1}]_{x,y} = \mathbb{E}^x \sum_{t=1}^{\infty} 1\{W(t) = y\} 1\{t < \tau_x\}$. Normalization of the distribution then implies that $1 = \pi(x) + \pi(x)(\mathbb{E}^x \tau_x - 1)$ and so $\pi(x) = 1/\mathbb{E}^x \tau_x$ and $\pi(y) = (1/\mathbb{E}^x \tau_x) \mathbb{E}^x \sum_{t=1}^{\infty} 1\{W(t) = y\} 1\{t < \tau_x\}$ for $y \ne x$.

Lemma 3.3. *The Markov chain is uniformly communicating to S_1 by which we mean that there exists an $\alpha' > 0$ and a time $s = 3(|V| - 1)$, called the communication time, such that*

$$\inf_{x \in S, x' \in S_1} M^s(x, x') \ge \alpha' > 0. \qquad (3.7)$$

Proof. The proof follows by noting that we can first: get into S_1, second: go from there into a state which has a prescribed maximum (possibly outside S_1), and third: go from that state into the corresponding state in S_1. In formulas it reads like this: Consider a starting configuration $y \in S^{(i)}$. Then, with the above construction we have

$$T_{a^{(i)}_{|V|-1}} \ldots T_{a^{(i)}_1} y = x^{(i)} \in S_1. \tag{3.8}$$

We note that $(T_j)^{|V|-1} x^{(i)} \in S^{(j)}$ since sufficiently many particle droppings at j are shifting the maximum to the point j. From that we get again by the first step that

$$x^{(j)} = T_{a^{(j)}_{|V|-1}} \ldots T_{a^{(j)}_1} (T_j)^{|V|-1} x^{(i)} = T_{a^{(j)}_{|V|-1}} \ldots T_{a^{(j)}_1} (T_j)^{|V|-1} T_{a^{(i)}_{|V|-1}} \ldots T_{a^{(i)}_1} y. \tag{3.9}$$

The proof is complete since $j \in V$ was arbitrary.   □

Remark. From the above definition of a communication set S_1 follows trivially that any subset is also a communication set since the inf has to be taken over fewer terms. While from a theoretical point of view it would be therefore sufficient to consider a single point $x_0 \in S_1$ in our example for our chain, returns are easiest understood when we talk about our definition of S_1. The remark will be clear after dealing with particle droppings according to a Markov chain (see Figure 4 and Figure 5).

We have from this the convergence to the invariant distribution in total variation:

Lemma 3.4.
$$\|M^s(x, \cdot) - \pi\|_{TV} \leq 1 - (\alpha')^2 |V|. \tag{3.10}$$

Proof. Call X_t the chain starting at x and Y_t the one starting with initial distribution π. Moreover call τ the random time of their first meeting in the product coupling. After they meet for the first time they stay together. The coupling inequality gives:

$$\|M^s(x, \cdot) - \pi\|_{TV} \leq \mathbb{P}_c(\tau > s)$$
$$\leq 1 - \sum_{y \in S} \mathbb{P}_x(X_s = y) \, \mathbb{P}_\pi(Y_s = y) = 1 - \sum_{y \in S} \mathbb{P}_x(X_s = y)\pi(y)$$
$$\leq 1 - \sum_{y \in S_1} \underbrace{\mathbb{P}_x(X_s = y)}_{\geq \alpha'} \underbrace{\pi_1(y)}_{\geq \alpha'} \leq 1 - (\alpha')^2 |V|. \tag{3.11}$$

<div align="right">□</div>

From the lemma follows the convergence by standard arguments, extending the lemma to $\|M^{sk}(x, \cdot) - \pi\|_{TV} \leq (1 - (\alpha')^2 |V|)^k$ for integer k and using that the total variation distance is decreasing in the time t.

3.2. Concentration properties of path functionals

Define, for $t' > t$, the coupling matrix

$$D_{t,t'} := \sup_{x,x'} \mathbb{P}_c\big(X(t') \neq X'(t') \mid X(t) = x, X'(t) = x'\big) \qquad (3.12)$$

where $\mathbb{P}_c$ is the product coupling mentioned above. We have for times which differ by the communication time s that

$$D_{t,t+s} \leq 1 - (\alpha')^2|V| \qquad (3.13)$$

and this implies for general times

$$D_{t,t'} \leq (1 - (\alpha')^2|V|)^{\lfloor (t'-t)/s \rfloor}. \qquad (3.14)$$

Lemma 3.5. *Let $g : S^n \to \mathbb{R}$ be a bounded measurable function. Then we have a Gaussian concentration bound of the form*

$$\mathbb{P}\big(|g - \mathbb{E}g| > y\big) \leq 2\exp\Big(-\frac{2y^2}{\|D\delta g\|^2} \Big) \qquad (3.15)$$

where

$$(\delta g)_u = \delta_u(g) = \max_{x_u, x'_u} g(x_1, \ldots, x_u, \ldots) - g(x_1, \ldots, x'_u, \ldots) \qquad (3.16)$$

is the variation at the time u.

Proof. In the following we give only the key steps in the proof of Lemma (3.5). We refer to [3] for details where the same proof in the context of models with finite state space was given. This is not a problem here since our observable is bounded, and, most importantly the coupling matrix D satisfies the nice bounds given above, in spite of our state space being unbounded, due to uniform coupling speed.

Using the standard decomposition into Martingale differences and the Markov property, we can write

$$g - \mathbb{E}g = \sum_{i=1}^{n} W_i(x_1, \ldots, x_i) \qquad (3.17)$$

with

$$W_i(x_1, \ldots, x_i) = \mathbb{E}\big(g \mid x_1, \ldots, x_i\big) - \mathbb{E}\big(g \mid x_1, \ldots, x_{i-1}\big)$$
$$\leq \sup_{\bar{x}_i \in S} \int \mathbb{P}\big(d\tilde{x}_{i+1}, \ldots, d\tilde{x}_n \mid \bar{x}_i\big) g(x_1, \ldots, \bar{x}_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n)$$
$$- \inf_{\bar{y}_i \in S} \int \mathbb{P}\big(d\tilde{x}_{i+1}, \ldots, d\tilde{x}_n \mid \bar{y}_i\big) g(x_1, \ldots, \bar{y}_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n)$$
$$=: X_i(x) - Y_i(x). \qquad (3.18)$$

Notice that the inf and sup appearing in the previous formula are well defined since g is bounded. Then, we use a simple telescoping identity to rewrite g as a sum of discrete gradients

$$g(x_1, \ldots, \bar{x}_i, \tilde{x}_{i+1}^{(1)}, \ldots, \tilde{x}_n^{(1)}) - g(x_1, \ldots, \bar{y}_i, \tilde{x}_{i+1}^{(2)}, \ldots, \tilde{x}_n^{(2)}) = \sum_{j=0}^{n-i} \nabla_{i,i+j}^{12} g \quad (3.19)$$

where $\nabla_{i,i+j}^{12} g$ is the difference between g's evaluated at two points that are the same except for the $(i+j)$th place. We define:

$$(\delta g)_u = \delta_u(g) = \max_{x_u, x_u'} g(x_1, \ldots, x_u, \ldots) - g(x_1, \ldots, x_u', \ldots), \quad (3.20)$$

and by construction we have

$$\nabla_{i,i+j}^{12} g \le \delta_{i+j}(g) \, 1\{\tilde{x}_{i+j}^{(1)} \ne \tilde{x}_{i+j}^{(2)}\}. \quad (3.21)$$

Then using (3.18) and (3.21), it follows that

$$
\begin{aligned}
X_i(x) - Y_i(x) &= \sup_{\bar{x}_i, \bar{y}_i \in S} \Big\{ \int \mathbb{P}\big(d\tilde{x}_{i+1}, \ldots, d\tilde{x}_n | \bar{x}_i\big) g(x_1, \ldots, \bar{x}_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n) \\
&\qquad \times \mathbb{P}\big(d\tilde{x}_{i+1}, \ldots, d\tilde{x}_n | \bar{y}_i\big) g(x_1, \ldots, \bar{y}_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n) \Big\} \\
&= \sup_{\bar{x}_i, \bar{y}_i \in S} \Big\{ \int \mathbb{P}_c\big(d\tilde{x}_{\ge i+1}^{(1)}, d\tilde{x}_{\ge i+1}^{(2)} \mid \tilde{x}_i^{(1)} = \bar{x}_i, \tilde{x}_i^{(2)} = \bar{y}_i\big) \\
&\qquad \times \big[g\big(x_1, \ldots, \bar{x}_i, \tilde{x}_{i+1}^{(1)}, \ldots, \tilde{x}_n^{(1)}\big) - g\big(x_1, \ldots, \bar{x}_i, \tilde{x}_{i+1}^{(2)}, \ldots, \tilde{x}_n^{(2)}\big) \big] \Big\} \\
&\le \sup_{\bar{x}_i, \bar{y}_i \in S} \sum_{j=0}^{n-i} \delta_{i+j}(g) \mathbb{P}_c\big(\tilde{x}_{i+j}^{(1)} \ne \tilde{x}_{i+j}^{(2)} \mid \tilde{x}_i^{(1)} = \bar{x}_i, \tilde{x}_i^{(2)} = \bar{y}_i\big) \\
&\le \sum_{j=0}^{n-i} D_{i,i+j} \delta_{i+j} g =: (D\delta g)_i. \quad (3.22)
\end{aligned}
$$

The last ingredient is the following lemma from [7].

Lemma 3.6. *Suppose $\mathcal{F}$ is a σ-field and Z_1, Z_2, W are random variables such that*

1. $Z_1 \le W \le Z_2$;

2. $\mathbb{E}(W \mid \mathcal{F}) = 0$;

3. Z_1 and Z_2 are $\mathcal{F}$-measurable.

Then, for all $\lambda \in \mathbb{R}$, we have the inequality

$$\mathbb{E}\big[\exp(\lambda W) \mid \mathcal{F}\big] \leq \exp\Big[\frac{\lambda^2 (Z_2 - Z_1)^2}{8}\Big]. \tag{3.23}$$

This lemma, in the present situation, works putting $W = W_i$, $Z_1 = X_i - \mathbb{E}[g \mid \mathcal{F}_{i-1}]$, $Z_2 = Y_i - \mathbb{E}[g \mid \mathcal{F}_{i-1}]$ and $\mathcal{F} = \mathcal{F}_{i-1}$. Since, from (3.22) and (3.18) we have

$$W_i \leq Y_i - X_i \leq (D\delta g)_i, \tag{3.24}$$

we obtain

$$\mathbb{E}\big[\exp(\lambda W_i) \mid \mathcal{F}_{i-1}\big] \leq \exp\Big[\frac{\lambda^2 (D\delta g)_i}{8}\Big]. \tag{3.25}$$

By the exponential Chebyshev inequality and iterating (3.25) by successive conditional expectations with respect to $\mathcal{F}_n$ we compute

$$\mathbb{P}\big(g - \mathbb{E}g \geq y\big) \leq \exp[-\lambda y]\exp\Big[\frac{\lambda^2}{8}\|D\delta g\|^2\Big]. \tag{3.26}$$

We choose the optimal $\lambda = 4y/\|D\delta g\|^2$ to obtain

$$\mathbb{P}\big(g - \mathbb{E}g \geq y\big) \leq \exp\Big[-\frac{2y^2}{\|D\delta g\|^2}\Big]. \tag{3.27}$$

The previous line of reasoning applies to $-g$ and $-W$, proving (3.5). $\qquad\square$

3.3. The total height as an additive path functional

Let us come back now to our main application and consider the maximum of the total height of the original process, started from the flat initial configuration at zero, given by

$$m_V(t) = \max_{j \in V} h_j(t). \tag{3.28}$$

The main idea is to write a formula as an additive functional of the Markov chain along the path:

$$m_V(t) = t - \sum_{u=1}^{t-1} \mathbb{1}\big\{\max_j h_j(u+1) = \max_j h_j(u)\big\}. \tag{3.29}$$

We will rewrite the functions under the sum in terms of the x-process instead of the original one using the following lemma using the following numbers.

Lemma 3.7.

$$\max_j h_j(u+1) = \max_j h_j(u) \iff \#\big\{j \in V \mid x_j(u) \neq x_j(u+1)\big\} = 1. \tag{3.30}$$

Proof. To see that the l.h.s. implies the r.h.s. note that under the assumption $\max_j h_j(u + 1) = \max_j h_j(u)$ we have $\#\{j \in V \mid x_j(u) \neq x_j(u + 1)\} = \#\{j \in V \mid h_j(u) \neq h_j(u + 1)\} = 1$.

To see that the r.h.s. implies the l.h.s., let us suppose that $\max_j h_j(u+1) = \max_j h_j(u) + 1$, and derive a contradiction. But indeed in that case we would have $\#\{j \in V \mid x_j(u) \neq x_j(u+1)\} = \#\{j \in V \mid h_j(u) \neq h_j(u+1)-1\} = |V|-1$ which is different from 1 if $|V| > 2$.   □

So we have

$$m_V(t) = t - \sum_{u=1}^{t-1} 1\big\{\#\{j \in V \mid x_j(u) \neq x_j(u + 1)\} = 1\big\}. \qquad (3.31)$$

In our case we have $\delta_v\big(\sum_{u=1}^{t-1} 1\big\{\#\{j \in V \mid x_j(u) \neq x_j(u+1)\} = 1\big\}\big) \leq 2$ giving us

$$\mathbb{P}\big(m_V(t) - \mathbb{E}m_V(t) > y\big) \leq \exp\Big(-\frac{y^2}{2\sum_{u=1}^t \big(\sum_{u':u<u'<t} D_{u,u'}\big)^2}\Big)$$

$$\leq \exp\Big(-\frac{y^2}{2t}\frac{|V|^2(\alpha')^4}{s^2}\Big) \qquad (3.32)$$

and the same bound for $\mathbb{P}(m_V(t) - \mathbb{E}m_V(t) < -y)$. Both bounds hold if we take for $\mathbb{P} = \mathbb{P}_\pi$ the chain in equilibrium or with a given initial condition, say $h = 0$. Denote this chain by $\mathbb{P}_0$. Since our original interest was in the latter one we need to note the closeness of the two expected values which follows again by using the uniform bound on the coupling to compare the two distributions in the second inequality of

$$\big|\mathbb{E}_\pi m_V(t) - \mathbb{E}_0 m_V(t)\big| \leq \sum_{u=1}^{t-1} \big|\mathbb{E}_0\big(1\big\{\#\{j \in V \mid x_j(u) \neq x_j(u + 1)\} = 1\big\}\big)$$

$$- \mathbb{E}_\pi\big(1\big\{\#\{j \in V \mid x_j(u) \neq x_j(u + 1)\} = 1\big\}\big)\big|$$

$$\leq \sum_{u=1}^{t-1} D_{0,u} \leq \frac{s}{|V|(\alpha')^2}. \qquad (3.33)$$

In particular we get the strong law of large numbers

$$\lim_{t\uparrow\infty} \frac{m_V(t)}{t} = 1 - \sum_{x,y} \pi(x)M(x,y)1\big\{\#\{j \in V|x_j \neq y_j\} = 1\big\}. \qquad (3.34)$$

This is a particular example of an Ergodic Theorem for path observables which enjoy the concentration property.

4. Markov chain particle droppings — the proof

We consider the mapping from $h(t)$ to $x(t)$ as above and remark that $(x(t), v(t))$ is a Markov chain again. Warning: It is not to be expected that the marginal process $x(t)$ is a Markov chain (of memory depth 1) now. It will be a chain with a depth of memory 2 since the position of $v(t)$ can be reconstructed looking by $(x(t), x(t-1))$.

It is useful to make explicit the graph (V, E) with undirected edges E defining the piling-up rule, and the graph (V, E_A) with directed edges $E_A = \{(i, j) \in V \times V : A(i, j) > 0\}$. The following considerations depend on A only through E_A.
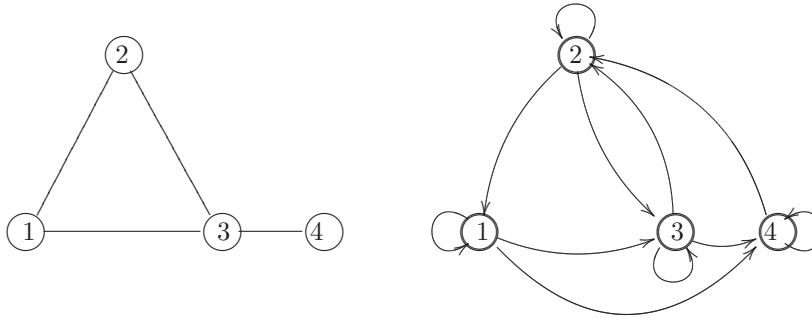

Figure 3. Not all edges in the graph (V, E) (on the left) correspond to two directed edges in the graph (V, E_A). This may forbid us to put particles neighboring each other in one step.

The first step is to extend the definition of S_1 to the present setup. A problem might be that the driving Markov chain forbids us to put balls neighboring each other *in one step* in the sense of the graph which defines our piling-up rules (see Figure 3). What we need is to make sure that we can define a finite set $\bar{S}_1$ to which the joint chain communicates uniformly. While in the independent case we could just take the configurations which were grown from nearest neighbor to nearest neighbor (along an i-ordering), here we have to add to it connecting strings of allowed transitions in between. A slight discomfort is that the maximum might change in a complicated way during this process of particle additions following this string. However, this is not really important. What is important is that a resulting configuration will only depend on the maximum of the initial configuration and otherwise be independent on its form. Now, we can ensure the latter by adding sufficiently many particles at the maximizing site initially. By laziness it is a chain of allowed transitions and it ensures that all influence of the configuration at any other site will be lost. This is formulated in the following lemmata. The first lemma is treating a situation where the

driving Markov chain drops a particle at the same position as the maximum. The next lemma shows how the situation where the driving Markov chain is in a different position than the maximum can be reduced to the first.

Lemma 4.1. *Suppose that $x \in S^{(i)}$. Then there exists a finite integer $s(i)$ and a sequence $(i = i_1, i_2, \ldots, i_{s(i)}) \in V^{s(i)}$ such that $(i_j, i_{j+1}) \in E_A$ is an allowed transition and the configuration*

$$x^{(i)} := T_{i_{s(i)}} \ldots T_{i_1} (T_i)^{s(i)} x \tag{4.1}$$

is independent of the choice of the initial configuration in $S^{(i)}$ and has a bounded depth $\min_{j \in V} x_j^{(i)} \geq -2s(i)$.

Proof. We choose for each vertex $i \in V$ an i-ordering $a^{(i)} = (a_1^{(i)}, \ldots, a_{|V|-1}^{(i)})$ of the sites in the set $V \setminus \{i\}$, which was defined above. We need to connect each of the occurring pairs of neighboring vertices $v = a_j^{(i)}$, $w = a_{j+1}^{(i)}$ with a chain of allowed transitions $(v_1 = v, v_2, \ldots, v_{s(v,w)} = w)$ where $s(v, w)$ is the shortest length of an oriented path in E_A. In particular every vertex in the string is visited only once. Let us denote the string from v to w which we obtain by the above by dropping the w from it by $c(v, w) = (v_1, \ldots, v_{s(v,w)-1})$. Then we concatenate the strings along the i-ordering and define

$$\begin{aligned}(i_1, &\ldots, i_{s(i)}) \\ &:= \big(c(i, a_1^{(i)}), c(a_1^{(i)}, a_2^{(i)}), c(a_2^{(i)}, a_3^{(i)})), \ldots, c(a_{|V|-2}^{(i)}, a_{|V|-1}^{(i)}), a_{|V|-1}^{(i)}\big).\end{aligned}$$

This string has the property that it contains the i-ordering as a substring and therefore erases the influence of an initial configuration $y \in S^{(i)}$ when applied to it, when the difference of the maximum at i and the configuration at any other site was bigger than any possible number of occurrences of a site j in $(i_1, \ldots, i_{s(i)})$ (see Figure 4). □

Lemma 4.2. *Suppose that $x \in S^{(i)}$ and $v \in V$. Then there exists a finite integer $\sigma(i)$ and a sequence $(i_1 = v, i_2, \ldots, i_{\sigma(i)} = i) \in V^{\sigma(i)}$ such that $(i_j, i_{j+1}) \in E_A$ is an allowed transition and*

$$T_{i_{\sigma(i)}} \ldots T_{i_1} x \in S^{(i)}. \tag{4.2}$$

The lemma says we can go from any initial position of the driving Markov chain and a height profile with maximum in i to a position with maximum again in i and driving Markov chain also in i, just as the first lemma assumed.

Proof. First drop $s(v, i)$ particles according to $c(v, i)$. Then drop $s(v, i)$ particles at i to be sure that the maximum will be again at i. This proves the lemma with $\sigma(i) = 2 \max_v s(v, i)$. □
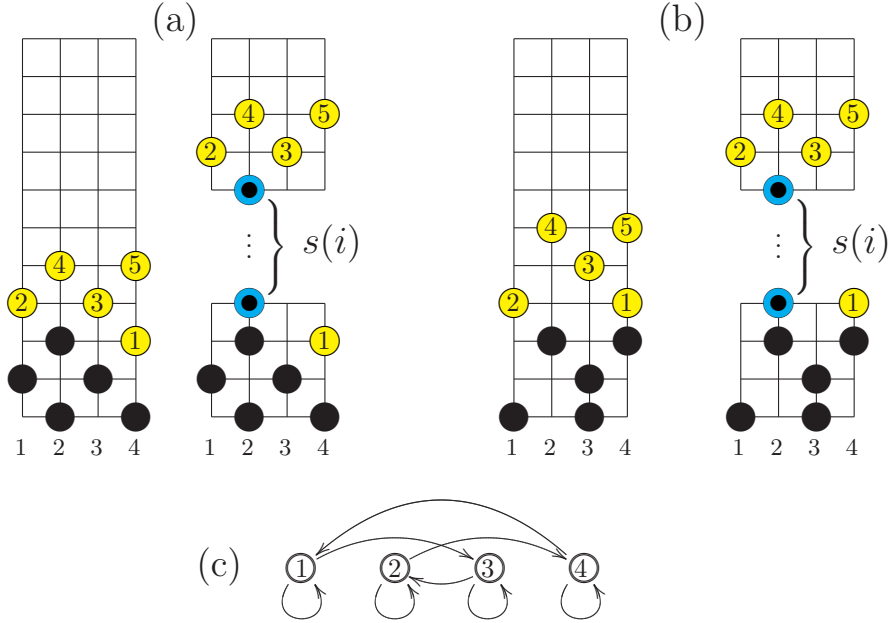
Figure 4. An example of the procedure described in the proof of Lemma 4.1 on the graph with edges $\{\{1,2\}, \{2,3\}, \{3,4\}\}$. Dropping particles according to the concatenated strings along the $i = 2$-ordering $(1,3,4)$ does not suffice to obtain the same configurations in $\bar{S}_1$ (see the left pictures of (a) and (b)). If we previously add $s(i)$ balls to the top of the height profiles the configurations are the same (see the right pictures of (a) and (b)).

In analogy to the independent case we put together these configurations and consider the finite subset (which this time however will live in the product space $\bar{S} = S \times V$) and define

$$\bar{S}_1 = \left\{ (x^{(i)}, i_{s(i)}) : i \in |V| \right\} \tag{4.3}$$

denoting the complement by $\bar{S}_2 = \bar{S} \setminus \bar{S}_1$.

Corollary 4.1. *The Markov chain M is uniformly communicating to $\bar{S}_1$ with a finite communication time $\bar{s}$.*

Proof. To prove that, for every joint configuration $\bar{x} = (x, v) \in \bar{S}_1$

$$\inf_{\substack{(x,v) \in \bar{S} \\ \bar{x} \in \bar{S}_1}} M^{\bar{s}}\big((x,v), \bar{x}\big) \geq \bar{\alpha}' > 0 \tag{4.4}$$

we repeat the argument of the independent case with a small modification: First
we get from (x,v) where $x \in S^{(i)}$ to a point (x',i) where $x' \in S^{(i)}$ in $\sigma(i)$ steps
by the second lemma. We warn the reader that the x' might be dependent on the
particular choice of x,v. Then we get from (x',i) to $\bar{x}^{(i)} = (x^{(i)}, a^{(i)}_{|V|-1})$ in $2s(i)$
steps by the first lemma. Then we get from there into the state j of the driving
Markov chain by means of the connecting string $(c(a^{(i)}_{|V|-1},j),j)$ and adding
particles at j in an i- and j-independent number of steps. This is equivalent
to saying that the driving chain is ergodic. Using now laziness we can go from
there into a state which has a prescribed maximum by adding sufficiently many
particles at j (which will typically be outside of $\bar{S}_1$) called (x'',j). In the third
step we go from that state into the corresponding state in $\bar{S}_1$ which has the
maximum at j (see Figure 5 for an illustration of this procedure). Note that
these procedures a priori might take a total number of particle droppings which
could depend on the (x,v). We can produce a number of particle droppings $\bar{s}$
which will do the job for all (x,v) by adding more particles, if necessary, at the
steps where particles are dropped at the same site. This proves the lemma with
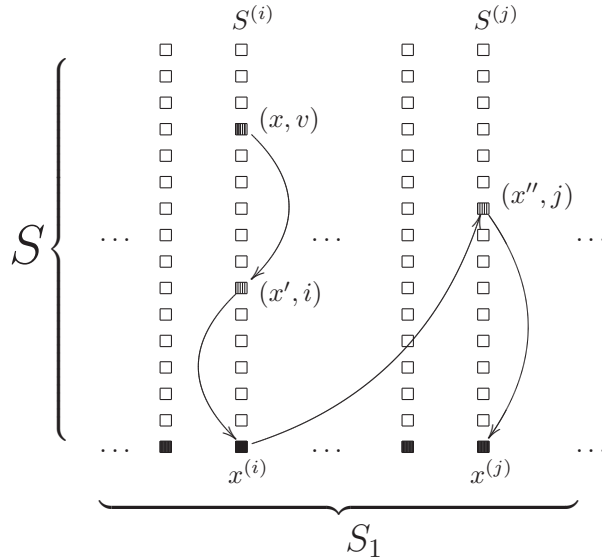$\bar{s} = 3 \max_{i \in V} 2(s(i) + \sigma(i))$.   □

Figure 5. A schematic representation of the path from $(x,v) \in \bar{S}$ to $\bar{x} \in \bar{S}_1$ in
the proof of Corollary 4.1.

Let us compare to the independent case. Then the $\sigma(i)$-term is not needed, one $s(i)$-term (needed to build up a sufficiently high maximum) can be dropped in the independent case, and one $s(i)$-term is just $|V|-1$ (the length of a covering string).

The previous considerations given in Section 2 give us now the existence of an invariant distribution $\bar{\pi}$ on $\bar{S}$, along with the convergence to it, and the bound on the coupling matrix

$$\bar{D}_{t,t'} := \sup_{\bar{x},\bar{x}'\in\bar{S}} \mathbb{P}_c\big(\bar{X}(t') \neq \bar{X}'(t') \mid \bar{X}(t) = \bar{x}, \bar{X}'(t) = \bar{x}'\big) \qquad (4.5)$$

where $\mathbb{P}_c$ is the coupling of $\bar{X}(t) = (X(v), v(t))$ with $\bar{X}'(t) = (X'(v), v'(t))$. We have for times at the distance of the communication time $\bar{s}$ that

$$\bar{D}_{t,t+\bar{s}} \leq 1 - (\bar{\alpha}')^2 |V| \qquad (4.6)$$

and this implies for general times

$$\bar{D}_{t,t'} \leq \big(1 - (\bar{\alpha}')^2 |V|\big)^{\lfloor (t'-t)/\bar{s} \rfloor}. \qquad (4.7)$$

The concentration lemma can be formulated for observables $\bar{g} : \bar{S}^n \to \mathbb{R}$ and otherwise stays the same.

All estimates on the maximal height $m_V(t)$ carry over when α' is replaced by $\bar{\alpha}'$ and s is replaced by $\bar{s}$. This finishes the proof of the theorem.

5. Extension to layer-dependent particle droppings

We will finally give an extension to a model of particle droppings which allows also for deposition of particles below the top layer, albeit only with a fixed finite depth. This however allows for a large class of deposition rules and we will be very general here. On the other hand, we want to assume a non-nullness condition of particle adsorption at any site to the top layer, independently of the configuration and the position of the last dropped particle. By the last requirement we exclude part of the difficulty dealt with in the case of Markov-chain droppings.

The idea of such models is to consider adsorption with the possibility of a finite depth penetration. Since this penetration has to depend also on the configuration below the top height, i.e. by the exclusion between particles, we are led to deal with a state space for our Markov chains, carrying more information as in the models considered above, as we will describe now.

Take the set $\bar{\Omega}$ of finite subsets of $V \times \mathbb{N}$. V is again the vertex set of a finite graph, and a site $v \in V$ should be viewed as a point in the base plane. Consequently the set $\Phi \in \bar{\Omega}$ is meant to describe a particle configuration by giving the places where particles are sitting, i.e. $(v, h) \in \Phi$ if and only if a

particle is sitting above the site v at the height h and the set $\Phi_v = \{h \in \mathbb{N}:$ $(v, h) \in \Phi\}$ describes the places where particles are sitting above the fixed site v. To each $\Phi \in \bar{\Omega}$ we associate the height function $h(\Phi) = (h_v(\Phi))_{v \in V}$ where $h_v(\Phi) = \max \Phi_v$. We introduce the configuration obtained by adding a particle at i applying the screening rule by $T_i\Phi = \Phi \cup \{(i, \max_{v:\mathrm{dist}(v,i) \leq 1} h_v(\Phi) + 1)\}$ (see Figure 6 for an illustration). With this notation we have compatibility with the previously defined action on the height profile, i.e. $h(T_i\Phi) = T_i h(\Phi)$. Denote the smaller set of configurations obeying nearest neighbor exclusion by $\Omega = \{\Phi \in \bar{\Omega}: (v, h) \in \Phi$ implies $(w, h) \notin \Phi$ if $w \sim v\}$. A growth process will be defined on Ω.
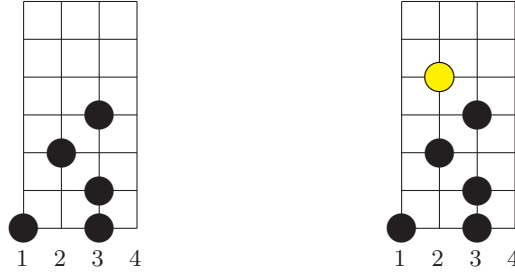

Figure 6. In this figure we demonstrate the operator T. Consider a configuration Φ (left figure). The application of $T_i(\Phi)$ basically puts a new particle on top of Φ above vertex i. As an example the figure on the right shows the new configuration after deposition of a new particle at 2, i.e. $T_2(\Phi) = \Phi \cup \{(2, \max_{v:\mathrm{dist}(v,2) \leq 1} h_v(\Phi) + 1)\} = \Phi \cup \{(2, 5)\}$.

Let $\Phi(t)$ denote a Markov chain with state space Ω and transition matrix $M(\Phi, \Phi')$ having the properties:

1. non-null screening rule: $\inf_{i \in V, \Phi \in \Omega} M(\Phi, T_i\Phi) \geq \varepsilon > 0$;

2. $M(\Phi, \Phi') = 0$ unless $\Phi' = \Phi \cup \{(v, h)\}$ for a single particle in the k-layer below the maximum, i.e. $h \in \{-k + h_v(\Phi), (T_v(h(\Phi)))_v\}$ (see Figure 7);

3. layer-k-depth memory: $M(\Phi_1, \Phi') = M(\Phi_2, \Phi')$ if $\Phi_1 \sim_k \Phi_2$. Here we have defined equivalence to the depth k, denoted by $\Phi_1 \sim_k \Phi_2$ if $h(\Phi_1) = h(\Phi_2)$ (the height profile coincides) and

$$(\Phi_1)_v \cap [-k + h_v(\Phi_1), h_v(\Phi_1)] = (\Phi_2)_v \cap [-k + h_v(\Phi_2), h_v(\Phi_2)]$$

that is the k-depth layer below the height profile coincides.

To formulate the last condition let us subtract the maximum and define $\Psi_v :=$ $\{x : x + \max_{w \in V} h_w(\Phi) \in \Phi_v\}$ and $\Psi = \cup_v (v \times \Psi_v)$ to be the set of occupations shifted by the maximum. As a result we have that the height function has the maximum zero, i.e.

$$\max_v h_v(\Psi) = 0.$$

We also want that

4. height-shift-invariance: $M(\Phi_1, \Phi_1') = M(\Phi_2, \Phi_2')$ if $\Psi(\Phi_1) = \Psi(\Phi_2)$ and $\Psi(\Phi_1') = \Psi(\Phi_2')$.

Denote by S the set of equivalence classes of images under Ψ w.r.t. looking at the k-depth layer. So it is the space of possible height-profiles enlarged by the information which sites below are occupied, up a depth k. It is clear that the process has a lift on S as a Markov process.
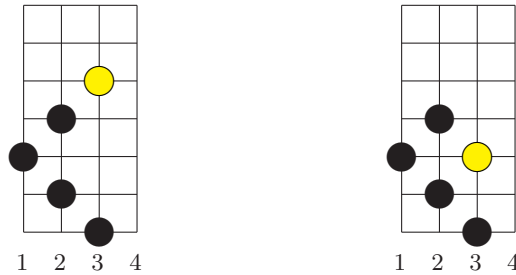

Figure 7. According to the non-null screening rule (property 1 of the transition matrix M) there is a positive probability that a dropping particle stops on the top of the configuration (left picture). At the same time the second property of M allows for a deposition below the top within a fixed depth k (right picture).

Theorem 5.1.

1. *The law of Markov process $\Psi(t)$ on the set of k-layer depth height-shift equivalence classes S converges in total variation to an invariant distribution $\bar{\pi}$ on S.*

2. *For each $g : S^n \to \mathbb{R}$ bounded, the random variable $g(\Psi(1), \ldots, \Psi(n))$ obeys the Gaussian concentration bound of Lemma 3.5, with a matrix $D_{t,t'} \leq A \exp\{-\lambda(t' - t)\}$ for all $t' \geq t$ and zero else.*

3. *In particular the function $m_V(t)$ obeys the bound (2.4) and (2.5) of Theorem 2.1 for suitable constants c and C.*

Outline of Proof. To prove the first assertion of the theorem we need to construct a coupling, starting from any two layer configurations $\Psi_1, \Psi_2 \in S$. Let us do this in several steps. Informally speaking one can go first to configurations with the property that the height profile takes values in S_1 (formulated for the top layer in the same way as we did in the section on independent particle droppings) and then create any desired allowed layer of thickness k by adding only particles which happen to feel the screening which happens with non-null probability. This can be done for any initial configuration, with the same outcome after sufficiently many steps. In this way one can produce a coupling between any two initial configurations with a uniform very small probability $\tilde{\alpha} > 0$ after some very large time $\tilde{s}$. From that point everything in the proof stays the same.

Now we give some details. Suppose that $h(\Psi) \in S^i$ (meaning that the top profile takes the maximum at i). Applying the sequence of particle additions we look at the resulting configuration

$$\tilde{\Psi}_j := T_{a_{|V|-1}^{(j)}} \ldots T_{a_1^{(j)}} (T_j)^{|V|-1} T_{a_{|V|-1}^{(i)}} \ldots T_{a_1^{(i)}} \Psi. \tag{5.1}$$

By the non-nullness screening condition we know that this has a probability which is bounded uniformly below by $\alpha_1 > 0$. We can be certain that $h(\tilde{\Psi})$ is equal to the previously defined $x^{(j)}$ independently of the initial condition. However, this might not hold for the k-layer below. To cure for this we take an arbitrary sequence $a = (v_1, v_2, v_3, \ldots, v_R)$ in which every vertex appears at least k times, and apply the corresponding particle additions using the map T_{v_j}. This creates a configuration $\Psi_j = T_{v_R} \ldots T_{v_1} \tilde{\Psi}_j$ whose k-depth layer is independent of the starting configuration Ψ. Define now the communication set in layer space by putting $\bar{S}_1 = \{\Psi_j : j \in V\}$. This has the desired properties, and by the previous argument proves the first part. The second part is a direct application of the concentration statement of Lemma 3.5. To prove the third part we write

$$m_V(t) = t - \sum_{u=1}^{t-1} 1\Big\{ \max_j h_j(\Phi(u+1)) = \max_j h_j(\Phi(u)) \Big\}$$

$$= t - \sum_{u=1}^{t-1} 1\Big\{ A(\Psi(u+1), \Psi(u)) \Big\} \tag{5.2}$$

where

$$A\big(\Psi(u+1), \Psi(u)\big) = \big\{ \#\{ j \in V \mid h(\Psi(u+1))_j - h(\Psi(u))_j \} \le 1 \big\}.$$

Note that we have written the inequality instead of equality in the last definition in order to account for particle depositions below the top layer. From here the proof of the concentration of the variable $m_V(t)$ stays the same as in the previous two cases. This concludes the proof of the theorem. □

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