GIBBS MEASURES OF DISORDERED SPIN SYSTEMS

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Abstract. We give a brief introduction to some aspects of the field of Gibbs measures of disordered (lattice) spin systems. We present a summary of some of the main results of our own contributions to the subject.

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0. INTRODUCTION

This is a review dealing with various related aspects of the Probability Theory of Gibbs measures of disordered systems. The models for disordered systems that will be considered here usually come from the statistical mechanics part of theoretical physics, but the desire to really understand them is a source of interesting mathematics.

The contributions we present here range from the more concrete to the more abstract. They are linked but can be loosely grouped in three parts. In Chapter 1 we give the proofs of long-range order for specific continuous spin lattice models. In Chapter 2 we focus on the conceptual novelties of the infinite volume description of a system that are *caused by*

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the disorder and are not present in deterministic systems. We discuss two examples as an illustration for that. In Chapter 3 we describe a more abstract contribution to Gibbsian theory. We investigate a general class of measures naturally appearing in this context and ask, whether and in what sense they can be interpreted as infinite volume Gibbs measures. For more details than we can provide here we refer the reader in particular to the references marked with stars.

We start with a brief reminder of some background material to Gibbs measures and their behavior at 'low temperature' where there is the possibility for phase transitions.

Background: Gibbs measures of lattice spin models

To put the questions and results about disordered systems in perspective it is good to recall the situation for lattice spin models without disorder. We start with the setup of Gibbs measures in this context and mention some of the important results concerning the low-temperature region of translation invariant systems. After that we come to disordered lattice spin models. We mention some known facts about the random field Ising model that will serve as a guiding example. We will refer to it in all of the three following chapters from different points of view. Readers who are familiar with these facts may want to go directly to Chapter 1 where we start to describe our own results.

Basic definitions

Take the lattice $\mathbb{Z}^d$ and consider the (so-called) spin variables $\sigma = (\sigma_x)_{x\in\mathbb{Z}^d} \in \Omega_0^{\mathbb{Z}^d}$. The latter space is called configuration space. We will consider only cases where the space Ω_0 (the 'local state space') is either finite, or given by the integers, or the real line, so that there is a natural topology and a corresponding σ-algebra. For the product-space one commonly uses the product topology and the product σ-algebra. Consider a collection of local functions $\Phi = (\Phi_A)_{A\subset\mathbb{Z}^d}$ indexed by the subsets of the lattice $\mathbb{Z}^d$, having the property that $\Phi_A(\sigma)$ depends on σ only through its value $\sigma_A \equiv (\sigma_x)_{x\in A}$. Φ is called (interaction) potential and the choice of Φ defines the model under consideration. One often encounters also the so-called formal Hamilton function (or energy function), given by the expression

$$H(\sigma) = \sum_{A\subset\mathbb{Z}^d} \Phi_A(\sigma) \tag{0.1}$$

This expression is meaningful only when restricted to a finite volume $\Lambda\subset\mathbb{Z}^d$, of course. The best known example of a lattice spin model is the usual nearest neighbor Ising model where $\Omega_0 = \{-1, 1\}$ and the Hamilton function is $H(\sigma) = -\sum_{<x,y>} J\sigma_x\sigma_y - \sum_x h\sigma_x$, where the first sum runs over all pairs of nearest neighbors x and y on the lattice. Here

J and h are two parameters having the meaning of a coupling constant and a magnetic field. Now, given some Φ, one forms the 'Gibbs measures in finite volume' $\Lambda \subset \mathbb{Z}^d$ with boundary condition $\sigma^{\text{b.c.}}$ which are the probability measures on Ω obtained by putting

$$\mu_\Lambda^{\sigma^{\text{b.c.}}}(f) := \frac{\sum_{\sigma_\Lambda} f(\sigma_\Lambda \sigma^{\text{b.c.}}_{\mathbb{Z}^d \setminus \Lambda}) \ \exp\left(-\sum_{A \cap \Lambda \neq \emptyset} \Phi_A(\sigma_\Lambda \sigma^{\text{b.c.}}_{\mathbb{Z}^d \setminus \Lambda})\right)}{\sum_{\sigma_\Lambda} \exp\left(-\sum_{A \cap \Lambda \neq \emptyset} \Phi_A(\sigma_\Lambda \sigma^{\text{b.c.}}_{\mathbb{Z}^d \setminus \Lambda})\right)} \tag{0.2}$$

for any bounded measurable observable $f : \Omega \to \mathbb{R}$. (Measurability is meant w.r.t the product σ-algebra.) The collection of the measures $\mu_\Lambda^{\sigma^{\text{b.c.}}}$ is also called 'local specification'. The finite-volume summation is over $\sigma_\Lambda \in \Omega_0^\Lambda$. The symbol $\sigma_\Lambda \sigma^{\text{b.c.}}_{\mathbb{Z}^d \setminus \Lambda}$ denotes the infinite volume configuration in Ω that is given by σ_x for $x \in \Lambda$ and by $\sigma_x^{\text{b.c.}}$ for $x \in \mathbb{Z}^d \setminus \Lambda$. For the sum to make sense, one needs some summability assumption on Φ (see e.g. page 6 of [K99b*], or Chapter 3). If one is dealing with continuous variables the sums must be replaced by integrals over a-priori measures. Now, most of the time in statistical mechanics, the task is the following:

Given an interaction potential Φ, characterize the corresponding infinite volume Gibbs measures μ!

Here, the infinite volume Gibbs measures μ are those probability measures on Ω whose finite volume conditional expectations coincide with the above finite-volume Gibbs measures given by (0.2), that is we have

$$\mu\left(\sigma_\Lambda \middle| \sigma_{\mathbb{Z}^d \setminus \Lambda}\right) = \frac{\exp\left(-\sum_{A \cap \Lambda \neq \emptyset} \Phi_A(\sigma_\Lambda \sigma_{\mathbb{Z}^d \setminus \Lambda})\right)}{\sum_{\tilde{\sigma}_\Lambda} \exp\left(-\sum_{A \cap \Lambda \neq \emptyset} \Phi_A(\tilde{\sigma}_\Lambda \sigma_{\mathbb{Z}^d \setminus \Lambda})\right)} \tag{0.3}$$

for any Λ and μ-a.e. σ. This equation for μ is called DLR equation. (DLR= Dobrushin, Lanford, Ruelle.)

Why do people care for infinite volume Gibbs measures? Usually one is given the potential Φ describing the interaction between the microscopic components of a system (like a piece of a ferromagnetic material, say) from theoretical physics and one asks for the resulting collective behavior in thermal equilibrium. Since one is dealing with a very large number of those microscopic components it is natural to investigate the limit $\Lambda \uparrow \mathbb{Z}^d$. While one might argue that it is physically more natural to stick with large but finite volumes, the notion of the infinite system is usually seen as an idealization where interesting properties one likes to study can be captured in a sharpened way. As we will see in Chapter 2, the question of the infinite volume limit has to be taken with more care in the case of (some) disordered systems.

What makes the DLR equation (and the physical systems it is supposed to describe) interesting is that one might encounter several solutions μ for the same Φ. For this to be the case Φ must describe a strong coupling between the spins in some sense. If non-unicity of the solutions happens one says that Φ allows for different phases. (The physically observable states of the system then correspond to the extreme elements of the simplex of solutions μ for a given Φ). A very clear probabilistic presentation of abstract Gibbsian theory is found in [Geo88], a softer pedagogical introduction without proofs in Chapter 2 of [EnFeSo93].

Translation-invariant systems at low temperature: Pirogov-Sinai theory

In the specific example of the nearest neighbor Ising model in $d \geq 2$ dimensions it is well-known that for $h = 0$ and J sufficiently large ('low temperature') there exist different translation-invariant Gibbs measures μ^+ (and μ^-) which describe small perturbations of the all-plus (respectively all-minus) spin-configuration. That is, a typical configuration of μ^+ looks like an infinite sea of plus spins with small and rare islands of minus-spins. If J is sufficiently small there is a unique Gibbs measure.

A similar suitably generalized low-temperature picture holds true for more general translation-invariant systems, where the spin variables may take a finite number of values, the interaction has finite range, but no symmetry of the interaction between the different spin-values is assumed. This is the content of the Pirogov-Sinai theory ([PS76a],[PS76b],[Si82],[Za84],[Za87],[Za98]). For a pedagogical description of the main results see e.g. the big review paper [EnFeSo93] Chapter B.4. To think of one concrete example where it applies take e.g. the Blume-Capel model, where $\sigma_x \in \{-1, 0, 1\}$ and $H(\sigma) = \beta \left(\sum_{<x,y>} (\sigma_x - \sigma_y)^2 - \sum_x g\sigma_x^2 - \sum_x h\sigma_x \right)$, and $\beta > 0$ (the 'inverse temperature') and g, h are parameters.

Here, depending on the values of the parameters, for large β the Hamiltonian admits either one, two, or three extremal translation-invariant Gibbs measures μ^q, $q \in \{-1, 0, 1\}$. These translation-invariant phases are 'q-like', i.e. $\mu^q [\sigma_x \neq q] \leq e^{-const\, \beta}$, with exponential decay of correlations, i.e. $|\mu^q [\sigma_x \sigma_y] - \mu^q [\sigma_x] \mu^q [\sigma_y]| \leq e^{-const\, \beta |x-y|}$. Furthermore, the 'lines of phase-coexistence' in the space of (g, h) where there are two extremal Gibbs measures, [for fixed β] deform in an analytic way as a function of β. This is true for (g, h) in a neighborhood of the origin.

For such results to hold in a general setup one needs that the interaction obey a 'Peierls condition'. The latter essentially demands that the energy difference of a perturbed configuration about the (candidate of a) ground state is at least as big as the volume where the perturbation occurs times a sufficiently large constant. This so-called Peierls constant then plays the role of an inverse temperature. E.g. in the Blume-Capel model the candidates for ground-states are potentially all three uniform spin-configurations. The regions on the lattice where changes in the spin-values relative to one of the 'ground-states' occur are termed (thick) *Peierls contours*. They play an important role as basic objects in the theory in that they describe the basic 'excitations' of the system. The proof of these results of Pirogov-Sinai theory is technically not simple. It is based on cluster-expansions (Taylor-expansions of logarithms of various types of 'partition functions' with an unbounded number of variables), and the solution of certain fixed point equations. In situations where the interaction is symmetric under permutation of the possible spin-values (like the standard Ising model in zero magnetic field) the situation simplifies considerably. Contours and cluster expansions can be a useful tool for the study of disordered models, too, and they also appear as important ingredients of our papers [K99a*], [K98d*], [K00].

More results in this spirit have been obtained and are still further developed for models possessing translation-invariance (at least in all but one directions). We mention here only: a general Pirogov-Sinai theory of interface states [HoZa97], finite size corrections [BoKo95], continuous spin systems [Za00], long-range interactions [BoZa01], the treatment of small quantum perturbations [DaFeRo96] etc. A generalization of analyticity results to the non-translation invariant situation, however still assuming uniform Peierls-estimates, is in work by the author [K01b].

We should mention that there is an approach to the low-temperature behavior alternative to expansion methods and Pirogov-Sinai theory, that is based on percolation techniques and the use of stochastic comparison inequalities [HGM00].

Gibbs measures of *disordered* lattice spin models: Basic definitions

Having recalled some of the properties of systems containing no disorder we will now come to disordered systems to which we will stick for the rest of the time. Now the picture will be more complicated: We are giving up translation-invariance of the interactions between the spins and make them random according to an external probability distribution.

Again we denote by $\Omega = \Omega_0^{\mathbb{Z}^d}$ the space of *spin-configurations* $\sigma = (\sigma_x)_{x\in\mathbb{Z}^d}$, where the single-spin space Ω_0 is as above. Similarly we denote by $\mathcal{H} = \mathcal{H}_0^{\mathbb{Z}^d}$ the space the *disorder variables* $\eta = (\eta_x)_{x\in\mathbb{Z}^d}$ take values in, where $\mathcal{H}_0$ is the real line, an interval or a finite set. Each copy of $\mathcal{H}_0$ carries a measure $\nu(d\eta_x)$ and $\mathcal{H}$ carries the product-measure over the sites, $\mathbb{P} = \nu^{\otimes_{\mathbb{Z}^d}}$. We denote the corresponding expectation by $\mathbb{E}$. The space of joint configurations $\Omega \times \mathcal{H} = (\Omega_0 \times \mathcal{H}_0)^{\mathbb{Z}^d}$ is called *skew space*. It is equipped with the product topology.

We consider disordered models whose formal *infinite volume Hamiltonian* can be written in terms of *disordered potentials* $(\Phi_A)_{A\subset\mathbb{Z}^d}$,

$$H^\eta(\sigma) = \sum_{A\subset\mathbb{Z}^d} \Phi_A(\sigma,\eta) \tag{0.4}$$

where Φ_A depends only on the spins and disorder variables in A. A lot of disordered models can be cast into this form.

A famous example of this is the random field Ising model where $\Omega_0 = \{-1,1\}$, $\mathcal{H}_0 = \{-1,1\}$ and the Hamilton function is $H^\eta(\sigma) = -\sum_{<x,y>} J\sigma_x\sigma_y - \sum_x h\eta_x\sigma_x$, where, again, the first sum runs over all pairs of nearest neighbors x and y on the lattice. The distribution of the 'random fields' is i.i.d. with symmetric distribution, say, e.g. symmetric Bernoulli, i.e. $\mathbb{P}[\eta_x = 1] = \mathbb{P}[\eta_x = -1] = \frac{1}{2}$.

For fixed realization of the disorder variable η we denote by $\mu_\Lambda^{\sigma^{\text{b.c.}}}[\eta]$ the corresponding *finite volume Gibbs measures* in $\Lambda \subset \mathbb{Z}^d$ with boundary condition $\sigma^{\text{b.c.}}$. They are the probability measures on Ω that are given by the formula

$$\mu_\Lambda^{\sigma^{\text{b.c.}}}[\eta](f) := \frac{\sum_{\sigma_\Lambda} f(\sigma_\Lambda \sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda}) \exp\left(-\sum_{A\cap\Lambda\neq\emptyset} \Phi_A(\sigma_\Lambda \sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda}, \eta)\right)}{\sum_{\sigma_\Lambda} \exp\left(-\sum_{A\cap\Lambda\neq\emptyset} \Phi_A(\sigma_\Lambda \sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda}, \eta)\right)} \tag{0.5}$$

for any bounded measurable observable $f : \Omega \to \mathbb{R}$.

Then the aim of the theory is usually

Given an interaction potential Φ^η, fix a realization η that is typical for $\mathbb{P}$ and characterize the corresponding infinite volume Gibbs measures $\mu[\eta]$!
Characterize the large volume behavior of Gibbs measures $\mu_\Lambda^{\sigma^{\text{b.c.}}}[\eta]$!

This can be much more difficult than in the translation invariant case. First of all, there are cases where arbitrarily small random perturbation may quantitatively change the behavior of a system and lead to new phenomena. Even if this is not the case, and disorder turns out to be 'irrelevant' in the sense that it does not fundamentally change the 'character' of the Gibbs measures, the analysis can be much harder than in the translation-invariant case. We will provide some concrete examples for this. Let us mention that, in particular there is no analogue of Pirogov-Sinai theory for disordered systems yet, although there is an outline of some ideas for such a project by Zahradnik. In fact, this would be a wonderful project.

The random field Ising model

Let us briefly discuss the concrete example of the random field Ising model (with symmetric non-degenerate distribution.) For this model it was proved in [AiWe90] that there is unicity of the Gibbs measure in 2-dimensions, at any fixed temperature, for $\mathbb{P}$-a.e. η. This is in contrast to the case of the model without disorder, which shows that the introduction of arbitrarily weak random perturbations can destroy a phase transition. It shows that randomness can potentially alter the behavior of the system in a fundamental way, and cannot always be treated as a small perturbation. The method of [AiWe90] is based on getting lower estimates on the fluctuations w.r.t. the distribution of $\mathbb{P}$ of certain extensive quantities that are related to free energies in finite volume (logarithms of partition functions). This method uses martingale techniques and is relatively soft and not too technical. We remark that it was later applied by [BoK96] to show the non-localization of interfaces in random environments in the framework of certain models for interfaces without overhangs in space dimensions less than $3 = 2 + 1$.

In the three or more dimensional random field Ising model, for small disorder, and small temperature, however, disorder does *not* destroy the ferromagnetic ordering. Here, [BrKu88] showed in their famous paper that there exist distinguished Gibbs measures $\mu^+[\eta]$ (and $\mu^-[\eta]$) which, for typical magnetic field configuration η, describe small perturbations around a plus-like (respectively a minus-like) infinite-volume ground state. A plus-like ground state looks like a sea of pluses with rare islands of minuses in those regions of space where the realizations of the magnetic fields happen to be mostly oriented to favor the minus spins. The method they used, the so-called 'renormalization group', is a multiscale method that consists in a successive application of a certain coarse-graining/rescaling procedure. This is necessary because there is no simple Peierls-condition for this model (say around the all-plus state.) The individual steps are controlled by expansion methods and probabilistic estimates of the undesirable event that regions of exceptionally large magnetic fields occur. This has to be done for all hierarchies occuring. This method is conceptually beautiful but technically hard to implement. It was later also applied by [BoK94] to show the stability of certain interface models in dimensions $d+1\geq 4$. (An analogous method was also used by [BrKu91] to show the diffusive behavior of random walks in asymmetric random environments in more than 2 dimensions.) We remark that the result of [BrKu88] was a nice example where a question that was truly under debate among theoretical physicists could be settled by mathematicians.

1. TWO DISORDERED MODELS OF CONTINUOUS SPINS

We will now come to the results of the first two papers. Besides lattice spin models taking a finite number of values, models of continuous spins have found a great interest. The reasons for this is that they are often taken by physicists as an ad hoc 'mesoscopic' description of physical phenomena. That is, they are meant by physicists to incorporate already an average over microscopic details of the physical world. (Taking this latter sentence serious from a probabilistic point of view also leads to a very interesting direction of research that we don't discuss here. Certain results of this sort can be obtained for models with long-range interations, see e.g. [K00], see also [BoZa01], [BuMePr97], [LMP98].)

The continuous spin random field model: ferromagnetic ordering in $d\geq 3$ (Results of [K99a*])

In the context of disordered systems the continuous spin version corresponding to the random field model is an important model to study. Here the spin variables m_x take values in $\mathbb{R}$ and the formal Hamiltonian for a spin-configuration $m_{\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ in the infinite volume is given by

$$E\left(m_{\mathbb{Z}^d}\right) = \frac{q}{2} \sum_{<x,y>} \left(m_x - m_y\right)^2 + \sum_x V(m_x) - \sum_x \eta_x m_x \tag{1.1}$$

where the first summation extends over all pairs of nearest neighbors $< x, y >$. (The finite volume Gibbs measures are then obviously formed by taking the exponential of the negative finite volume restriction of (1.1) as the non-normalized Lebesgue-density.) The potential V has a symmetric double-well structure. The most popular choice is that of a polynomial of fourth order. For concreteness we will stick to it. We choose a scaling where the potential has unity curvature in the minima $\pm m^*$ that is

$$V(m_x) = \frac{\left(m_x^2 - (m^*)^2\right)^2}{8 {m^*}^2}$$

and investigate the Gibbs measures for $q \geq 0$ sufficiently small and $q\,(m^*)^2$ sufficiently large. The latter quantity gives the order of magnitude of the minimal energetic contribution to the Hamiltonian (1.1) caused by neighboring spins in different wells. Thus it corresponds to a Peierls constant.

Here the $(\eta_x)_{x \in \mathbb{Z}^d}$, are i.i.d. symmetrically distributed random variables that satisfy the probabilistic bound $\mathbb{P}\left[\eta_x \geq t\right] \leq e^{-\frac{t^2}{2\sigma^2}}$ where the $\sigma^2 \geq 0$ governing the smallness of the random variables has to be sufficiently small. Moreover we impose a fixed uniform bound on $|\eta_x|$, independent of σ^2. This is for technical reasons. In this context we show that there is in fact a 'ferromagnetic' phase transition, in dimensions $d \geq 3$, for sufficiently small σ^2 (meaning small disorder), sufficiently large $q(m^*)^2$, and not too big $q(m^*)^{\frac{2}{3}}$ (controlling the 'anharmonicity' of the minima, as it can be seen from the proof). We prove the following: The [random] Gibbs-probability (w.r.t. to the finite volume-measure with plus-boundary conditions) of finding the spin left to the positive potential well is very small, uniformly in the volume, on a set of realizations of η of a size [w.r.t $\mathbb{P}$] of at least $1 - e^{-\frac{const}{\sigma^2}}$. The precise statement is found in Theorem 1 p.1272 of [K99a*]. For more information and explanation we refer to the introduction of [K99a*]. Let us however mention the following: The particular form of the potential as a fourth order polynomial is of no importance, as well as the requirement of uniform boundedness on the random fields and the restriction to nearest neighbor couplings in the Hamiltonian (instead of finite-range interactions) could be given up.

The novelty of the proof is the use of a stochastic mapping of the continuous spins to their sign-field (independently over the sites). We choose it such that the probability that a continuous spin m_x is mapped to its sign is given by $\frac{1}{2}(1 + \tanh(am^*|m_x|))$. (Here a is a parameter close to one that needs to be tuned in a useful way.) The image measure of a particular sign-configuration then gives the approximate weights of finding continuous spins in the neighborhood of the potential wells indexed by these signs. Using a suitable combination between high temperature and low temperature expansions it is shown that the resulting model has the form of an Ising model with exponentially decaying interactions. (These expansions are related to those used by [Za00] in the translation-invariant context where however, due to the lack of positivity, no probabilistic interpretation can be given.) This can be seen as a 'single-site-coarse-graining'-method. Next, having constructed the Ising-system, it can be cast into a contour model representation for which the renormalization group of [BrKu88] can be used.

This mapping is really compatible with the infinite volume limit in the sense that the infinite system under consideration is mapped to an infinite volume *Gibbs* measure of an Ising model (see Theorem 2 of [K99a*], p.1273). So, this stochastic map also provides an interesting example of a 'coarse-graining without pathologies'. This means that the coarse-graining produces no 'artificial' non-local dependencies in the conditional expectations of the resulting measure. Let us remind the reader that this need not be the case in general in the sense that there are many examples of 'innocent transformations' acting on 'innocent' infinite volume Gibbs measures that produce non-Gibbsian measures as images.

These example mainly come from the coarse-graining transformations motivated by the 'renormalization group'. Maybe the simplest example of such a transformation is taking marginals on a sublattice of the Gibbs distribution of an ordinary nearest neighbor Ising model in the plus phase at low temperatures in zero magnetic field. In Chapter 3 we will come back to the question whether and to what extent certain natural measures arising in the context of disordered systems can be interpreted as Gibbs measures, when we discuss in more detail the papers [K99b*],[K01a*].

Stability for a continuous SOS-interface model in a randomly perturbed periodic potential in $d+1 \geq 3+1$ (Results of [K98d*])

The result of the second paper [K98d*] concerns the stability of a (so-called) continuous interface model. In this model an interface without overhangs is modelled by a continuous-valued height-configuration $(m_x)_{x\in\mathbb{Z}^d}$ over the d-dimensional lattice that is subjected to a weakly disordered *random* potential $V_x(m_x)$. The Hamiltonian reads

$$E\left(m_{\mathbb{Z}^d}\right) = \frac{q}{2} \sum_{<x,y>} (m_x - m_y)^2 + \sum_x V_x(m_x) \tag{1.2}$$

For this model the random single-site potential $V_x(m_x)$ is site-wise independent again, and chosen so that it becomes periodic under the shift orthogonal to the base plain in the limiting case of vanishing disorder. For technical reasons the particular choice as a logarithm of an infinite sum of Gaussian terms with random parameters is most convenient (see page 2 of [K98d*]). For simplicity we restrict the analysis to this case, although perturbations around this form could also be treated.

In [K98d*] we prove that, for almost all realizations of the random potentials, the model possesses Gibbs measures that describe localized interfaces in a fixed height, in dimensions $d+1 \geq 3+1$, for a choice of parameters corresponding to low-temperatures and small disorder. (See [K98d*] Theorem 1, page 3.)

For the proof we generalize the method of stochastic mapping from continuous variables to discrete variables that was used in [K99a*]. While we had to deal with a double-well potential therein we must now take care of an infinite number of wells. Thus we must use a suitable $\mathbb{Z}$-valued stochastic map (corresponding to the smoothed map to the sign-field of [K99a*]). This allows to use the discrete renormalization group method for the contour model representation of the image model that was developed in [BoK94]. It was used there to treat the contour model representation of a similar (slightly simpler nearest neighbor) $\mathbb{Z}$-valued model.

Given our special choice of the potential it turns out that the (relevant) continuous-variable infinite volume random Gibbs measures μ can then be written in a nice representation as superpositions of massive Gaussian fields in the infinite volume (see Theorem 2 of [K98d*], page 5): Denoting by $\mathcal{N}[m;(1-q\Delta)^{-1}]$ the Gaussian field with covariance matrix $(1-q\Delta)^{-1}$ and expected value m we have that

$$\mu = \int \nu(dh)\mathcal{N}[\hat{m}(h),(1-q\Delta)^{-1}] \tag{1.3}$$

where the continuous infinite volume configuration $\hat{m}(h)$ is an (approximately) local function on the discrete configurations $h=(h_x)_{x\in\mathbb{Z}^d} \in \mathbb{Z}^{\mathbb{Z}^d}$. The measure $\nu(dh)$ is a Gibbs measure of the random (w.r.t. disorder variables) integer-valued model arising as image under the stochastic transformation. In particular the formula applies to those Gibbs measure $\nu=\nu^k$ that describe a localized discrete interface at given height k, carrying over the localization property to the continuous model. It is not difficult to formally obtain the decomposition formula (1.8) given the particular definition of the potential, but to prove in the infinite volume, one needs certain localization properties of the discrete model $\nu(dh)$. (These hold in particular for the measures $\nu=\nu^k$.)

2. VOLUME DEPENDENCE AND METASTATES

Background

We now come to the second aspect of Gibbs measures for disordered systems that we want to focus on. We start with some motivation. Having just described the continuous spin version of it, let us come back again to the random field Ising model that was already described in the last part of Chapter 0. We look at it in three or more dimensions, in the regime of 'small disorder' and low temperatures, as an example of a disordered system that shows distinguished phases. For this model it follows from the proof of [BrKu88] that, for $\mathbb{P}$-a.e. η, the finite volume Gibbs measures with all +-boundary conditions $\mu_\Lambda^+[\eta]$ converge weakly (that is on local observables) to the infinite volume plus-state $\mu^+[\eta]$ as $\Lambda \uparrow \mathbb{Z}^d$ (say, along a sequence of nested cubes). In the same way we have that $\mu_\Lambda^-[\eta]$ converges to $\mu^-[\eta]$. This behavior is an example of a *simple scenario* that can happen for the volume dependence of disordered systems (even though it might not be simple to prove). Here the boundary condition *preselects the particular infinite volume Gibbs measure.* This situation is of course the standard situation for low-temperature systems without disorder. For systems falling into the realm of Pirogov-Sinai theory the situation can be analysed in great detail. Here, when there are different q-like (q'-like) infinite volume Gibbs states μ^q (and $\mu^{q'}$) for the same interaction potential, they can be constructed as a weak limit of the finite volume Gibbs measures μ_Λ^q with the appropriate all q-boundary condition. Moreover, the speed of the approach to the limit on given observables can be controlled by cluster-expansions.

There are however natural cases of disordered systems where one is interested in boundary conditions that do not preselect a particular infinite volume Gibbs state.

Spinglasses

Let us deviate a little and talk about spin-glasses for a motivation of what follows. We won't discuss any result for a real spin-glass model in any of our papers and the reader who is not interested in them may directly go to 'Metastates'.

A situation where the connection between boundary condition and infinite volume Gibbs state is complicated can be expected e.g. in the famous Edwards-Anderson spin-glass. [This model has the Hamiltonian $H^J(\sigma) = \sum_{<x,y>} J_{x,y}\sigma_x\sigma_y$ where $\sigma_x \in \{-1,1\}$ and the $J_{x,y}$ are i.i.d. mean zero Gauß variables.] Unfortunately, little is rigorously known about this model, none of the mentioned methods can be applied to it, and we won't discuss it here. There is however agreement in the belief that there are multiple phases at sufficiently high dimensions.

There are also more detailed conjectures about the Gibbs measures that are based on the heuristic solution by Parisi (see [MePaVi87]) of the corresponding so-called mean-field model, which is known as *Sherrington-Kirkpatrick model.* Generally, in the definition of a mean-field model corresponding to a lattice model, the lattice $\mathbb{Z}^d$ is replaced by the complete graph with vertices $\{1,\dots,n\}$. Nearest neighbor interactions are replaced by 'corresponding' interactions between all pairs of spins. For this to make sense in the limit $n \uparrow \infty$ of a large number of spins, one needs the strength of the interactions to scale appropriately with n. In the case of the EA-spinglass this leads to the corresponding definition

$$\mu_n[J]\left((\sigma_i)_{i=1,\dots,n}\right) = \frac{1}{Norm.} \exp\left(\frac{\beta}{2\sqrt{n}} \sum_{1\le i,j\le n} J_{i,j}\sigma_i\sigma_j\right) \tag{2.1}$$

for the *finite volume Gibbs measures of the Sherrington-Kirkpatrick model*, where $\sigma_i = \pm 1$ are Ising spins and the $J_{i,j}$ are i.i.d. standard Gauß variables. Now, the famous heuristic solution of this SK-model by Parisi is however still far from being mathematically justifiable (although generally accepted by physicists). Worse than that, not all of its predictions can be unambiguously interpreted in terms of meaningful mathematical objects. Despite of this all it is taken as a basis in the physics literature to conjecture that there are infinitely many pure states at low temperatures, in sufficiently high dimensions also in the lattice model. However, this so-called 'SK-picture' put forward by Parisi and co-workers is not undisputed among physicists [BDM98] with numerical simulations giving no clear evidence. There is still no mathematical understanding of the low-temperature phase in the SK model. There has however been made remarkable progress in particular in the mathematical analysis of simpler related mean-field spinglass-type models (like the Hopfield model and the so-called p-spin model) and also progress for the SK model itself ([BoGa98a],[BoSz98],[Ta98],[Ta00a-d]).

Now, a different approach was that of Newman and Stein ([NS96a,b], [NS98a], [Ne99]) whose aim was to rule out some of the conjectures for the lattice spin-glass with the use of softer arguments by carefully examining the notion of the infinite system. Newman and Stein noted that a phenomenon they called 'chaotic size dependence' is likely to occur. By this it is meant that, for boundary conditions that are not specially chosen to pick a pure phase, it is possible to have many different limiting states along a subsequences of volumes tending to $\mathbb{Z}^d$ while the realization of the disorder variables is fixed. Examples of such boundary conditions are all-plus, open, or periodic boundary conditions in the EA model.

Metastates

To account for such situations in the general context of disordered systems and define a meaningful limiting object that describes the asymptotic large-volume behavior Newman and Stein proposed the following: Look at a sequence of finite volume Gibbs measures $\mu_{\Lambda_n}[\eta]$ (for a given fixed boundary condition) in terms of their *empirical average*

$$\kappa_N(\eta) = \frac{1}{N} \sum_{n=1}^{N} \delta_{\mu_{\Lambda_n}[\eta]} \tag{2.2}$$

taken along the 'trajectory' Λ_n (say, a sequence of cubes). See, if it converges with $N \uparrow \infty$! This is in analogy to the construction of invariant measures for dynamical systems. Now the role of the time is taken by the given sequence of volumes. They called the resulting object *empirical metastate*. It will thus be a probability measure on the Gibbs measures of the system that depends on the particular realization of the disorder variables η. The interpretation is the following: The metastate gives the likelihood of finding a disordered system in a particular Gibbs measure when we choose a very large system.

There are general existence results about the convergence for $\mathbb{P}$-a.e. η that follow from compactness arguments but these are only for sparse enough subsequences of n's and N's (see [Ne99]). These results hold, if one sticks to a *local* notion of convergence for all measures appearing, where convergence of expectations of local functions has to be checked.

**Metastates in Disordered Mean-Field Models:
Random Field and Hopfield Models (Results of [K97*] and [K98b*])**

After Newman and Stein had proposed the metastate-formalism we gave the first two rigorous examples of non-trivial metastates of disordered systems. These examples are simple and well-known mean-field systems (see [K97*],[K98b*]). They showed in particular that it is really necessary in general to take a subsequence of a given sequence of volumes Λ_n to get a.s. convergence for the empirical mean (2.2). This phenomenon is in contrast to an earlier conjecture. Later, also metastates for more complicated (however mean-field) models were constructed ([BoGa98b], [BEN99], [To99], [BoMa01].)

Our first example is the easier one of the two, and it is probably the easiest system showing nontrivial behavior of the metastate. It is the *Curie Weiss Random Field Ising Model* (CWRFIM) whose *Gibbs measures* in the finite volume $\Lambda_n \equiv \{1, \ldots, n\}$ are given by

$$\mu_n[\eta]\left((\sigma_i)_{i=1,\ldots,n}\right) = \frac{1}{Norm.} \exp\left(\frac{\beta}{2n}\sum_{1\le i,j\le n}\sigma_i\sigma_j + \beta\sum_{1\le i\le n}\eta_i\sigma_i\right) \tag{2.3}$$

Here $\sigma_i = \pm 1$ are Ising spins and η_i are taken as i.i.d. variables with $\mathbb{P}\left[\eta_i = \pm\epsilon\right] = \frac{1}{2}$. Our second example will be the *Hopfield Model with finite number M of patterns*, to be described below. The advantage of these mean field models is that they allow rigorously to make sense out of an approximate extreme decomposition of the form

$$\mu_n[\eta] \approx \sum_m p_n^m(\eta)\mu_\infty^m[\eta] \tag{2.4}$$

Here η is a generic notation for the quenched disorder variable, $\mu_\infty^m[\eta]$ are the 'extremal infinite volume Gibbs measures' describing the m' th phase, and $p_n^m(\eta)$ are the random weights whose large n-behavior contains the phenomenon of size dependence.

The phase diagram of the CWRFIM is well known. At low temperatures $1/\beta$ and small ϵ the model is ferromagnetic, i.e. there exist two 'pure' phases, a ferromagnetic $+$ phase $\mu_\infty^+[\eta]$ and a $-$ phase $\mu_\infty^-[\eta]$. This is the same picture as for the lattice model in 3 or more dimensions, but it is much easier to obtain than in the lattice model.

In this situation we have Theorem 1 of [K97*] that gives the additional information about the corresponding *metastate*. It says that the empirical metastate taken along the sequence $\{1, \ldots, n\}$ does *not* converge for a.e. realization. However it *does converge in* *distribution.* Looking at its expectation of a local function F on the states of the system we have

$$\lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=1}^{N}F(\mu_n[\eta]) =^{\text{law}} n_\infty F\left(\mu_\infty^+[\eta]\right)+(1-n_\infty)F\left(\mu_\infty^-[\eta]\right) \tag{2.5}$$

where n_∞ is a 'fresh' random variable, independent of η on the r.h.s., with arcsine-distribution (that is $\mathbb{P}\left[n_\infty < x\right] = \frac{2}{\pi}\arcsin\sqrt{x}$). A simple heuristic explanation of this result can be found in the introduction of [K97*], below the statement of Theorem 1. Let us remark here that we expect the non-convergence of the empirical metastate for fixed realization to occur also in the lattice random field Ising model in the phase transition regime if we use a sequence of nested boxes $(\Lambda_n)_{n=1,2,\dots}$ containing $|\Lambda_n|\sim n^d$ spins (see e.g. the explanation in [K98a].) On the other hand, if one takes as Λ_n a deterministic sequence of volumes that is sufficiently sparse, convergence of the l.h.s. of (2.5) takes place to $\frac{1}{2}F\left(\mu_\infty^+[\eta]\right)+\frac{1}{2}F\left(\mu_\infty^-[\eta]\right)$ for $\mathbb{P}$-a.e. η.

In our second example, the *Hopfield model with finite number M of patterns*, the metastate structure is richer. We mention this example because it shows in particular that it is possible that the metastate gives mass also to non-trivial mixtures of extremal Gibbs measures (at least in a mean-field model). For this model the finite volume Gibbs measure is given by

$$\mu_n[\xi]\left((\sigma_i)_{i=1,\dots,n}\right)=\frac{1}{Norm.}\exp\left(\frac{\beta}{2n}\sum_{1\le i,j\le n}\sum_{1\le\nu\le M}\xi_i^\nu\xi_j^\nu\sigma_i\sigma_j\right) \tag{2.6}$$

The 'disorder' enters through the so-called patterns $\xi^\mu=(\xi_i^\mu)_{i\in\mathbb{N}}$ with i.i.d. bits with $\mathbb{P}\left[\xi_i^\mu=\pm1\right]=\frac{1}{2}$. It is well-known that the role of the plus and the minus state as extremal Gibbs measures in the CWRFIM is now played by M symmetric mixtures of pairs of extremal measures, the so-called Mattis states $\mu_\infty^\nu[\xi]$. (The measure with the index ν has typical spin-configurations that resemble the pattern with index ν or its global spin-flip.) It turns out that, again, the empiricial metastate taken along the sequence $\{1,\dots,n\}$ does *not* converge for a.e. realization, but it does converge in distribution. The limiting expression looks more complicated than that of the CWRFIM. We have

$$\lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=1}^{N}F(\mu_n[\xi]) =^{\text{law}} \int_0^1 dt\ F\left(\sum_{\nu=1}^{M}p^\nu\left(\frac{W_t}{\sqrt{t}}\right)\mu_\infty^\nu[\xi]\right) \tag{2.7}$$

Here W_t is a 'fresh' Brownian motion in a space of $M\times M$-matrices, independent of ξ on the r.h.s. The probability vector $\left(p^\nu(\,\cdot\,)\right)_{\nu=1,\dots,M}$ is a known function on this matrix space. Obviously, the t-integral just arises as a continuous version of the sum on the l.h.s. If one takes as Λ_n a deterministic sequence of volumes that is sufficiently sparse, convergence for a.e. pattern ξ takes place to the expression $\mathbb{E}_g F\left(\sum_{\nu=1}^{M}p^\nu\left(g\right)\mu_\infty^\nu[\xi]\right)$.

Here $\mathbb{E}_g$ is the expectation of the variable g w.r.t. a Gaussian distribution in the matrix-space. We note that in fact all mixtures of Mattis states appear with positive probability density. This is in contrast to the situation in the CWRFIM where the metastate gives mass only to the pure phases.

In the paper [K98b*] (which is a follow-up to [K97*]) we also proved refinements of those convergence results in the two above mean-field models. Therein we constructed the limiting processes of the whole paths $t \mapsto \mu_{[tN]}[\eta]$ as N tends to infinity, obtaining an object that was termed 'superstate' by [BoGa98b].

A random energy model for size dependence: recurrence vs. transience (Results of [K98c*])

For systems with infinitely many pure Gibbs states new phenomena can be expected to appear. In [K98c*] we defined a simple heuristic model to understand possible different scenarios in the behavior of such systems.

The model consists of a simple ansatz for the form of the weights appearing on the r.h.s. of (2.4) in a hypothetical extreme decomposition for the large volume Gibbs measures of a disordered system. To make this ansatz, we simply assume that in the volume labeled by N the system is in a superposition of only the 'first' M_N states $\nu = 1, \dots, M_N$. The function M_N thus gives the maximal number of states that can be 'seen' by a system of size N. For us it will be just a parameter of our effective model.

The precise definition of the model is the following. For each N we define a random probability distribution $q_N \equiv (q_N^\nu)_{\nu \in \mathbb{N}}$ supported on $\{1, 2, \dots, M_N\} \subset \mathbb{N}$ by putting

$$q_N^\nu := \frac{e^{\lambda X_N^\nu}}{\sum_{\mu=1,\dots,M_N} e^{\lambda X_N^\mu}} \tag{2.8}$$

for $\nu = 1, \dots, M_N$. Here $\lambda > 0$ is a constant, and $(X_N^\nu)_{\nu \in \mathbb{N}, N \in \mathbb{N}}$ are Gaussian random walks in the index N, with standard normal increments, independent over the index ν (labelling the state). We ask: How does this random probability distribution on the integers behave for large N, for typical realizations? Here we focus on the large N-behavior of the *paths* $(q_N)_{N \in \mathbb{N}}$ in the space of probability distributions on $\mathbb{N}$. We also investigate a slightly more complicated version of the model where the family of independent random walks is replaced by branching random walks.

This model can be seen as a generalization of Derrida's random energy model. As a motivation, let us mention that it should describe an approximation for the true weights appearing in the approximate extreme decomposition of a certain modification of a Hopfield model with external magnetic field, and in that of a model for interfaces in a random environment in a particular geometry.

The model given by (2.8) can be analysed in detail. First of all, the weights are concentrated on their maximum over ν, with large probability. This is true for reasonable choices of the parameters, see [K98c*] Theorem 1, page 64. More interestingly, it turns out that there is a transition between *recurrence* and *transience*, depending on the *growth* of the function M_N (see [K98c*] Theorem 2, page 65). 'Transience' means here: the weights q_N^ν of all states ν converge to zero with the volume label N tending to infinity (for almost every realization of the random walks X_N^ν). The interpretation of this is that the system takes any given state only for a finite number of volumes. 'Recurrence' means here: existence of subsequences of volumes N_K such that the weight $q_{N_K}^\nu$ converges to one when K is tending to infinity, for all states ν. This means, the system returns to every possible state an infinite number of times. As we prove in [K98c*], the 'critical regime' for the growth (where the behavior switches) turns out to be $M_N \sim (\log N)^p$ (with critical point $p = 1$). In this regime we compute the almost sure large N asymptotics of the relative weights for finding a particular state (see [K98c*] Theorem 2', page 66). We also compute the set of a.s. cluster points of the corresponding occupation times (corresponding to the empirical metastate, see [K98c*] Theorem 3, page 67).

3. THE GIBBSIAN NATURE OF THE JOINT MEASURES

In Chapter 1 we have already investigated the question whether a given measure could be interpreted as a Gibbs measure in the infinite volume. The measure under consideration was the image of the continuous-spin random field measure under the stochastic map to Ising spins. The answer was: yes, and the interaction potential could be explicitly constructed. Such a result has to be seen in the context of the long discussion in the mathematical statistical mechanics community about the appearance of non-Gibbsian measures.

Let us recall that Gibbs measures of an infinite volume lattice spin system are characterized by the fact that their conditional expectations can be written in terms of an absolutely summable potential. [This is to be understood in the sense of formula (0.3). It must be tested for all finite volumes Λ outside of which the conditioning takes place.] When we ask for Gibbsianness we are thus faced with the task

Given a measure μ on a lattice system, find a corresponding interaction potential Φ!

By an old result of Kozlov the existence of a potential is equivalent to the continuity of the conditional expectations $\mu\left(\sigma_\Lambda \middle| \sigma_{\mathbb{Z}^d\setminus\Lambda}\right)$ as a function of the conditioning $\sigma_{\mathbb{Z}^d\setminus\Lambda}$ (w.r.t. the product topology.) The possibility that simple transformations can produce non-Gibbsian measures from Gibbsian ones was first observed in the context of the so-called renormalization group transformations (the first examples were discovered by [GrPe79]). Being faced with the possibility that the transformed system could not be described in terms of a regular interaction potential seemed to be frightening. In fact, in theoretical physics the existence of a 'renormalized Hamiltonian' was always taken for granted, and taken as a starting point for numerous approximation schemes. For a clear presentation of various mechanism leading to non-Gibbsian measures, see [EnFeSo93]. For a discussion of the relevance of this phenomenon see also the more recent [En99],[Fe98].

Joint measures in product space

In the next two papers we investigated the Gibbsian nature of a large class of measures that appear in the context of disordered lattice spin systems. We consider a disordered lattice spin model fitting into the setup of Chapter 0. That is, spin-variables as well as disorder variables take values in corresponding finite sets. The range of the interaction is finite. As before, we denote the Gibbs measures in finite volume Λ by $\mu_\Lambda[\eta](d\sigma)$. The spin lattice-variable is $\sigma = (\sigma_x)_{x\in\mathbb{Z}^d}$ and $\eta = (\eta_x)_{x\in\mathbb{Z}^d}$ is a lattice random variable with product distribution $\mathbb{P}$ (describing the disorder of the model.) It is good to think here again of the random field Ising model as a concrete example. Our aim is then to look at the *joint measures* in the infinite volume that are given by the possible limits of $\lim_\Lambda \mathbb{P}(d\eta)\mu_\Lambda[\eta](d\sigma)$ as Λ tends to $\mathbb{Z}^d$. Here we assume that we have fixed a particular boundary condition. It is suppressed in the notation. These measures are then probability measures $K(d\xi)$ on the space of *joint spin configurations* $\xi = (\sigma,\eta) = (\sigma_x,\eta_x)_{x\in\mathbb{Z}^d}$. We ask

Can these joint measures be interpreted as (generalized) Gibbs measures on the product space of spin-variable and disorder variable ?

Despite the analogy with the problem of renormalization group pathologies there was no systematic mathematical investigation of the problem so far. Our present general investigation was motivated by the special recent example of the Ising ferromagnet with site dilution. For this example [EMSS00] discovered that the corresponding joint measure at low temperature, low dilution is not a Gibbs measure in the product space. To ask for Gibbsianness more generally is then a natural mathematical question. It is also of some physical relevance. In fact, the formal interpretation of the joint measures as Gibbs measure is known in the physics literature as the starting point of the so-called Morita approach to the description of disordered systems ([Ku96], [Mo64], [EKM00]).

(Non-) Gibbsianness and phase transitions in random lattice spin models (Results of [K99b*])

Now, on the 'negative side', it turns out as a consequence of [K99b*] that for many systems in a low-temperature region the ordinary Gibbs property fails. The ordinary Gibbs property demands that the conditional expectations can be written in terms of an absolutely summable potential. [Recall: A potential $(U_A(\xi))_{A\subset\mathbb{Z}^d}$ is called absolutely summable, iff $\sum_{A \ni x} \sup_\xi |U_A(\xi)| < \infty$ for all lattice sites x.] In the paper [K99b*] we give criteria that explain the link between phase transitions of the disordered system for fixed realizations, and Gibbs property in product space: Loosely speaking, a discontinuity in the quenched Gibbs expectation $\mu[\eta]$ can destroy the Gibbs property in product space, if it can be observed for the spin-observables that are conjugate to the local disorder variables. This is best understood in the example of the random field Ising model where the corresponding observable is just the magnetization.

For the random field Ising model we show more precisely the following: In every dimension, the so-called almost sure Gibbs property for the joint system holds precisely in the single-phase region of the phase diagram. The almost sure Gibbs property for the joint system does not hold in the multi-phase region of the phase diagram.

Here, a measure $K(d\xi)$ is called *almost Gibbs*, iff the set of discontinuity points of its conditional expectations $K(\xi_\Lambda \big| \cdot)$ has zero measure w.r.t. the measure K itself. So, the notion of 'almost Gibbsianness' is one natural possibility of a relaxation of the usual Gibbs property, where one demands that the set of discontinuities is empty. It was proposed in the context of RG-pathologies, for a discussion see [MRM99].

Hence, the example of the Ising ferromagnet in a weak random magnetic field, at low temperature, in 3 or more dimensions gives a 'strong pathology' since these condition imply the existence of ferromagnetic order. This kind of 'strong pathology' does not hold for the example of [EMSS00], by the way, where there is still almost Gibbsianness.

Weakly Gibbsian representations for joint measures of quenched lattice spin models (Results of [K01a*])

A different generalization of the classic Gibbs property is the so-called *weak Gibbs property.* It goes back to Dobrushin. Here one asks only for the existence of a potential $(U_A(\xi))_{A \subset \mathbb{Z}^d}$ that converges K-almost everywhere. [That is, the sums $\sum_{A \ni x} |U_A(\xi)|$ need to be finite only for K-a.e. ξ and not necessarily for all ξ.] Intuitively speaking, one allows for potentials with a 'configuration-dependent range of interaction'. Now, on the 'positive' side we prove that there is always a potential (depending on both spin and disorder variables) for the joint measure that converges absolutely on a set of full measure w.r.t. the joint measure ('weak Gibbsianness'). This is somewhat surprising. The proof is soft and exploits the specific structure of the joint measures whose marginals on the η are product measures. It uses a generalization of Kozlov's construction and a martingale argument. However, if one is interested in more specific properties of the potential one likes to construct, more assumptions are needed. We also provide general conditions giving the convergence of vacuum potentials, conditions for the decay of the joint potential in terms of the decay of the disorder average over certain quenched correlations, and finally discuss some applications to models with random couplings.

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