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Non-Atomicity of the extremal decomposition of the free state for finite-spin models on Cayley trees

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Abstract. We prove the non-atomicity of the extremal decomposition measure of the free state of low temperature Potts models, and more generally of ferromagnetic finite-spin models, on a regular tree, including general clock models. The decomposition is supported on uncountably many inhomogeneous extremal states, that we call *glassy states*. The method of proof provides explicit concentration bounds on *branch overlaps*, which play the role of an order parameter for typical extremal states. The result extends to the counterpart of the free state (called *central state*) in a wide range of models which have no symmetry, allowing also the presence of sufficiently small field terms. Our work shows in particular that the decomposition of central states into uncountably many glassy states in finite-spin models on trees at low temperature is a generic phenomenon, and does not rely on symmetries of the Hamiltonian.

Résumé. Nous prouvons la non-atomicité de la mesure de décomposition extrémale de l'état libre pour les modèles de Potts à basse température, et plus généralement les modèles ferromagnétiques à spin fini, modèles d'horloge généraux compris, sur un arbre régulier. La décomposition est portée par d'indénombrables états extrémaux inhomogènes, que nous appelons *états vitreux*. La méthode de preuve fournit des bornes de concentration explicites sur les *recouvrements de branches*, qui jouent le rôle de paramètres d'ordre pour les états extrémaux typiques. Le résultat s'étend à l'analogue de l'état libre (appelé *état central*) dans une grande famille de modèles sans symétrie, permettant notamment la présence de champs suffisamment petits. Nos travaux montrent en particulier que, dans des modèles à spin fini sur des arbres réguliers, à basse température, le phénomène de décomposition extrémale des états centraux portée par d'indénombrables états vitreux est un phénomène générique, et ne repose pas sur les symétries du hamiltonien.

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1. Introduction

Models of statistical mechanics supported on trees or non-amenable graphs, are known to show rather interesting behavior with multiple critical values, which displays complex phenomenology and challenges for a rigorous understanding. The free state of the Ising model in zero external magnetic field on a regular tree, obtained as the infinite-volume Gibbs measure with free (open) boundary conditions, is such an example. This homogenous (tree-automorphism invariant) Gibbs measure indeed displays a complex structure at low temperature, which took a long history to understand.

First recall [5] that the free state $\mu_\beta^{\mathbb{Z}^d}$ of the Ising model on the lattice $\mathbb{Z}^d$ at inverse temperature β decomposes into the symmetric convex combination

$$\mu_\beta^{\mathbb{Z}^d} = (\mu_\beta^{+,\mathbb{Z}^d} + \mu_\beta^{-,\mathbb{Z}^d})/2, \quad \text{for all } \beta > 0. \tag{1}$$

where $\mu_\beta^{+,\mathbb{Z}^d}$ (respectively $\mu_\beta^{-,\mathbb{Z}^d}$) is obtained as the infinite-volume Gibbs measure with $+$ (respectively $-$)-boundary conditions. The extremal decomposition of this form becomes non-trivial below the critical temperature of the model, i.e. for all $\beta > \beta_c^{\mathbb{Z}^d} := \inf\{\beta > 0, \mu_\beta^{+,\mathbb{Z}^d} \neq \mu_\beta^{-,\mathbb{Z}^d}\}$. By abstract Gibbs theory [19] for any spin-model on countable graphs, a unique decomposition into extremal (pure, non-decomposable) infinite-volume Gibbs measures always holds.

On the tree $\mathcal{T}^d$ with $d+1>2$ nearest neighbors however,

$$\mu_\beta^{\mathcal{T}^d} \neq (\mu_\beta^{+,\mathcal{T}^d} + \mu_\beta^{-,\mathcal{T}^d})/2 \tag{2}$$

in the whole non-uniqueness region $d(\tanh\beta)>1$ corresponding to $\beta>\beta_c^{\mathcal{T}^d}$ [29]. Moreover, $\mu_\beta^{\mathcal{T}^d}$ is extremal in the intermediate low temperature region $d(\tanh\beta)^2<1$, which was shown in the sequence of works [2, 4, 20–22].

Note that on a branching plane $\mathcal{T}^d\times\mathbb{Z}$, investigated in [28], there are three parameter regimes corresponding to: (i) uniqueness, (ii) non-uniqueness with tree-like structure (2) and (iii) non-uniqueness with lattice-like structure (1).

Understanding the extremal decomposition of the Ising free state on a tree is known to be particularly complex as it involves uncountably many Gibbs measures on which the extremal decomposition measure is supported, see the much more recent work of [15]. These states have a broken translational symmetry, and show characteristics of glassy behavior. To the best of our knowledge there are no results concerning the extremal decomposition measure of the Ising free state in an external field, nor the one of the Potts free state without a field or in a field. Further, there are also no results on the sensitivity of the decomposition measure w.r.t. particularities and symmetries of the model. Our purpose in the present work is to investigate this complexity in the general framework of ferromagnetic finite-spin models with nearest neighbor interactions.

In the paper [10], we prove the existence of an uncountable family of extremal inhomogeneous states for a large class of models on regular trees, including finite-spin models. More precisely, we give an explicit description of a family of local ground state configurations Ω_{GS} with a sparse enough set of broken bonds giving rise to extremal low temperature states, in the sense that for any $\omega\in\Omega_{GS}$, the weak limit of finite volume measures with boundary condition ω exists and is an extremal state. Note that many other extremal states exist, see discussions in [10, 17, 24] and references therein.

The purpose of the current paper is to show that uncountably many states, which can be seen as perturbations of the above inhomogeneous states, enter the extremal decomposition of the free state. The result extends to the counterpart of the free state (called *central state*), in a wide range of asymmetric models *e.g.* where small inhomogenous field terms are added. Our approach has been inspired by the work of Gandolfo, Maes, Ruiz and Shlosman [15], who discuss these questions in the case of the Ising model in zero external field. Our approach is nevertheless essentially different as we develop a new method based on *concentration of branch overlaps*, see below.

In the Ising case, soon after the characterization of the homogeneous Gibbs measures as Markov chains by Spitzer [31], Higuchi [20] proved the non-extremality of the 'third Markov chain' (the free state) for $d(\tanh\beta)^2>1$. These works were complemented by Bleher *et al.* [4] and Ioffe [22] who proved extremality of the free state for $d(\tanh\beta)^2<1$. In the latter, and already in [9, 13], non-extremality has been related to fluctuations of the spin-glass order paramater on trees used as a discriminating tail observable, the so-called *Edwards-Anderson parameter*, which is nothing but the variance of the root magnetization. Note that non-extremality of the free state was also proved by Markov chains Kesten-Stigum techniques in [27], who also conjectured independently the extremality at intermediate temperatures.

For general finite spin models, including q-clock models, we introduce a generalised version of the Edwards-Anderson parameter measuring the non-degeneracy of the law of the root-spin when the boundary condition at infinity is sampled according to the free/central state μ:

$$q_{EA}=\frac{1}{q}\sum_{a\in\{1,\dots,q\}}\mathrm{Var}_\mu\big(\pi(\sigma_0=a|\cdot)\big)$$

where the π-kernel $\pi(\cdot|\omega)$, so-called *boundary condition at infinity kernel*, is defined to be the tail-measurable kernel

$$\pi(\cdot|\omega)=\lim_{\Lambda\uparrow\mathcal{T}^d}\gamma_\Lambda(\cdot|\omega) \tag{3}$$

if the limits exists for all cylinder events (generating $\mathcal{F}$), and $\pi(\cdot|\omega):=\nu$ to be a fixed arbitrary probability measure else. Standard measure theory arguments [19] shows that the limit exists and is an extremal Gibbs measure for μ-a.e. ω.

We prove q_{EA} to be strictly positive at large enough β (see Theorem 3.2), implying that the free/central state is not extremal at low temperature. From an information-theoretic view, this means the model is reconstruction-solvable [26] as the π-kernel is able to restore information which was sent from the root to infinity by means of the measure μ.

Our main results concern the extremal decomposition of the free/central state μ. Together with Backward Martingale Theorem, DLR equations imply

$$\mu(\cdot)=\int_\Omega\pi(\cdot|\omega)d\mu(\omega)=\int_{\mathrm{ex}\mathcal{G}(\gamma)}\nu(\cdot)\mathrm{d}\alpha_\mu(\nu) \tag{4}$$

where α_μ is the *extremal decomposition measure* of the free state. In Theorem 3.3, we prove that if we sample independently at random two boundary conditions ω and ω' under the free state μ, then the states $\pi(\cdot|\omega)$ and $\pi(\cdot|\omega')$ are almost surely singular with respect to each other:

$$\mu \otimes \mu(\{(\omega,\omega') : \pi(\cdot|\omega) \perp \pi(\cdot|\omega')\}) = 1.$$

which implies that the measure α_μ has no atom. The extremal decomposition of the free state μ is thus supported on an uncountable family of extremal states.

To this purpose, we introduce the following tail measurable observable

$$\underline{\phi}^\omega = \liminf_{n\uparrow\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} 1_{\sigma_v=\omega_v}$$

which we call *branch overlap*, and which measures how much a configuration σ agrees with ω on a (sparse enough) sequence of vertices lying on a branch of the tree. We prove that $\underline{\phi}^\omega$ has different expectations under $\pi(\cdot|\omega)$ and $\pi(\cdot|\omega')$, which is enough to imply almost sure singularity. More precisely, we prove that $\underline{\phi}^\omega$ has an expectation tending to 1 under $\pi(\cdot|\omega)$, and tending to $\sum_a \mu(\sigma_0 = a)^2$ (which equals $1/q$ in the case of the q state Potts model) under $\pi(\cdot|\omega')$, as $\beta \to \infty$, with explicit bounds. This can be viewed as a quantitative statement of some "boundary condition resampling chaos".

The paper is organized as follows: In Section 2 we recall the necessary backgrounds on Gibbs measures, extremal decompositions and Markov chains on Cayley trees. We state our results in Section 3 and provide the proofs in Section 4.

2. Definitions and tools

2.1. Models

Let $\mathcal{T}^d = (V,E)$ denote the Cayley tree of order d, on which any vertex $v \in V$ has exactly $d+1$ neighbors. Pairs $\{v,w\} \in E$ of nearest-neighbors are written $v \sim w$. To any vertex $v \in V$, we attach a spin, which is a random variable σ_v taking values in $\Omega_0 = \mathbb{Z}_q = \{0,\dots,q-1\}$, for $q \in \{2,3,\dots\}$. Let Ω_0 be equipped with the σ-algebra $\mathcal{E} = \mathcal{P}(\Omega_0)$ of all subsets of Ω_0. We are interested in probability measures on the product space $(\Omega,\mathcal{F}) = (\Omega_0^V, \mathcal{E}^{\otimes V})$. For any subset $\Lambda \subset V$ and $\omega \in \Omega$ we define the finite-volume configuration (or projection) $\sigma_\Lambda(\omega) = \omega_\Lambda = (\omega_v)_{v\in\Lambda}$. If $\Lambda \subset V$ is a finite subset, we also write $\Lambda \Subset V$. We write $\partial\Lambda = \{v \in V\backslash\Lambda : \exists w \in \Lambda, v \sim w\}$. We also denote by $\mathcal{F}_\Lambda$ the σ-algebra generated by the variables $\sigma_\Lambda = (\sigma_v)_{v\in\Lambda}$, or equivalently by the cylinders denoted $\{\sigma_\Lambda = \omega_\Lambda\}$. Another important σ-algebra will be the tail σ-algebra of asymptotic events $\mathcal{F}_\infty = \cap_{\Lambda\Subset V}\mathcal{F}_{\Lambda^c}$ where the intersection runs over finite subsets.

We introduce an interaction potential Φ and consider equilibrium states to be Gibbs measures built with the DLR framework, [19]: they are the probability measures μ consistent with the Gibbsian specification γ^Φ in the sense that a version of their conditional probabilities w.r.t. the outside of any finite set Λ of the tree is provided by the *DLR equations*,

$$\forall \Lambda \Subset V,\ \forall \omega_\Lambda \in \Omega_\Lambda,\ \mu[\sigma_\Lambda = \omega_\Lambda \mid \mathcal{F}_{\Lambda^c}](\cdot) = \gamma_\Lambda^\Phi(\omega_\Lambda \mid \cdot),\ \mu - a.s. \tag{5}$$

Here, the Gibbs specification $\gamma^\Phi = (\gamma_\Lambda^\Phi)_{\Lambda\Subset V}$ is made of the probability kernels γ_Λ^Φ from Ω_{Λ^c} to $\mathcal{F}_\Lambda$ defined as

$$\gamma_\Lambda^\Phi(\omega_\Lambda \mid \omega_{\Lambda^c}) = \frac{1}{Z_\Lambda^\omega} e^{-\beta H(\omega_\Lambda \omega_{\Lambda^c})}. \tag{6}$$

The partition function Z_Λ^ω is the normalization constant for a boundary condition ω_{Λ^c}, at finite volume Λ, and the corresponding Hamiltonian with b.c. ω_{Λ^c} is provided by

$$H_\Lambda^\omega(\omega) = H(\omega_\Lambda\omega_{\Lambda^c}) = \sum_{A\cap\Lambda\neq\emptyset} \Phi_A(\omega_\Lambda\omega_{\Lambda^c}) \tag{7}$$

where $\omega_\Lambda\omega_{\Lambda^c}$ denotes the concatenation of ω_Λ and ω_{Λ^c}. We write H for the Hamiltonian with free boundary conditions:

$$H_\Lambda(\omega) = \sum_{A\subset\Lambda} \Phi_A(\omega_\Lambda). \tag{8}$$

and define the *free state* μ to be the Gibbs measure obtained by taking the infinite-volume limit with free boundary conditions. The limit exists in particular, when all finite volume measures are (Kolmogorov) consistent, which is e.g. the case for the Ising model or the Potts model in zero external field. In general, in a non-zero field one could think to consider sub-sequences with free boundary conditions, but it is necessary to put suitable non-deterministic boundary conditions generalizing the free ones, for which we will need the theory of boundary laws, see Section 4.1 and Theorem 4.1.

Throughout the paper we consider ferromagnetic nearest-neighbor (n.n.) potentials of the form

$$\Phi_{\{v,w\}}(\omega) = \sum_{i,j=1}^{q} u_{i,j} \cdot \mathbf{1}_{\{(\omega_v,\omega_w)=(i,j)\}} \cdot \mathbf{1}_{v\sim w} \qquad \text{where} \qquad u_{i,i}=0 \text{ and } 0<u:=\min_{i\neq j} u_{i,j} \leq \max_{i,j} u_{i,j} =: U. \tag{9}$$

The strict positivity of the lower bound expresses that homogeneous spin configurations are energetically favored by the pair interaction, while the energies of excitations w.r.t. different ground states may depend in general on the ground state. The latter includes the q-state Potts model (the Ising model corresponding to $q=2$), for which

$$\Phi^{\text{Potts}}_{\{v,w\}}(\omega) = \mathbf{1}_{\omega_v\neq\omega_w} \cdot \mathbf{1}_{v\sim w}. \tag{10}$$

More generally than the Potts model, we consider q-state *clock models* as an important sub-class. In these models the pair potential has a discrete rotational symmetry which means that there is a positive function $\bar{u}$ for which

$$u_{i,j} = \bar{u}(|i-j|) \tag{11}$$

where $|i-j|$ is the distance between i and $j \in \mathbb{Z}_q$. As the finite-volume Gibbs measures with free boundary conditions are consistent measures in clock-models, they immediately yield a well-defined free state in infinite volume.

Finally we extend our framework from the free state in clock models to *central states* (see the definition in Section 4.1.3), which are constructed as (small) deformations of a free state. Here we do not assume the strict discrete clock-symmetry of the interaction. Observe that the simplest example of such a central state which is not a free state, is obtained for the low-temperature Ising model in a small field.

As our proofs do not rely on symmetry considerations, we can indeed extend them to Hamiltonians of the form

$$H(\omega) = \sum_{v\sim w} \Phi_{\{v,w\}}(\omega) + \sum_{v\in V} \Psi(\omega_v). \tag{12}$$

with a pair potential $\Phi_{\{v,w\}}$ fulfilling (9) uniformly in $\{v,w\}$ and homogeneous single site potential $\Psi : \mathbb{Z}_q \to \mathbb{R}$ s.t.

$$\|\Psi\|_\infty \leq u(d-1)/8. \tag{13}$$

We moreover need a suitable assumption on the pair potential and the single-site potential imposed by the 'lazyness' condition (see Definition 2.2). In the sequel, we shall use the terminology *class (10)* or *class (12)* for models whose Hamiltonians are of the corresponding form.

2.2. *Choquet simplex of DLR measures*

We denote by $\mathcal{G}(\gamma)$ the set of DLR measures satisfying DLR equations (5) for a general specification γ, see [19, 30]. This convex set has in general a particularly interesting structure of being a *Choquet simplex*, that is a compact convex set possessing a subset of extremal elements $\text{ex}\mathcal{G}(\gamma)$ such that any $\mu \in \mathcal{G}(\gamma)$ has a unique convex decomposition onto $\text{ex}\mathcal{G}(\gamma)$. These extremal elements are mutually singular and considered to be the physical *states* of the system, see the discussions in [14, 19, 25]. Note that spatially homogeneous states may decompose into extremal but non-homogeneous states. It is the precisely the purpose of this work, to describe how this happens for free and central states.

To briefly formalize this, denote $\mathcal{M}_1^+(\Omega)$ to be the set of probability measures on $(\Omega,\mathcal{F})$ and let the tail σ-algebra of asymptotic events be denoted by $\mathcal{F}_\infty = \cap_{\Lambda\Subset V}\mathcal{F}_{\Lambda^c}$.

Definition 2.1 (see [19]). *Any $\mu \in \text{ex}\mathcal{G}(\gamma)$ is characterized by the equivalent items:*

1. *Tail-triviality: $A \in \mathcal{F}_\infty \Longrightarrow \mu(A) \in \{0,1\}$.*
2. *Short-range correlations*

$$\lim_{\Lambda\uparrow V} \sup_{B\in\mathcal{F}_{\Lambda^c}} \left|\mu(A\cap B) - \mu(A)\mu(B)\right| = 0. \tag{14}$$

By DLR consistency, for any $\Lambda \Subset V$, the kernel γ_Λ is a regular version of conditional probability of μ and DLR Equations read $\mu = \mu\gamma_\Lambda$. Using the well-defined π-kernel (3) and standard Backward Martingale Limit theorem with respect to the filtration $(\mathcal{F}_{\Lambda_n^c})_n$, one eventually gets the formal simplicial decomposition

(15)
$$\forall \mu \in \mathcal{M}_1^+(\Omega),\ \mu(\cdot) = \int \mu[\cdot\,|\mathcal{F}_\infty](\omega)\, d\mu(\omega) = \int \pi(\cdot|\omega) d\mu(\omega).$$

To work with probability measures on spaces $\mathcal{M}$ of measures, one endows such spaces with a canonical measurable structure. For any subset of probability measures $\mathcal{M} \subset \mathcal{M}_1^+(\Omega)$, the natural way to do so is to evaluate any $\mu \in \mathcal{M}$ via the *evaluation maps* on $\mathcal{M}$, the maps $e_A : \mu \longmapsto e_A(\mu) := \mu(A)$. The *evaluation σ-algebra* $e(\mathcal{M})$ is then the smallest σ-algebra on $\mathcal{M}$ that makes these evaluation maps measurable, see [19] for details.

Theorem 2.1 ([12, 14, 19, 25])**.** *Assume that $\mathcal{G}(\gamma) \neq \emptyset$. Then $\mathcal{G}(\gamma)$ is a convex subset of $\mathcal{M}_1^+(\Omega)$ whose extreme boundary is denoted $\mathrm{ex}\mathcal{G}(\gamma)$, and satisfies the following properties:*

(16)
$$\forall \mu \in \mathcal{G}(\gamma),\ \mu = \int_{\mathrm{ex}\mathcal{G}(\gamma)} \nu \cdot \alpha_\mu(d\nu)$$

where $\alpha_\mu \in \mathcal{M}_1^+\Big(\mathrm{ex}\mathcal{G}(\gamma), e(\mathrm{ex}\mathcal{G}(\gamma))\Big)$ is defined for all $M \in e(\mathrm{ex}\mathcal{G}(\gamma))$ by $\alpha_\mu(M) = \mu\Big[\big\{\omega \in \Omega : \pi(\cdot|\omega) \in M\big\}\Big]$.

Our goal in this work is to rigorously establish that for the low temperature free state μ (and for central states) the measure α_μ has no atom.

2.3. *Tree-indexed Markov chains*

A probability measure μ on $(\mathbb{Z}_q)^V$ is a homogeneous tree-indexed Markov chain if and only if it allows the following iterative construction:

1. Sample σ_0 at (arbitrary) root 0 according to single-site marginal of μ.
2. Sample σ_w via some transition matrix $P(\omega_v, \omega_w)$ from inside to outside.

The abstract general definition of tree-indexed Markov chain which does not assume any invariance under any tree-automorphism requires that

(17)
$$\mu(\sigma_w = \cdot|\mathcal{F}_{\text{past of }(v,w)}) = \mu(\sigma_w = \cdot|\mathcal{F}_v)$$

holds for all oriented edges (v, w), and the past of an oriented edge is the set of vertices which are closer to v than to w.

We note that in the trivial case $d = 1$ (that is excluded for the most part in the analysis of this paper) μ is a homogeneous Markov chain if it is shift-invariant and *reversible*.

Any extremal Gibbs measure is a tree-indexed Markov chain. The converse is not true, as the famous example of the free state for the Ising model in zero external field shows (Higuchi [20]). In the whole article we only suppose is that μ is a homogeneous tree-indexed Markov chain in the class (12), with a certain lazyness property in the following sense.

Definition 2.2 (Lazyness parameter)**.** *Let μ be a homogeneous tree-indexed Markov chain. Denote by P its transition matrix (which is then homogeneous, too) and define the corresponding maximal jump probability to be*

(18)
$$p_1 = p_1(\mu) := \max_{i \in \mathbb{Z}_q} \sum_{j \neq i} P(i, j).$$

Equivalently, for all $v \sim 0$, we have $p_1 = \max_{i \in \mathbb{Z}_q} \mu(\sigma_v \neq i|\sigma_0 = i)$.

In the course of the proof of our main theorem in Section 3, which asserts the non-atomicity of the extremal decomposition measure of a tree-indexed Markov chain μ, we ask for sufficient smallness of the quantity $p_1(\mu)$. This requirement means that all states are sufficiently lazy, i.e. with large probability the chain stays in each of the states.

Below we will see that the examples of *central states* obtained by small perturbations of low temperature free states of clock models keep the Markov chain property and the small p_1 property. The reason is that p_1 deforms smoothly under perturbations, and so lazyness carries over from the unperturbed model.

3. Results

3.1. Reconstruction bounds with asymptotic errors for central states

We have the following reconstruction statement, for the central states in any model of the class (12).

Theorem 3.1. *Consider any central state μ in the class (12), fulfilling the bounds (9) and (13). Then, there exists $\beta_0 > 0$ large enough such that for any $\beta > \beta_0$, for any $a \in \mathbb{Z}_q$,*

$$\mu\Big(\{\omega : \pi(\sigma_0 = a|\omega) \le 1 - \epsilon_1(\beta)\}\Big|\sigma_0 = a\Big) \le \epsilon_2(p_1) \tag{19}$$

where $\epsilon_1(\beta) = \epsilon_1(\beta, u, U, d) \downarrow 0$ as $\beta \uparrow \infty$ and $\epsilon_2(p_1) = \epsilon_2(p_1, u, U, d) \downarrow 0$ as $p_1 \downarrow 0$.
In particular, if μ is the free state of a clock model, the reconstruction bound holds at large enough β:

$$\int d\mu(\omega|\sigma_0 = a)\pi(\sigma_0 = a|\omega) > \frac{1}{q} = \mu(\sigma_0 = a). \tag{20}$$

From a signal recontruction point of view (see [26]), the statement says that a signal a which is sent from the origin to infinity through noisy canals can be almost-surely restored by the best tail-measurable predictor $\pi(\sigma_0 = a|\omega)$ up to thermal fluctuations $\epsilon_1(\beta)$ up to an error probability $\epsilon_2(p_1)$.

The proof is a direct consequence of the Key Lemma 4.3 proved in Section 3.1 which relies on a good site/bad site decomposition for boundary conditions ω, together with Peierls bounds under disorder.

3.2. Positivity of the Edwards-Anderson parameter and non-extremality

The so-called Edwards-Anderson parameter is usually defined in spin glass models as a quantity measuring the degree of randomness of the (random) spin magnetisation at the origin (see e.g. [9]). In our context, the analogue should be a quantity measuring the degree of randomness of the probability vector $(\pi(\sigma_0 = a|\omega))_{a\in\mathbb{Z}_q}$, when the boundary condition at infinity ω is distributed according to μ. Let us thus define the following quantity.

Definition 3.1 (Central state Edwards-Anderson parameter)**.**

$$q^{\mu}_{\mathrm{EA}} := \frac{1}{q}\sum_{a\in\mathbb{Z}_q} \mathrm{Var}_\mu\big(\pi(\sigma_0 = a|\cdot)\big). \tag{21}$$

Note that when μ is the free state of a clock model, by symmetry the above definition boils down to

$$q^{\mathrm{Clock}}_{\mathrm{EA}} := \mathrm{Var}_\mu\big(\pi(\sigma_0 = a|\cdot)\big) = \mu(\pi(\sigma_0 = a|\cdot)^2) - \frac{1}{q^2} \tag{22}$$

for any $a \in \mathbb{Z}_q$, whereas for the free state of the Ising model in zero field, Definition 3.1 writes $q^{\mathrm{Ising}}_{\mathrm{EA}} = \frac{1}{4}\mathrm{Var}\big(\pi(\sigma_0|\cdot)\big)$.

Clearly, having $q_{\mathrm{EA}} > 0$ implies that there exists some $a \in \mathbb{Z}_q$ such that $\mathrm{Var}_\mu(\pi(\sigma_0 = a|\cdot)) > 0$ and thus the tail-measurable random variable $\mu(\sigma_0 = a|\mathcal{F}_\infty)(\cdot)$ is not μ-a.s. constant. Thus μ cannot be tail-trivial. We have the following quantitative lower bound.

Theorem 3.2. *Consider any central state μ in the class (12) fulfilling the bounds (9) and (13). Then, there exists $\beta_0 > 0$ large enough such that for any $\beta > \beta_0$, there exist two functions $\epsilon_1(\beta) \downarrow 0$ as $\beta \uparrow \infty$ and $\epsilon_2(p_1) \downarrow 0$ as $p_1 \downarrow 0$ such that*

$$q^{\mu}_{\mathrm{EA}} \ge \frac{1}{q}((1 - \epsilon_1(\beta))^2(1 - \epsilon_2(p_1)) - \max_{a\in\mathbb{Z}_q}\mu(\sigma_0 = a)). \tag{23}$$

In particular, for large enough β and small enough $p_1 = p_1(\mu)$, μ is not extremal since $q^{\mu}_{\mathrm{EA}} > 0$.

Note that for the free state of clock models $\max_{a\in\mathbb{Z}_q}\mu(\sigma_0 = a)$ is equal to $1/q$ by symmetry. More generally, for a central state this term is close to $1/q$ (see (35) and Definition 4.2 below).

Theorem 3.2 follows from Theorem 3.1. The proof is elementary and can be found in Section 4.3.

3.3. Non-atomicity of the extremal decomposition of the free state

We prove in the following theorem that uncountably many extremal Gibbs measures enter the decomposition of the central state at low enough temperature. In the sequel, we simply call *extremals* these extremal Gibbs measures.

Theorem 3.3 (Almost-sure singularity of extremals)**.** *Consider any central state μ in the class (12) fulfilling the bounds (9) and (13). Then, there exist $\beta_0 > 0$ large enough and $p_1^0 = p_1^0(\beta, u, U, d) > 0$ small enough such that for any $\beta > \beta_0$, $p_1(\mu) < p_1^0$ and for $\mu \otimes \mu$-a.e. pair (ω, ω') the extremals $\pi(\cdot|\omega)$ and $\pi(\cdot|\omega')$ are singular with respect to each other, i.e.*

$$\mu \otimes \mu\big(\{(\omega, \omega') \in \Omega \times \Omega : \pi(\cdot|\omega) \perp \pi(\cdot|\omega')\}\big) = 1. \tag{24}$$

Corollary 3.1. *The decomposition measure α_μ has no atoms, i.e. $\alpha_\mu(\{\nu\}) = 0$ for all $\nu \in \mathrm{ex}\mathcal{G}(\gamma)$. In particular there are uncountably many extremal states which enter the extremal decomposition of μ.*

Proof. Suppose the opposite, namely that $\mu(\pi(\cdot|\omega) = \mu_0) > 0$, for some atom μ_0. Then

$$\mu \otimes \mu(\{(\omega, \omega') : \pi(\cdot|\omega) \perp \pi(\cdot|\omega')\}) = 1 - \mu \otimes \mu(\{(\omega, \omega') : \pi(\cdot|\omega) = \pi(\cdot|\omega')\}) \leq 1 - \mu(\pi(\cdot|\omega) = \mu_0)^2 < 1$$

which is a contradiction. □

The idea to prove almost sure singularity of typical extremals taken from the product measure is to produce a tail-measurable order parameter, which carries enough information to distinguish two different typical extremal Gibbs measures. Indeed, as the infinite-volume kernel $\pi(\cdot|\omega)$ is supported on the extremals, and extremals are uniquely described by their restriction to the tail-sigma algebra, it suffices to find a tail-measurable observable ϕ on which the expectations $\pi(\phi|\omega)$ and $\pi(\phi|\omega')$ differ.

For this purpose we construct ϕ by looking at empirical sequences of overlaps of the spin variables σ with ω, and show that its expectation becomes big in $\pi(\phi|\omega)$ on the one hand, and small in $\pi(\phi|\omega')$ on the other hand, for typical choices of (ω, ω'). Theorem 3.3 will thus follow from a control of so-called *branch overlaps* defined in the section below.

3.4. Concentration of branch-overlap for typical extremals

Let $n \in \mathbb{N}$ and $\mathrm{r} = (r_1, r_2, \dots)$ be an increasing sequence of positive integers. Let

$$\Lambda_n = \Lambda_n^{\mathrm{r}} = \{v_1, v_2, \dots, v_{n^2}\} \subset V \tag{25}$$

so that $|\Lambda_n| = n^2$, and the vertices v_i are chosen along a branch of the tree, in such a way that their spacing in graph distance is given by the sequence r, i.e. $|v_{i+1} - v_i| =: r_i$ for all $i \in \{1, \dots, n^2\}$.

Definition 3.2 (Thinned branch overlaps)**.** *Define the tail-measurable function, called thinned branch-overlap, measuring how much the configuration σ matches with ω on the increasing sparse volumes Λ_n, as*

$$\underline{\phi}^\omega = \liminf_{n \uparrow \infty} \frac{1}{|\Lambda_n^{\mathrm{r}}|} \sum_{v \in \Lambda_n^{\mathrm{r}}} 1_{\sigma_v = \omega_v}. \tag{26}$$

For any fixed configuration ω, this is a tail-measurable observable w.r.t. the dependence on the spins σ, which takes values on the interval $[0, 1]$. Analogous tail-measurability also holds w.r.t. the parameter ω.

Note that there is thinning in two ways: the volume becomes increasingly sparse, and the liminf is taken along volumes of n^2 sites. The following theorem can be viewed as a quantitative statement of the glassiness of the states $\pi(\cdot|\omega)$, for μ-almost every ω: it describes quantitatively how much typical configurations σ sampled from $\pi(\cdot|\omega)$ are ω-like.

Theorem 3.4 (Branch overlap)**.** *Consider any central state μ in the class (12) fulfilling the bounds (9) and (13). Then there are two functions $\epsilon_1(\beta) \downarrow 0$ as $\beta \uparrow \infty$ and $\epsilon_2(p_1) \downarrow 0$ as $p_1 \downarrow 0$ such that for sparse enough sets $\Lambda_n = \Lambda_n^{\mathrm{r}}$ the following holds. For μ-a.e. ω, the thinned branch-overlap $\underline{\phi}^\omega$ is $\pi(\cdot|\omega)$-a.s. lower bounded by*

$$\underline{\phi}^\omega = \liminf_{n \uparrow \infty} \frac{1}{|\Lambda_n|} \sum_{v \in \Lambda_n} 1_{\sigma_v = \omega_v} \geq 1 - \epsilon_1(\beta) - \epsilon_2(p_1). \tag{27}$$

The proof can be found in Section 4.4.4. From Theorem 3.4, we deduce that the tail-measurable observable $\underline{\phi}^\omega$ has different expectations under $\pi(\cdot|\omega)$ and $\pi(\cdot|\omega')$ if $\omega \neq \omega'$, through the following corollary.

Corollary 3.2. *Let β be large enough and p_1 be small enough such that $\epsilon_1(\beta)+\epsilon_2(p_1)<\frac{1}{2}(1-\sum_{a\in\mathbb{Z}_q}\mu(\sigma_0=a)^2)$. Then there is a sequence of integers $\mathrm{r}=(r_i)_{i\in\mathbb{N}}$ such that for the correspondingly defined tail-measurable observable $\underline{\phi}^\omega$ we have the strict inequality valid for $\mu\otimes\mu$-a.e. (ω,ω'):*

$$\pi(\underline{\phi}^\omega|\omega)>\pi(\underline{\phi}^\omega|\omega'). \tag{28}$$

The proof can be found in Section 4.5. From Corollary 3.2 the statement of Theorem 3.3 is immediate.

3.5. Structure of the proofs

To prove the concentration bound of Theorem 3.4 it is useful to adopt a quenched-disordered systems view (we refer to [7] for a general introduction to statistical mechanics of disordered systems). A configuration ω drawn from the free/central state μ corresponds to a boundary condition at infinity, but also plays the role of disorder. The intuition behind is that the extremal measure $\pi(\cdot|\omega)$, which is in general inhomogeneous and plays the role of a quenched disordered state, mostly resembles ω locally, and for μ-typical ω.

To make this precise, we will need contours, as introduced in [10], and Peierls bounds relative to the reference configuration ω. Typically ω will contain a small density of broken bonds (along which ω changes). However the latter will not be uniformly sparse, as needed in [10] and [16] to ensure the excess energy estimate leading to Peierls bounds. To treat the rare but arbitrarily large regions where the Peierls bound locally fails, we introduce a notion of "bad site" (see Definition 4.4), which is an essential tool. Our good/bad site decomposition is somewhat reminiscent to that invented by Chayes-Chayes-Fröhlich in [8], to treat lattice Ising models with i.i.d. random bonds, which are mostly but not strictly ferromagnetic. However, we work on the tree, and in a regime where the "disorder-measure" μ is not tail-trivial, it is far from an i.i.d. disorder measure, making things more intricate.

Nevertheless, as we shall see, there is one-dimensional correlation decay, conditionally on the state of the root, along a branch of the tree. This will be exploited to prove exponential decorrelation of bad sites in their distance, see Lemma 4.4, leading to concentration of thinned branch overlaps around their means, under the "quenched measures" $\pi(\cdot|\omega)$, for μ-almost every ω, see Lemma 4.6.

4. Proofs

We first recall some tools about tree-indexed Markov chains and boundary laws.

4.1. Markov chains on trees and boundary laws.

Being n.n., our class of models (12) lead to Gibbs measures that are (spatial) Markov fields. As we work on trees, there is an important class of Gibbs measures (including extremals) which has a direct transcription in terms of Markov chains on trees. These are described via so-called boundary laws introduced by Zachary [32]. Boundary laws are (non-normalized) positive measures which are invariant under an interaction-dependent non-linear map along the tree. Moreover they are closely related (but not equal) to the invariant single-site probability measures for the associated one-step Markov chain transition matrix, see below, and see Georgii [19, Chapter 12] for details.

In our case of a homogeneous n.n. interaction on the tree, the specification (6) is equivalently described by a positive *transfer matrix (or transfer operator)* $Q:\mathbb{Z}_q\times\mathbb{Z}_q\mapsto(0,\infty)$ via the prescription at finite volume $\Lambda\Subset V$ and b.c. ω,

$$\gamma_\Lambda(\omega_\Lambda\mid\omega_{\Lambda^c})=\frac{1}{Z^\omega_\Lambda}\prod_{\substack{\{v,w\}\cap\Lambda\neq\emptyset\\ v\sim w}}Q(\omega_v,\omega_w).$$

where, writing $b=\{v,w\}$, with pair potential Φ, and single-site potential Ψ as in (12), and

$$Q(\omega_v,\omega_w)=:Q_b(\omega)=e^{-\beta(\Phi_b(\omega)+(\Psi(\omega_v)+\Psi(\omega_w))/(d+1))}.$$

4.1.1. Boundary laws

Definition 4.1. *A boundary law λ for a transfer matrix Q is a family of row vectors $\lambda_{vw}\in\,]0,\infty[^{\mathbb{Z}_q}$ which satisfy, for all oriented pairs of n.n. $v,w\in V$, the following consistency equation: there exists some $c_{vw}>0$ such that for all $i\in\mathbb{Z}_q$,*

$$\lambda_{vw}(i)=c_{vw}\prod_{z\in\partial\{v\}\setminus\{w\}}\sum_{j\in\mathbb{Z}_q}Q(i,j)\lambda_{zv}(j).$$

Theorem 4.1 ([32]). *There is a one-to-one relation between Gibbs measures μ which are also tree-indexed Markov chains and boundary laws λ, then described via its finite-volume marginals in any $\Lambda \Subset V$*

$$\mu(\omega_{\Lambda\cup\partial\Lambda}) = (Z^{\omega}_{\Lambda})^{-1} \prod_{w\in\partial\Lambda} \lambda_{ww_{\Lambda}}(\omega_w) \prod_{b\cap\Lambda\neq\emptyset} Q_b(\omega), \tag{29}$$

where w_{Λ} is the unique neighbor of $w \in \partial\Lambda$ which lies in Λ. The Markov chain transition operator is given for all $v, w \in V$ by the stochastic matrix

$$P_{vw}(\omega_v, \omega_w) = \frac{Q(\omega_v, \omega_w)\lambda_{wv}(\omega_w)}{\sum_{j\in\mathbb{Z}_q} Q(\omega_v, j)\lambda_{wv}(j)}. \tag{30}$$

We note that no homogeneity of the Gibbs measure, boundary law, and transition operator are assumed. It is perfectly possible and very relevant that homogeneous Q allow for non-trivial non-homogeneous Gibbs measures μ (as for example the Bleher–Ganikhodjaev states [3], which can be seen as some analogues of the Dobrushin states [11] on the tree).

In the special case of homogeneous boundary laws $\lambda_{vw}(i) \equiv u(i)$ on regular trees of degree d, considered here, note that these must satisfy the homogeneous equation

$$u(i) = c\Big(\sum_{j\in\mathbb{Z}_q} Q(i,j)u(j)\Big)^d, \tag{31}$$

which we may write in short notation as $u = c(Qu)^d$. The constant c can be chosen to our convenience, a possible and often convenient choice is $c = 1$. The single-site marginal of the measure given in (29) then becomes[1] $u^{\frac{d+1}{d}}/\|u^{\frac{d+1}{d}}\|_1$ and the transition matrix given in (30) becomes

$$P^u(i,j) = \frac{Q(i,j)u(j)}{\sum_{k\in\mathbb{Z}_q} Q(i,k)u(k)}, \text{ for } i,j \in \mathbb{Z}_q. \tag{32}$$

In our non-hard core context, homogeneous boundary laws u can also be characterized by consistent effective boundary fields $h = (h_1, \dots, h_q)$, defined as $u_i = e^{h_i}$, for $i \in \mathbb{Z}_q$, that themselves satisfy a consistency mean-field equation.

In the Ising case, this equation involving hyperbolic tangent possibly has 3 independent solutions and gives rise to three homogeneous Markov chains μ^+, μ^- and the free state μ (see the work of Spitzer [31], or the one of Higuchi [20], who called μ the 'third Markov chain', and already noticed that it was not necessarily extremal). The case of the free state μ is discussed in the next section.

For the Potts model, but also very generally for q-clock models in the case of absence of an external field, this equation has a unique solution at small β and correspondingly uniqueness of the Gibbs measure. For large β, the situation for general clock models in the class (11) is already more interesting and looks as follows (see [23, 24] and [1]): For each subset $A \subset \mathbb{Z}_q$ of the local spin-space, at sufficiently large $\beta \geq \beta_0(|A|)$ there is a spatially homogeneous Markov chain μ_A whose single-site marginals $\pi_A = \mu_A \circ \sigma_0^{-1}$ are concentrated on the spin-values in A. Moreover, the restriction of π_A to A approximately equals the equidistribution on A, with explicit β-dependent error bounds. It is important to realize that these states contain the q states with singleton-localization centers $|A| = 1$, constructed in [18], but at sufficiently low temperatures there are always independent states with non-singleton localization centers A.

4.1.2. Free states of clock models

Clock models are defined by the requirement that the transfer matrix depends only on the distance in $\mathbb{Z}_q$ between i and j and hence has the form $Q(i,j) = Q_0(j-i)$ with an even function Q_0 on $\mathbb{Z}_q$ assumed to be strictly positive. In that case the homogeneous boundary law equation (31) can be written in terms of the discrete convolution as $u = (Q_0 * u)^d$. This in particular shows that the constant boundary law $u(i) = a$ for all $i \in \mathbb{Z}_q$ solves the equation for the non-zero value of a given by $a = (a\|Q_0\|_1)^d$. The Markov chain transition operator (32) obtained for constant boundary law is then the normalized transition operator $P^u(i,j) = Q_0(i-j)/\|Q_0\|_1$ itself, which has as unique invariant distribution the equidistribution on $\mathbb{Z}_q$. The formula (29) for the Gibbs measure μ obtained for constant boundary law then reduces to

$$\mu(\omega_{\Lambda\cup\partial\Lambda}) = (Z^{\omega}_{\Lambda})^{-1} \prod_{b\cap\Lambda\neq\emptyset} Q(\omega_b), \tag{33}$$

[1] where the ℓ^p norm of a function $f : \mathbb{Z}_q \to \mathbb{R}$ is defined by $\|f\|_p := (\sum_{i\in\mathbb{Z}_q} |f(i)|^p)^{1/p}$.

from which we also see that our first definition of a homogeneous tree-indexed Markov chain is satisfied. The r.h.s. of (33) on the other hand is the obvious formula for the open boundary condition finite-volume Gibbs measure in Λ. In this way we see that the free state in clock models is the tree-indexed Markov chain with constant boundary law.

4.1.3. Central states of perturbed clock models
We now consider more generally models which are perturbations of clock models, by which we mean that $Q = Q_0 + \tilde{Q}$ where $\tilde{Q}$ is small in (some) matrix norm. Such a situation arises for example in the important special case of a clock model which is perturbed by additional single-site terms coming from a non-trivial potential $\Psi_{\{v\}} \equiv \Psi_0$. In this case the transfer operator does not describe a clock model anymore but takes the general matrix form

$$Q(i,j) = a(i)Q_0(i,j)a(j),$$

where $a(i) = e^{-\beta\Psi(i)/(d+1)}$ is close to one for all i, and $Q_0(i,j)$ only depends on $|i-j|$. Assume without loss of generality the normalization $\|Q_0\|_1 = 1$, let $x := u^{\frac{1}{d}}$ and write the homogeneous equation (31) as $F(x,Q) = 0$ with

$$F_i(x,Q) := x(i) - \sum_j Q(i,j)x(j)^d \text{ for all } i \in \mathbb{Z}_q. \tag{34}$$

A direct application of the implicit function theorem implies the following[2]

Lemma 4.1. *Suppose that all eigenvalues of Q_0 are different from $1/d$. Then there is a neighborhood of Q_0 such that for all Q in this neighborhood there exists a continuously differentiable solution $Q \mapsto \bar{x}(Q)$ of the boundary law equation (34) for Q, which has the property that $\bar{x}(Q_0) = 1$.*

Note that the eigenvalues of the matrix Q_0 are given by discrete Fourier transform of the vector $(Q_0(j))_{j\in\mathbb{Z}_q}$ and hence directly computable. We note in particular that for the β-dependent normalized form

$$Q_0(j) = e^{-\beta\Phi_0(j)}/\|e^{-\beta\Phi_0}\|_1$$

with a potential $\Phi_0 : \mathbb{Z}_q \mapsto [0,\infty)$ which has a minimum at 0, and satisfies $0 < u \le \Phi_0(i)$ for $i \ne 0$, the condition of Lemma 4.1 is valid for large enough β. More quantitatively, this is ensured by $(d-1)/(d+1) > (q-1)e^{-\beta u}$ which follows from the Lemma 4.2 below.

Definition 4.2. *(Central State) Consider the continuously differentiable solution $\bar{x}$ of Lemma 4.1. We call central state the Markov chain Gibbs state associated to the pair $(\bar{x}(Q), Q)$ solving (34).*

As a consequence of Lemma 4.1, the corresponding single-site marginal $Q \mapsto \pi_Q = \bar{x}(Q)^{d+1}/\|\bar{x}(Q)^{d+1}\|_1$ and transition matrix $Q \mapsto P_Q(i,j) = Q(i,j)\bar{x}(Q)^d_j/(\sum_k Q(i,k)\bar{x}(Q)^d_k)$ are continuously differentiable perturbations of the values $\pi_{Q_0} = 1/q$ and $P_{Q_0} = Q_0/\|Q_0\|_1$ of the free state of the reference clock model Q_0. Moreover, in this normalization, the perturbed solution is close to the free state of the clock model as

$$\bar{x}(Q) = 1 + (id_q - dQ_0)^{-1}(Q - Q_0)1 + o(\|Q - Q_0\|) \tag{35}$$

in any matrix norm $\|Q - Q_0\|$.

We now give lower bounds on the eigenvalues of the transition operator of a clock model, which are larger than $1/d$ for β large enough, ensuring the condition in Lemma 4.1.

Lemma 4.2. *Consider a clock model in the class (11) fulfilling the bound (9). We have lower bounds on the eigenvalues λ_j of the transition operator $Q_0 = \frac{e^{-\beta\Phi}}{\|e^{-\beta\Phi}\|_1}$ of the form*

$$\lambda_j \ge \frac{1-(q-1)e^{-\beta u}}{1+(q-1)e^{-\beta u}}.$$

Proof. Clearly $\|e^{-\beta\Phi}\|_1 \le 1 + (q-1)e^{-\beta u}$. To estimate the spectral radius of the symmetric matrix $e^{-\beta\Phi} - id_q$, observing that for v with $\|v\|_2 = 1$ we have

$$|\langle v, (e^{-\beta\Phi} - id_q)v\rangle| \le e^{-\beta u}\sum_{i\ne j}|v_i||v_j| = e^{-\beta u}((\sum_i |v_i|)^2 - 1) \le e^{-\beta u}(q-1). \tag{36}$$

[2] Denote by 1 the vector with all coefficients equal to 1.

This implies that the eigenvalues of $e^{-\beta\Phi}$ are bounded below by $1-e^{-\beta u}(q-1)$, which proves the lemma. □

4.1.4. Lazyness assumption

Recall the lazyness parameter $p_1(\mu)$ in Definition 2.2. For our main Theorem 3.4 to be meaningful we need sufficient smallness of the quantity $p_1(\mu)$.

From the above follows that the *central states* obtained by small perturbations of clock models have an associated transition matrix which keep the small p_1 property. Indeed, assume again the normalization of Q_0 such that $\|Q_0\|_1 = 1$, and $Q_0(0) = 1$. Suppose that all eigenvalues of Q_0 are different from $1/d$, and let μ_Q denotes the central state, defined in a sufficiently small neighborhood of Q_0. Then $Q \mapsto p_1(\mu_Q)$ is a continuously differentiable function. In particular, whenever the reference clock model Q_0 has small

$$p_1(\mu_{Q_0}) = \frac{\sum_{i\neq 0} Q_0(i)}{\|Q_0\|_1} \leq (q-1)e^{-\beta u}, \tag{37}$$

this smallness carries over to $p_1(Q)$ for Q in a sufficiently small neighborhood around Q_0. Specifically for the Potts model as a reference clock model we have

$$p_1(Q_0) = \frac{q-1}{\mathrm{e}^{\beta}+q-1} \tag{38}$$

which tends to 0 as β tends to infinity.

4.2. Typical contours and Key Lemma

In our previous paper [10], we exhibit low temperature local ground states which give rise to a wide family of inhomogenous extremal states, since Peierls bounds hold. Here we will need to treat rare but arbitrary large regions where Peierls bounds locally fail, and prove that they are exponentially dumped in the infinite-volume limit. Let us recall a few definitions from [10], used afterwards for μ-typical b.c ω^0.

Definition 4.3 (Contour with respect to a fixed configuration ω^0)**.**
Let $\omega^0 \in \Omega_0^V$ be a fixed reference configuration. A contour for the spin configuration $\omega \in \Omega_0^V$ relative to ω^0 is a pair

$$\bar{\gamma} = (\gamma, \omega_\gamma)$$

where the support $\gamma \subset \{v \in V : \omega_v \neq \omega_v^0\}$ is a connected component, and $\omega_\gamma = (\omega_v)_{v\in\gamma}$.

We define then set of broken bonds of the configuration $\omega \in \Omega$ by $D(\omega) = \{\{v,w\} \in E : \omega_v \neq \omega_w\}$ and denote the set of edges attached to $\gamma \subset V$ by $E(\gamma) = \{\{x,y\} \in E, \{x,y\} \cap \gamma \neq \emptyset\}$. Inspired by the Excess Energy Lemma satisfied by our inhomogeneous ground states (see [10], Lemma 2), we introduce the following definition of bad events/contours.

Definition 4.4 (Bad events)**.** *Consider any model in the class (12). Let*

$$\delta_0 := \frac{1}{2}\cdot\frac{(d-1)u}{(u+U)} \tag{39}$$

where $u, U \in \mathbb{R}^+$ are the bounds on energy costs defined in (9) and d is the branching number of the tree.
Denote by B_v the bad event that there exists a contour around v with respect to ω^0 which does not have good enough excess energy in the sense that

$$B_v := \bigcup_{\gamma:\gamma\ni v} B(\gamma) \text{ where } B(\gamma) := \{\omega^0 : |D(\omega^0)\cap E(\gamma)| \geq \delta_0|\gamma|\}. \tag{40}$$

Before describing our Key Lemma, we need the following

Definition 4.5. *Suppose that μ is a homogeneous tree-indexed Markov chain for a Hamiltonian in the class (12) fulfilling the bounds (9) and (13). Let $p_1 = p_1(\mu)$ be its lazyness parameter as in Defintion 2.2. Let δ_0 as in (39). Define $\lambda(p_1) = \lambda(p_1, u, U, d)$ as*

$$e^{-\lambda(p_1)} := \inf_{t\geq 0} e^{-t\delta_0}(p_1 e^t + 1 - p_1)^{(d+1)} = \left(\frac{p_1}{\delta_0}\right)^{\delta_0}\left(\frac{1-p_1}{d+1-\delta_0}\right)^{1-\delta_0}. \tag{41}$$

Note that $\lambda(p_1)\uparrow\infty$ or equivalently $e^{-\lambda(p_1)}\downarrow 0$ as $p_1\downarrow 0$. In particular, if μ_β is the free state of the q-clock model at inverse temperature β (see (38)),

$$\lambda(p_1(\mu_\beta))\uparrow\infty \text{ as } \beta\uparrow\infty.$$

Lemma 4.3 (Key Lemma)**.** *Let μ be the central state of any model in the class (12) fulfilling the bounds (9) and (13). Recall the Definition 4.4 of the event B_v and the Definition 2.2 of p_1. Then for β large enough, for μ-almost every ω, there exists $\epsilon_1(\beta)\downarrow 0$ as $\beta\uparrow\infty$ such that*

$$\pi(\sigma_v\neq\omega_v|\omega)\le 1_{B_v}(\omega)+\epsilon_1(\beta). \tag{42}$$

Moreover, for p_1 small enough, for any $v\in V$, there exists $\epsilon_2(p_1)\downarrow 0$ as $p_1\downarrow 0$ such that

$$m(p_1):=\mu(B_v)\le\epsilon_2(p_1). \tag{43}$$

Proof. We decompose

$$\pi(\sigma_v\neq\omega_v|\omega)=\pi(\sigma_v\neq\omega_v|\omega)(1_{B_v}(\omega)+1_{B_v^c}(\omega)) \tag{44}$$

and we use the infinite-volume version of the Peierls bound of [10] applied to the complement of a bad event at v to bound the probability of a mismatch at v by the sum over contours attached at v. This gives

$$\pi(\sigma_v\neq\omega_v|\omega)\cdot 1_{B_v^c}(\omega)\le 1_{B_v^c}(\omega)\sum_{\bar\gamma,\gamma\ni v}\rho[\omega](\bar\gamma) \tag{45}$$

where the sum is over pairwise compatible contours $\bar\gamma$ with activities

$$\rho[\omega](\bar\gamma)=\exp(-\beta(H_{\gamma\cup\partial\gamma}(\sigma_{\gamma\cup\partial\gamma})-H_{\gamma\cup\partial\gamma}(\omega_{\gamma\cup\partial\gamma}))).$$

Note that ω is not assumed to be a local ground state (in the sense of [10]), but Definition 4.4 is tailored to ensure the Peierls bound on the event B_v^c. This delivers, using the assumptions on the potential in the class (12),

$$\begin{aligned}\rho[\omega](\bar\gamma)1_{B_v^c}(\omega)&\le\exp(-\beta(d-1)u|\gamma|+\beta(u+U)|D(\omega)\cap E(\gamma)|+\|\Psi\|_\infty|\gamma|)1_{B_v^c}(\omega)\\&\le\exp(-\beta|\gamma|((d-1)u-(u+U)\delta_0-\|\Psi\|_\infty))=\exp(-\beta|\gamma|(d-1)u/4).\end{aligned} \tag{46}$$

Using the following standard upper bound on the number of connected subsets of vertices of $\mathcal{T}^d=(V,E)$ (see e.g. [16]):

$$\#\{\gamma\subset V:\gamma\text{ connected },\gamma\ni 0,|\gamma|=\ell\}\le(d+1)^{2(\ell-1)}, \tag{47}$$

we deduce that there exists $C',c'>0$ such that

$$\pi(\sigma_v\neq\omega_v|\omega)1_{B_v^c}(\omega)\le\sum_{\bar\gamma,\gamma\ni v}e^{-\beta c|\gamma|}\le\sum_{\ell\ge 1}(d+1)^{2(\ell-1)}(q-1)^\ell e^{-\beta\ell(d-1)u/4}\le C'e^{-c'\beta}=:\epsilon_1(\beta) \tag{48}$$

for $\beta>4\log((q-1)(d+1)^2)/(u(d-1))$, which proves (42).

Next we use the exponential Markov inequality to bound $\mu(B(\gamma))$:

$$\mu(B(\gamma))=\mu\big(|D(\omega)\cap E(\gamma)|\ge\delta_0|\gamma|\big)\le\inf_{t\ge 0}e^{-t\delta_0|\gamma|}\mu(e^{t|D(\omega)\cap E(\gamma)|}). \tag{49}$$

It suffices to fix any $x\in\gamma$, look at the conditional measure $\mu_a(\cdot):=\mu(\cdot|\sigma_x=a)$ for any fixed $a\in\mathbb{Z}_q$, and find an upper bound which does not depend on a. Using the Markov chain property of the measure μ_a, successive applications of the homogeneous transition matrix P yield

$$\mu_a(e^{t|D(\omega)\cap E(\gamma)|})\le(p_1e^t+1-p_1)^{(d+1)|\gamma|} \tag{50}$$

which follows from

$$\max_{a\in\mathbb{Z}_q}\sum_{b\neq a}P(a,b)e^t+P(a,a)\le p_1e^t+1-p_1$$

and the bound $|E(\gamma)|\le(d+1)|\gamma|$. Optimizing over t leads to the bound

$$\mu(B(\gamma))\le\left(\inf_{t\ge 0}e^{-t\delta_0}(p_1e^t+1-p_1)^{(d+1)}\right)^{|\gamma|}=e^{-\lambda(p_1)|\gamma|} \tag{51}$$

since Definition 4.5 of $\mathrm{e}^{-\lambda(p_1)}$ is made to be the value of the above infimum. Finally using the entropy bound (47) on the number of contours of fixed length which are attached to a given point, the estimate for $\mu(B_v)$ follows. Indeed, there exists $C>0$ such that for any $c\in(0,1)$ we have

$$\mu(B_v)\le\sum_{\gamma:\gamma\ni v}\mu(B(\gamma))\le\sum_{\gamma:|\gamma|\ge 1}e^{-\lambda(p_1)|\gamma|}\le\sum_{\ell\ge 1}(d+1)^{2(\ell-1)}(q-1)^{\ell}e^{-\lambda(p_1)\ell}\le Ce^{-c\lambda(p_1)}=:\epsilon_2(p_1) \tag{52}$$

for p_1 sufficiently small, which proves (43). □

4.3. Positivity of the Edwards-Anderson parameter

Proof of Theorem 3.2. The proof is short and elementary, given the reconstruction bound of Theorem 3.1. Observe that

$$\begin{aligned}q^{\mu}_{\mathrm{EA}}&=\frac{1}{q}\sum_{a\in\mathbb{Z}_q}\left(\mu\Big(\pi(\sigma_0=a|\cdot)^2\Big)-\mu(\sigma_0=a)^2\right)=\frac{1}{q}\sum_{a\in\mathbb{Z}_q}\left(\sum_{b\in\mathbb{Z}_q}\mu(\sigma_0=b)\mu\Big(\pi(\sigma_0=a|\cdot)^2|\sigma_0=b\Big)-\mu(\sigma_0=a)^2\right)\\&\ge\frac{1}{q}\sum_{a\in\mathbb{Z}_q}\left(\mu(\sigma_0=a)\mu\Big(\pi(\sigma_0=a|\cdot)^2|\sigma_0=a\Big)-\mu(\sigma_0=a)^2\right)\end{aligned}$$

which, by the quantitative reconstruction Theorem 3.1 gives the lower bound

$$q^{\mu}_{\mathrm{EA}}\ge\frac{1}{q}\sum_{a\in\mathbb{Z}_q}\left(\mu(\sigma_0=a)(1-\epsilon_1(\beta))^2(1-\epsilon_2(p_1))-\mu(\sigma_0=a)^2\right)\ge\frac{1}{q}((1-\epsilon_1(\beta))^2(1-\epsilon_2(p_1))-\max_{a\in\mathbb{Z}_q}\mu(\sigma_0=a))$$

which proves the theorem. □

4.4. Overlap control for typical extremals

4.4.1. Exponential decorrelation of bad sites

Suppose that μ is a homogeneous tree-indexed Markov chain. We do not need to assume any invariance of the measure μ under joint transformations of the local spin-spaces (like a discrete clock-rotation or even permutation symmetry). We only assume that the jump probability p_1 (see Definition 2.2), controlling also the density of broken bonds in the configuration drawn from μ, is small enough. Using this assumption, we derive an upper bound on the decorrelation between two bad events occurring at the sites u and v (see Definition 4.4), that is exponentially small in the distance between u and v.

Lemma 4.4. *Consider the central state μ of any model in the class (12) fulfilling the bounds (9) and (13). Suppose that the transition matrix P of the state μ is irreducible and aperiodic. Recall the Definition 4.4 of a bad event B_v at $v\in V$ and the notation for its expectation $\mu(B_v)=:m(p_1)$ (which does not depend on v). Let*

$$\mathrm{Cov}(u,v):=\mu\Big((1_{B_v}(\omega)-m(p_1))(1_{B_u}(\omega)-m(p_1))\Big). \tag{53}$$

Then for small enough p_1, there exist $C>0$ such that for any $c_1\in(0,\frac{1}{3})$ and $c_2\in(0,1)$, for any $u,v\in V$,

$$|\mathrm{Cov}(u,v)|\le Ce^{-c|v-u|} \tag{54}$$

where $|v-u|$ is the graph distance in the tree and $\mathrm{e}^{-c}=\max\{e^{-c_1\lambda(p_1)},|\lambda_2(P)|^{c_2}\}$, where $\lambda(p_1)$ is defined in Definition 4.5 and $\lambda_2(P)$ is the second largest eigenvalue (in modulus) of the transition matrix P of μ.

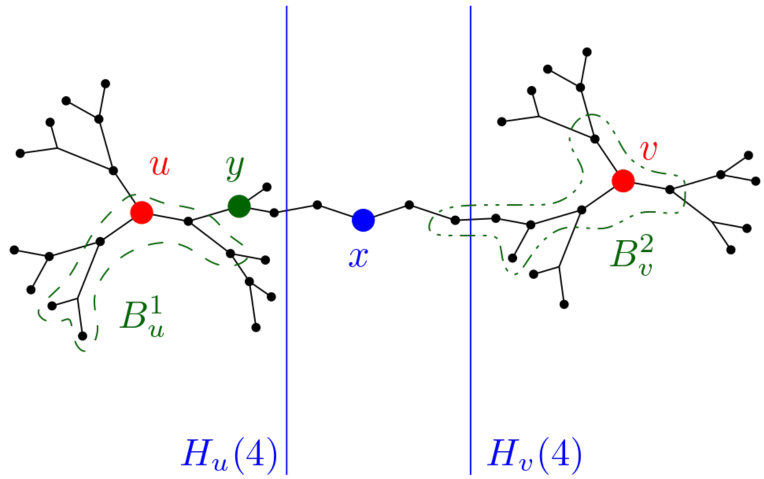

Figure 1: Objects appearing in the proof of Lemma 4.4 of exponential decorrelation of bad events at u and v. On the picture $d = 2, |u - v| = 11, r = 4$. Half-trees $H_u(4)$ and $H_v(4)$ are delimited by the two vertical lines. One of the midpoints between u and v is denoted by x. The support of the bad event B_u^1, depicted with a dashed line, stays inside $H_u(4)$, while the support of the bad event B_v^2, depicted with a dotted line, exits into $H_v(4)$. The vertex y is the closest to u on the path from u to x which does not touch the support of B_u^1.

The proof uses a combination of two facts. First, long bad contours attached to a given site are exponentially improbable, see Lemma 4.3. This explains the occurence of $\lambda(p_1)$. Second there is exponential relaxation of the one-dimensional Markov chain obtained by restriction of μ to a branch, conditionally on a root, which explains the appearance of the second largest eigenvalue $\lambda_2(P)$ of the transition matrix. Of course $|\lambda_2(P)| < 1$.

Proof. We first define several objects and events depicted in Figure 1. Denote by the half-trees $H_u(r)$ and $H_v(r)$ the sets

$$H_u := H_u(r) := \{x \in V, |x-u| \le |x-v| - r\} \quad \text{and} \quad H_v := H_v(r) := \{x \in V, |x-v| \le |x-u| - r\}. \tag{55}$$

Choose $r = \lfloor |u-v|/3 \rfloor$. Clearly H_u and H_v are disjoint, with their distance growing proportionally to $|u-v|$. Define

$$B_u^1 := \bigcup_{\gamma : H_u \supset \gamma \ni u} B(\gamma), \quad B_u^2 := B_u \backslash B_u^1 \quad \text{and} \quad B_v^1 := \bigcup_{\gamma : H_v \supset \gamma \ni v} B(\gamma), \quad B_v^2 := B_v \backslash B_v^1. \tag{56}$$

The sets B_u^1 and B_v^1 describe bad events at u (resp. v) which are produced by contours included in H_u (resp. H_v). The sets B_u^2 and B_v^2 describe bad events which are not produced by the previous contours and may connect u and v.

Bad events caused by long contours are exponentially improbable in the minimal length of such contours. Namely, by the Key Lemma 4.3 and the entropy bound, there exists $C_1 > 0$ such that for any $c_1 \in (0, \frac{1}{3})$, we have

$$\begin{aligned} \mu(B_u^2) = \mu(B_v^2) &\le \sum_{\gamma : \gamma \ni u, \gamma \cap H_u^c \neq \emptyset} \mu(B(\gamma)) \le \sum_{\gamma : \gamma \ni u, |\gamma| \ge |u-v|/3} e^{-\lambda(p_1)|\gamma|} \\ &\le \sum_{\ell \ge |u-v|/3} (d+1)^{2(\ell-1)} (q-1)^{\ell} e^{-\lambda(p_1)\ell} \le C_1 e^{-c_1 \lambda(p_1)|u-v|}. \end{aligned} \tag{57}$$

Spelling out the correlation function between the bad events in terms of the decomposition (56), we therefore have

$$\begin{aligned} |\mathrm{Cov}(u,v)| &= |\mu(B_u^1; B_v^1) + \mu(B_u^1; B_v^2) + \mu(B_u^2; B_v^1) + \mu(B_u^2; B_v^2)| \le |\mu(B_u^1; B_v^1)| + 3 \max\{\mu(B_u^2), \mu(B_v^2)\} \\ &\le |\mu(B_u^1; B_v^1)| + 3C_1 \exp(-c_1 \lambda(p_1)|u-v|). \end{aligned} \tag{58}$$

It remains to control the decay of $|\mu(B_u^1; B_v^1)|$. Exact decorrelation $\mu(B_u^1; B_v^1) = 0$ holds for free states but for an non-symmetric model there is no reason to assume it. Nevertheless, we may use the Markov chain property of μ along the path from u to v and the exponential convergence of the corresponding 1d-Markov chain, to arrive at an expontential decorrelation in the distance $|u-v|$ (with our irreducibility and aperiodicity assumptions).

Indeed, choose $x = x(u,v)$ to be the (or one of the at most two) middle sites between u and v (which means that the distances $|u-x|$ and $|v-x|$ differ at most by one). From the tree-indexed Markov chain property of μ, using conditional independence, we have

$$\mu(B_u^1 \cap B_v^1) = \sum_{a\in\mathbb{Z}_q} \mu(B_u^1|\sigma_x = a)\mu(B_v^1|\sigma_x = a)\mu(\sigma_x = a). \tag{59}$$

Let $\Delta_{u,x}(a) := \mu(B_u^1|\sigma_x = a) - \mu(B_u^1)$ and similarly for v. We have

$$\begin{aligned}|\mu(B_u^1; B_v^1)| &= \Big|\sum_{a\in\mathbb{Z}_q} \Big(\mu(B_u^1)\Delta_{v,x}(a) + \mu(B_v^1)\Delta_{u,x}(a) + \Delta_{u,x}(a)\Delta_{v,x}(a)\Big)\mu(\sigma_x = a)\Big| \\ &\le 2\max_{a\in\mathbb{Z}_q}|\Delta_{u,x}(a)| + \max_{a\in\mathbb{Z}_q}|\Delta_{v,x}(a)|.\end{aligned} \tag{60}$$

By symmetry it suffices to bound $|\Delta_{u,x}(a)|$. Choose y on the path $[x,u]$, between x and u, such that it is the closest to u with the property that all the contours appearing in the definition of B_u^1 do not touch $(x,y]$. To control the difference between $\mu(B_u^1|\sigma_x = a)$ and $\mu(B_u^1)$ we use the exponential relaxation of the homogenous 1d Markov chain which arises as restriction of μ, on the path from x to y. There exists $C_2 > 0$ such that for any $c_2 \in (0,1)$,

$$\begin{aligned}|\Delta_{u,x}(a)| &= \Big|\sum_{b\in\mathbb{Z}_q} \mu(B_u^1|\sigma_y = b)(\mu(\sigma_y = b|\sigma_x = a) - \mu(\sigma_y = b))\Big| \le \sum_{b\in\mathbb{Z}_q} |\mu(\sigma_y = b|\sigma_x = a) - \mu(\sigma_y = b))| \\ &= \sum_{b\in\mathbb{Z}_q} |P^{|x-y|}(a,b) - \mu(\sigma_y = b))| \le C_2|\lambda_2(P)|^{c_2|x-y|}\end{aligned} \tag{61}$$

where the last line follows e.g. from [6, Example 4.3.9]. As by construction $|y-x| \sim \frac{1}{6}|u-v|$, as $|u-v| \uparrow \infty$, this implies the desired exponential decorrelation of $|\mu(B_u^1; B_v^1)|$, and concludes the proof. □

4.4.2. Almost sure convergence of empirical means of bad sites

As a consequence of the exponential decorrelation of bad events, we harvest the following lemma.

Lemma 4.5. *Consider the central state μ of any model of the class (12) fulfilling the bounds (9) and (13). For any sequence of subsets $\Lambda_n = \Lambda_n^r$ of a branch of the tree such that $\sum_{n=1}^\infty |\Lambda_n|^{-1} < \infty$ and for μ-a.e. ω,*

$$\lim_{n\uparrow\infty} \frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n} 1_{B_v}(\omega) = \mu(B_0) = m(p_1). \tag{62}$$

While the limit in question, if it exists, is necessarily tail-trivial, the statement is non-trivial as the measure μ is not extremal at low temperature. So it must be proved manually.

Proof. The almost sure limit holds, if we can show that for any fixed $\delta > 0$ the number of indices n for which

$$\frac{1}{|\Lambda_n|}\Big|\sum_{v\in\Lambda_n}(1_{B_v}(\omega) - m(p_1))\Big| \ge \delta \tag{63}$$

is finite, for μ-almost every ω. By Borel-Cantelli it suffices to show that

$$\sum_{n=1}^\infty \mu\Big(\frac{1}{|\Lambda_n|}\Big|\sum_{v\in\Lambda_n}(1_{B_v}(\omega) - m(p_1))\Big| > \delta\Big) < \infty. \tag{64}$$

For this we use the quadratic Chebychev inequality to bound the probability in the above sum by

$$\frac{1}{|\Lambda_n|^2\delta^2}\sum_{v,u\in\Lambda_n} \mu\Big((1_{B_v}(\omega) - m(p_1))(1_{B_u}(\omega) - m(p_1))\Big). \tag{65}$$

Recalling the definition (53) of site covariances $Cov(u,v)$ and using Lemma 4.4, we deduce that there exists $C>0$ such that

(66) $$\sum_{n=1}^{\infty}\frac{1}{|\Lambda_n|^2}\sum_{v,u\in\Lambda_n}Cov(u,v)\le\sum_{n=1}^{\infty}\left(\frac{1}{|\Lambda_n|}Cov(0,0)+\frac{2}{|\Lambda_n|^2}\sum_{v\in\Lambda_n}\sum_{u\in\Lambda_n,u>v}Cov(u,v)\right)\le C\sum_{n=1}^{\infty}\frac{1}{|\Lambda_n|}<\infty$$

since for any vertex $v\in V$, the sum $\sum_{u>v}Cov(u,v)\le C'<\infty$ uniformly in v by Lemma 4.4. This proves the existence of the a.s. limit of empirical sums of indicators of bad points, along the quadratic volume sequences defined in (25). □

4.4.3. Concentration of spin-overlaps under typical extremals

We have the following concentration statement on the branch overlap.

Lemma 4.6. *Consider the central state μ of any model in the class (12) fulfilling the bounds (9) and (13). There exist a sequence of spacings $r=(r_i)_{i\in\mathbb{N}}$, in general depending on model parameters, such that for any sequence of subsets $\Lambda_n=\Lambda_n^r$ of a branch of the tree such that $\sum_{n=1}^{\infty}|\Lambda_n|^{-1}<\infty$, the following holds. For μ-almost every ω, for $\pi(\cdot|\omega)$-almost every realizations of σ we have*

(67) $$\limsup_{n\uparrow\infty}\frac{1}{|\Lambda_n|}\Big|\sum_{v\in\Lambda_n}(1_{\sigma_v\neq\omega_v}-\pi(\sigma_v\neq\omega_v|\omega))\Big|=0.$$

Proof. It suffices to prove that, for any $\delta>0$ we have, for μ-a.e. ω,

(68) $$\pi\left(\frac{1}{|\Lambda_n|}\Big|\sum_{v\in\Lambda_n}(1_{\sigma_v\neq\omega_v}-\pi(\sigma_v\neq\omega_v|\omega))\Big|>\delta\text{ for infinitely many }n\,\middle|\,\omega\right)=0.$$

Using the Borel-Cantelli lemma it suffices to show that, for μ-a.e. ω,

(69) $$\sum_{n=1}^{\infty}\pi\left(\frac{1}{|\Lambda_n|}\Big|\sum_{v\in\Lambda_n}(1_{\sigma_v\neq\omega_v}-\pi(\sigma_v\neq\omega_v|\omega))\Big|>\delta\,\middle|\,\omega\right)<\infty.$$

Now by the quadratic Chebychev inequality, it is enough to show that

(70) $$\sum_{n=1}^{\infty}\frac{1}{|\Lambda_n|^2}\sum_{v,u\in\Lambda_n}|\pi(\sigma_v\neq\omega_v;\sigma_u\neq\omega_u|\omega)|<\infty$$

holds for μ-a.e. ω. We would like to use the abstract fact that $\pi(\cdot|\omega)$ is an extremal Gibbs measure, and hence decorrelates in the sense of (14). Indeed, by extremality of $\pi(\cdot|\omega)$ at fixed ω, for any fixed u and any sequence $v\uparrow\infty$ (meaning that v leaves any finite ball) the following non-quantative result holds: For μ-a.e. ω,

(71) $$\lim_{v\uparrow\infty}|\pi(\sigma_v\neq\omega_v;\sigma_u\neq\omega_u|\omega)|=0.$$

The problem is that the rate of convergence may depend on ω, while we want a speed which is uniform in all the extremals (in order for the limit (67) to hold along the same sparse sequence of volumes for μ-a.e. ω). An explicit analysis is difficult, due to the lack of spatial homogeneity. To bypass this difficulty, note that to ensure (70) it is sufficient to ensure that

(72) $$\int\mu(d\omega)\sum_{n=1}^{\infty}\frac{1}{|\Lambda_n|^2}\sum_{v,u\in\Lambda_n}|\pi(\sigma_v\neq\omega_v;\sigma_u\neq\omega_u|\omega)|<\infty.$$

Using monotone convergence for non-negative functions we can deduce this from

(73) $$\sum_{n=1}^{\infty}\frac{1}{|\Lambda_n|^2}\sum_{v,u\in\Lambda_n}\underbrace{\int\mu(d\omega)|\pi(\sigma_v\neq\omega_v;\sigma_u\neq\omega_u|\omega)|}_{=:c(u,v)}<\infty$$

while by dominated convergence we have $\lim_{v\uparrow\infty}c(u,v)=0$ for any $u\in V$, Looking at the off-diagonal terms in the double-sum (73) as in (66), we can now achieve the desired summability

$$\sum_{n=1}^{\infty}\frac{1}{|\Lambda_n|^2}\sum_{i=1}^{|\Lambda_n|}\sum_{j=1}^{i-1}c(v_j,v_i)<\infty \tag{74}$$

by iteratively choosing v_i given $v_1,\dots,v_{i-1}$ on the same branch so large that $\sum_{j=1}^{i-1}c(v_j,v_i)<i^{-2}$. The above choice provides the following upper bound of (74) as $\sum_{n=1}^{\infty}\frac{1}{|\Lambda_n|}\sum_{i=1}^{\infty}i^{-2}$ which is finite. □

4.4.4. Proof of Theorem 3.4

Proof. Applying Lemma 4.6, the Key Lemma 4.3, and Lemma 4.5, we deduce that for sparse enough sets $\Lambda_n=\Lambda_n^{\mathrm{r}}$ the following holds. For μ-almost every ω, $\pi(\cdot|\omega)$-almost surely,

$$\begin{aligned}\liminf_{n\uparrow\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n}1_{\sigma_v=\omega_v}&\geq 1-\limsup_{n\uparrow\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n}\pi(\sigma_v\neq\omega_v|\omega)\geq 1-\liminf_{n\uparrow\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n}1_{B_v}(\omega)-\epsilon_1(\beta)\\&=1-\epsilon_1(\beta)-m(p_1)\geq 1-\epsilon_1(\beta)-\epsilon_2(p_1)\end{aligned}$$

with $\epsilon_1(\beta)=C'e^{-c'\beta}$, see (48), and $\epsilon_2(p_1)=Ce^{-c\lambda(p_1)}$, see (52) and Definition 2.2. This proves the theorem. □

4.5. From overlap control to almost sure singularity of extremals

Proof of Corollary 3.2. On the one hand we have the following lower bound from Theorem 3.4 and monotone convergence. For sparse enough sets $\Lambda_n=\Lambda_n^{\mathrm{r}}$, for μ almost every ω,

$$\pi(\underline{\phi}^{\omega}|\omega)=\liminf_{n\uparrow\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n}\pi(\sigma_v=\omega_v|\omega)\geq 1-\epsilon_1(\beta)-\epsilon_2(p_1). \tag{75}$$

On the other hand, we use $1_{\sigma_v=\omega_v}\leq 1_{\omega_v=\omega'_v}+1_{\sigma_v\neq\omega'_v}$ to write the upper bound

$$\pi(\underline{\phi}^{\omega}|\omega')\leq\limsup_{n\uparrow\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n}1_{\omega_v=\omega'_v}+\pi\left(\limsup_{n\uparrow\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n}1_{\sigma_v\neq\omega'_v}\middle|\omega'\right). \tag{76}$$

The first term in the r.h.s. is equal to $1/q$ for $\mu\otimes\mu$-a.e. pair (ω,ω'), in the case of clock models. Indeed, the Borel-Cantelli argument, together with the Chebychev inequality as in the proof of Lemma 4.5 shows that the desired strong law of large numbers follows (since exponential decorrelation of the local events $(1_{\omega_v=a})_{v\in\Lambda_n^r}$ follows from exponential decorrelation of the 1d Markov chains as in (61)). In the general case of central states this term equals $\sum_{a\in\mathbb{Z}_q}\mu(\sigma_0=a)^2$; it is close to $1/q$ for perturbed free states of clock models, see Section 4.1.4.

The second term in the r.h.s. is $\pi(\cdot|\omega')$-a.s. controlled via Theorem 3.4 by

$$\limsup_{n\uparrow\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n}1_{\sigma_v\neq\omega'_v}\leq\epsilon_1(\beta)+\epsilon_2(p_1). \tag{77}$$

So the statement of the theorem follows once $1-\epsilon_1(\beta)-\epsilon_2(p_1)>\sum_{a\in\mathbb{Z}_q}\mu(\sigma_0=a)^2+\epsilon_1(\beta)+\epsilon_2(p_1)$. □

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