The Ising model: Highlights and perspectives

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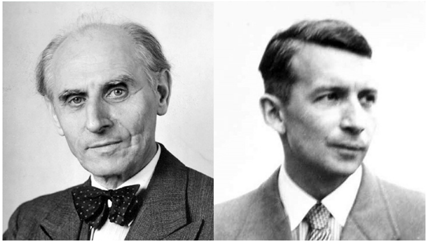
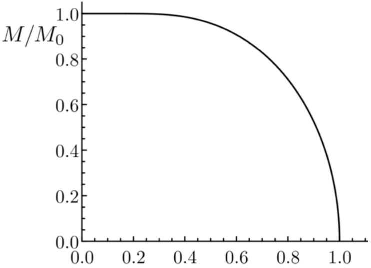
A theory for ferromagnetic ordering?



Wilhelm Lenz (1888 Frankfurt - 1957 Hamburg) was Ph.D. advisor to

Ernst Ising (1900 Cologne - 1998 Peoria, Illinois) at Hamburg



Spontaneous magnetization of a ferromagnet as a function of temperature.

Can this be explained by a short-range probabilistic model, following statistical mechanics?

Definition of the Ising model

Model invented by Wilhelm Lenz in 1920 with the idea:

Only minimal local structure, only nearest neighbor interaction

Magnetic moments (spins)

$\sigma_i \in \{1, -1\}$ are sitting at sites $i \in \Lambda \subset \mathbb{Z}^d$,

(Other choices of base spaces of interest today: networks,

other interpretations of magnetic moments e.g. as "opinions")

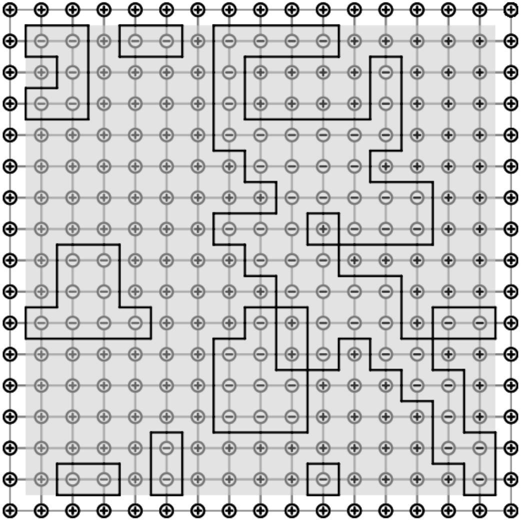
Gibbs distribution

(Probability measure depending on inverse temperature β):

Probability to see a configuration $\sigma_\Lambda = (\sigma_i)_{i \in \Lambda}$ given by

$$\frac{1}{Z_\Lambda} \exp(\beta \sum_{i \sim j} \sigma_i \sigma_j + h \sum_i \sigma_i)$$

Definition of the Ising model



Two-dimensional Ising model with plus-boundary condition

picture from Friedli-Velenik book

Phase Transitions and long-range order

Questions:

- Spontaneous magnetization at large β?
- Does the model have a magnetic phase transition ?
- What is actually a phase transition mathematically?
- Behavior of other physical quantities?

Ernst Ising (Ph.D. thesis 1924 Hamburg): Explicit computations show that the model has **no phase transition in spatial dimension** $d = 1$, published in

Zeitschrift für Physik (since 1997 European Physical Journal) 1925

Phase Transitions and long-range order in lattice dimensions $d \geq 2$

Rudolph Peierls
(1907 Berlin - 1995 Oxford)

Proof of long-range order in space dimensions $d \geq 2$

given in the year 1936

Phase Transitions and long-range order in lattice dimensions $d = 2$

Lars Onsager (1903 Oslo - 1976 Florida),

Nobel prize chemistry 1968 for non-equilibrium statistical mechanics

L. Onsager, Phys. Rev. (1944): The **free energy** of the Ising model in zero external field $h = 0$ in lattice dimension $d = 2$ is

$$\lim_{\Lambda} \frac{1}{|\Lambda|} \log Z_\Lambda = \log 2$$

$$+ \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log\Big(\cosh(2\beta)^2 - \sinh(2\beta)\big(\cos\vartheta_1 + \cos\vartheta_2\big)\Big) d\vartheta_1 d\vartheta_2$$

Onsager uses combinatorics, algebra, determinants

Modern proof via fermionic integration

Phase Transitions and long-range order in lattice dimensions $d = 2$

Critical inverse temperature $2\beta_c = \log(1+\sqrt{2})$
at which specific heat diverges

Hendrik Kramers - Gregory Wannier: self-duality temperature (1941)

Onsager/Kaufmann: Power law behavior for magnetisation via computation
$m(\beta) = \langle \sigma_0 \rangle \sim (\beta - \beta_c)^{\frac{1}{8}}$ as $\beta \downarrow \beta_c$

Critical behavior and universality: a physical perspective

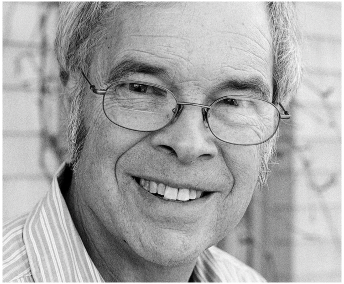
Universality conjecture (supported by physical experiments):

Critical behavior and **critical exponents** for short-range systems depend only on spatial dimension d, spin dimension (universality classes)

$$m(\beta) = \langle \sigma_0 \rangle \sim \begin{cases} (\beta - \beta_c)^{0.3264\ldots} & d = 3 \\ (\beta - \beta_c)^{\frac{1}{2}} & d \geq 4 \end{cases}$$

Many materials behave like prototypical models, e.g. like the Ising model!

Critical behavior and universality: a physical perspective



Kenneth Wilson
(1936 Massachusetts - 2013 Maine)

Nobel prize in theoretical physics 1982

Universality and power laws explained by (non-rigorous) renormalization group theory

Renormalization group theory investigates the system under scaling (zooming out) looking for non-standard limit theorems

What is a phase transition? Infinite systems and Gibbs simplex

Roland Dobrushin
(1929 St Petersburg - 1995 Moscow)

Oscar Lanford
(1940 NYC -
2013 Switzerland)

David Ruelle
(1935 Ghent, Belgium -)

Rigorous theory of phase transition for **infinite systems**

What is a phase transition? Infinite systems and Gibbs simplex

DLR-formalism

$$\mu\gamma_\Lambda = \mu$$

DLR consistency equation for μ probability measure on $\{-1,1\}^{\mathbb{Z}^d}$

Multiple Gibbs measures μ (multiple macroscopic states)
for the same model parameters (temperature, external fields, ...)
indicate a phase transition

Aizenman, Fröhlich, Lebowitz, ...
Varadhan (**large deviations**)

Scaling limits of critical systems in $d = 2$: Stochastic Loewner evolution

Ising model, (percolation, random walks with interaction, ...)
at critical point of the model, after a **scaling limit**
give rise to **fractal random curves described by SLE**κ

$\kappa \in (0, 8)$ captures all model-dependence,
e.g. Ising interfaces $\kappa = 3$

κ relates to critical exponents

Scaling limits of critical systems in $d = 2$: Stochastic Loewner evolution

Oded Schramm
(1961 Jerusalem - 2008 Washington)

Invented SLE 2000

Wendelin Werner
(1968 Cologne -)

Fields medal 2006

Stanislav Smirnov
(1970 Leningrad -)

Fields medal 2010

Hugo Duminil-Copin
(1985 Île-de-France -)

Fields medal 2022

$d = 3, 4$

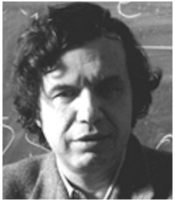
Disordered Ising models

Random "disordered" Gibbs distribution with spin probabilities

$$\frac{1}{Z_\Lambda} \exp(\beta \sum_{i \sim j} J_{ij} \sigma_i \sigma_j + \sum_i h_i \sigma_i)$$

Random field systems, spin glasses $J_{ij} \sim \mathcal{N}(0,1)$

- When does randomness of the interaction **destroy a phase transition**?
- When does randomness of the interaction **create new complex phases**?



Giorgio Parisi (1948 Rome -)

(mathematically non-rigorous) solution of quenched mean-field spin-glass

Nobel Prize in Physics (2021) "interplay of disorder and fluctuations in physical systems"

Disordered Ising models

Michel Talagrand (1952 Béziers, France -)

Concentration of measure, chaining

Validity of Parisi Formula (2006)

Abel Prize 2024

- What about influence of **Spatial Disorder** on collective spin-behavior?

Dilute lattices, random networks, point clouds,

spin dynamics,

dynamical networks, ... ?