5-FUNCTOR FORMALISM FOR SOLID SHEAVES ON SCHEMES (PRELIMINARY VERSION – DO NOT CITE)

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We introduce the *(pro-)unramified site* of a scheme and study its properties. It turns out that it posesses enough contractible objects, similar to the pro-étale site of Bhatt–Scholze [BS15]. In general, the (pro-)unramified site is less well-behaved than the (pro-)étale site. However, we show that for a certain class of schemes, which we call *combs* and which form a basis for the v-topology on the category of schemes, the pro-unramified site behaves in a nice way. In particular, in a sufficiently v-local setup we prove an unconditional base change result for unramified sheaves in Theorem 4.2, which parallels [Sch18, Corollary 16.10] for étale sheaves on diamonds (note that this can fail for étale sheaves on schemes when the base change map is not flat). We use this in §5 to develop a 5-functor formalism for solid sheaves on schematic v-stacks, analogous to the 5-functor formalism in the work of Fargues–Scholze [FS, Chapter VII].

Outline. In §1 we define the pro-unramified site of a scheme, describe its contractible objects (in Theorem 1.33) and prove that for geometrically unibranch schemes, satisfying an additional condition, the pushforward from (pro-)unramified to (pro-)étale abelian sheaves is exact (Theorem 1.40). In §2 we compare the pro-étale site with the v-site of a scheme and prove that pullback from pro-étale sheaves to v-sheaves is fully faithful (on the level of abelian categories); this is a schematic analogue of [Sch18, Proposition 14.7] for diamonds. In §3 we restrict to combs (i.e., sufficiently v-local schemes) and prove that for them the pullback from pro-unramified sheaves to v-sheaves is fully faithful. In §4 we use all the theory set up before to prove the unconditional base change (Theorem 4.2) for unramified torsion sheaves (whose order is coprime to the characteristic) on combs, and to study the usual four functors for unramified sheaves. Finally, in §5 we exploit the above results to define (by v-descent from combs) the category of solid sheaves on a schematic v-stack and to define the five functors on it, closely following the approach of [FS, Chapter VII]. ;

Notation and prelimilaries. We abbreviate quasi-compact and quasi-separated by "qcqs", and quasi-compact open by "qc open". For a scheme X we sometimes denote by $|X|$ the underlying topological space.

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1. The pro-unramified site of a scheme

1.1. **Some notation and preliminaries.** Let $\mathcal{S}$ be the category of spectral spaces with spectral maps and $\mathcal{S}_f$ the full subcategory of finite spectral spaces. Recall that $\mathcal{S} \cong \mathrm{Pro}(\mathcal{S}_f)$. For $X \in \mathcal{S}$, let X^c (resp. X^{gen}, resp. $\pi_0(X)$) denote the set of closed points (resp. generic points of irreducible components, resp. the set of connected components) of X. We write X^{cons} for X endowed with the constructible topology. Recall that for $X \in \mathcal{S}$, a subset $Z \subseteq X$ is closed in the constructible topology if and only if it is pro-constructible, i.e., intersection of constructible

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subsets. Recall that for $X \in \mathcal{S}$, $\pi_0(X)$ is naturally a profinite set, and the map $\pi_X \colon X \twoheadrightarrow \pi_0(X)$, sending a point of X to its connected component is a quotient map [Sta14, 0906]. If X is clear from the context, we also write π for π_X. Finally, recall that the image of a spectral map between spectral spaces is pro-constructible [Sta14, 0A2S].

We will use these facts without further reference below.

1.2. **Straight spectral spaces.** To any spectral space X, Hochster attached a *dual spectral space* X^*, which has the same underlying set, but in which the closed sets are precisely the pro-(qc open) subsets of X [Hoc69, Prop. 8]; $X \mapsto X^* \colon \mathcal{S} \to \mathcal{S}$ is a (covariant) duality functor. In particular, $X^{**} = X$ and if $f \colon X \to Y$ is a map in $\mathcal{S}$, then the induced map $f^* \colon X^* \to Y^*$ is again spectal. Note that if $X \in \mathcal{S}_f$, then the opens in X^* are precisely the closed subsets in X.

Lemma 1.1. *Let $X \in \mathcal{S}$. Then $\pi_0(X) = \pi_0(X^*)$ and $X^{\mathrm{gen}} = (X^*)^c$.*

Proof. For the first claim, it suffices (by duality) to check that if $Y \subseteq X$ is connected, then Y is also connected in X^*. For the second claim, note that for $x \in X^{\mathrm{gen}}$, the intersection of all quasi-compact opens containing x equals $\{x\}$, which implies $x \in (X^*)^c$. □

Recall from [BS15, Lemma 2.1.8] that $\mathcal{S}$ admits all small limits, and the forgetful functor $\mathcal{S} \to \mathrm{Sets}$ commutes with them.

Lemma 1.2. *$X \mapsto X^*$ commutes with all small limits.*

Proof. As $\mathcal{S} = \mathrm{Pro}(\mathcal{S}_f)$, it is enough to show that (i) if $X = \varprojlim_i X_i \in \mathcal{S}$ with $X_i \in \mathcal{S}_f$, then the natural continuous bijection $X^* \to \varprojlim_i X_i^*$ is a homeomorphism, and (ii) fiber products in $\mathcal{S}_f$ commute with $(\cdot)^*$. (i) follows by noticing that $T \subseteq X^*$ is closed if and only if $T = \bigcap_i (X \to X_i)^{-1}(U_i)$ for some $U_i \subseteq X_i^*$ closed, which implies that T is closed in $\varprojlim_i X_i^*$. For (ii), let $X \to Z \leftarrow Y$ be maps in $\mathcal{S}_f$. Recall from [BM21, proof of 2.1.8], that the topology on $X \times_Z Y$ is induced from the product topology on $X \times Y$. We must show that the continuous bijection $(X \times_Z Y)^* \to X^* \times_{Z^*} Y^*$ is closed. Any closed subset of $(X \times_Z Y)^*$ is a (finite) union of (finite) intersections of sets of the form $T_{A,B} = (A \times B) \cap (X \times_Z Y) (\subseteq X \times Y)$ with $A \subseteq X$, $B \subseteq Y$ open. It suffices to show that $T_{A,B}$ is closed in $X^* \times_{Z^*} Y^*$, which is obvious. □

In contrast to $X \to \pi_0(X)$ there is no such natual map for $\mathrm{Irr}(X)$. Instead, the situation is somewhat dual. We have the injection $\mathrm{Irr}(X) \xrightarrow{\sim} X^{\mathrm{gen}} \hookrightarrow X$ sending an irreducible component to its generic point, and we may give $\mathrm{Irr}(X)$ the induced topology. With this topology, $\mathrm{Irr}(X)$ does not need to be quasi-compact (see Example 1.4). However, if it is, then it is a profinite set as the next lemma shows.

Lemma 1.3. *Let X be a spectral space with X^{gen} quasi-compact. Then X^{gen} is pro-(qc open) and hence pro-constructible, and a profinite set.*

Proof. We claim that for any $y \in X \smallsetminus X^{\mathrm{gen}}$, there is some quasi-compact open $U \supseteq X^{\mathrm{gen}}$ with $y \notin U$. As $y \notin X^{\mathrm{gen}}$, we have $\overline{\{y\}} \cap X^{\mathrm{gen}} = \varnothing$. As qc opens form a basis for the topology, we may for any $x \in X^{\mathrm{gen}}$ find some quasi-compact open $x \in V_x \subseteq X \smallsetminus \overline{\{y\}}$. Then $\{V_x \cap X^{\mathrm{gen}}\}_{x \in X^{\mathrm{gen}}}$ is an open covering of X^{gen}, hence it has a finite subcovering. Thus there are some $x_1, \ldots, x_n \in X^{\mathrm{gen}}$, such that $\bigcup_{i=1}^n V_{x_i} \supseteq X^{\mathrm{gen}}$. As $\bigcup_{i=1}^n V_{x_i}$ is also quasi-compact, the claim follows. The claim now implies that X^{gen} is the intersection of all quasi-compact opens containing it, hence pro-(qc open). As X^{gen} is spectral (by the first claim and [Sta14, 0902]) and its points do not admit non-trivial specialization relations, it follows from [Sta14, 0905] that it is profinite. □

Note that for the set of closed points $X^c \subseteq X$ the situation is somewhat dual: X^c is always quasi-compact [Sta14, 00ZM], and it is pro-constructible in X if and only if it is closed.

Example 1.4. (The first part of this example was explained to us by P. Scholze.) For $n \geq 1$, let X_n be the finite T_0-space, with $n+1$ points $x_1, \dots, x_n$ and η_n, such that $\{x_i\}$ is clopen for $1 \leq i < n$; $\{x_n, \eta_n\}$ is clopen and irreducible with generic point η_n. For $n \geq 2$, let $X_n \to X_{n-1}$ be the map given by $x_i \mapsto x_i$ $(1 \leq i \leq n-2)$, $x_{n-1} \mapsto \eta_{n-1}$, $\eta_n \mapsto \eta_{n-1}$, $x_n \mapsto x_{n-1}$. Let $X = \varprojlim_n X_n$. Then X^c is not closed (and hence not pro-constructible). Consequently, $(X^*)^{\mathrm{gen}}$ is not pro-constructible and hence not quasi-compact.

We have a dual version of [BS15, Lemma 2.1.4].

Lemma 1.5. *For a spectral space X the following are equivalent:*

(i) X^{gen} is quasi-compact and any surjection $\coprod_i V_i \to X$, with all V_i open in X^*, admits a section.
(ii) X^{gen} is quasi-compact and any connected component of X has a unique generic point.

For such a space the composition $X^{\mathrm{gen}} \to X \to \pi_0(X)$ is a homeomorphism.

Proof. Condition (i) is equivalent to X^* being w-local. Indeed, by Lemma 1.3, quasi-compactness of X^{gen} implies its pro-constructibility, and (as the constructible topologies on X and X^* agree) Lemma 1.1 then implies that $(X^*)^c$ is pro-constructible. Being stable under specialization, $(X^*)^c$ is then also closed. Similarly, Lemma 1.1 shows that condition (ii) for X is equivalent to the assertions that each connected component of X^* has a unique closed point and that $(X^*)^c$ is closed. The result now follows from [BS15, Lemma 2.1.4] for X^*. □

Recall that $V \subseteq X$ is closed constructible if and only if $X \setminus V$ is qc open. As $V \subseteq X$ is open in X^* if and only if its complement is the union of quasi-compact opens of X, each map $\coprod_i V_i \to X$ as in Lemma 1.5(i) admits a refinement with all $V_i \subseteq X$ closed constructible. We have the following dual version of [BS15, Def. 2.1.1].

Definition 1.6. A spectral space is *straight* if it satisfies the equivalent conditions of Lemma 1.5. A map $X \to Y$ of straight spectral spaces is *straight*, if it is spectral and $f(X^{\mathrm{gen}}) \subseteq Y^{\mathrm{gen}}$. Denote by $i\colon \mathcal{S}^{str} \to \mathcal{S}$ the subcategory of straight spaces with straight maps.

Remark 1.7. The functor $X \mapsto X^*$ restricts to a covariant duality $\mathcal{S}^{wl} \to \mathcal{S}^{str}$. (Note that $X \mapsto X^*$ does not preserve closed subsets, but does preserve pro-constructibles.) By duality, basic properties of w-local spaces ([BS15, 2.1.3, 2.1.4, 2.1.6, 2.1.9, 2.1.10] respectively) carry over to straight spaces:

(1) Let $X \in \mathcal{S}^{str}$, $Z \subseteq X$ a pro-(qc open) subspace. Then $Z \in \mathcal{S}^{str}$.
(2) Let $Z \subseteq X$ be pro-(qc open). We say that X is *straight along* Z, if $X^{\mathrm{gen}} \subseteq Z$. The *straightification of X along Z* is the set $\widetilde{Z}$ of all specializations in X of points in Z. As Z is pro-constructible, $\widetilde{Z}$ is closed in X.
(3) A spectral space X, which is straight along a straight pro-(qc open) subspace $Z \subseteq X$ with $\pi_0(Z) \cong \pi_0(X)$ is also straight.
(4) $\mathcal{S}^{str}$ admits all small limits and the inclusion $\mathcal{S}^{str} \to \mathcal{S}$ preserves these limits.
(5) Recall that the inclusion $\mathcal{S}^{wl} \to \mathcal{S}$ admits a right adjoint $X \mapsto X^{wl}$, and that X^{wl} is a pro-(Zariski localization) of X [BS15, 2.1.10-12] (in *loc. cit.*, X^{wl} was denoted X^Z). Dually, $\mathcal{S}^{str} \to \mathcal{S}$ admits a right adjoint $X \mapsto X^{str}$. On the level of $\mathcal{S}_f$ it is given by $X^{str} = \coprod_{x \in X} \overline{\{x\}}$, and in general by passing to the pro-category as in [BS15, proof of Lemma 2.1.10]; the composite $(X^{str})^{\mathrm{gen}} \to X^{str} \to X$ is a homeomorphism for the constructible topology on X. Moreover, $X^{str} = ((X^*)^{wl})^*$.
(6) A map $Y \to X$ is a *closed constructible localization* if $Y = \coprod_i Y_i$ with $Y_i \to X$ (isomorphic to) an immersion of a closed constructible subset. A *pro-(closed constructible*

localization) is a cofiltered limit of such maps. Dualizing the fact that $X^{wl} \to X$ is a pro-(Zariski localization), we get that $X^{str} \to X$ is surjective and a pro-(closed constructible localization).

Lemma 1.8. *We have* $\mathcal{S}^{wl} = \mathrm{Pro}(\mathcal{S}^{wl} \cap \mathcal{S}_f)$ *and* $\mathcal{S}^{str} = \mathrm{Pro}(\mathcal{S}^{str} \cap \mathcal{S}_f)$.

Proof. By duality it suffices to show the first equality. We claim that any $X \in \mathcal{S}^{wl}$ can be written as a cofiltered limit of finite w-local spaces along w-local maps. Write $X = \varprojlim_i X_i$ as a cofiltered limit with $X_i \in \mathcal{S}_f$. Then $X^{wl} = \varprojlim_i X_i^{wl}$ (with all transition maps w-local). By adjunction and as X is w-local, the natural pro-open cover $X^{wl} \to X$ admits a w-local section $s \colon X \to X^{wl}$, which is necessarily an isomorphism onto a closed subspace (e.g., as this holds for affine schemes and by [Hoc69, Theorem 6]). For each i, let Y_i be the closure of the image of $X \overset{s}{\hookrightarrow} X^{wl} \to X_i^{wl}$. Being closed in a w-local space, Y_i is w-local itself; moreover, it is clear that $\{Y_i\}_i$ form an inverse system of finite w-local spaces, along w-local transition maps. Clearly, s factors through an inclusion $X \to \varprojlim_i Y_i (\subseteq \varprojlim X_i^{wl} = X^{wl})$, which is bijective as $X \subseteq X^{wl}$ is closed. As $\varprojlim_i Y_i$ also has the subspace topology of X^{wl}, this is an isomorphism, and the claim is proven. □

1.3. *w*-**local straight spaces.** We say that a map $Y \to X$ in $\mathcal{S}$ is a *topological v-cover* if any specialization relation $x \rightsquigarrow y$ in X lifts to Y (in this section we simply write "v-cover", because there is no risk of confusion). The combination of w-locality and straightness splits all v-covers by locally closed constructible subsets:

Lemma 1.9. *For a spectral space X the following are equivalent:*

- *(i)* X^{gen} *is quasi-compact,* X^c *is closed and any surjection* $\coprod_i C_i \to X$ *splits if it satisfies the following two conditions:*
  - *(a)* $C_i = U_i \cap V_i$ *with* $U_i \subseteq X$ *open, and* $V_i \subseteq X^*$ *open, and*
  - *(b)* $\coprod_i C_i \to X$ *is a v-cover.*
- *(ii) X is w-local and straight*

Proof. (ii) follows from (i) by [BS15, Lemma 2.1.4] and Lemma 1.5, as (b) is vacuous for open (resp. closed constructible) covers. Now assume (ii), and let $\coprod_i C_i \to V$ be as in (i). We may find $U_i = \bigcup_k U'_{ik}$ and $V_i = \bigcup_j V'_{ij}$ with all U'_{ik}, $X \smallsetminus V'_{ij}$ quasi-compact open. Then $C_i = \bigcup_{j,k} (U'_{ik} \cap V'_{ij})$. It is easy to check that the refinement $\{U'_{ik} \cap V'_{ij}\}_{ijk}$ of our covering satisfies condition (b). Replacing $\coprod_i C_i \to X$ by this refinement, we may assume that U_i and $X \smallsetminus V_i$ are quasi-compact open in X. For $x \in \pi_0(X)$, let x^c (resp. η_x) denote the closed (resp. generic) point of the connected component $X_x \subseteq X$ corresponding to x. By condition (b), we may, for any $x \in \pi_0(X)$ find an $i = i(x)$, such that $\eta_x \rightsquigarrow x^c$ lifts to $C_i = U_i \cap V_i$. Then, $U_i \supseteq X_x$ (as U_i is open and contains x^c) and similarly, $V_i \supseteq X_x$. Moreover, recall that X^{gen} is profinite by Lemma 1.3. We claim that in the above situation $C_i \cap X^{\mathrm{gen}}$ is a clopen neighboorhood of $\eta_x \in X^{\mathrm{gen}}$. Indeed, $U_i \cap X^{\mathrm{gen}}$ is open and, in fact, quasi-compact neighboorhood of η_x in X^{gen} (indeed, recall that by Lemma 1.3, X^{gen} is pro-constructible, hence retrocompact in X). Thus $U \cap X^{\mathrm{gen}} \subseteq X^{\mathrm{gen}}$ is clopen. Similarly, $(X \smallsetminus V_i) \cap X^{\mathrm{gen}} \subseteq X^{\mathrm{gen}}$ is quasi-compact open, hence clopen subset of X^{gen}. Hence its complement $V_i \cap X^{\mathrm{gen}}$ is a clopen neighboorhood of η_x in X^{gen}, and the claim follows. Further, note that X^c is profinite. Dually to the preceeding claim, one shows that $C_i \cap X^c$ is a clopen neighboorhood of x^c in X^c. Both compositions $X^c \to X \to \pi_0(X)$ and $X^{\mathrm{gen}} \to X \to \pi_0(X)$ are continuous bijections of profinite sets, hence homemorphisms [Sta14, 08YE]. Projecting $C_i \cap X^{\mathrm{gen}}$ and $C_i \cap X^c$ down to $\pi_0(X)$ and intersecting the images, we get a (still clopen) neighboorhood $N_x \subseteq \pi_0(X)$ of x, with the property that for all $y \in N_x$, $X_y \subseteq C_i$ (this inclusion follows by the same argument which showed $X_x \subseteq C_i$ above). Finally, $\{N_x \colon x \in \pi_0(X)\}$ is a clopen covering of the profinite set $\pi_0(X)$. We may refine it by a finite disjoint clopen covering $\{N_j\}_j$ [Sta14, 08ZZ]. This induces a clopen covering $\{X_{N_j}\}_j$ of X. Over each X_{N_j}, $\coprod_i C_i \to X$ obviously admits a section, and these sections glue to one section over the whole of X. □

Definition 1.10. Denote by $\mathcal{S}^{wls} \subseteq \mathcal{S}$ the subcategory of w-local straight spectral spaces with w-local straight spectral maps.

Example 1.11. Any profinite set is w-local straight. The spectrum of any local domain is w-local straight. Finite disjoint unions of w-local straight spaces are w-local straight.

Clearly, we have $\mathcal{S}^{wls} = \mathcal{S}^{wl} \cap \mathcal{S}^{str}$ within $\mathcal{S}$. Moreover, $X \mapsto X^*$ restricts to a duality functor on $\mathcal{S}^{wls}$. For any $X \in \mathcal{S}^{wls}$, the inclusions $X^{\mathrm{gen}} \hookrightarrow X \hookleftarrow X^c$ induce homeomorphisms $X^{\mathrm{gen}} \cong \pi_0(X) \cong X^c$.

Lemma 1.12. *$\mathcal{S}^{wls}$ admits all small limits, and the inclusion $\mathcal{S}^{wls} \to \mathcal{S}$ preserves these limits.*

Proof. This follows formally from [BS15, Lemma 2.1.9] and Lemma 1.2. □

We now construct the right adjoint of the inclusion $\mathcal{S}^{wls} \to \mathcal{S}$. Recall that $\pi_0(X^{wl}) \cong X^{cons}$ [BS15, Lemma 2.1.10]. Similarly, let

$$T(X) := \{(x, y) \in X \times X : x \rightsquigarrow y\}$$

be the set of all specialization relations in X. We define the *constructible topology on* $T(X)$ as follows. For a presentation $X = \varprojlim_i X_i$ with all $X_i \in \mathcal{S}_f$, note that the natural map $T(X) \to \varprojlim_i T(X_i)$ is bijective. Via this bijection, we declare the constructible topology on $T(X)$ to be the inverse limit of discrete topologies on all $T(X_i)$. This is independent of the choice of the presentation $X = \varprojlim_i X_i$.

We will abbreviate locally closed constructible by *l.c.c.*

Lemma 1.13. *The inclusion $\mathcal{S}^{wls} \to \mathcal{S}$ has a right adjoint $X \mapsto X^{wls}$. The counit $X^{wls} \to X$ is a pro-(l.c.c.) v-cover. We have $\pi_0(X^{wls}) \cong T(X)$, where $T(X)$ is equipped with the constructible topology.*

Proof. As in [BS15, proof of Lemma 2.1.10], for the construction of the adjoint it suffices to work with finite T_0-spaces and to construct, for each $X \in \mathcal{S}_f$, a functorial l.c.c. cover $X^{wls} \to X$ with X^{wls} w-local and straight, such that (a) $X \mapsto X^{wls}$ carries maps to w-local straight maps, (b) $\pi_0(X^{wls}) \cong T(X)$. We take

$$X^{wls} = \coprod_{x \rightsquigarrow y \in T(X)} T_{x \rightsquigarrow y}.$$

where $T_{x \rightsquigarrow y} := \{z \in X : x \rightsquigarrow z \rightsquigarrow y\} = \overline{\{x\}} \cap X_y$ (where X_y is the set of all generalizations of y) is locally closed in X. Given a map $f : Y \to X$ in $\mathcal{S}_f$, we send it to the map induced by $T_{x \rightsquigarrow y} \to T_{f(x) \rightsquigarrow f(y)}$. The conditions (a),(b) are easily checked.

To prove that $X \mapsto X^{wls}$ is indeed an adjoint to inclusion, it suffices (as in *loc. cit.*) to show that any spectral map $h : Y \to X$ with $Y \in \mathcal{S}^{wls}$, $X \in \mathcal{S}$ factors uniquely through a w-local straight map $h' : Y \to X^{wls}$. For $? \in \{c, \mathrm{gen}\}$, let $g_? : Y^? \hookrightarrow Y \xrightarrow{h} X$ and $s_? : Y \twoheadrightarrow \pi_0(Y) \xrightarrow{\sim} Y^?$ be the induced maps. Pick $x \rightsquigarrow y$ in X with $g_{\mathrm{gen}}^{-1}(x) \neq \varnothing$ and $g_c^{-1}(y) \neq \varnothing$. Then replacing Y by the clopen (cf. *loc. cit.*) subset $s_{\mathrm{gen}}^{-1}(g_{\mathrm{gen}}^{-1}(x)) \cap s_c^{-1}(g_c^{-1}(y))$, we may assume that h factors through $T_{x \rightsquigarrow y}$. Then we define the lift h' by sending Y to $T_{x \rightsquigarrow y} \subseteq X^{wls}$. □

Lemma 1.14. *The functor $X \mapsto X^{wl} : \mathcal{S} \to \mathcal{S}^{wl}$ from Remark 1.7 restricts to the right adjoint of the inclusion $\mathcal{S}^{wls} \to \mathcal{S}^{str}$. The analogous statement holds for $X \mapsto X^{str} : \mathcal{S} \to \mathcal{S}^{str}$.*

Proof. This follows from Lemma 1.8 by checking that the construction of X^{wl} in [BS15, 2.1.10] preserves straight spaces and straight maps. The second claim follows by duality. ☐

Remark 1.15. Lemma 1.14 gives another construction of X^{wls}: the functors $X \mapsto X^{wls}$, $X \mapsto X^{str} \mapsto (X^{str})^{wl}$ and $X \mapsto X^{wl} \mapsto (X^{wl})^{str}$ are naturally isomorphic (it suffices to check this on finite T_0-spaces). In particular, $T(X) \cong \pi_0(X^{wls}) \cong (X^{wl})^{cons} \cong (X^{str})^{cons}$. Moreover, using this and (the dual of) [BS15, Remark 2.1.11], we can describe $X^{wls}, (X^{wls})^{\mathrm{gen}}$ as follows:

$$X^{wls} = \varprojlim_{\{Y_i \hookrightarrow X^{wl}\}} \coprod_i \overline{Y_i} \supseteq (X^{wls})^{\mathrm{gen}} = \varprojlim_{\{Y_i \hookrightarrow X^{wl}\}} \coprod_i Y_i$$

where the limit is taken over the cofiltered category of all constructible stratifications $\{Y_i \hookrightarrow X^{wl}\}$. The space $X^{str} \in \mathcal{S}^{str}$ from Remark 1.7(5) admits a similar description.

1.4. **Quasi-(pro-unramified) morphisms.** Following [Sta14, 02G5], we call a morphism of schemes *G-unramified*, if it is formally unramified and locally of finite presentation. Recall also the notion of schematic v-covers from [BS17].

Definition 1.16. Let $f\colon Y = \operatorname{Spec} B \to \operatorname{Spec} A = X$ be a morphism of affine schemes.

(1) f is called *pro-unramified*, if $B = \varinjlim B_i$ is a filtered colimit of G-unramified A-algebras.
(2) f is called *quasi-(pro-unramfied)* if there exists a pro-unramified morphism $Y' = \operatorname{Spec} C \to Y$ which is a v-cover, such that the composition $Y' \to Y \to X$ is pro-unramified.

We have the corresponding notions of *ind-unramified* and *quasi-(ind-unramified)* maps of rings.

Lemma 1.17. *(1) Pro-unramified (resp. quasi-(pro-unramified)) maps are stable under composition and base change.*
(2) Quasi-(pro-unramified) maps are local on the target and source, with respect to the topology given by quasi-(pro-unramified) v-covers of affine schemes. In particular, they are so Zariski-locally.
(3) Any immersion of affine schemes is pro-unramified.
(4) Let X be an affine scheme, Y a pro-unramified X-scheme and Z any affine X-scheme. Any X-map $Y \to Z$ is pro-unramified.
(5) Pro-unramified maps have flat diagonal (i.e., are weakly unramified, cf. Definition 1.18).
(6) Any map $Y \to X$ with flat diagonal is formally unramified, i.e., $\Omega^1_{Y/X} = 0$.
(7) A map is G-unramified if and only it is pro-unramified and of finite presentation.

Proof. (1): The claim for composition of pro-unramified maps follows from the same claim for unramified maps [Sta14, 02G9] along with approximation results [Sta14, 01ZM and 0C4W][1]. The claim for base change of pro-unramified maps follows fom the same claim for unramified maps [Sta14, 02GA]. The assertions for quasi-(pro-unramified) maps follow formally from the assertions about pro-unramified maps (and similar ones for v-covers). (2): This formally follows from the definitions, (1) and stability of v-covers under composition and base change. (3): By (1) it suffices to show it for closed and for open immersions separatly. Moreover, any closed immerison can be written as an inverse limit of of finitely presented ones, so that we are done by [Sta14, 02GB and 02GC]. (4): factor $f\colon Y \to Z$ as the graph followed by second projection $Y \to Y \times_X Z \to Z$. Now $Y \to Y \times_X Z$ is pro-unramified by (3), and $Y \times_X Z \to Z$ is pro-unramified by (1). We conclude by (1). (5): If $Y = \lim_i Y_i \to X$ with all Y_i G-unramified X-schemes, then the diagonal of $Y_i \to X$ is open, hence flat. Inverse limit of these is the diagonal of $Y \to X$, which is hence also flat. (6): The question is local on the source, so we may assume Y and X are affine. Let $f\colon A \to B$ be a map of rings, such that $B \otimes_A B \to B$ is flat.

[1] Already here it becomes important to use G-unramified maps (instead of unramfied ones) in Definition 1.16(1).

By [Sta14, 08S2], $\Omega_{B/A}$ is the conormal sheaf of the diagonal Δ_f, which is a closed immersion. In general, if $\alpha\colon Z \hookrightarrow X$ is a flat closed immersion of schemes, then the conormal sheaf $\mathscr{C}_{Z/X} = 0$ vanishes. Indeed, we may assume $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/I$ are affine; then $\mathscr{C}_{Z/X} \cong I/I^2$. Flatness of $Z \hookrightarrow X$ is equivalent to I being a pure ideal of A (by [Sta14, 04PU,04PW]). But then, by [Sta14, 04PS (2)], $I^2 = I$ and we are done. (7): This follows from (5), (6) and [Sta14, 02G5]. □

By Lemma 1.17(2) the notion introduced in part (2) of the following definition is well-behaved.

Definition 1.18. Let $f\colon Y \to X$ be a morphism of schemes.

(1) f is called *weakly unramified*, if the diagonal $\Delta_f\colon Y \to Y \times_X Y$ is flat.
(2) f is called *quasi-(pro-unramified)* if for any $y \in Y$ there are open affine neighboorhoods $y \in V \subseteq Y$ and $f(x) \in U \subseteq X$ with $f(V) \subseteq U$, such that $f\colon V \to U$ is quasi-(pro-unramified).
(3) We denote by X_{pu} the full subcategory of X-schemes, whose objects are quasi-(pro-unramified) morphisms $Y \to X$. We endow it with the structure of a site by declaring covers to be those in the v-topology, cf. [Sta14, 0ETB and 0ETH].
(4) We call an object $U \in X_{\mathrm{pu}}$ *pro-unramified affine* if we can write $U = \varprojlim_i U_i$ as a small cofiltered limit of G-unramified maps $U_i \to X$, such that all U_i are affine schemes. We denote the full subcategory of X_{pu} spanned by pro-unramified affine objects by $X_{\mathrm{pu}}^{\mathrm{aff}}$.

Remark 1.19. As in [BS15, Rem.4.1.2], we can avoid set-theoretic issues by (re)defining X_{pu} using only quasi-(pro-unramified) maps $Y \to X$ with $|Y| < \kappa$ for a fixed uncountable strong limit cardinal, larger than $|X|$.

Lemma 1.20. *Let X be a scheme.*

(1) Any immersion and any weakly étale map is quasi-pro-unramified.
(2) Quasi-(pro-unramified) maps are stable under composition and base change.
(3) All X-maps between quasi-pro-unramified X-schemes are quasi-pro-unramified.
(4) The category X_{pu} has finite limits. The subcategory spanned by affine quasi-pro-unramified maps $Y \to X$ has all small limits. All limits agree with those in X-schemes.
(5) Any map in $X_{\mathrm{pu}}^{\mathrm{aff}}$ is pro-unramified.

Proof. (1): the claims are local on target and source, and hence follows from Lemma 1.17(3) resp. [BS15, Theorem 2.3.4] respectively. (2): follows from Lemma 1.17(1),(2). (3): the claim is Zariski-local on X, the target and the source. For affine schemes, it follows from 1.17(4). (4): X is a final object of X_{pu} and if $Y_1 \to Y_2 \leftarrow Y_3$ are maps in X_{pu}, then the X-scheme $Y_1 \times_{Y_2} Y_3$ is in X_{pu}, as follows from parts (2) and (3) of the lemma. (5): similar as [BS15, Lemma 4.2.2] (using [Sta14, 02GG] to ensure unramifiedness of maps at the finite level). □

General quasi-pro-unramified maps can be quite complicated. For example, if $Y \to X$ is quasi-pro-unramified, and $Y \hookrightarrow Y'$ is any nilpotent thickening of X-schemes, then Y' is also quasi-pro-unramified over X. This is compensated by the following (probably well-known) fact.

Lemma 1.21. *Let $\mathscr{F}$ be a sheaf on X_v (resp. X_{pu}). Let $Z \to Y$ be a universal homeomorphism in X_v (resp. X_{pu}). Then $\mathscr{F}(Y) \to \mathscr{F}(Z)$ is bijective.*

Proof. This follows as $Z \to Y$ is a v-cover and the diagonal $Z \to Z \times_Y Z$ is a surjective closed immersion (cf. the proof of [Sta14, 0F6V]), hence also a v-cover. □

Remark 1.22. Instead of X_{pu}, one might also consider the category of all weakly unramified X-schemes. This, however, seems not to lead to a well-behaved site in general. E.g., there is no reason for Lemma 1.20(3) to hold for weakly unramified maps. Note also that quasi-(pro-unramified) does not imply weakly unramified (as it would be the case for the analogous notions of quasi-(pro-étale) and weakly étale maps).

Lemma 1.23. *Let X be any scheme. The topos $X_{\mathrm{pu}}^{\sim}$ of small sheaves is algebraic. A generating system of qcqs objects stable under fiber products is given by $X_{\mathrm{pu}}^{\mathrm{aff}}$. If X is affine, $X_{\mathrm{pu}}^{\mathrm{aff}}$ is the category of all pro-unramified X-schemes, it has the structive of a site (with v-covers) and $(X_{\mathrm{pu}})^{\sim} \cong (X_{\mathrm{pu}}^{\mathrm{aff}})^{\sim}$.*

Proof. The first two facts follow from the finitary nature of v-covers and the fact that for any $Y \in X_{\mathrm{pu}}$, there is a v-cover $\coprod_i Y_i \to Y$ with $Y_i \in X_{\mathrm{pu}}^{\mathrm{aff}}$, which is a formal consequence of the definitions. For affine X, [BS15, Remark 4.2.5] applies in our situation. □

Remark 1.24. In contrast to the pro-étale site of a scheme X, X_{pu} is not subcanonical. However, it follows from descent results of Rydh [Ryd10] that it is subcanonical up to relative seminormalization, i.e., up to universal homeomorphisms inducing isomorphisms on residue fields. More precisely, for $Y \in X_{\mathrm{pu}}$ let $h_Y(T) = \mathrm{Hom}_X(T, Y)$, and let $h_Y^{\#}$ denote its sheafification. If T is reduced, then $h_Y(T) \hookrightarrow h_Y^{\#}(T)$ is injective by [Ryd10, Proposition 7.2]. Moreover, for a pro-unramified v-cover $T' \to T$ in $X_{\mathrm{pu}}^{\mathrm{aff}}$, we have the *weak normalization* $T' \to T^{T'/\mathrm{wn}} \to T$, i.e., $T^{T'/\mathrm{wn}} \to T$ is the maximal separated universal homeomorphism, such that $T' \to T^{T'/\mathrm{wn}}$ is schematically dominant, cf. [Ryd10, 7.3 and Appendix B][2]. Note that $T^{T'/\mathrm{wn}} \in X_{\mathrm{pu}}$. (Relative) weak normalization is functorial in T', so we may form

$$T^{\mathrm{pu}/\mathrm{wn}} := \varprojlim_{T' \to T} T^{T'/\mathrm{wn}},$$

where the cofiltered limit is taken over all (fine enough) covers of T in $X_{\mathrm{pu}}^{\mathrm{aff}}$. (Note that by Lemma 1.21, we have $h_Y^{\#}(T) = h_Y^{\#}(T^{\mathrm{pu}/\mathrm{wn}})$ for all T.) It is clear that if $T = T^{\mathrm{pu}/\mathrm{wn}}$, then T is reduced and for any cover $T' \to T$ in X_{pu}, $T' \to T$ is weakly normal. Thus, [Ryd10, Theorem 7.4 and Remark 7.5] show that for any $Y \in X_{\mathrm{pu}}^{\mathrm{aff}}$, we have $h_Y^{\#}(T) = h_Y(T^{\mathrm{pu}/\mathrm{wn}})$.

Lemma 1.25. *A presheaf $\mathscr{F}$ on X_{pu} is a sheaf if and only if it satisfies the two conditions:*

(i) For any v-cover $V \to U$ in $X_{\mathrm{pu}}^{\mathrm{aff}}$, the sequence $F(U) \to F(V) \rightrightarrows F(V \times_U V)$ is exact.
(ii) $\mathscr{F}$ is a Zariski sheaf.

Proof. Same as the proof of [BS15, 4.2.6] (or [Sta14, 0ETM] for the v-topology). □

Over a field the situation is as nice as possible:

Lemma 1.26. *Let k be a field and let $X \in (\mathrm{Spec}\, k)_{\mathrm{pu}}$. Then $X_{\mathrm{red}} \in (\mathrm{Spec}\, k)_{\mathrm{proet}}$, cf. [BS15, Example 4.1.10]. We have $(\mathrm{Spec}\, k)_{\mathrm{pu}}^{\sim} \cong (\mathrm{Spec}\, k)_{\mathrm{proet}}^{\sim}$.*

Proof. The last claim follows from the first and Lemma 1.21. Write $f\colon X \to \mathrm{Spec}\, k$ for the structure map. We may assume X is affine. By assumption there exists a pro-unramified v-cover $X' \xrightarrow{g} X$, such that X' is affine and fg is pro-unramified. By Lemma 1.17(5) and as k is absolutely flat, fg is weakly étale. Thus, by [Sta14, 092E and 092F], X' is the spectrum of an absolutely flat ring (and in particular, reduced). As $X' \to X$ is a v-cover, X has no non-trivial specialization relations, i.e., X_{red} is the spectrum of an absolutely flat ring by [Sta14, 092F]. As X' is reduced, g factors through $g_{\mathrm{red}}\colon X' \to X_{\mathrm{red}}$. As g_{red} is surjective and X_{red} absoultely flat, g_{red} is faithfully flat. As g_{red} and fg are weakly unramified (by Lemma 1.17), we conclude by [Sta14, 092K (1)] that $f_{\mathrm{red}}\colon X_{\mathrm{red}} \to \mathrm{Spec}\, k$ also is. As f_{red} is automatically flat, it is weakly étale. □

[2]In fact, it is easy to see that (as $T' \to T$ is pro-unramified) $T^{T'/\mathrm{wn}} \to T$ induces isomorphisms on residue fields, and hence coincides with the semi-normalization of T in T'. Cf. [Sta14, 0EUL].

1.5. **Straight affine schemes.** It turns out that the pro-unramified site has enough contractible objects, just as the pro-étale site in [BS15, §2.2].

Definition 1.27. Let A, B be rings, and $f: A \to B$ a homomorphism.

(1) We call A *straight* if A is reduced and $\operatorname{Spec} A$ is straight. If A, B are straight, we call f *straight*, if $\operatorname{Spec} f$ is straight.
(2) We call f a *closed constructible localization* if $B = \prod_{i=1}^n A/I_i$, where for all i, $I_i \subseteq A$ is a finitely generated ideal.[3] An *ind-(closed constructible localization)* is a filtered colimit of closed constructible localizations.

Thus, an affine scheme $X = \operatorname{Spec} A$ is w-local straight if it is reduced, X^{gen} is quasi-compact, $X^c \subseteq X$ is closed, and each connected component of X is the spectrum of a local domain. We note that rings A with $(\operatorname{Spec} A)^{\mathrm{gen}}$ quasi-compact were studied by Olivier [Oli68].

Lemma 1.28. *The inclusion of the category of straight (resp. w-local straight) rings and maps into the category of all rings has a left adjoint $A \mapsto A^{str}$ (resp. $A \mapsto A^{wls}$). For $? \in \{str, wls\}$ we have $\operatorname{Spec}(A^?) = (\operatorname{Spec} A)^?$. The unit $A \to A^{str}$ (resp. $A \to A^{wls}$) is an ind-(closed constructible v-cover) (resp. an ind-(l.c.c.) v-cover).*

Proof. Using Remark 1.15, $(\operatorname{Spec} A)^{str}$, $(\operatorname{Spec} A)^{wls}$ admit natural scheme structures as inverse limits of the reduced induced scheme structures on all finite steps. As straight rings are reduced, the claims about adjunction follow in the same way as [BS15, Lemma 2.2.4]. It remains to prove that $A \to A^{str}$ and $A \to A^{wls}$ are v-covers. As $A \to A^{wl}$ is pro-etale surjective, hence v-cover, it suffices to do this for $A \mapsto A^{str}$. By Remark 1.15, $\operatorname{Spec} A^{str} \to \operatorname{Spec} A$ is surjective and an inverse limit of finite disjoint unions of closed immersions. Note that these properties hold after any base change. Thus, by [Ryd10, Remark 2.5(1)], it suffices to show that for any surjection f of affine schemes, which is an inverse limit of finite disjoint unions of closed immersions, f is specializing and f^{cons} is submersive. Any surjective spectral map is submersive in the constructible topology [Ryd10, Remark 2.3]. Also, if $f_i: Y_i \to Y$ is an inverse system of specializing maps between affine schemes, then a compactness argument (for the constructible topology) shows that $\varprojlim_i f_i: \varprojlim_i Y_i \to Y$ also is. As closed immersions are specializing, this finishes the proof. ☐

Although $A \to A^{wls}$ is a v-cover, not every topological v-cover (in the sense of §1.3) in $(\operatorname{Spec} A)_{\mathrm{pu}}$ is a v-cover, as the following example shows.

Example 1.29. Let X be an affine line with one node x over a field. Then $\operatorname{Spec} \mathcal{O}_{X,x}^{sh}$ has two branches, and let Y be one of them (so $Y \subsetneq \operatorname{Spec} \mathcal{O}_{X,x}^{sh}$ is an irreducible component). Then $(X \smallsetminus \{x\}) \coprod Y \to X$ is a topological v-cover, but not a v-cover, as its base change to $\operatorname{Spec} \mathcal{O}_{X,x}^{sh}$ is not. Using [Sta14, 0395], it is not hard to show that if X is a qcqs geometrically unibranch scheme, then for all $f: Y \to X$ in X_{pu}, f is a topological v-cover $\Leftrightarrow$ f is a v-cover.

We call a subset Y of an affine scheme X *pro-(principal open)* if it is the intersection of the principal open subsets $D(f) \subseteq X$ containing it.

Proposition 1.30. *Let $X = \operatorname{Spec} A$ be an affine scheme. Let $Z \subseteq X$ be a subset, such that $\pi: X \to \pi_0(X)$ restricts to a homeomorphism $Z \xrightarrow{\sim} \pi_0(X)$. Let $\widetilde{Z} \subseteq X$ be the set of all generalizations of points in Z. Then $\widetilde{Z}$ is a pro-(principal open) of X. Moreover, Z is pro-constructible in X if and only if $\widetilde{Z}$ is w-local.*

[3]Recall that a closed subset $Z \subseteq \operatorname{Spec} A$ is constructible if and only if there exist a finitely generated ideal $I \subseteq$ with $Z = V(I)$.

The assumption on Z is equivalent to the following two conditions: (i) π induces a bijection between Z and $\pi_0(X)$, and (ii) Z with its subspace topology induced from X is quasi-compact.

Proof. The last assertion holds by [Sta14, 08YE]. For $x \in \pi_0(X)$ denote by $\mathfrak{p}_x$ the unique point of Z lying over x. The connected component $\pi^{-1}(x)$ is closed in X, equipped with its canonical scheme structure (cf. [Sta14, 04PX]); let A_x denote its global sections. Let $\bar{\mathfrak{p}}_x$ be the image of $\mathfrak{p}_x$ under $A \twoheadrightarrow A_x$. Consider the multiplicative subset $S = A \smallsetminus \bigcup_{x\in\pi_0(X)} \mathfrak{p}_x$ of A. We show that $\widetilde{Z} = \operatorname{Spec} S^{-1}A$, which is pro-(principal open). The inclusion $\widetilde{Z} \subseteq \operatorname{Spec} S^{-1}A$ is clear. To show the converse, note that S consists of all elements $a \in A$ such that for all $x \in \pi_0(X)$ the image $a_x \in A_x$ of a lies in $A_x \smallsetminus \bar{\mathfrak{p}}_x$. We claim that for all $x \in \pi_0(X)$, S maps surjectively onto $A_x \smallsetminus \bar{\mathfrak{p}}_x$ under $A \twoheadrightarrow A_x$. Assuming this for a moment, we can prove $\operatorname{Spec} S^{-1}A \subseteq \widetilde{Z}$: let $\mathfrak{q} \in X$, let $x = \pi(\mathfrak{q})$ and let $\bar{\mathfrak{q}}$ be the image of $\mathfrak{q}$ in A_x. Assume that $\mathfrak{q} \notin \widetilde{Z}$, i.e., $\bar{\mathfrak{q}} \not\subseteq \bar{\mathfrak{p}}_x$. Then there is some $\alpha \in \bar{\mathfrak{q}} \smallsetminus \bar{\mathfrak{p}}_x$. By the claim, there is a lift $a \in S$ of α along $A \twoheadrightarrow A_x$. But all such lifts lie in $\mathfrak{q}$ (as $\ker(A \twoheadrightarrow A_x) \subseteq \mathfrak{q}$). Thus $\mathfrak{q} \cap S \neq \varnothing$, i.e, $\mathfrak{q} \notin \operatorname{Spec} S^{-1}A$.

Next, we prove our claim about S. Let $a_x \in A_x \smallsetminus \bar{\mathfrak{p}}_x$. Let $a \in A$ be an arbitrary lift of a_x. The set $U_a = \{\mathfrak{q} \in X : a \neq 0 \text{ in } \kappa(\mathfrak{q})\}$ is an open neighbourhood of $\mathfrak{p}_x$ in X. Then $U_a \cap Z \subseteq Z$ is an open neighbourhood of $\mathfrak{p}_x$. The assumptions imply that $V_a = \pi(U_a \cap Z)$ is open in $\pi_0(X)$, and for each $y \in V_a$, we have $a_y \in A_y \smallsetminus \bar{\mathfrak{p}}_y$. Now choose any quasi-compact open (hence clopen) subset $x \in V \subseteq V_a$, and let $e_V \in A$ denote the idempotent element corresponding to V (so, $e_V + e_{\pi_0(X)\smallsetminus V} = 1$). Replacing the lift a by $e_{\pi_0(X)\smallsetminus V} + e_V a$ we obtain a new lift a of a_x which satisfies $a_y \in A_y \smallsetminus \bar{\mathfrak{p}}_y$ for all $y \in \pi_0(X)$. With other words, $a \in S$. This proves the claim.

Finally, suppose additionally that Z is pro-constructible in X. Then Z is also pro-constructible in $\widetilde{Z}$, as follows, e.g., from [Sta14, 09YF] by writing Z as an intersection of constructible subsets of X ([Sta14, 09YF] applies because $\widetilde{Z}$ and X are both affine and hence $\widetilde{Z} \subseteq X$ is retrocompact). As by construction Z is also stable under specializations in $\widetilde{Z}$, Z is closed in $\widetilde{Z}$ [Sta14, 0903]. As π induces a homeomorphism between Z and $\pi_0(X) = \pi_0(\widetilde{Z})$, it follows that $\widetilde{Z}$ is w-local. For the coverse direction, note that w-locality of $\widetilde{Z}$ implies that Z is closed in $\widetilde{Z}$, which is itself pro-constructible in X. Hence Z is pro-constructible in X. ☐

The closed set of closed points of a w-local scheme with its reduced structure is cut out by the Jacobson ideal [BS15, Lemma 2.2.3]. This has the following dual version:

Corollary 1.31. *Let $X = \operatorname{Spec} A$ be straight. Then $X^{\mathrm{gen}} = \operatorname{Spec} S^{-1}A$ is a pro-(principal open) subset of X, where $S = A \smallsetminus \bigcup_{\mathfrak{p} \text{ minimal}} \mathfrak{p}$. Moreover, $S^{-1}A$ is an absolutely flat ring.*

Proof. This follows from Proposition 1.30, and the characterization of absolutely flat rings as reduced ones with Hausdorff spectrum. ☐

The proof of Proposition 1.30 shows that if X is qcqs, all connected components of X are irreducible, and the set X^{gen} of their generic points is quasi-compact, then $X \to \pi_0(X)$ is open. This is not an equivalence, so we may ask the following question:

Question 1.32. Under which assumptions on a (qcqs) scheme X is the quotient map $X \to \pi_0(X)$ open?

We have the following analogue of [BS15, 2.3.4, 2.3.7 and 2.4.9] for the pro-unramified site.

Theorem 1.33. *(1) Any qcqs scheme X admits a pro-unramified v-cover $X' \to X$, such that X' is a w-contractible straight affine scheme. Moreover, if $Y \to X$ is a map of affine schemes, then one may choose X', Y' as above, such that there exists a commutative* *diagram*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

with $Y' \to X'$ w-local and straight.

(2) Let X be an affine scheme. Then X is w-contractible straight if and only if any any quasi-(pro-unramified) v-cover $f\colon Y \to X$ admits a splitting.

Proof. (1): We may assume X, Y are affine. By [BS15, Lemma 2.3.7] we may find faithfully flat pro-étale maps $X_1 \to X$ and $Y_1 \to Y$ with a w-local map $Y_1 \to X_1$ lifting $Y \to X$ and such that X_1, Y_1 are w-(strictly local). Passing to straightification as in Lemma 1.28, we obtain a w-local straight map $Y_2 := (Y_1)^{str} \to (X_1)^{str} =: X_2$ lifting f, where $X_2 \to X$ is a pro-unramified v-cover, and similarly for Y. Next, choose any continuous surjection $T \to \pi_0(X_2)$ with T extremally disconnected. Replacing X_2 by $(X_2)_T$ (cf. §) and Y_2 by $Y_2 \times_{X_2} (X_2)_T$, we have achieved, by [BS15, Lemma 2.4.9], that now X_2 is w-contractible straight (note that the map $Y_2 \to X_2$ remains w-local straight after this replacement). Similarly, we may replace Y_2 by $(Y_2)_{T'}$ for some $T' \twoheadrightarrow \pi_0(Y_2)$ with T' extremally disconnected, to make Y_2 w-contractible.

(2): First, we prove the 'only if' direction. It suffices to treat the case that $Y \to X$ is a pro-unramified v-cover. By part (1) we may also assume that Y is w-contractible straight. Using w-locality (resp. straightness) of Y and X, one checks that the union Y_1 (resp. Y_2) of all connected components of Y, whose closed (resp. generic) point hits X^c (resp. X^{gen}), is closed in Y. As f is a v-cover, it follows that under $Y_1 \cap Y_2 \to X$ all specialization relations $x_{gen} \rightsquigarrow x_c$ (where x_{gen} (resp. x_c) is the generic (resp. closed) point of a connected component of X) lift. In particular, $Y_1 \cap Y_2 \to X$ is surjective on sets of connected components. Replacing Y by the closed reduced subscheme $Y_1 \cap Y_2$, we get a w-local straight map $Y \to X$. As $\pi_0(X)$ is extremally disconnected, we may find a section s of $\pi_0(Y) \twoheadrightarrow \pi_0(X)$. Replacing Y by the closed reduced subscheme $Y_{s(\pi_0(X))}$, we get $f\colon Y \to X$ which is pro-unramified (by Lemma 1.20(1)), w-local straight, and $\pi_0(f)$ is a homeomorphism. We claim that then f is an isomorphism. It suffices to check this on each connected component, i.e., we may assume that $f\colon Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ is a local dominant pro-unramified map, with A striclty Henselian. Write $B = \varinjlim_i B_i$ with B_i (G-)unramified A-algebra. We may assume all B_i are reduced. By [Sta14, 04GL], $A \to B_i$ is surjective, and hence an isomorphism as $\operatorname{Spec} B_i \to \operatorname{Spec} A$ is dominant. Thus also $A \to B$ is an isomorphism.

Conversely, assume any quasi-pro-unramified v-cover of X splits. Then, in particular, any pro-étale cover splits, so X is w-contractible. Moreover, by assumption (and by Lemma 1.28), the straigthification $X^{str} \to X$ admits a section s. By duality (cf. §1.2), we may regard s as a section s^* to the map of spectral spaces $(X^*)^{wl} \to X^*$. As this map comes from some map of rings, the section s^* will be a homeomorphism onto a closed subset. Dualilizing back, s is thus induces a homeomorphism onto a pro-(qc open) subset of X^{str}. Thus, by Remark 1.7, X is straight. □

Corollary 1.34. *For any scheme X, the topos $X^{\sim}_{\mathrm{pu}}$ is locally weakly contractible.*

1.6. **Comparison with the unramified site.** Now we discuss the subsite of X_{pu} of objects which are locally of finite presentation.

Lemma 1.35. *Let $f\colon Y \to X$ be a map of schemes. Then the following are equivalent:*

(i) f is quasi-(pro-unramified) and locally of finite presentation.

(ii) There exists a G-unramified v-cover $g\colon Z \to Y$, such that the composition $Z \to X$ is G-unramified.

Proof. (ii) $\Rightarrow$ (i) is clear. (i) $\Rightarrow$ (ii): By assumption there is some pro-unramified v-cover $Z \to Y$, such that $Z \to X$ is pro-unramified. Write $Z = \varprojlim_i Z_i$ as a cofiltered limit with all Z_i G-unramified. As f is of locally of finite presentation, $Z \to X$ factors through $Z_i \to Y$ for some i [Sta14, 01ZC]. As $Z_i \to X$ is G-unramified and f locally of finite presentation, [Sta14, 02GG] implies that $Z_i \to Y$ is G-unramified. As $Z \to Y$ is a v-cover, also $Z_i \to Y$ is. □

Definition 1.36. Let X be any scheme. We denote by X_{unr} (resp. $X^{\mathrm{aff}}_{\mathrm{unr}}$) the full category of all objects in X_{pu} (resp. $X^{\mathrm{aff}}_{\mathrm{pu}}$) which are locally of finite presentation over X.

Clearly, X_{unr} is a site with respect to the the topology from X_{pu}. Also, if X is affine, $X^{\mathrm{aff}}_{\mathrm{unr}}$ is a site and it generates the same topos as X_{unr}. For any X we have a map of sites

$$\nu\colon X_{\mathrm{pu}} \to X_{\mathrm{unr}},$$

which induces a map of associated topoi.

Lemma 1.37. *Let X be a scheme.*

(i) Let $\mathscr{F}$ be a sheaf on X_{unr}. Then for any $U \in X^{\mathrm{aff}}_{\mathrm{pu}}$ with a presentation $U = \varprojlim_i U_i$, one has $\nu^*\mathscr{F}(U) = \varinjlim_i F(U_i)$.

(ii) The functor $\nu^*\colon X^{\sim}_{\mathrm{unr}} \to X^{\sim}_{\mathrm{pu}}$ is fully faithful. Its essential image consists of exactly those sheaves $\mathscr{F}$ which satisfy $\mathscr{F}(U) = \varinjlim_i \mathscr{F}(U_i)$ for any $U \in X^{\mathrm{aff}}_{\mathrm{pu}}$ with presentation $U = \varprojlim_i U_i$.

(iii) Let $\mathscr{F}$ be a sheaf on X_{pu}. If $\{Y_i \to X\}$ is a cover in X_{pu}, such that $\mathscr{F}|_{Y_i}$ is in the essential image of ν^* for each i, then $\mathscr{F}$ is also in the essential image of ν^*.

Proof. (i): We follow the proof of [BS15, Lemma 5.1.1], exploiting Lemma 1.25. One has to be careful at one point: Let A be a ring, $B \to C$ a G-unramified map of ind-unramified A-algebras, which is a v-cover. Let $B = \varinjlim_i B_i$ be a presentation of B with all B_i G-unramified over A. As $B \to C$ is of finite presentation, we may assume that $B \to C$ comes as base change of some map $B_0 \to C_0$ of finite presentation; let also $C_i = B_i \times_{B_0} C_0$. Then by [Sta14, 0C4W], $B_i \to C_i$ is G-unramified for $i \gg 0$. Now (crucially) by [Ryd10, Theorem 6.4], as $B \to C$ is a v-cover, $B_i \to C_i$ is too for $i \gg 0$. The rest of the proof of [BS15, Lemma 5.1.1] applies *verbatim.* (ii), (iii): same as [BS15, Lemma 5.1.2 and 5.1.4]. □

We have the analogue of [BS15, Corollary 5.1.6] (and [Sch18, 14.8]).

Lemma 1.38. *Let $K \in D^+(X_{\mathrm{unr}})$. Then the adjunction map $K \to R\nu_*\nu^*K$ is an equivalence. If $U \in X^{\mathrm{aff}}_{\mathrm{pu}}$ with presentation $U = \varprojlim_i U_i$ with $U_i \in X^{\mathrm{aff}}_{\mathrm{unr}}$, then $R\Gamma(U, \nu^*K) \cong \varinjlim_i R\Gamma(U_i, K)$.*

Proof. The proof of [BS15, Corollary 5.1.6] applies (instead of [BS15, Theorem 2.3.4] we use Lemma 1.23). □

The following consequence of Lemma 1.37 allows to reduce many questions about unramified sheaves to connected components.

Corollary 1.39. *Let X be a scheme. Then the collection of pullback functors $f_x^*\colon \mathrm{Shv}(X_{\mathrm{unr}}) \to \mathrm{Shv}(x_{\mathrm{unr}})$ for all $x \in \pi_0(X)$ is conservative.*

Proof. We may assume that X is affine. Let $\alpha\colon \mathscr{F} \to \mathscr{G}$ be a map of sheaves on X_{unr} such that the pulled back maps $\alpha_x\colon f_x^*\mathscr{F} \xrightarrow{\sim} f_x^*\mathscr{G}$ are isomorphisms for all $x \in \pi_0(X)$. We first show that α is injective. Suppose it is not the case. Then there are some $U \in X_{\mathrm{unr}}$ and some $s, t \in \mathscr{F}(U)$ such that $s \neq t$ and $\alpha(s) = \alpha(t)$. Note that for all $u \in \pi_0(U)$ the map $\alpha_u\colon f_u^*\mathscr{F} \xrightarrow{\sim} f_u^*\mathscr{G}$ is an isomorphism, where $f_u\colon u \to U$ is the inclusion (namely, every such u admits a map to $U \times_X x$ for some $x \in \pi_0(X)$). Thus we can replace X by U and assume that $s, t \in \mathscr{F}(X)$ are global sections from now on. Pick any $x \in \pi_0(X)$ and write it as the intersection $x = \bigcap_i X_i$ of its clopen neighbourhoods in X. By Lemma 1.37 we have $\mathscr{F}(x) = \varinjlim_i \mathscr{F}(X_i)$ (where we view $\mathscr{F}$ as a sheaf on X_{pu}) and by assumption on α we know that the images of s and t in $\mathscr{F}(x)$ must agree. It follows that there is some i such that s and t agree on X_i. Repeating this argument for all $x \in \pi_0(X)$ we deduce that s and t agree on an open cover of X and are thus equal. This proves injectivity of α. Surjectivity is proven similarly (using the established injectivity). □

1.7. **Comparison with pro-étale site.** For a scheme X we have the map of sites

$$\mu = \mu_X\colon X_{\mathrm{pu}} \to X_{\mathrm{proet}} \quad \text{and} \quad \mu = \mu_X\colon X_{\mathrm{unr}} \to X_{\mathrm{et}}.$$

Theorem 1.40. *Let X be a geometrically unibranch and straight scheme. Then, for both versions of the functor μ, we have $\mu_*\mu^* \cong \mathrm{id}$ and μ_* is exact on abelian sheaves.*

Corollary 1.41. *If X is straight and geometrically unibranch, $R\Gamma(X_{\mathrm{pu}}, \mu^*\mathscr{F}) = R\Gamma(X_{\mathrm{proet}}, \mathscr{F})$ for any sheaf $\mathscr{F}$ on X_{proet}.*

This corollary applies in particular to all integral normal schemes. To prove Theorem 1.40, we first show some lemmas.

Lemma 1.42. *Let $f\colon Y \to X$ be a weakly étale map of affine schemes with X straight. Then Y^{gen} is quasi-compact, hence a pro-(principal open) subscheme of Y (cf. Corollary 1.31). Moreover, $Y^{\mathrm{gen}} = X^{\mathrm{gen}} \times_X Y$.*

Proof. By Corollary 1.31, X^{gen} is a pro-(principal open) subscheme of X. In particular, it is affine, and hence the spectrum of an absoltely flat ring. Thus $X^{\mathrm{gen}} \times_X Y \subseteq Y$ is also a pro-(principal open) affine subscheme of Y, and it suffices to show that topologically it agrees with Y^{gen}. As f is flat, generalizations must lift along f, and it follows that $Y^{\mathrm{gen}} \subseteq X^{\mathrm{gen}} \times_X Y$ On the other hand, $X^{\mathrm{gen}} \times_X Y \to X^{\mathrm{gen}}$ is the base change of f, hence weakly étale. Thus, as X^{gen} is absolutely flat, also $X^{\mathrm{gen}} \times_X Y$ is. In particular, $X^{\mathrm{gen}} \times_X Y$ admits no non-trivial specialization relations, and as it contains Y^{gen} and is pro-open in Y, it cannot contain any other point of Y. □

Lemma 1.43. *Let X be a geometrically unibranch scheme. Let $Y \to X$ be weakly étale. Then Y is geometrically unibranch. If moreover, X is straight and Y is w-local, then Y is also straight.*

Proof. Let $y \in Y$ with image $x \in X$. For the first claim it suffices to show that $\mathcal{O}^{sh}_{Y,y}$ has a unique minimal prime ideal. There is a local map $\mathcal{O}^{sh}_{X,x} \to \mathcal{O}^{sh}_{Y,y}$ [Sta14, 04GU]. The composition $\operatorname{Spec} \mathcal{O}^{sh}_{Y,y} \to Y \to X$ is weakly étale (as both maps are). As also $\operatorname{Spec} \mathcal{O}^{sh}_{X,x} \to X$ is weakly étale, also $\mathcal{O}^{sh}_{X,x} \to \mathcal{O}^{sh}_{Y,y}$ is [BS15, Prop.2.3.3(4)]. By Olivier's theorem [BS15, Theorem 2.3.5], this map is an isomorphism. Thus $\mathcal{O}^{sh}_{Y,y}$ is a geometrically unibranch local ring, as $\mathcal{O}^{sh}_{X,x}$ is.

Now assume additionally that X is straight and Y w-local. By Lemma 1.42, Y^{gen} is quasi-compact. Moreover, Y is reduced by [Sta14, 094Y]. It remains to show that the local ring at any closed point y of Y is a domain. If $x \in X$ is the image of y, we have $\mathcal{O}^{sh}_{X,x} \cong \mathcal{O}^{sh}_{Y,y}$ (by Olivier's theorem). But as X was geometrically unibranch, this is a domain by [Sta14, 06DM], hence $\mathcal{O}_{Y,y}$ also is. □

Proof of Theorem 1.40. First we deal with $\mu\colon X_{\mathrm{pu}} \to X_{\mathrm{proet}}$. For exactness of μ_*, let $\mathscr{F} \to \mathscr{G}$ be a surjection in $\mathrm{Ab}(X_{\mathrm{pu}})$. By [BS15, Lemma 2.4.9] it suffices to show that $\mathscr{F}(U) \to \mathscr{G}(U)$ is surjective for all w-contractible $U \in X_{\mathrm{proet}}$. As straight w-contractible schemes are contractible objects for X_{pu} by Theorem 1.33(2), it simply suffices to show that U itself is straight. But this follows from Lemma 1.43. The isomorphism $\mu_*\mu^* \cong \mathrm{id}$ follows by a similar argument.

Now we deal with $\mu\colon X_{\mathrm{unr}} \to X_{\mathrm{ét}}$. To show exactness of μ_*, it suffices to show that for any geometric point $\bar{x} \in X$, $H^i((\operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{X,\bar{x}})_{\mathrm{unr}}, \mathscr{F})$ for any $i > 0$ and any abelian sheaf $\mathscr{F}$ on X_{unr}. But $\operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{X,\bar{x}}$ is w-contractible (by construction) and straight (by Lemma 1.43). Hence it is contractible in X_{pu} by Theorem 1.33(2), and all higher cohomology vanishes. Finally, $\mu_*\mu^* \cong \mathrm{id}$ follows from the same claim for $X_{\mathrm{pu}} \to X_{\mathrm{proet}}$, [BS15, Lemma 5.1.2] and Lemma 1.37. ☐

Let us single out one argument used in the proof of Theorem 1.40:

Lemma 1.44. *Let X be a straight affine scheme. Let $Y \in X^{\mathrm{aff}}_{\mathrm{pu}}$ be straight. Then the union of all connected components Y_2 of Y, whose image in X contains the generic point of the corresponding connected component of X, is closed. If, moreover, X is geometrically unibranch, the composition $Y_2 \to Y \to X$ is pro-étale.*

Proof. For the first claim, cf. the proof of Theorem 1.40. The second claim follows from the first and Lemma 1.45. ☐

Lemma 1.45. *Let X be an affine integral and geometrically unibranch scheme. Let $f\colon Y \to X$ be a pro-unramified map with Y connected. Suppose the generic point of X lies in the image of f. Then f is pro-étale. Moreover, one can write $Y = \varprojlim_i Y_i$ with Y_i irreducible étale X-schemes.*

Proof. Write $Y = \varprojlim_i Y_i$ with all Y_i G-unramified X-schemes. For each i, replacing Y_i by the connected component of Y_i containing the image of Y, we may assume that Y_i is connected. Let $\eta \in Y$ be such that $f(\eta)$ is the generic point of X. The irreducible component of Y_i which contains the image of η dominates X. Then [Sta14, 0GS9] shows that $Y_i \to X$ is in fact étale and Y_i is irreducible. ☐

For $Y \in X_{\mathrm{pu}}$, let $h^{\#}_Y$ be as in Remark 1.24. In some cases we have a simple formula for the restriction of a $h^{\#}_Y$ to X_{proet}:

Lemma 1.46. *Let X be geometrically unibranch straight affine scheme. For straight $Y \in X^{\mathrm{aff}}_{\mathrm{pu}}$, let Y_2 be as in Lemma 1.44. Then $\mu_*(h^{\#}_Y) = h^{\mathrm{proet}}_{Y_2}$.*

Proof. Let $T \in X^{\mathrm{aff}}_{\mathrm{proet}}$ be w-local (those form a basis of X_{proet}). By Lemma 1.43, T is itself straight and geometrically unibranch. By Lemma 1.44, any cover of T in $X^{\mathrm{aff}}_{\mathrm{pu}}$ admits a pro-étale refinement. Then faithfully flat descent implies that $h_Y(T) \to h^{\#}_Y(T)$ is bijective. As X_{proet} is subcanonical and μ_* is given by restriction, it remains to show that any X-map $T \to Y$ factors (necessarily uniquely) through a map $T \to Y_2$. But if T_0 is a connected component of T, whose image in Y (resp. X) is contained in the connected component Y_0 (resp. X_0), then –as $T \to X$ is proetale,– the image of $T_0 \to X_0$ contains the generic point of X_0, and hence the image of $Y_0 \to X_0$ does too. Thus $Y_0 \subseteq Y_2$, so $T \to Y$ factors through Y_2 at least topologically. But T is straight, hence reduced, i.e., it factors also scheme-theoretically. ☐

Lemma 1.47. *Let X be a geometrically unibranch straight affine scheme. Then μ_* preserves qcqs sheaves.*

Proof. A sheaf on X_{pu} is qcqs if and only if it can be realized as the coequalizer of some maps $\coprod_{j\in J} h^{\#}_{Z_j} \rightrightarrows \coprod_{I \in I} h^{\#}_{Y_i}$ with I, J finite. Changing the presentation if necessary, we may replace Y_i's and Z_j's by some v-covers. Hence we may assume all of them to be straight. Now the result immediately follows from Lemma 1.46 and Theorem 1.40. ☐

2. Pro-étale and pro-unramified sites vs. the v-site

In this section we compare the sites X_{proet} (and X_{pu}) to X_v. In full generality, our results for X_{pu} are not as good as for X_{proet}, and we will remedy this in §3 by studying X_{pu} for a very nice class of schemes (which form a basis for the v-topology).

First we need some preparations. In §2.1 and §2.2 we show some refinement results about v-covers, and in §2.3 we investigate the Henselization of a morphism.

If X is a qcqs scheme, and $T \to \pi_0(X)$ is a continuous map from a profinite set, we denote by X_T the scheme $T \times_{\pi_0(X)} X$, constructed in [BS15, Lemma 2.2.8]. It is a pro-(Zariski localization) of X, $\pi_0(X_T) = T$, and $|X_T|$ is the fiber product $T \times_{\pi_0(X)} X$ in topological spaces.

2.1. **Refining v-covers by w-local v-covers.** The following proposition tells that any v-cover $Y \to X$ with X, Y w-local can be refined by a w-local v-cover.

Proposition 2.1. *Let X, Y be w-local schemes and $f\colon Y \to X$ a v-cover. The union Y_1 of all connected components of Y, whose image in X meets X^c is a closed (reduced) subscheme of Y. Moreover, $Y_1 \to X$ is a w-local map and a v-cover.*

Proof. For $y \in \pi_0(Y)$, let y^c denote the unique closed point of y. Consider the set

$$S := \{y \in \pi_0(Y)\colon f(y^c) \in X^c\}$$

Note that $S \subseteq \pi_0(Y)$ is closed, as it is equal to the preimage of the closed subset $X^c \subseteq X$ under the composition $\pi_0(Y) \xrightarrow{\sim} Y^c \hookrightarrow Y \to X$. Thus $Y_1 \subseteq Y$, which is the pull-back of S under $Y \to \pi_0(Y)$, is also closed, and hence w-local [BS15, 2.1.3]. We equip Y_1 with the reduced subscheme structure. Moreover, the map $Y_1 \to X$ is w-local by construction and it remains to show that it is a v-cover. It suffices to do so after pullback to all connected components of X. Hence we may assume that X is connected (and hence local).

Lemma 2.2. *Let $X = \operatorname{Spec} A$ for a local ring A. Let V a valuation ring and let $g\colon \operatorname{Spec} V \to X$ be a morphism. Then there exists a valuation ring $W \subseteq V$ with the same fraction field, such that g extends to a map $\operatorname{Spec} W \to X$, whose image contains the closed point of X.*

Proof. Replacing X by X^{red} we may assume X is reduced. Replacing X by the irreducible component containing the image of $\operatorname{Spec} V \to X$, we may assume X is integral. Replacing X by the closure of the image of $\operatorname{Spec} V \to X$, we may assume that $\operatorname{Spec} V \to X$ dominant. Now $\operatorname{Spec} V \to X$ corresponds to an injection $A \hookrightarrow V$, and the result follows from Lemma 2.3. ☐

Lemma 2.3. *Let $A \subseteq V$ be a local subring of a valuation ring. Then there is a valuation subring $W \subseteq V$ with the same fraction field, which dominates A.*

Proof. For a local ring R, denote by $\mathfrak{m}_R$ its maximal ideal, and by κ_R its residue field. Let $\bar{A}$ be the image of $A \to V \to \kappa_V$. Clearly, $\bar{A}$ is non-zero and hence a local ring. Let $\bar{W}$ any valuation ring in κ_V, dominating $\bar{A}$, i.e., $\bar{A} \subseteq \bar{W}$ and $\mathfrak{m}_{\bar{A}} \subseteq \mathfrak{m}_{\bar{W}}$. Now, put $W = \{x \in V\colon x \bmod \mathfrak{m}_V \in \bar{W}\}$. Then $W \subseteq V$ is a valuation ring in $\operatorname{Frac}(V)$, and $\mathfrak{m}_W$ is the preimage of $\mathfrak{m}_{\bar{W}}$. Thus $A \subseteq W$ and $\mathfrak{m}_A \subseteq \mathfrak{m}_W$, i.e., W dominates A. ☐

Now we can finish the proof of Proposition 2.1. Let V be a valuation ring and $h\colon \operatorname{Spec}(V) \to X$ a morphism. Applying Lemma 2.2 we can extend h to some map $\operatorname{Spec} W \to X$, whose image contains the closed point x_0 of X. As the fraction fields of V, W coincide, V is necessarily a Zariski localization of W, i.e., $V = W[w_i^{-1}\colon i \in I]$ for some elements $w_i \in W$. By assumption on $Y \to X$, there is a faithfully flat extension of valuation rings $W \subseteq W'$, and a map $\operatorname{Spec} W' \to Y$ lifting $\operatorname{Spec} W \to X$. The (set-theoretic) image Z of $\operatorname{Spec} W' \to Y$ is contained in a connected component Y_0 of Y, and the image of Z in X contains x_0. Thus the image of Y_0 in X contains x_0, and hence, by continuity, the closed point of Y_0 maps to x_0. With other words, $Y_0 \subseteq Y_1$.

Hence $\operatorname{Spec} W' \to Y$ factors through Y_1. Now let $V' = W'[w_i^{-1}: i \in I](= V \otimes_W W')$, which is a Zariski localization of W, hence a valuation ring itself. Clearly, $\operatorname{Spec} V' \to \operatorname{Spec} V$ is surjective, and being the restriction of $\operatorname{Spec} W' \to \operatorname{Spec} W$, it is also flat. As $\operatorname{Spec} V' \hookrightarrow \operatorname{Spec} W' \to Y_1$ lifts $\operatorname{Spec} V \hookrightarrow \operatorname{Spec} W \to X$, we are done. □

Remark 2.4. The last claim of Proposition 2.1 becomes false, when "v-cover" is replaced by "arc-cover" everywhere. Indeed, if V is a valuation ring of rank ≥ 2 and $\mathfrak{p} \subseteq V$ is a prime ideal, which is neither 0, nor maximal, then $Y = \operatorname{Spec} V/\mathfrak{p} \sqcup \operatorname{Spec} V_{\mathfrak{p}} \to \operatorname{Spec} V = X$ is an arc-cover by [BM21, Cor. 2.9], and $Y_1 = \operatorname{Spec} V/\mathfrak{p} \to X$ is not surjective.

2.2. **Componentwise dominant refinement of v-covers.** We also can prolong valuations to the generic point. We use this to prove a statement which is in a sense dual to Proposition 2.1. Recall that a quasi-compact morphism is dominant if and only if its image contains the generic points of all irreducible components of the target [Sta14, 01RL].

Proposition 2.5. *Let X, Y be affine schemes, such that X_{red}, Y_{red} are straight. Let $f\colon Y \to X$ be a v-cover. The union of all connected components Y_2, whose image in X meets X^{gen} is a closed (reduced) subscheme of Y. Moreover, $Y_2 \to X$ is a v-cover.*

Proof. As X^{gen} is quasi-compact and all connected components are irreducible, $X^{\mathrm{gen}} \to X \to \pi_0(X)$ is a homeomorphism (and similarly for Y). By Lemma 1.3, $X^{\mathrm{gen}}, Y^{\mathrm{gen}}$ are pro-constructible. For $y \in \pi_0(Y)$, let η_y denote the generic point of the corresponding component. Let $S = \{y \in \pi_0(Y)\colon f(\eta_y) \in X^{\mathrm{gen}}\}$. Then $S \subseteq \pi_0(Y)$ is closed, as it is the image under the homeomorphism $Y^{\mathrm{gen}} \xrightarrow{\sim} \pi_0(Y)$ (from Lemma 1.5) of the subset $Y^{\mathrm{gen}} \cap f^{-1}(X^{\mathrm{gen}})$, which is pro-constructible, and hence quasi-compact. Now, Y_2 is simply the preimage of S under $Y \to \pi_0(Y)$, hence closed. It remains to show that $Y_2 \to X$ is a v-cover. Therefore we need a lemma on extension of valuations.

Lemma 2.6. *Let A be a local domain and $\mathfrak{p} \in \operatorname{Spec} A$. Let V be a valuation ring and $\operatorname{Spec} V \to \operatorname{Spec} A/\mathfrak{p}$ a dominant map. Then there exists a valuation ring W, a dominant map $\operatorname{Spec} W \to \operatorname{Spec} A$, a prime ideal $\mathfrak{q}$ of W and a faithfully flat map $V \to W/\mathfrak{q}$ of valuation rings, such that the diagram*

$$\begin{array}{ccccc}
\operatorname{Spec} W & \longrightarrow & & \longrightarrow & \operatorname{Spec} A \\
\uparrow & & & & \uparrow \\
\operatorname{Spec} W/\mathfrak{q} & \longrightarrow & \operatorname{Spec} V & \longrightarrow & \operatorname{Spec} A/\mathfrak{p}
\end{array}$$

commutes and the square is Cartesian up to reduction, i.e., $\operatorname{Spec} W/\mathfrak{q} = (\operatorname{Spec} W/\mathfrak{p}W)_{\mathrm{red}}$.

Proof. Applying [Sta14, 03C3] to $A_{\mathfrak{p}}$ and the field extension $\operatorname{Frac}(A/\mathfrak{p}) \subseteq \operatorname{Frac}(V)$ gives a local ring B with residue field $\operatorname{Frac}(V)$ together with a local and flat map $A_{\mathfrak{p}} \to B$, such that $\mathfrak{m}_B = \mathfrak{p}B$. Being flat, $\operatorname{Spec} B \to \operatorname{Spec} A_{\mathfrak{p}}$ is dominant. Choosing an irreducible component $\operatorname{Spec} B' \subseteq \operatorname{Spec} B$, whose image contains the generic point of $\operatorname{Spec} A_{\mathfrak{p}}$ and replacing B by B', we get a local domain B with a dominant map $\operatorname{Spec} B \to \operatorname{Spec} A_{\mathfrak{p}}$ such that $\mathfrak{m}_B = \mathfrak{p}B$ (however, we may loose flatness). Let now W_1 be some valuation ring with fraction field $\operatorname{Frac}(B)$ dominating B. The closed subscheme $\operatorname{Spec}(W_1/\mathfrak{m}_B W_1)$ of $\operatorname{Spec} W_1$ has a generic point, which corresponds to a prime ideal $\mathfrak{p}_1$ of W_1, and replacing W_1 by $(W_1)_{\mathfrak{p}_1}$, we may assume that $\sqrt{\mathfrak{m}_B W_1} = \mathfrak{m}_{W_1}$, i.e., the fiber of $\operatorname{Spec} W_1 \to \operatorname{Spec} B$ over the closed point has exactly one point. Now, the local map $B \hookrightarrow W_1$ induces an extension of residue fields $\operatorname{Frac}(V) = \kappa_B \subseteq \kappa_{W_1}$. Choose $V' \subseteq \kappa_{W_1}$ to be any valuation ring containg V such that $V \to V'$ is faithfully flat (this is possible by [Sta14, 00IA]). Let now W be the composition of W_1 and V', i.e., $W = \{x \in W_1\colon x \bmod \mathfrak{m}_{W_1} \in V'\}$ (cf. [Sta14, 088Z]). As $\mathfrak{m}_1$ maps to $\mathfrak{p} \in \operatorname{Spec} A$, the composition $A \to B \to W_1$ factors through a map $A \to W$.

By construction it is clear that the generic point of $\operatorname{Spec} W$ (which coincides with the generic point of $\operatorname{Spec} W_1$) goes to the generic point of $\operatorname{Spec} A$. Note that $\mathfrak{m}_{W_1} \subseteq W$ and $W/\mathfrak{m}_{W_1} = V'$. Let $\mathfrak{q} = \sqrt{\mathfrak{p}W} \in \operatorname{Spec} W$ (cf. Lemma 3.1(1)). It is enough to show that $\mathfrak{q} = \mathfrak{m}_{W_1}$. Clearly, $\mathfrak{q} \subseteq \sqrt{\mathfrak{m}_B W_1} \subseteq \mathfrak{m}_{W_1}$. Thus $\mathfrak{q}$ corresponds to a point in $\operatorname{Spec} W_1$. Thus $\mathfrak{q}$ maps to $\mathfrak{p} \in \operatorname{Spec} A$ (indeed, if $\mathfrak{p}' \in \operatorname{Spec} A$ is the image of $\mathfrak{q}$, we clearly have $\mathfrak{p}' \supseteq \mathfrak{p}$; but as $\mathfrak{q} \in \operatorname{Spec} W_1$, we have $\mathfrak{p}' \in \operatorname{Spec} A_{\mathfrak{p}}$, i.e., $\mathfrak{p}' \subseteq \mathfrak{p}$). Now, by construction, the preimage of $\mathfrak{p}$ in $\operatorname{Spec} B$ is the point $\mathfrak{m}_B$, and its preimage in $\operatorname{Spec} W_1$ is the point $\mathfrak{m}_1$, i.e., we have $\mathfrak{m}_1 = \mathfrak{q}$. ☐

Returning to the proof of Proposition 2.5, we may assume that X is connected. Let V be a valuation ring and $\operatorname{Spec} V \to X$ a map. We apply Lemma 2.6 to the local domain obtained by localizing X at the image of the closed point of $\operatorname{Spec} V$. This gives $W, V', \mathfrak{q}$ and the dominant map $\operatorname{Spec} W \to X$ as in the lemma. As $Y \to X$ is a v-cover, there is some faithfully flat map $W \to W'$ of valuation rings, and a map $\operatorname{Spec} W' \to Y$ lifting $\operatorname{Spec} W \to X$. The map $\operatorname{Spec} W' \to Y$ factors over a connected component of Y, which necessarily must lie in Y_2, as its image in X contains the generic point of X. ☐

Combining Propositions 2.1 and 2.5 we deduce the following.

Corollary 2.7. *Let $Y \to X$ be a v-cover of affine w-local schemes, whose reductions are straight. Then the union Z of all connected components of Y, whose image in X meets X^{gen} and X^c is a closed (reduced) subscheme of Y, and $Z \to X$ is a v-cover.*

2.3. **Henselization.** To prove fully faithfullness of pullback $\lambda_X^* \colon X^\sim_{\mathrm{proet}} \to X^\sim_v$ in §2.4, we need to study Henselizations of morphisms. *All schemes in this section are affine, except the converse is explicitly stated.* Let X be a scheme. For any X-scheme $Y \to X$ we have the Henselization

$$Y \to \lambda_{X^\circ}(Y) := \mathrm{Hens}_X(Y) \to X$$

of Y, which is defined as $\lim_{Y \to U \to X} U$, with U affine étale over X, cf. [BS15, Definition 2.2.10]. Then $Y \mapsto \lambda_{X^\circ}(Y)$ is a functor, and any X-map $Y \to Z$ with Z pro-étale over X factors through a unique X-map $\lambda_{X^\circ}(Y) \to Z$. As one should expect, Henselization only depends on a pro-étale neighboorhood:

Lemma 2.8. *Let $Y \to X' \to X$ be morphisms with $X' \to X$ pro-étale. Then $\lambda_{X'^\circ}(Y) \cong \lambda_{X^\circ}(Y)$ canonically.*

Proof. Exploiting the universal property of $\lambda_{X^\circ}(Y)$ (and [BS15, 2.3.3(4)]) one checks that $\lambda_{X^\circ}(Y)$ satisfies the universal property of $\lambda_{X'^\circ}(Y)$, proving the lemma. ☐

Next, we study how to compute the Henselization.

2.3.1. *Reduction to case $\pi_0(Y) = \pi_0(X)$.* Let $f \colon Y \to X$ be any morphism. It induces a map $\pi_0(Y) \to \pi_0(X)$ and hence factors through a map $\tilde{f} \colon Y \to X_{\pi_0(Y)}$. Then $\pi_0(\tilde{f})$ is the identity. As $X_{\pi_0(Y)} \to X$ is a pro-(Zariski localization), Lemma 2.8 shows that $\lambda_{X^\circ}(Y) = \lambda_{X_{\pi_0(Y)}^\circ}(Y)$. With other words, when computing the Henselization of f, we may without loss of generality assume that $\pi_0(f)$ is an isomorphism. We have the following consequence.

Lemma 2.9. *For any morphism $f \colon Y \to X$ we have $\pi_0(\lambda_{X^\circ}(Y)) = \pi_0(Y)$.*

Proof. By the above discussion we may assume that $\pi_0(f)$ is a homeomorphism. Write $T := \pi_0(\lambda_{X^\circ}(Y))$. Then $\pi_0(f)$ factors as $\pi_0(Y) \xrightarrow{g} T \xrightarrow{h} \pi_0(X)$, and it follows that h is surjective. As continuous bijections of profinite sets are homeomorphisms, it suffices to show that h is bijective. Suppose this is not the case. Consider two maps $\alpha, \beta \colon T \to T$, where $\alpha := \mathrm{id}_T$ and $\beta \colon T \xrightarrow{h} \pi_0(X) \xrightarrow{\pi_0(f)^{-1}} \pi_0(Y) \xrightarrow{g} T$. As h is not injective, we have $\alpha \neq \beta$. Now note that we have a canonical map $\mathrm{can}_{\lambda_{X^\circ}(Y)}\colon \lambda_{X^\circ}(Y) \to T \times_{\pi_0(X)} X$, corresponding to $\pi_0(\mathrm{can}_{\lambda_{X^\circ}(Y)}) = \mathrm{id}_T$, and that α, β induce maps $\alpha \times \mathrm{id}_X, \beta \times \mathrm{id}_X \colon T \times_{\pi_0(X)} X \to T \times_{\pi_0(X)} X$. Composing these maps with can, we obtain two X-morphisms $\tilde{\alpha}, \tilde{\beta}\colon \lambda_{X^\circ}(Y) \to T \times_{\pi_0(X)} X$. Note that $\tilde{\alpha} \neq \tilde{\beta}$, as $\pi_0(\tilde{\alpha}) = \alpha \neq \beta = \pi_0(\tilde{\beta})$. Let $\tau\colon Y \to \lambda_{X^\circ}(Y)$ denote the canonical map. One easily checks (using $\mathrm{can} \circ \tau = (g \times \mathrm{id}_X) \circ \mathrm{can}_Y$) that $\tilde{\alpha}\tau = \tilde{\beta}\tau$. With other words, $\tilde{\alpha}, \tilde{\beta}$ are two different maps, through which the X-map $\tilde{\alpha}\tau = \tilde{\beta}\tau\colon Y \to T \times_{\pi_0(X)} X$ factors. As $T \times_{\pi_0(X)} X$ is pro-étale over X, this contradicts the universal property of $\lambda_{X^\circ}(Y)$. Thus h must be bijective. □

2.3.2. *Reduction to w-local maps.* First we need a lemma.

Lemma 2.10. *Let $f\colon Y \to X$ be a morphism such that Y is w-local and $\pi_0(f)$ is a homeomorphism. Then $f(Y^c)$ is pro-constructible in X and profinite, and $Y^c \to f(Y^c)$ is a homeomorphism.*

Proof. Being the image of $Y^c \to Y \to X$, $f(Y^c) \subseteq X$ is pro-constructible. By [Sta14, 0902], $f(Y^c)$ is itself a spectral space. As Y is w-local and $\pi_0(Y) \cong \pi_0(X)$, it is clear that $Y^c \to f(Y^c)$ is a continuous bijection. Moreover, $f(Y^c)$ does not admit any non-trivial specialization relations, as all points lie in different connected components of X. It follows that $f(Y^c)$ is a profinite space [Sta14, 0905]. □

We can reduce the computation of Henselizations with w-local source to the case of w-local maps inducing a homeomorphism on connected components.

Proposition 2.11. *Let $f\colon Y \to X$ be a morphism, such that Y is w-local and $\pi_0(f)$ is a homeomorphism. Then the set $\widetilde{Y}^X$ of all generalizations of $f(Y^c)$ is a w-local pro-(principal open) affine subscheme of X, satisfying $\pi_0(\widetilde{Y}^X) \cong \pi_0(Y)$ and $(\widetilde{Y}^X)^c = f(Y^c)$. Moreover, f factors through a map $\tilde{f}\colon Y \to \widetilde{Y}^X$, which is w-local. We have $\lambda_{X^\circ}(Y) = \lambda_{\widetilde{Y}^X{}_\circ}(Y)$.*

Proof. Proposition 1.30 applied to $f(Y^c) \subsetneq X$ along with Lemma 2.10 show the first claim. The (topological) image of f is contained in $\widetilde{Y}^X$. Clearly, f factors through $Y \to U$ for any open neighborhood U of $\widetilde{Y}^X$. As $\widetilde{Y}^X$ is the limit over all such neighboorhoods, f factors through a unique map $\tilde{f}\colon Y \to \widetilde{Y}^X$. The w-locality of $\tilde{f}$ is clear, as by construction we have $\tilde{f}(Y^c) = (\widetilde{Y}^X)^c$. (Note also that we have a distinguished section $\pi_0(\widetilde{Y}^X) = \pi_0(X) \xrightarrow{\sim} \pi_0(Y) \xrightarrow{\sim} Y^c \xrightarrow{\sim} \tilde{f}(Y^c)$.) The last claim follows from Lemma 2.8 as $\widetilde{Y}^X \to X$ is pro-etale. □

2.3.3. *Henselization over a w-strictly local base.* We have the following lemmas.

Lemma 2.12. *Let X be the spectrum of a strictly Henselian ring, let Z be a connected affine scheme. Any weakly étale map $Z \to X$, whose image contains the closed point of X, is an isomorphism.*

Proof. First assume Z that is pro-étale over X. Write $Z = \lim_n Z_n \to X$ with $Z_n \to X$ étale. Replacing Z_n by the connected component containing the image of $Z \to Z_n$, we may assume that each Z_n is connected. As the image of $Z_n \to X$ contains the closed point of X, it admits a section $s_n\colon X \to Z_n$. The image of the section is open (as $Z_n \to X$, and hence s_n, is étale) and closed (as $Z_n \to X$ is separated). Hence, as Z_n is connected, s_n is surjective, i.e., provides an inverse to $Z_n \to X$. Thus $Z_n \to X$ is an isomorphism for each n, and hence $Z \to X$ is too.

In general, by [BS15, Theorem 2.3.4] we may find some faithfully flat pro-étale map $Z' \to Z$, such that $Z' \to Z \to X$ is pro-étale. By the above $Z' \to X$ is an isomorphism, i.e., $Z \to X$ admits a surjective section, which implies that $Z \to X$ is an isomorphism. □

Lemma 2.13. *Let X be w-strictly local and let $Y \to X$ be a map such that the image of each connected component of Y contains the closed point of the corresponding component of X. Then $\lambda_{X^\circ}(Y) = X_{\pi_0(Y)}$.*

Proof. By §2.3.1, we may assume that $\pi_0(Y) = \pi_0(X)$. By Lemma 2.9, $\pi_0(\lambda_{X^\circ}(Y)) = \pi_0(Y)$. The assumption on $Y \to X$ remains valid for $Y \to X_{\pi_0(Y)}$. Lemma 2.12 now shows that $\lambda_{X^\circ}(Y) \to X$ is an isomorphism. □

2.3.4. *Henselization of w-contractible schemes.* As a w-contractible scheme is w-local [BS15, 2.4.2], the w-locality claim in the following lemma makes sense.

Lemma 2.14. *Let $f\colon Y \to X$ be any map with Y w-contractible. Then $\lambda_{X^\circ}(Y)$ is w-contractible and the map $h\colon Y \to \lambda_{X^\circ}(Y)$ is w-local.*

Proof. The first claim is a formal consequence of the definition of w-contractibility and the fact that faithfully flat and pro-étale maps are stable under base change. We omit the details. Now we prove that h is w-local. Suppose that $h(Y^c) \nsubseteq \lambda_{X^\circ}(Y)^c$. Similar as in §2.3.2, let $Y' \subseteq \lambda_{X^\circ}(Y)$ be the pro-open subscheme of all generalizations of $h(Y^c)$. Then $\iota\colon Y' \overset{\neq}{\hookrightarrow} \lambda_{X^\circ}(Y)$ and h factors through a map $Y \to Y'$. Via $t\iota$, Y' is a pro-étale X-scheme, and the universal property of $\lambda_{X^\circ}(Y)$ gives a map $\gamma\colon \lambda_{X^\circ}(Y) \to Y'$. Now one verifies using the universal property of $\lambda_{X^\circ}(Y)$ that $\iota\gamma = \mathrm{id}_{\lambda_{X^\circ}(Y)}$, which is absurd. This gives a contradiction and we are done. □

It is not clear to us whether the implication "Y w-strictly local $\Rightarrow \lambda_{X^\circ}(Y)$ w-strictly local" holds, as we cannot show that $\lambda_{X^\circ}(Y)$ is w-local in this case.

Corollary 2.15. *Let $Y \to X$ be a morphism with Y w-contractible. Then $\lambda_{X^\circ}(Y^c) \cong \lambda_{X^\circ}(Y)$ canonically.*

Proof. By functoriality, we have a map $g\colon \lambda_{X^\circ}(Y^c) \to \lambda_{X^\circ}(Y)$ in X_{proet}, which by Lemma 2.9 induces an isomorphism on connected components. By Lemma 2.14, source and target are w-contractible. Moreover, as any closed point of $\lambda_X(Y^c)$ lifts to Y^c, Lemma 2.14 also implies that g is w-local. Thus g admits a section. Each connected component of the source and the target is the spectrum of a strictly Henselian ring, hence g is an isomorphism componentwise by Lemma 2.12. This implies that g is an isomorphism. □

Lemma 2.16. *Let X be a scheme. Let Z, Y be w-contractible X-schemes. Let $f\colon Z \to Y$ be an X-morphism, which is a w-local v-cover. Then the following hold:*

(i) $\lambda_{X^\circ}(Z) \to \lambda_{X^\circ}(Y)$ is faithfully flat pro-étale.
(ii) $\lambda_{X^\circ}(Z \times_Y Z) \to \lambda_{X^\circ}(Z) \times_{\lambda_{X^\circ}(Y)} \lambda_{X^\circ}(Z)$ is faithfully flat pro-étale.

Proof. (i): It suffices to show that $\lambda_{X^\circ}(Z) \to \lambda_{X^\circ}(Y)$ is surjective. Lemmas 2.14, 2.9 imply (as Z, Y are w-local) that $Z \to \lambda_{X^\circ}(Z)$ maps Z^c bijectively to $\lambda_{X^\circ}(Z)^c$, and similarly for Y. It follows that $\lambda_{X^\circ}(f)$ maps $\lambda_{X^\circ}(Z)^c$ into $\lambda_{X^\circ}(Y)^c$. Thus $\lambda_{X^\circ}(f)$ is w-local. Moreover, $\pi_0(\lambda_{X^\circ}(f))$ is surjective by Lemma 2.9, as $Z \to Y$ is a v-cover. Now we conclude by Lemma 2.12.

(ii): By Lemma 2.8, it is harmless to replace X by $\lambda_{X^\circ}(Y)$. In particular, by Lemma 2.14, we may assume that X is w-contractible and $Y \to X$ is w-local. Let $T \to Z \times_Y Z$ be any v-cover of $Z \times_Y Z$ with T w-local (which exists by [BS17, Lemma 6.2]). Then we have surjections

$$\pi_0(T) \twoheadrightarrow \pi_0(Z \times_Y Z) \twoheadrightarrow \pi_0(Z) \times_{\pi_0(Y)} \pi_0(Z).$$

(first map is surjective as $T \to Z \times_Y Z$ is a v-cover; for the surjectivity of the second map, note that if Z_1, Z_2 are two connected components of Z lying over the same component of Y, then $Z_1 \times_Y Z_2 \neq \varnothing$). The composition $Z \to Y \to X$ is w-local, as both maps are. Thus $\lambda_{X^\circ}(Z) = X_{\pi_0(Z)}$ by Lemma 2.13. Next, let T_1 be the closed subscheme of T defined in Proposition 2.1, relative to the composed map $T \to Z \times_Y Z \to Y$, which is a v-cover. Thus $T_1 \to Y$ is w-local, hence the composition $T_1 \to Y \to X$ is too, and we deduce $\lambda_{X^\circ}(T_1) = X_{\pi_0(T_1)}$ by Lemma 2.13. By functoriality of λ_{X° we have maps

$$X_{\pi_0(T_1)} = \lambda_{X^\circ}(T_1) \to \lambda_{X^\circ}(Z \times_Y Z) \to \lambda_{X^\circ}(Z) \times_X \lambda_{X^\circ}(Z) = X_{\pi_0(Z)\times_{\pi_0(X)}\pi_0(Z)}$$

Thus it suffices to show that the composition $\pi_0(T_1) \hookrightarrow \pi_0(T) \twoheadrightarrow \pi_0(Z)\times_{\pi_0(Y)}\pi_0(Z)$ is surjective. With other words, we have to show that for any connected components Z_1, Z_2 of Z lying over the same component Y_0 of Y, there is a connected component T' of T, mapping to Z_i under the i-th projection $p_i \colon Z \times_Y Z \to Z$, whose image in Y meets the closed point of Y_0. There is a distinguished connected component Z' of $Z_1 \times_{Y_0} Z_2$, which contains the closed irreducible (cf. the proof of Corollary 3.6) subscheme $z_1 \times_{y_0} z_2$ (z_i = closed point of z_i, y_0 closed point of Y_0). Now, any component T' of T which lies over $Z' \subseteq Z_1 \times_{Y_0} Z_2$, and whose image contains an arbitrary closed point of $z_1 \times_{y_0} z_2$, does the job. □

Remark 2.17. Lemma 2.16 suffices for our purposes. However, note that stronger result –the scheme-theoretic analog of [Sch18, Lemma 14.5]– does not hold. Indeed, $Y \to \lambda_{X^\circ}(Y)$ can fail to be surjective, even if X is the spectrum of a strictly Henselian ring, such that $|X|$ is not homeomorphic to the spectrum of a valuation ring. Indeed, under these assumptions, let $Y \to X$ be a w-local v-cover, such that all connected components are spectra of valuation rings (it exists by [BS17, Lemma 6.2]). Then $\#\pi_0(Y) > 1$, $\lambda_{X^\circ}(Y) = \pi_0(Y) \times_{\pi_0(X)} X$ (by Lemma 2.13). It follows that $\alpha \colon Y \to \lambda_{X^\circ}(Y)$ is not surjective (in fact, $\pi_0(\alpha)$ is a homeomorphism by Lemma 2.9, and restricted to each of the connected components α cannot be surjective).

2.4. **Pullback from X_{proet} to X_v.** From now on, we again consider arbitrary (not necessarily affine) schemes. The identity of a scheme X induces a map of topoi $\lambda = \lambda_X \colon X_v^\sim \to X_{\mathrm{proet}}^\sim$.

Lemma 2.18. *Suppse X is affine. Let $\mathscr{F} \in X_{\mathrm{proet}}^\sim$. For any affine w-contractible $Y \in X_v$, we have $\lambda^*\mathscr{F}(Y) = \mathscr{F}(\lambda_{X^\circ}(Y))$.*

Proof. Let λ^p denote the presheaf pullback along λ. For any $Y \in X_v$ we have $\lambda^p\mathscr{F}(Y) = \mathscr{F}(\lambda_{X^\circ}(Y))$ by the universal property of $\lambda_{X^\circ}(Y)$. We have the natural map $\alpha \colon \lambda^p\mathscr{F}(Y) \to \lambda^*\mathscr{F}(Y)$. Suppose Y is w-contractible. We have to show that $\alpha_{Y,\mathscr{F}}$ is an isomorphism.

Let $s \in \ker(\alpha)$. Then there is some v-cover $g \colon Z \to Y$, such that $g^*s = 0 \in \lambda^p\mathscr{F}(Z) = \mathscr{F}(\lambda_{X^\circ}(Z))$. Refining Z, we may assume that it is w-contractible. Let $Z_1 \subseteq Z$ be the closed subscheme from Proposition 2.1 (relative to $Z \to Y$). Then Z_1 is w-strictly local, and if $T \twoheadrightarrow \pi_0(Z_1)$ is any surjection with T extremally disconnected, $Z_2 := T \times_{\pi_0(Z_1)} Z_1$ will be w-contractible, and the composition $Z_2 \to Z_1 \to Y$ is a w-local v-cover (as both maps are). Thus, replacing Z by Z_2, we may assume that Z is w-contractible and that $Z \to Y$ is w-local v-cover. Now, by Lemma 2.16(i), $\lambda_{X^\circ}(Z) \to \lambda_{X^\circ}(Y)$ is faithfully flat pro-étale. Thus $s = 0$, as $\mathscr{F}$ is a sheaf.

For surjectivity of α, note that to give a section $t \in \lambda^*\mathscr{F}(Y)$ is equivalent to give some v-cover $Z \to Y$ (which, as above, we may assume to be w-local with Z being w-contractible) plus a section $t_Z \in \lambda^p\mathscr{F}(Z)$, such that $p_1^*t_Z = p_2^*t_Z$, where p_i is the ith projection $Z \times_Y Z \to Z$ and p_i^* are induced by functoriality of λ_{X° in the commutative diagram below:

$$\begin{array}{ccccc}
\lambda^p\mathscr{F}(Y) & \longrightarrow & \lambda^p\mathscr{F}(Z) & \underset{p_1^*}{\overset{p_2^*}{\rightrightarrows}} & \lambda^p\mathscr{F}(Z \times_Y Z) = \mathscr{F}(\lambda_{X^\circ}(Z \times_Y Z)) \\
\| & & \| & & \uparrow \beta \\
\mathscr{F}(\lambda_{X^\circ}(Y)) & \longrightarrow & \mathscr{F}(\lambda_{X^\circ}(Z)) & \underset{\pi_1^*}{\overset{\pi_2^*}{\rightrightarrows}} & \mathscr{F}(\lambda_{X^\circ}(Z) \times_{\lambda_{X^\circ}(Y)} \lambda_{X^\circ}(Z))
\end{array}$$

The right square is commutative (with the lower horizontal maps induced by the two projections $\pi_i \colon \lambda_{X^\circ}(Z) \times_{\lambda_{X^\circ}(Y)} \lambda_{X^\circ}(Z) \to \lambda_{X^\circ}(Z)$). By Lemma 2.16(ii), β is injective, and hence $\pi_1^*t_Z =$

$\pi_2^* t_Z$. By Lemma 2.16(i), $\lambda_{X^\circ}(Z) \to \lambda_{X^\circ}(Y)$ is a pro-étale covering, thus t_Z comes from a section in $\mathscr{F}(\lambda_{X^\circ}(Y)) = \lambda^p \mathscr{F}(Y)$. This finishes the proof. □

Proposition 2.19. *Let X be an arbitrary scheme. The following hold:*

(i) For any $\mathscr{F} \in X^\sim_{\mathrm{proet}}$, the adjunction map $\mathscr{F} \to \lambda_* \lambda^* \mathscr{F}$ is an isomorphism. The pullback functor $\lambda^* \colon X^\sim_{\mathrm{proet}} \to X^\sim_v$ is fully faithful.

(ii) Suppose X is qcqs. The essential image of λ^* consists of all v-sheaves $\mathscr{F}$ on X_v satisfying the following two conditions:

(a) For any w-contractible X-scheme Y, the restriction $\mathscr{F}(Y) \to \mathscr{F}(Y^c)$ is bijective.

(b) For any X-morphism $g \colon Z \to Y$ between two absolutely flat w-contractible X-schemes, such that $\pi_0(g)$ is an homeomorphism, the induced map $\mathscr{F}(Y) \to \mathscr{F}(Z)$ is bijective.

(iii) λ^* preserves all small limits.

(iii)' Let $f \colon Y \to X$ be a map of schemes. Then $f^* \colon Y^*_{\mathrm{proet}} \to X^*_{\mathrm{proet}}$ preserves all small limits.

(iv) Let $f \colon Y \to X$ be a v-cover of schemes. Then $f^*_{\mathrm{proet}} \colon X^\sim_{\mathrm{proet}} \to Y^\sim_{\mathrm{proet}}$ is faithful.

(v) Let $f \colon Y \to X$ be a v-cover. Let $\mathscr{F} \in X^\sim_v$. If $f^* \mathscr{F}$ comes by pullback from Y_{proet}, then $\mathscr{F}$ satisfies (b) and the injectivity part of (a) from part (ii).

Proposition 2.19(ii) is the pro-étale analogue of Gabber's criterion for a v-sheaf to be étale [HS21, Lemma 5.5]. Also, a result of Gabber [HS21, ...] shows that if a v-sheaf is étale after some v-cover, then it is étale itself. We do not know whether there is a similar statement for pro-étale sheaves, but Proposition 2.19(v) at least generalizes the easy part of Gabber's result.

Proof. (i): The second claim follows from the first, using adjunction. The first claim is pro-étale local on X, so we may assume X is affine. As w-contractible affine $Y \in X_{\mathrm{proet}}$ form a base for X_{proet}, it suffices to check that $\mathscr{F}(Y) \to \lambda_* \lambda^* \mathscr{F}(Y)$ is an isomorphism for all such Y. But this follows from Lemma 2.18.

(ii): $\mathscr{F} \in X^\sim_v$ comes from the pro-étale site if and only if the natural map $\lambda^* \lambda_* \mathscr{F} \to \mathscr{F}$ is an isomorphism. As w-contractible X-schemes Y, such that each connected component is the spectrum of an aic valuation ring form a basis for v-topology (see Lemma 3.12), this is equivalent to $\lambda^* \lambda_* \mathscr{F}(Y) \to \mathscr{F}(Y)$ being an isomorphism for all such Y. By Lemma 2.18, this is equivalent to $\mathscr{F}(\lambda_{X^\circ}(Y)) \to \mathscr{F}(Y)$ being bijective for all such Y.

Suppose now that (a) and (b) hold. Let Y be as above. Then we have

$$\mathscr{F}(Y) \cong \mathscr{F}(Y^c) \cong \mathscr{F}(\lambda_{X^\circ}(Y)^c) \cong \mathscr{F}(\lambda_{X^\circ}(Y)),$$

by (a) for Y, (b) for $Y^c \to \lambda_{X^\circ}(Y)^c$, and (a) for $\lambda_{X^\circ}(Y)$ respectively This gives sufficiency. Conversely, assume that $\mathscr{F}$ comes from X_{proet}. By Corollary 2.15 $\mathscr{F}(Y) \cong \mathscr{F}(\lambda_{X^\circ}(Y)) \cong \mathscr{F}(\lambda_{X^\circ}(Y^c)) \cong \mathscr{F}(Y^c)$, whence (a). For (b), note that $\lambda_{X^\circ}(Z)$ and $\lambda_{X^\circ}(Y)$ coincide.

(iii): The claim is pro-étale local on X, so we may assume that X is affine. Then the claim follows from Lemma 2.18 and the fact that evaluation of a sheaf commutes with limits. To deduce (iii)' consider the commutative diagam of sites

$$\begin{array}{ccc} Y_v & \xrightarrow{f_v} & X_v \\ \downarrow{\scriptstyle \lambda_Y} & & \downarrow{\scriptstyle \lambda_X} \\ Y_{\mathrm{proet}} & \xrightarrow{f_{\mathrm{proet}}} & X_{\mathrm{proet}} \end{array}$$

As f_v^* and $\lambda_{Y,*}$ are just restriction functors, we have

$$\lambda_X^* \mathscr{F}(Y) = \lambda_{Y,*} f_v^* \lambda_X^* \mathscr{F}(Y) = \lambda_{Y,*} \lambda_Y^* f^*_{\mathrm{proet}} \mathscr{F}(Y) = f^*_{\mathrm{proet}} \mathscr{F}(Y),$$

where the last equality holds by (i). As the functor $\mathscr{F} \mapsto \lambda_X^*\mathscr{F}(Y)$ preserves small limits by (iii), the same holds for $f_{\mathrm{proet}}^*\mathscr{F}(Y)$, and (iii)' follows.

(iv): As above, we have $f_{\mathrm{proet}}^*\mathscr{F} = \lambda_{Y,*}\lambda_Y^* f_{\mathrm{proet}}^*\mathscr{F} = \lambda_{Y,*}f_v^*\lambda_X^*\mathscr{F}$. Using this, for any $\mathscr{F}, \mathscr{G} \in X_{\mathrm{proet}}^{\sim}$, we have the maps on Hom-spaces:

$$(\mathscr{F},\mathscr{G})_{X_{\mathrm{proet}}} \stackrel{\lambda_X^*}{\to} (\lambda_X^*\mathscr{F},\lambda_X^*\mathscr{G})_{X_v} \stackrel{f_v^*}{\to} (f_v^*\lambda_X^*\mathscr{F}, f_v^*\lambda_X^*\mathscr{G})_{Y_v} \stackrel{\lambda_{Y,*}}{\to} (f_{\mathrm{proet}}^*\mathscr{F}, f_{\mathrm{proet}}^*\mathscr{G})_{Y_{\mathrm{proet}}}$$

The first map is bijective by (i). The second map is injective as f is a v-cover and $\lambda_X^*\mathscr{F}$, $\lambda_X^*\mathscr{G}$ are v-sheaves. The last map is bijective because the restriction of $\lambda_{Y,*}$ to the essential image of λ_Y^* is a quasi-inverse to λ_Y^* (again, by (i)). □

(v): By part (ii), we know that $f^*\mathscr{F} = \mathscr{F}|_{Y_v}$ satisfies the conditions (a) and (b), and we have to show that $\mathscr{F}$ satisfies the same conditions. First we check (b). Therefore, we may assume that X is itself absolutely flat and w-contractible, and that $g\colon Z \to X$ is a map such that Z is absolutely flat and w-contractible and $\pi_0(g)$ is a homeomorphism. We then have to show that $\mathscr{F}(X) \to \mathscr{F}(Z)$ is bijective. Refining Y by a w-contractible v-cover and then replacing Y by Y^c, we still have that $Y^c \to X$ is a v-cover (as X is absolutely flat, any surjection is a v-cover). Thus we may assume that Y is also absolutely flat and w-contractible. As $\pi_0(g)$ is a homeomorphism, we have $\pi_0(Z) \times_{\pi_0(X)} \pi_0(Y) \cong \pi_0(Y)$. Consider the pullback $g'\colon Z' := Z \times_X Y \to Y$ of g. We have the map

$$\pi_0(Z') \to \pi_0(Z) \times_{\pi_0(X)} \pi_0(Y) \cong \pi_0(Y)$$

This map is bijective, since the the spectrum of the tensor product of two fields over a third, which is separably closed, is non-empty and connected. Hence it is a homeomorphism, i.e., $\pi_0(Z') \cong \pi_0(Y)$. As $Z' \to Z$ and $Y \to X$ are v-covers, we have $\mathscr{F}(X) = \mathrm{Eq}(\mathscr{F}(Y) \rightrightarrows \mathscr{F}(Y \times_X Y))$ and $\mathscr{F}(Z) = \mathrm{Eq}(\mathscr{F}(Z') \rightrightarrows \mathscr{F}(Z' \times_Z Z'))$. Now we have

$$\mathscr{F}(Z') = \mathscr{F}(\lambda_Y(Z')) = \mathscr{F}(Y),$$

by (b), respectively by Lemma 2.13, as $\pi_0(\lambda_Y(Z')) \cong \pi_0(Z') \cong \pi_0(Y)$. In particular, we deduce $\mathscr{F}(X) \to \mathscr{F}(Z)$ is injective. To show the surjectivity of this map it suffices to check that the natural map $\mathscr{F}(Z' \times_Z Z') \to \mathscr{F}(Y \times_X Y)$ is injective. Via the first projection, $Y \times_X Y$ is a Y-scheme, and thus by Lemma 2.13 (and condition (b) for $\mathscr{F}|_{Y_v}$),

$$\mathscr{F}(Y \times_X Y) = \mathscr{F}(\lambda_Y(Y \times_X Y)) = \mathscr{F}(\pi_0(Y \times_X Y) \times_{\pi_0(Y)} Y),$$

where the map $\pi_0(Y \times_X Y) \to \pi_0(Y)$ is induced by the first projection. Similarly, $\mathscr{F}(Z' \times_Z Z') = \mathscr{F}(\pi_0(Z' \times_Z Z') \times_{\pi_0(Y)} Y)$. Thus it suffices to check that the natural map $\pi_0(Z' \times_Z Z') \to \pi_0(Y \times_X Y)$ is surjective. Let $y_1, y_2 \in Y$ be two points with the same image $x \in X$, corresponding to $\bar{y}_1, \bar{y}_2 \in \pi_0(Y)$. Let $z \in Z$ be the unique point lying over x. For $i = 1, 2$, pick any point z_i' of Z' lying over z and y_i. Then $z_1' \times_z z_2'$ is a non-empty subscheme of $Z' \times_Z Z'$ lying over $y_1 \times_x y_2 \subseteq Y \times_X Y$, and hence mapping to the connected component $(\bar{y}_1, \bar{y}_2) \in \pi_0(Y) \times_{\pi_0(X)} \pi_0(Y) = \pi_0(Y \times_X Y)$ (this last equality follows in the same way as $\pi_0(Z') \cong \pi_0(Y)$ above). This proves (b) for $\mathscr{F}$. Now we prove injectivity in (a). We may assume that X is w-contractible, and have to show that $\mathscr{F}(X) \to \mathscr{F}(X^c)$ is injective. Using Proposition 2.1 to refine the v-cover $Y \to X$, we may assume that Y is w-contractible and $Y \to X$ is a w-local v-cover. As X^c is absolutely flat, $Y^c \to X^c$ is a v-cover. Thus, $\mathscr{F}(X) = \mathrm{Eq}(\mathscr{F}(Y) \to \mathscr{F}(Y \times_X Y))$ and $\mathscr{F}(X^c) = \mathrm{Eq}(\mathscr{F}(Y^c) \to \mathscr{F}(Y^c \times_{X^c} Y^c))$. As, by assumption, $\mathscr{F}(Y) = \mathscr{F}(Y^c)$, we deduce that $\mathscr{F}(X) \to \mathscr{F}(X^c)$ is injective.

Remark 2.20. Let X be a scheme. As in §1.7 we have the map of sites $\mu\colon X_{\mathrm{pu}} \to X_{\mathrm{proet}}$. The same arguments as in Lemma 2.18 and Proposition 2.19(i) show that for any sheaf $\mathscr{F}$ on X_{proet} and any w-contractible $Y \in X_{\mathrm{pu}}$, one has $\mathscr{F}(Y) = \mathscr{F}(\lambda_{X,\circ}(Y))$, and that $\nu^*\colon X_{\mathrm{proet}}^{\sim} \to X_{\mathrm{pu}}^{\sim}$ is fully faithful.

Without additional assumptions on X, the situation with $\lambda^*\colon X_{\mathrm{pu}}^{\sim} \to X_v^{\sim}$ is worse than with X_{proet}. First, there is an obvious analogue of Henselization for the pro-unramified site. For any X-scheme $Y \to X$ we put

$$Y \to \lambda_{\mathrm{pu},X^\circ}(Y) := \lim_{Y \to U \to X} U \to X \tag{2.1}$$

where the limit is taken over all maps $Y \to U$ with U affine and unramified over X. Moreover, Lemmas 2.8-2.14 and Lemma 2.16(i) admit appropriate generalizations. For later use, we record one of them.

Lemma 2.21. *Let $X = \operatorname{Spec} A$ with A local strictly Henselian domain. Let Y be a connected affine scheme. Any pro-unramified map $f\colon Y \to X$, whose image contains the closed and the generic point of X, is an isomorphism.*

Proof. Write $Y = \operatorname{Spec} B = \varprojlim_i Y_i$, with $Y_i = \operatorname{Spec} B_i$ connected and $Y_i \to X$ unramified. As the closed point of X is in the image of $Y_i \to X$, and Y_i is connected [Sta14, 04GL] shows that $A \to B_i$ is a surjection. It follows that $A \to B$ is surjective, i.e., $Y \to X$ is a closed immersion. As its image contains the generic point of X, it is a nil-immersion. As A is a domain, it is an isomorphism. □

On the other hand, Lemma 2.16(ii) fails in the context of pro-unramified maps:

Example 2.22. Let $X = Y = \operatorname{Spec} A$, where A is the strict Henselization of the local ring of $\mathbb{A}^2$ (over a field) at the origin, and let $Z = \operatorname{Spec} \prod_{i \in I} V_i$, where the product is indexed over the discrete set I of equivalence classes of all rank two valuations on A, and V_i is the corresponding valuation ring. Then $\lambda_{\mathrm{pu},X^\circ}(Z) \cong X_{\beta I}$. Pick $i \neq j \in I$. We have a corresponding connected component $Z_0 \cong X$ of $\lambda_{\mathrm{pu},X^\circ}(Z) \times_X \lambda_{\mathrm{pu},X^\circ}(Z)$. The pullback of Z_0 along $\lambda_{\mathrm{pu},X^\circ}(Z \times_Y Z) \to \lambda_{\mathrm{pu},X^\circ}(Z) \times_X \lambda_{\mathrm{pu},X^\circ}(Z)$ is isomorphic to $\lambda_{\mathrm{pu},X^\circ}(\operatorname{Spec}(V_i \otimes_A V_j))$. But $\operatorname{Spec}(V_i \otimes_A V_j) \to \operatorname{Spec} A$ factors through $\operatorname{Spec}(\kappa_A) \coprod \operatorname{Spec}(\operatorname{Frac}(A)) \to \operatorname{Spec} A$ (where κ_A is the residue field of A), so cannot be a v-cover.

3. Combs

In this section we consider a class of affine schemes, which we call combs, which serve as a basis for the schematic v-topology. If X is a comb, X_{pu} is rather well-behaved and we are able to show analoga of the (most) results for the proetale site from §2.

3.1. **Valuation rings.** First we recollect some (well-known or easy) facts about spectra of valuation rings.

Lemma 3.1. *Let V be a valuation ring and $X = \operatorname{Spec} V$. Then the following hold:*

1. *Any radical ideal of V is either prime or the unit ideal.*
2. *For any non-empty subset $S \subseteq X$, there is a unique point in X, which specializes (resp. generalizes) to all $x \in S$ ("specializing/generalizing limit of S").*
3. *(cf. [BM21, Remark 2.2]) A proper subset $S \subset X$ is open (resp. closed) constructible if and only if S is stable under generalization (resp. specialization) and there exists an immediate specialization $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ in X –i.e., $(V/\mathfrak{p})_{\mathfrak{q}}$ has rank one– with $\mathfrak{p} \in S$ and $\mathfrak{q} \notin S$ (resp. $\mathfrak{p} \notin S$ and $\mathfrak{q} \in S$). Moreover, constructible opens coincide with principal opens.*
4. *(cf. [BM21, Remark 2.2]) For any two distinct points $\mathfrak{p} \rightsquigarrow \mathfrak{q}$, there exists points $\mathfrak{p} \rightsquigarrow \mathfrak{p}_1 \rightsquigarrow \mathfrak{q}_1 \rightsquigarrow \mathfrak{p}$, such that $\mathfrak{p}_1 \rightsquigarrow \mathfrak{q}_1$ is an immediate specialization.*
5. *Let A be a V-algebra, which is a local domain. Assume that $f\colon Y = \operatorname{Spec} A \to X$ maps the closed (resp. generic) point to the closed (resp. generic) point. Then A is faithfully flat over V.*

(6) *Let W be a valuation ring and let $f\colon \operatorname{Spec} W \to X$ be a map. Then f is surjective $\Leftrightarrow$ f faithfully flat $\Leftrightarrow$ f is a v-cover $\Leftrightarrow$ the image of f contains the closed and the generic points of $\operatorname{Spec} V$.*
(7) *Let $\varnothing \neq S \subseteq X$ be pro-constructible. Then S contains both of its limit points (see (2)).*

Proof. (1) is well-known, and we omit a proof. (2) easily follows (from (1) and) the fact that all ideals of $\mathfrak{p}$ are totally ordered by inclusion. (3): first claim is easy, and the second follows from the following obersevation (which uses (1)): For $f \in V$ non-zero and non-unit, the ideal $\mathfrak{p} = \bigcup_n f^n V$ is the maximal prime ideal contained in fV, $\mathfrak{q} = \sqrt{fV}$ is the minimal prime ideal containing f. Moreover, $\mathfrak{p} \neq \mathfrak{q}$, so that $(V/\mathfrak{p})_{\mathfrak{q}}$ has rank 1, and f is a pseudo-uniformizer in $(V/\mathfrak{p})_{\mathfrak{q}}$. (4): Apply the last claim of (3) to $D(f)$ for any $f \in \mathfrak{q} \smallsetminus \mathfrak{p}$. (5): By assumption $V \to A$ is injective. As A is a domain, it follows that A is torsion-free V-module, and hence flat over V [Sta14, 0539]. By the going-down theorem [Sta14, 00HS], $f(Y)$ is thus stable under generalization. As $f(Y)$ contains the closed point of X, we see that $f(Y) = X$, and we are done. (6): this is well-known, cf. [Ryd10, Prop. 2.7]. (7): being pro-constructible, S is itself spectral [Sta14, 0902]. Now, S is irreducible, and if the generalizing limit of S is not in S, then S does not contain a generic point. If the specializing limit of S is not in S, then S is not quasi-compact. □

Lemma 3.2. *Let V denote a valuation ring and $X = \operatorname{Spec} V$.*

(1) *In X, the quasi-compact open subsets are precisely the principal opens.*
(2) *Following subsets of X coincide:*
    (a) *quasi-compact pro-open subsets*
    (b) *pro-(quasi-compact open) subsets*
    (c) *pro-(principal open) subsets*
    (d) *subsets of the form $\operatorname{Spec} V_{\mathfrak{p}}$ for $\mathfrak{p} \in X$.*
(3) *Any subset of X, which is the intersection of a closed with a quasi-compact pro-open subset is of the form $\operatorname{Spec}(V/\mathfrak{p})_{\mathfrak{q}}$ for some points $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ of X.*

Proof. (1) is clear as the principal opens $D(f)$ are totally ordered by inclusion. (2): that (b) = (c) follows from (1) and (d) = (c) is clear. A pro-(principal open) is pro-constructible, hence quasi-compact, whence (c) $\subseteq$ (a). To show that (a) $\subseteq$ (d) it suffices to show that a quasi-compact pro-open is the set of generalizations of a given point. Being pro-open it is closed under generalization, and if it does not have a point to which all other specialize, one easily shows that it is not quasi-compact. (3) follows from (2). □

Recall that a ring is *absolutely integrally closed* (or *aic*) if every monic polynomial over it admits a root [BM21, Def. 3.22], and that for a valuation ring this is equivalent to the fraction field being algebraically closed. Note that if V is an aic valuation ring, then for any $\mathfrak{p} \rightsquigarrow \mathfrak{q} \in \operatorname{Spec} V$, $(V/\mathfrak{p})_{\mathfrak{q}}$ is also an aic valuation ring.

Lemma 3.3. *Let $X = \operatorname{Spec} V$ be an aic valuation ring and let $f\colon Y \to X$ be a map with Y connected affine.*

(1) *If f is weakly étale, then f induces an isomorphism $Y \xrightarrow{\sim} \operatorname{Spec} V_{\mathfrak{p}}$, for some prime ideal $\mathfrak{p}$ of V.*
(2) *Suppose that f is pro-unramified, or that f is weakly unramified and Y is path-connected.[4] Then f induces an isomorphism $Y \xrightarrow{\sim} \operatorname{Spec} V_{\mathfrak{p}}/\mathfrak{q}V_{\mathfrak{p}}$ for some prime ideals $\mathfrak{q} \rightsquigarrow \mathfrak{p}$ of V.*

In particular, in both cases Y is itself an aic valuation ring.

[4]There exist connected affine schemes, which are not path-connected, cf. [Ele].

Proof. (1): By 3.1(7), $f(Y)$ contains a unique closed point $\mathfrak{p}$. Thus f factors through the pro-open subscheme $\operatorname{Spec} V_{\mathfrak{p}}$ of X. Note that $V_{\mathfrak{p}}$ is an aic valuation ring and hence strictly Henselian. As Y is connected affine, $f'\colon Y \to \operatorname{Spec} V_{\mathfrak{p}}$ is weakly étale (by [BS15, Proposition 2.3.3(4)]) and its image contains the closed point of $\operatorname{Spec} V_{\mathfrak{p}}$, the map f' is an isomorphism by Lemma 2.12.

(2): By Lemma 3.1(7), $f(Y)$ contains a unique closed point $\mathfrak{p}$ and a unique generic point $\mathfrak{q}$. Replacing V by the aic valuation ring $V_{\mathfrak{p}}/\mathfrak{q}V_{\mathfrak{p}}$, we may assume that $f(Y)$ contains the closed and the generic point of X. If f is pro-unramified, we conclude by Lemma 2.21. If Y is irreducible, then f is flat (by Lemma 3.1(5)), and hence weakly étale. Then we conclude by (1). Finally, assume that f is weakly unramified and Y path-connected. If suffices to show that Y is irreducible. If not, then by path-connectedness, there are two irreducible components Y_1, Y_2 with $Y_1 \cap Y_2 \neq \varnothing$. We may replace Y by $Y_1 \cup Y_2$ (closed reduced subscheme), as $Y_1 \cup Y_2 \hookrightarrow Y \to X$ is weakly unramified. Let $\mathfrak{q}_i \rightsquigarrow \mathfrak{p}_i$ $(i = 1, 2)$ be the generic resp. closed points of $f(Y_i)$. By the above, $f\colon Y_i \xrightarrow{\sim} \operatorname{Spec} V_{\mathfrak{p}_i}/\mathfrak{q}_i V_{\mathfrak{p}_i}$. As $Y_1 \cap Y_2$ is closed in Y_1, Y_2, $\mathfrak{p}_1 = \mathfrak{p}_2$. Wlog, assume $\mathfrak{q}_2 \rightsquigarrow \mathfrak{q}_1$. Let y_i be the point of Y_i be the point of Y_i lying over $\mathfrak{q}_1$. As $Y_1 \cap Y_2$ is a proper closed subset of Y_i, $y_i \notin Y_1 \cap Y_2$. Thus, the fiber of $Y \times_X Y \to X$ over $\mathfrak{q}_1$ consists of four points, mapping to (y_i, y_j), $i, j \in \{1, 2\}$ under the both projections to Y. Only two of these four points lie in the image of Δ_f. But all four specialize to the unique point of $Y \times Y$ lying over $\mathfrak{p}_1$. Thus the image of Δ_f is not stable under generalization, i.e., f is not weakly unramified. □

The following approximation property generalizes [BM21, Lemma 2.20].

Lemma 3.4. *Let $f\colon V \to W$ be a faithfully flat map of valuation rings. Then f can be written as a filtered colimit of maps $f_i\colon V_i \to W_i$, where*

(1) All $V_i \subseteq V$, $W_i \subseteq W$ are valuation subrings of finite rank and $V = \varinjlim_i V_i$, $W = \varinjlim_i W_i$.
(2) All f_i and all transition maps $V_i \to V_j$, $W_i \to W_j$ are faithfully flat.

If in addition V and W are aic, V_i and W_i may be chosen aic too.

Proof. Let K, L be the fraction fields of V, W. For each subfield $L_i \subseteq L$ of finite transcendence degree over the prime field, $K_i := L_i \cap K$ has finite transcendence degree over the prime field by [Sta14, 030H]. Let $V_i = V \cap K_i$ and $W_i = W \cap L_i$. As in the proof of [BM21, Lemma 2.20], V_i, W_i are valuation subrings of V, W of finite rank and all transition maps $V_i \to V_j$, $W_i \to W_j$ are faithfully flat. As $\operatorname{Spec} V = \varprojlim_i |\operatorname{Spec} V_i|$, $\operatorname{Spec} V \to \operatorname{Spec} V_i$ is surjective, as all transition maps are (and similarly for W). Clearly, we have $V_i \subseteq W_i$. Moreover, as all maps $\operatorname{Spec} V \to \operatorname{Spec} V_i$, $\operatorname{Spec} W \to \operatorname{Spec} W_i$ and $\operatorname{Spec} W \to \operatorname{Spec} V$ are surjective, the same holds for $\operatorname{Spec} f_i$, with other words, f_i is faithfully flat. In the case of aic valuation rings V, W, a similar argument –with L_i varying through all algebraically closed subfields of L of finite transcendence degree over the prime field– applies, once we note that the intersection of two algebraically closed subfields of a field is again algebraically closed. □

The following lemma presents a property which is particular for aic valuation rings only, and which we crucially will use below.

Lemma 3.5. *Let V be a domain with separably closed field of fractions. Let A_1, A_2 be two domains, which are flat V-algebras. Then* $\operatorname{Spec}(A_1 \otimes_V A_2)$ *is irreducible.*

In particular, this applies when V is an aic valuation ring and A_1, A_2 are domains and torsion-free V-algebras.

Proof. Over a valuation ring, any torsion-free module is flat, hence the last claim. For the first, put $\eta = \operatorname{Spec} \operatorname{Frac}(V)$, $\eta_i = \operatorname{Spec} \operatorname{Frac}(A_i)$. As $\operatorname{Frac}(V)$ is separably closed, $\eta_1 \times_\eta \eta_2$ is irreducible by [Sta14, 037Q and 038F]. Thus it suffices to show that any point of $\operatorname{Spec}(A_1 \otimes_V A_2)$ admits a generalization in $\eta_1 \times_\eta \eta_2$. As $A \otimes_V B$ is flat over V, going-down shows that any point of $\operatorname{Spec}(A_1 \times_V A_2)$ admits a generalization which lies over $\eta \in \operatorname{Spec} V$. Pulling back along $\eta \hookrightarrow \operatorname{Spec} V$, we may assume that V is a field. Next, applying going-down to the flat extension $A_1 \to A_1 \otimes_V A_2$, we see that any point of $\operatorname{Spec}(A_1 \otimes_V A_2)$ generalizes to a point in $\eta_1 \times_\eta \operatorname{Spec} A_2$. Similarly, applying going-down to the flat extension $A_2 \to \operatorname{Frac}(A) \otimes_V A_2$, any point of $\eta_1 \times_\eta \operatorname{Spec} A_2$ generalizes to a point of $\eta_1 \times_\eta \eta_2$, and we are done. □

Corollary 3.6. *Let $W_1 \leftarrow W_3 \to W_2$ be maps of aic valuation rings, such that $W_1 \otimes_{W_3} W_2 \neq 0$. Then $\operatorname{Spec}(W_1 \otimes_{W_3} W_2)$ is irreducible.*

Proof. For $i = 1, 2$ let $\mathfrak{p}_i = \ker(W_3 \to W_i)$. As W_i is a domain, $\mathfrak{p}_i$ is a prime ideal. As ideals of W_3 are totally ordered by inclusion, we may by symmetry assume $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$. We have

$$\operatorname{Spec}(W_1 \otimes_{W_3} W_2) = \operatorname{Spec}(W_1 \otimes_{W_3/\mathfrak{p}_1} W_2/\mathfrak{p}_1 W_2) \leftarrow \operatorname{Spec}(W_1 \otimes_{W_3/\mathfrak{p}_1} W_2/\sqrt{\mathfrak{p}_1 W_2}),$$

where the right map is a homeomorphism (as $S_{\mathrm{red}} \to S$ is a universal homeomorphism for any scheme S). Note that $W_3/\mathfrak{p}_1 \to W_2/\sqrt{\mathfrak{p}_1 W_2}$ is injective (indeed, $\alpha\colon W_3/\mathfrak{p}_1 \to W_2/\mathfrak{p}_1 W_2$ is injective, and so if $x \in \ker(W_3/\mathfrak{p}_1 \to W_2/\sqrt{\mathfrak{p}_1 W_2})$, then $\alpha(x)^n = 0$ for some $n > 0$ and hence also $x^n = 0$; but $W_3/\mathfrak{p}_1$ is a domain, so $x = 0$.) By Lemma 3.1(1), $W_2/\sqrt{\mathfrak{p}_1 W_2}$ is an aic valuation ring. Thus, replacing W_3 by $W_3/\mathfrak{p}_1$ and W_2 by $W_2/\sqrt{\mathfrak{p}_1 W_2}$, we may assume that the maps $W_1 \leftarrow W_3 \to W_2$ are injective. Now the result follows from Lemma 3.5. □

Example 3.7. Corollary 3.6 fails for (aic) striclty Henselian rings W_3, even if W_1, W_2 are aic valuation rings. In fact, let $W_3 = \mathcal{O}^{\mathrm{sh}}_{\mathbb{A}^2_k,0}$ be the strict Henselization of the local ring of the affine plane over a field k at the origin, let C_1, C_2 be two different irreducible curves in $\mathbb{A}^2_k$ passing through 0 with generic points η_1, η_2, and let $W_3 \to W_1$, $W_3 \to W_2$ be two local and dominant maps into rank 2 valuation rings, such that for $i = 1, 2$ the image of $\operatorname{Spec} W_i \to \mathbb{A}^2_k$ contains η_i. Then $\operatorname{Spec}(W_1 \otimes_{W_3} W_2)$ will not be connected. This can be adapted to the case that W_3 is aic.

We will also need the following result.

Proposition 3.8. *Let $V \to W$ be a flat map of aic valuation rings. Let X be a flat scheme over V. Assume that X is geometrically unibranch and irreducible, and that $X \times_V \operatorname{Frac}(V)$ is normal. Then the pullback $X_W = X \times_V W$ is geometrically unibranch.*

Proof. Let s (resp. η) denote the special (resp. generic) point of $\operatorname{Spec} W$; let s_0, η_0 be their images in $\operatorname{Spec} V$. By flatness, η_0 is the generic point of $\operatorname{Spec} V$. It suffices to show that for any geometric point x of X_W, $\operatorname{Spec} \mathcal{O}^{sh}_{X_W,x}$ is irreducible. We may assume that x lies over s (otherwise localizing W at the image of x). Let $n > 1$ be any integer invertible in V and let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ denote the constant étale sheaf on $\operatorname{Spec} V$. We have the functors of vanishing cycles $R\Psi_V = R\Psi_{(V,X,\eta_0,s_0)}$, $R\Psi_W = R\Psi_{(W,X_W,\eta,s)}$ as in [Hub96, §4.2], and by [Hub96, Prop. 4.2.4], $R\Psi$ commutes with base change along $V \to W$. In particular, we have $R^0\Psi_W(\Lambda) = R^0\Psi_V(\Lambda)$. As X is geometrically unibranch, $R^0\Psi_V(\Lambda) = \Lambda$. It thus follows that $R^0\Psi_W(\Lambda) = \Lambda$. Computing the stalk at x, this implies that $\Gamma(\operatorname{Spec} \mathcal{O}^{sh}_{X_w,x} \times_{X_W} X_{W,\eta}, \Lambda) = \Lambda$, i.e., $\operatorname{Spec} \mathcal{O}^{sh}_{X_w,x} \times_{X_W} X_{W,\eta}$ is connected (where $X_{W,\eta}$ is the generic fiber of X_W). Now $X_{W,\eta} = X_{\eta_0} \times_{\eta_0} \eta$ is normal by [Sta14, 038O], as η_0 is algebraically closed and as X_{η_0} is normal by assumption. Hence $X_{W,\eta}$ is geometrically unibranch. By Lemma 3.5, X_W and hence also $X_{W,\eta}$ are irreducible. Noting that $\operatorname{Spec} \mathcal{O}^{sh}_{X_w,x} \times_{X_W} X_{W,\eta} \to X_{W,\eta}$ is pro-étale, Lemma 3.9 shows that $\operatorname{Spec} \mathcal{O}^{sh}_{X_w,x} \times_{X_W} X_{W,\eta}$ is in fact irreducible, and we are done. □

It remains to prove the following lemma.

Lemma 3.9. *Let S be an irreducible and geometrically unibranch scheme. Let $T \to S$ be pro-étale with T connected and qcqs. Then T is irreducible.*

Proof. Write $T = \varprojlim_i T_i$ as a cofiltered limit with T_i étale over S. We may assume all T_i are qcqs. As T is connected, we may assume that all T_i are too. As $T_i \to S$ is étale, all generic points of T_i lie over the generic point of S; but as T_i is qcqs, it follows that T_i has only finitely many generic points. In particular, T_i has only finitely many connected components, which are, a posteriori, clopen in T_i. Replacing each T_i with the connected component containing the image of T, we may thus assume that all T_i are connected. As S is geometrically unibranch, [Sta14, 0CB4] implies that irreducible components of T_i are disjoint. But as there are only finitely many of them, it follows that each of them is clopen in T_i. By connectedness, this implies that T_i is irreducible. Now, $T \to T_i$ is pro-étale, hence maps generic points to generic ones. As T_i are irreducible and $|T| = \varprojlim_i |T_i|$, T has at most one point, which maps to the generic point of T_i for each i. With other words, T is irreducible. □

3.2. Geometry of combs.

Definition 3.10. (i) An affine scheme is a *comb*, if each of its connected components is the spectrum of an aic valuation ring. It is an *extremally disconnected comb* if in addition the space of connected components is extremally disconnected.

(ii) A *product comb* is an affine scheme of the form $\operatorname{Spec}\left(\prod_{i\in I} A_i\right)$ where I is a (discrete) set and A_i are aic valuation rings for all $i \in I$.

(iv) Let $f\colon Y \to X$ be a map of combs. We call f *w-local* (resp. *straight*) if $f(Y^c) \subseteq X^c$ (resp. $f(Y^{\mathrm{gen}}) \subseteq X^{\mathrm{gen}}$).

Note that the affine scheme as in (ii) is indeed a comb, cf. [BS17, Lemma 6.2] and [BM21, Lemma 3.24]. An aic valuation ring is strictly Henselian. Thus any w-local comb is w-strictly local; it is w-contractible if and only if its space of connected components is extremally disconnected. We also have:

Lemma 3.11. *Let $f\colon Y \to X$ be a map of combs. Then the following hold.*

(i) If X is a product comb, then X is w-contractible and straight.

(ii) $f(Y^{\mathrm{gen}}) \subseteq X^{\mathrm{gen}}$ if and only if f is flat.

(iii) Suppose that f is surjective. If Y and f are w-local, then X is. If Y is straight and f flat, then X is straight.

(iv) If $\pi_0(f)$ is a homemorphism, then f surjective $\Leftrightarrow$ f faithfully flat $\Leftrightarrow$ f is a v-cover.

(v) X is geometrically unibranch.

Proof. (i): w-contractibility follows from [BS17, proof of Lemma 6.2] and [BS15, Lemma 2.4.8]. To show that a product comb $\operatorname{Spec} A$ with $A = \prod_{i\in I} A_i$ is straight, consider the absolutely flat ring $A^{\mathrm{gen}} = \prod_{i\in I} \operatorname{Frac}(A_i)$. We have $\pi_0(\operatorname{Spec} A^{\mathrm{gen}}) \cong \pi_0(\operatorname{Spec} A) = \beta I$ (the Stone–Čech compactification of the discrete set I), the first map induced by $A \to A^{\mathrm{gen}}$. Let $t \in \beta I$, corresponding to the ultrafilter $\mathscr{U}$ on I. Let A_t resp. A_t^{gen} denote the global sections of the (affine) connected component of $\operatorname{Spec} A$ resp. $\operatorname{Spec}(A^{\mathrm{gen}})$ corresponding to t. The natural map

$$A_t = \varinjlim_{W\in\mathscr{U}} \prod_{i\in W} A_i \to A_t^{\mathrm{gen}} = \varinjlim_{W\in\mathscr{U}} \prod_{i\in W} \operatorname{Frac}(A_i) = A_t^{\mathrm{gen}}$$

is the filtered colimit of injective ring maps, hence is itself injective. Thus the image of $\operatorname{Spec}(A_t^{\mathrm{gen}}) \to \operatorname{Spec} A_t$ is precisely the generic point of $\operatorname{Spec} A_t$. It follows that the image of $\operatorname{Spec}(A^{\mathrm{gen}}) \to \operatorname{Spec} A$ is precisely $(\operatorname{Spec} A)^{\mathrm{gen}}$. As $\operatorname{Spec}(A^{\mathrm{gen}})$ is qcqs, it follows that $(\operatorname{Spec} A)^{\mathrm{gen}}$ pro-constructible.

(ii): Flatness can be checked componentwise, and a map $V \to W$ of valuation rings is flat if and only if it is injective. (iii): Suppose Y, f are w-local. As f is surjective and w-local, we have $f(Y^c) = X^c$. As Y is w-local, $f(Y^c)$ is pro-constructible, hence closed in X. The proof of the second assertion is similar. (iv) is clear by Lemma 3.1(6). (v) follows directly from [Sta14, 06DM]. ☐

Note that by Lemma 3.11(v), all results of §1.7 apply to straight combs. It is well-known that locally in the v-topology, any qcqs scheme is a product comb:

Lemma 3.12. *Any qcqs scheme admits a v-cover which is a product comb.*

Proof. This follows from (the proof of) [BM21, Prop. 3.26] (or, the combination of [BM21, Lemma 3.23] and the proof of [BS17, Lm.6.2]). ☐

We record the following convenient statement about refinement of v-covers of nice combs.

Lemma 3.13. *Let X be a w-contractible straight comb. Then any v-cover $Y \to X$ admits a refinement by a v-cover $f\colon Z \to X$ such that Z is a w-contractible straight comb, $\pi_0(f)$ is a homeomorphism and f is faithfully flat.*

Proof. By Lemma 3.12 there exists a v-cover $Y' \to Y$ with Y' a product-like comb. Then $Y' \to X$ is a v-cover. By Corollary 2.7 and Lemma 3.1(5) the union of connected components $Y'' \subseteq Y'$, which are faithfully flat over their respective component in X, is a v-cover of X and a closed reduced subscheme of Y'. As $\pi_0(X)$ is extremally disconnected, the surjection $\pi_0(Y'') \to \pi_0(X)$ admits a splitting s. The image of s is closed and as $Y'' \to X$ is componentwise faithfully flat, it is clear that $Z := (Y'')_{s(\pi_0(X))} \to X$ is still a v-cover. ☐

Lemma 3.2 admits a (partial) generalization to combs:

Lemma 3.14. *Let X be a comb. The following subsets of X coincide:*

(a) quasi-compact pro-open subsets
(b) pro-(quasi-compact open) subsets
(c) pro-(principal open) subsets.

Proof. Any principal open is quasi-compact, hence (c) $\subseteq$ (b). Any pro-(quasi-compact open) is pro-constructible, hence quasi-compact, hence (b) $\subseteq$ (a). To show (a) $\subseteq$ (c), let $U \subseteq X$ be a quasi-compact pro-open subset. The image $\pi_0(U)$ of U under $X \to \pi_0(X)$ is quasi-compact, hence closed. Note that $X_{\pi_0(U)} \subseteq X$ is pro-(principal open) (indeed, $\pi_0(X) \smallsetminus \pi_0(U)$ is covered by clopen subsets V which it contains; if $e_V + e_{\pi_0(X) \smallsetminus V} = 1$ are the corresponding idempotents on X, $X_{\pi_0(U)} = \bigcap_V D(e_{\pi_0(X) \smallsetminus V})$). Thus we may assume that $\pi_0(U) = \pi_0(X)$ by replacing X by $X_{\pi_0(U)}$. Now, any connected component of U is quasi-compact pro-open, hence contains a (relatively) closed point (by Lemma 3.2(2)). If U^c is the set of those points, then it is clear that U is precisely the set of generalizations of U^c. Using that U is quasi-compact, it is easy to check directly that U^c is quasi-compact (cf. Lemma 3.15). Now we conclude by Proposition 1.30. ☐

Lemma 3.15. *Let T be a spectral topological space. Then the subset $T^c \subseteq T$ of closed points is quasi-compact. If, additionally, any connected component of T has a unique closed point, then T^c is a profinite set and the composition $T^c \to T \to \pi_0(T)$ is a homeomorphism.*

Note that in the last statement of the lemma, T^c is itself spectral, but the inclusion $T^c \to T$ is spectral if and only if $T^c \subseteq T$ is closed (which does always not hold). This lemma shows that the last claim of [BS15, Lemma 2.1.4] holds even without the assumption that T^c is closed.

Proof. The first claim is [Sta14, 00ZM]. For the second assertion of the lemma, note that the assumption implies that $T^c \to T \to \pi_0(T)$ is a bijection. As T^c is quasi-compact by the above and $\pi_0(T)$ is profinite [Sta14, 0906], the result follows by [Sta14, 08YE]. ☐

By Lemma 3.2(1), any intersection of a quasi-compact open U of the comb X with a connected component of X is a principal open. However, it is not clear whether U itself must be principal.

3.3. **The pro-unramified site of a comb.** The pro-unramified site of a comb is well-behaved. As any comb X is affine, the sites $X_{\mathrm{pu}}^{\mathrm{aff}}$ and X_{pu} generate the same topos. We consider $X_{\mathrm{pu}}^{\mathrm{aff}}$, and provide a quite explicit topological description of it.

Lemma 3.16. *Let X be a comb and $Y \in X_{\mathrm{pu}}^{\mathrm{aff}}$. Then Y is a comb. Moreover, Y is isomorphic to the intersection of a pro-(qc open) with a closed subset of $X_{\pi_0(Y)}$, which meets every connected component of $X_{\pi_0(Y)}$. If, in addition, $Y \in X_{\mathrm{unr}}^{\mathrm{aff}}$, then this subset of $X_{\pi_0(Y)}$ is l.c.c.*

Proof. By Lemma 1.17(3),(4), any connected component Y_0 of Y is pro-unramified over the corresponding connected component X_0 of X. Thus, by Lemma 3.3(2), Y_0 is itself the spectrum of an aic valuation ring, and identifies with a subset of X_0. As Y is affine, it is a comb. For the second claim, let $f\colon Y \to X$ be the structure map. Replacing X by $X_{\pi_0(Y)}$ we may assume that $\pi_0(f)$ is an isomorphism. By the above, Y identifies with the subset $f(Y)$ of X. First, $f(Y)$ is pro-constructible, hence its intersection with any connected component X_0 of X is. By Lemma 3.1(7), $f(Y) \cap X_0$ contains a closed point. The collection $f(Y)^c$ of all those points (for varying X_0) is precisely the set of all closed points of $f(Y)$, and the above shows that $f(Y)^c$ is in bijection with $\pi_0(X)$. As $f(Y)$ is pro-constructible, it is spectral [Sta14, 0902]. Thus, by Lemma 3.15, $Z := f(Y)^c$ is quasi-compact. Applying Proposition 1.30 we get a pro-(principal open) set $\widetilde{Z} \subseteq X$ of all generalizations of points in Z, and it is clear that $Y \to X$ factors through a map $\tilde{f}\colon Y \to \widetilde{Z}$. We claim that $\tilde{f}(Y) \subseteq \widetilde{Z}$ is closed. As it is pro-constructible, it suffices to show that it is stable under specialization, which can be done componentwise. But this is clear as, by Lemma 3.3(2), the intersection of $f(Y)$ with a connected component $\operatorname{Spec} V$ of $\widetilde{Z}$ is isomorphic to $\operatorname{Spec} V/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of V.

Finally, assume that $Y \in X_{\mathrm{unr}}^{\mathrm{aff}}$. This remains to hold after the above replacement of X by $X_{\pi_0(Y)}$ (as $\pi_0(Y) \to \pi_0(X)$ is the base change of a map of finite sets, and hence $X_{\pi_0(Y)} \to X$ is of finite presentation, so that also $Y \to X_{\pi_0(Y)}$ is of finite presentation). But then, with notation as in the previous paragraph, it suffices to show that $f(Y) \subseteq X$ is l.c.c. By the above, $f(Y) = \widetilde{Z} \cap C$ for some pro-(qc open) $\widetilde{Z} \subseteq X$ and closed $C \subseteq X$, both uniquely determined by $f(Y) \subseteq X$. By Chevalley's theorem [Sta14, 054K], $f(Y)$ is constructible, hence $f(Y) = \bigcup_{i=1}^n (\widetilde{Z}_i \cap C_i)$ with $\widetilde{Z}_i$, $X \smallsetminus C_i$ qc open. Note that $f(Y) = \bigcup_{i=1}^n \widetilde{Z}_i \smallsetminus \bigcup_{i=1}^n C_i$, as this clearly holds in each connected component of X. As $\widetilde{Z}, C$ are determined by $f(Y)$, it follows that $\widetilde{Z} = \bigcup_{i=1}^n \widetilde{Z}_i$ and $C = \bigcup_{i=1}^n C_i$, so that $\widetilde{Z}$ and $X \smallsetminus C$ are qc open. The lemma is proved. □

We call a topological space a *topological comb*, if it is homeomorphic to the topological space underlying a comb. For any topological comb S and any continuous map $T \to \pi_0(S)$ with T profinite, we have the fiber product $S_T = S \times_{\pi_0(S)} T$ in topological spaces. If $S = |X|$ for a comb X, then $S_T = |X_T|$.

The following definition is very similar to that of affinoid pro-étale maps over a topological comb [Sch18, Def. 7.20]. The difference is that maps between schemes (unlike maps between adic spaces) need not be generalizing.

Definition 3.17. Let S be a topological comb. Let S_{pu} be the category of all triples (T, α, Y), where T is a profinite set, $\alpha\colon T \to \pi_0(S)$ a continuous map and $Y \subseteq S_T$ a subset, which is the intersection of a closed subset and a quasi-compact pro-open subset, subject to the condition that Y meets any connected component of S_T. Morphisms $(T, \alpha, Y) \to (T', \alpha', Y')$ are continuous maps $\beta\colon T \to T'$, satisfying $\alpha_1 = \alpha_2\beta$ and – if $\beta_S\colon S_T \to S_{T'}$ denotes the induced map – $\beta_S(Y) \subseteq Y'$. We make S_{pu} a site by declaring covers to be those families of maps, which can be refined by a finite family, which is a topological v-cover.

Moreover, we define S_{unr} to be the subsite consisting of those triples (T, α, Y) where Y is l.c.c. in S_T, with the same covers.

Lemma 3.18. *Let X be a comb and $|X|$ the underlying topological comb. Then the functor*

$$X^{\mathrm{aff}}_{\mathrm{pu}} \to |X|_{\mathrm{pu}}, \quad Y \mapsto (\pi_0(Y), \pi_0(Y) \to \pi_0(X), |Y|)$$

defines an equivalence of categories. This restricts to an equivalence of categories $X^{\mathrm{aff}}_{\mathrm{unr}} \to |X|_{\mathrm{unr}}$.

Proof. By Lemma 3.16 we get a functor (note that by Lemma 3.14, pro-(qc open) agrees with quasi-compact pro-open). Fully faithfullness follows from Lemma **??**. It remains to show essential surjectivity. Given $(T, \alpha, Y) \in |X|_{\mathrm{pu}}$, we may pullback to X_T and so assume that $T = \pi_0(X)$ and $\alpha = \mathrm{id}$. It now suffices to show that any quasi-compact pro-open subset $U \subseteq X$ meeting every connected component of X is in fact affine. But this follows from Lemma 3.14. The last statement is immediate from the last claim of Lemma 3.16. □

The unramified site of an arbitrary comb is well-behaved:

Lemma 3.19. *Let X be a comb. Any cover in X_{unr} splits.*

Proof. We can apply the proof of [Sch18, 7.16]: It is enough to show that any v-cover $f\colon Y \to X$ with $Y \in X^{\mathrm{aff}}_{\mathrm{unr}}$ splits. Restricted to any connected component of X, f admits a section by Lemma 3.3. By the finiteness assumption, we may spread it out, and so deduce that $Y \to X$ splits Zariski locally on X. But any Zariski cover of X splits (with Fargues's argument, [Sch18, Lemma 7.2]). □

Recall the unramified analogue $\lambda_{\mathrm{pu},X^\circ}(\cdot)$ of Henselization from (2.1).

Lemma 3.20. *If $f\colon Y \to X$ is a map of combs, then $\lambda_{\mathrm{pu},X^\circ}(Y)$ corresponds under the equivalence of Lemma 3.18 to $(\pi_0(Y), \pi_0(f), \tilde{f}(Y))$, where $\tilde{f}\colon Y \to X_{\pi_0(Y)}$ is the natural map.*

Proof. Observe that the argument in the proof of Lemma 3.16 applies equally good to $\tilde{f}(Y)$, showing that it is the intersection of a pro-(qc open) with a closed subset of $X_{\pi_0(Y)}$. Now, Lemma 3.18 implies that the universal property characterizing $\lambda_{\mathrm{pu},X^\circ}(Y)$ holds for $\tilde{f}(Y) \in X^{\mathrm{aff}}_{\mathrm{pu}}$, which corresponds to $(\pi_0(Y), \pi_0(f), \tilde{f}(Y))$. □

Lemma 3.21. *Let X be a comb.*

(i) Let Y be a comb and let $f\colon Y \to X$ be any map. Then $Y \to \lambda_{\mathrm{pu},X^\circ}(Y)$ is a v-cover. In particular, if $Y_1 \to Y_2$ is a v-cover of combs over X, then $\lambda_{\mathrm{pu},X^\circ}(Y_1) \to \lambda_{\mathrm{pu},X^\circ}(Y)$ is a v-cover.

(ii) Let $Y_1 \to Y_3 \leftarrow Y_2$ be maps of combs over X. Then $\lambda_{\mathrm{pu},X^\circ}(Y_1 \times_X Y_2) \to \lambda_{\mathrm{pu},X^\circ}(Y_1) \times_{\lambda_{\mathrm{pu},X^\circ}(Y_3)} \lambda_{\mathrm{pu},X^\circ}(Y_2)$ is an isomorphism in each of the following cases:
  - *(a) for $i = 1, 2$, $Y_i \to Y_3$ is w-local, or*
  - *(b) for $i = 1, 2$, $Y_i \to Y_3$ is flat, or*
  - *(c) $Y_3 = X$.*

Proof. (i): The map $g\colon Y \to \lambda_{\mathrm{pu},X^\circ}(Y)$ is surjective (cf. Lemma 3.20) and $\pi_0(g)$ is a homeomorphism. Thus the first assertion follows from Lemma 3.1(6). The second assertion is a consequence of the first. (ii): Let us prove case (a). From Lemma 3.22 it follows that $\pi_0(Y_1 \times_{Y_3} Y_2) \to \pi_0(Y_1) \times_{\pi_0(Y_3)} \pi_0(Y_2)$ is a homeomorphism, and we may reduce to the case when Y_i $(i = 1, 2, 3)$ and X are connected. Then the right hand side of the claimed isomorphism corresponds (via Lemma 3.18) to the subset $\mathrm{im}(Y_1 \to X) \cap \mathrm{im}(Y_2 \to X)$ of X. On the other side, as $Y_1 \times_{Y_3} Y_2$ is connected, $\lambda_{\mathrm{pu},X^\circ}(Y_1 \times_{Y_3} Y_2)$ is connected, and corresponds to some subset of $X = \operatorname{Spec} V$, of the form $\operatorname{Spec} V_{\mathfrak{p}}/\mathfrak{q}V_{\mathfrak{p}}$, which (at least) contains the image of $Y_1 \times_{Y_3} Y_2 \to X$ (in fact, it is equal to this image). Thus, it suffices to show that topologically $\mathrm{im}(Y_1 \times_{Y_3} Y_2) = \mathrm{im}(Y_1 \to X) \cap \mathrm{im}(Y_2 \to X)$ in X. The inclusion $\subseteq$ is clear, so let us prove the other one. Using locality of the maps, for $i = 1, 2$ the image Z_i of Y_i in Y_3 is a closed subset of

Y_3 (e.g., by Lemma 3.1(7)). Then either $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$. Wlog, assume $Z_1 \subseteq Z_2$. Hence the same holds for the images of Y_i in X, i.e., $\mathrm{im}(Y_1 \to X) \cap \mathrm{im}(Y_2 \to X) = \mathrm{im}(Y_1 \to X)$. Any $x \in \mathrm{im}(Y_1 \to X)$ may be lifted to a point $y_1 \in Y_1$, and if y_3 is the image of y_1 in Y_3, then $y_3 \in Z_1 \subseteq Z_2$; so y_3 lifts to a point $y_2 \in Y_2$, and $y_1 \times_{y_3} y_2 \subseteq Y_1 \times_{Y_3} Y_2$ is non-empty. This proves the claim, and hence (a). The proof of case (b) is similar; (c) straightforwardly follows from Lemma 3.18 and Lemma 3.22 (we will not make use of (b) and (c), so we omit the details). □

Lemma 3.22. *Let $Y_1 \to Y_3 \leftarrow Y_2$ be maps of X-combs. Then $\pi_0(Y_1 \times_{Y_3} Y_2) \to \pi_0(Y_1) \times_{\pi_0(Y_3)} \pi_0(Y_2)$ is a homeomorphism onto the closed subset of those $(y_1, y_2) \in \pi_0(Y_1) \times_{\pi_0(Y_3)} \pi_0(Y_2)$, for which the (topological) images of the components y_1 and y_2 in Y_3 intersect non-trivially.*

Proof. Clearly, the image is precisely the claimed subset. As the left side is profinite, the image is quasi-compact, and as the right side is profinite, the image is a closed subset, hence itself profinite. By [Sta14, 08YE], it suffices to show the injectivity of the map. It is a direct consequence of Corollary 3.6. □

Remark 3.23. The analogue of [Sch18, Lemma 14.5(ii)] (which is a slightly stronger version of Lemma 3.21(ii)) does not hold in our setup. E.g., let $X = \mathrm{Spec}\, V$, $Y_3 = \mathrm{Spec}\, W$ be spectra of aic valuation rings, and let $f\colon Y_3 \to X$ be a map, such that for some point $\mathfrak{p} \in X$, the preimage in Y_3 contains more than one point. Let $\mathfrak{p}_1 \rightsquigarrow \mathfrak{p}_2 \rightsquigarrow \mathfrak{p}_3 \rightsquigarrow \mathfrak{p}_4$ be distinct points of Y_3, such that $f(\mathfrak{p}_2) = f(\mathfrak{p}_3) = \mathfrak{p}$. Let $Y_1 = \mathrm{Spec}(W/\mathfrak{p}_1)_{\mathfrak{p}_2}$ and $Y_2 = \mathrm{Spec}(W/\mathfrak{p}_3)_{\mathfrak{p}_4}$ be connected combs over Y_3. Then $Y_1 \times_{Y_3} Y_2 = \varnothing$, but $\mathfrak{p} \in \lambda_{\mathrm{pu},X^\circ}(Y_1) \times_{\lambda_{\mathrm{pu},X^\circ}(Y_3)} \lambda_{\mathrm{pu},X^\circ}(Y_2) \subseteq X$.

Remark 3.24. If X is a comb, then for an X-scheme the map $Y \to \lambda_{\mathrm{pu},X^\circ}(Y)$ does not in general need to be a v-cover.

3.4. **Relation between v- and pro-unramified sites of a comb.** Let X be a comb. We have the morphism

$$\lambda_{\mathrm{pu}} = \lambda_{X,\mathrm{pu}}\colon X_v^\sim \to X_{\mathrm{pu}}^\sim$$

of topoi.

Lemma 3.25. *Let X be a comb and $\mathscr{F} \in X_{\mathrm{pu}}^\sim$. Let $Y \in X_v$ be a comb. If Y is either w-local or straight, then $\lambda_{\mathrm{pu}}^*\mathscr{F}(Y) = \mathscr{F}(\lambda_{\mathrm{pu},X^\circ}(Y))$. Moreover, for any $Z \in X_v$, such that $Z \to \lambda_{\mathrm{pu},X^\circ}(Z)$ is a v-cover, the natural map $\mathscr{F}(\lambda_{\mathrm{pu},X^\circ}(Z)) \to \lambda_{\mathrm{pu}}^*\mathscr{F}(Z)$ is injective.*

Proof. This follows from Lemma 3.21 in the same way as the analogous claim for diamonds follows from [Sch18, Lemma 14.5], cf. the proof of [Sch18, Lemma 14.6]. To be able to apply Lemma 3.21(ii), we use the fact that any v-cover $S' \to S$ of w-local (resp. straight) schemes can be refined by a w-local (resp. straight) v-cover by Proposition 2.1 (resp. Proposition 2.5). For the last assertion, note that the assumption on $Z \to \lambda_{\mathrm{pu},X^\circ}(Z)$ to be a v-cover implies that for any v-cover $T \to Z$, the induced map $\lambda_{\mathrm{pu},X^\circ}(T) \to \lambda_{\mathrm{pu},X^\circ}(Z)$ is a v-cover. □

Now we are able to prove an analogue of [Sch18, Prop. 14.7] for the pro-unramified site of a comb. Moreover, part (v) of the proposition below gives an important descent statement.

Proposition 3.26. *Let X be a comb. Then the following hold.*

- *(i) For any $\mathscr{F} \in X_{\mathrm{pu}}^\sim$, the adjunction map $\mathscr{F} \to \lambda_{\mathrm{pu},*}\lambda_{\mathrm{pu}}^*\mathscr{F}$ is an isomorphism. The pullback functor $\lambda_{\mathrm{pu}}^*\colon X_{\mathrm{pu}}^\sim \to X_v^\sim$ is fully faithful.*
- *(ii) The essential image of λ_{pu}^* is the full subcategory of all $\mathscr{F} \in X_v^\sim$ satisfying the following condition: for any surjective map $g\colon Z \to Y$ of w-local combs over X with $\pi_0(g)$ homeomorphism, the pullback $\mathscr{F}(Y) \to \mathscr{F}(Z)$ is bijective. Here, one can replace 'w-local' by 'straight' resp. 'w-local straight'; also one can assume Z to be a product comb.*
- *(iii) λ_{pu}^* preserves all small limits.*

(iii)' If $f\colon Y\to X$ is a map of combs, then $f_{\rm pu}^*\colon X_{\rm pu}^\sim\to Y_{\rm pu}^\sim$ preserves all small limits.

(iv) Let $\mathscr{F}$ be a sheaf of abelian groups (resp. groups) that comes via pullback from $X_{\rm et}$. Then $R^i\lambda_{{\rm pu}*}\mathscr{F}=0$ for all $i>0$ (resp. for $i=1$).

(v) Let X be a comb. Let $\mathscr{F}$ be a v-sheaf on X. Let $f\colon Y\to X$ be a v-cover such that $f^*\mathscr{F}$ comes via pullback from $Y_{\rm pu}$. Then $\mathscr{F}$ comes via pullback from $X_{\rm pu}$.

Proof. (i): This follows from Lemma 3.25, cf. the proof of Proposition 2.19(i). (ii): Suppose $\mathscr{F}$ is in the essential image. By Lemma 3.25, it suffices to show that for $Z\to Y$ as in the statement, $\lambda_{{\rm pu},X^\circ}(Z)\to\lambda_{{\rm pu},X^\circ}(Y)$ is an isomorphism. But this holds by Lemma 3.20. For the converse direction, it suffices to handle the case of w-local straight combs, and moreover (by Lemma 3.11(i)) we may additionally assume that Z in the condition is a product comb. By Lemma 3.12, it suffices to show that $(\lambda_{\rm pu}^*\lambda_{{\rm pu}*}\mathscr{F})(Z)\to\mathscr{F}(Z)$ is a bijection for each product comb $Z\in X_v$. But by Lemma 3.20 the condition in (ii) applies to $Z\to\lambda_{{\rm pu},X^\circ}(Z)$ (note that $\lambda_{{\rm pu},X^\circ}(Z)$ is w-local straight, cf. Proposition **??**), so exploiting Lemma 3.25 and the condition in (ii), we get $(\lambda_{\rm pu}^*\lambda_{{\rm pu}*}\mathscr{F})(Z)=\mathscr{F}(\lambda_{{\rm pu},X^\circ}(Z))=\mathscr{F}(\lambda_{{\rm pu},X^\circ})(Z)$.

(iii): For an inverse system $\mathscr{F}_i$ of sheaves on $X_{\rm pu}$, we have to show that $\lambda_{\rm pu}^*(\varprojlim_i\mathscr{F}_i)=\varprojlim_i\lambda_{\rm pu}^*\mathscr{F}_i$. It suffices to show that equality holds after evaluation at each w-local comb in X_v. But this follows from Lemma 3.25 and the fact that evaluation commutes with inverse limits of sheaves. (iii)' follows from (iii) in the same way as in Proposition 2.19.

(iv): It suffices to show that $H^i(X_v,\mathscr{F})=0$. As $i>0$, there is some v-cover $X'\to X$, such that $s|_{X'}=0$. We may write $X'=\varprojlim_j X_j$ for $X_j\to X$ finitely presented v-cover for each j. As $\mathscr{F}$ is étale, $H^i(X',\mathscr{F})=\varinjlim_j H^i(X_j,\mathscr{F})$. Thus, replacing X' by X_j for sufficiently big j, we may assume that $X'\to X$ is finitely presented. Restricted to each connected component $X'\to X$ has a section, as follows (for example) from [Ryd10, Prop. 2.7(viii)] and the fact that any faithfully flat finitely presented morphism over a strictly henselian ring admits a section. As $X'\to X$ is finitely presented, each such section $X_t\to X'\times_X X_t$ ($t\in\pi_0(X)$) extends to an open neighboorhood of the closed point of X_t, which necessarily contains X_T for some $t\in T\subseteq\pi_0(X)$ quasi-compact open (by Lemma 3.15). Then $s|_{X_T}=0$. As finitely many of such X_T's cover X, we have $s=0$.

(v): It suffices to check the condition from (ii). I.e., we have to show the following: if $g\colon X'\to X$ is a surjective map of w-local straight combs with $\pi_0(g)$ homeomorphism, and $f\colon Y\to X$ is a v-cover such that $f^*\mathscr{F}$ comes by pullback from $Y_{\rm pu}$, then $\mathscr{F}(X)\to\mathscr{F}(X')$ is bijective. Refining $Y\to X$, we may by Lemma 3.13 assume that Y is w-local straight, that $\pi_0(f)$ is a homeomorphism and that f is faithfully flat. Let $Y'=Y\times_X X'$ and let $Y''\to Y'$ be any v-cover by a product comb. Let $Y'''\subseteq Y''$ be the union of all connected components of Y'' which are faithfully flat over their respective image components in X' and in Y. As X' and Y are w-contractible and straight, Y''' is closed in Y'' by Propositions 2.1 and 2.5. We claim that $Y'''\to Y$ and $Y'''\to X'$ are v-covers. Therefore, let $X_0\subseteq X$ be a connected component and let $X_0'\subseteq X'$ and $Y_0\subseteq Y$ be the connected components lying over it. Both are faithfully flat over X_0. By Corollary 3.6, $X_0'\times_{X_0}Y_0$ is irreducible. Let η be its generic point (which lies over the generic points of X_0' and Y_0) and let x be any point in the closed subscheme $(X_0')^c\times_{X_0^c}Y_0^c$ (which lies over the closed points of X_0' and Y_0). By irreducibility, $\eta\rightsquigarrow x$ is a specialization relation in $X_0'\times_{X_0}Y_0$. It lifts to a specialization relation $\eta'\rightsquigarrow x'$ in a connected component Y_0'' of the v-cover Y''. This component necessarily lies in Y''', and our claim follows from Lemma 3.1(5). Finally, using that $\pi_0(Y)$ is extremally disconnected, choose a continuous section $s\colon\pi_0(Y)\to\pi_0(Y''')$; the image of s is closed. Moreover, from the claim (and as $\pi_0(Y)\to\pi_0(X)\cong\pi_0(X')$ is surjective) it follows that the closed subscheme $Z:=(Y''')_{s(\pi_0(Y))}$ of Y''' is still a v-cover of Y and of X'. Thus we have $\mathscr{F}(X)={\rm Eq}(\mathscr{F}(Y)\rightrightarrows\mathscr{F}(Y\times_X Y))$ and

$\mathscr{F}(X') = \mathrm{Eq}(\mathscr{F}(Z) \rightrightarrows \mathscr{F}(Z \times_{X'} Z))$. As $\pi_0(Z) = \pi_0(Y)$, the assumption on $\mathscr{F}$ and part (ii) imply that $\mathscr{F}(Y) = \mathscr{F}(Z)$. Thus $\mathscr{F}(X) \to \mathscr{F}(X')$ is injective. For surjectivity, regard $Y \times_X Y$ as an Y-scheme via the first projection, and note that $\mathscr{F}(\lambda^{\mathrm{pu}}_{Y^\circ}(Y \times_X Y)) = \lambda^{p}_{Y,\mathrm{pu}}(\mathscr{F}|_{Y_{\mathrm{pu}}})(Y \times_X Y) \to \mathscr{F}(Y \times_X Y)$ is injective by Lemma 3.25, as $Y \times_X Y \to \lambda^{\mathrm{pu}}_{Y^\circ}(Y \times_X Y)$ is a v-cover (by Corollary 3.6 and as each connected component of Y is faithfully flat over X. As also $\mathscr{F}(Y) \to \mathscr{F}(Y \times_X Y)$ factors through this injection, we have $\mathscr{F}(X) = \mathrm{Eq}(\mathscr{F}(Y) \rightrightarrows \mathscr{F}(\lambda^{\mathrm{pu}}_{Y^\circ}(Y \times_X Y))$. Similarly, we have $\mathscr{F}(X') = \mathrm{Eq}(\mathscr{F}(Z) \rightrightarrows \mathscr{F}(\lambda^{\mathrm{pu}}_{Z^\circ}(Z \times_{X'} Z)))$. It suffices to show that $\mathscr{F}(\lambda^{\mathrm{pu}}_{Z^\circ}(Z \times_{X'} Z)) \to \mathscr{F}(\lambda^{\mathrm{pu}}_{Y^\circ}(Y \times_X Y))$ is injective, or that $\lambda^{\mathrm{pu}}_{Z^\circ}(Z \times_{X'} Z) \to \lambda^{\mathrm{pu}}_{Y^\circ}(Y \times_X Y)$ is a v-cover. But this is clear as $\lambda^{\mathrm{pu}}_{Y^\circ}(Y \times_X Y) = Y_{\pi_0(Y) \times_{\pi_0(X)} \pi_0(Y)}$ and similarly for the other side. □

4. Unramified sheaves

4.1. **Base change for unramified sheaves.** Let V be an aic valuation ring. By Lemma 3.1(3), the locally closed constructible subsets of $X = \operatorname{Spec} V$ are precisely the subsets of the form $\operatorname{Spec}(V/\mathfrak{p})_{\mathfrak{q}}$, where $\mathfrak{p}$, $\mathfrak{q}$ are radical ideals of principal ideals.

Lemma 4.1. *Let $X = \operatorname{Spec} V$ for an aic valuation ring V. Any $Y \in X^{\mathrm{aff}}_{\mathrm{unr}}$ is isomorphic over X to a finite disjoint union of locally closed constructible subsets of X.*

Proof. By Lemma 1.17(7), $Y \to X$ is G-unramified. Using [BM21, Lemma 2.20] and that $Y \to X$ is finitely presented, we can reduce to the case that V has finite rank, i.e., $|X|$ is finite. But in this case it is clear that Y has only finitely many irreducible components, as the same holds for all fibers of $Y \to X$. In particular, Y has finitely many connected components, and we may assume that Y is connected. But then we are done by Lemma 3.3(2) and the fact that the image of the finitely presented map $Y \to X$ is constructible. □

Theorem 4.2. *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

where X, Y and Y' are combs. Let $\mathscr{F}$ be an abelian torsion sheaf on Y_{unr} whose torsion is prime to $\mathrm{char}(Y)$. Then the natural map

$$g^* Rf_* \mathscr{F} \xrightarrow{\sim} Rf'_* g'^* \mathscr{F}$$

is an isomorphism of sheaves on X'_{unr}.

Proof. In the following proof we will repeatedly make use of the fact that the claimed base-change property is stable under cofiltered limits of cartesian diagrams in the sense of [Sta14, Lemma 0EZT]. Namely, the proof of loc. cit. holds verbatim for the unramified site in place of the étale site if we replace the reference to [Sta14, Lemma 0EYM] by Corollary 5.2.

If g is in X_{unr} then the claim is clear (see [Sta14, 0D6G]). By the previous paragraph the claim also follows in the case that g is a cofiltered inverse limit of qcqs objects in X_{unr}. In particular the claim is true if g is the inclusion of a connected component. Combining this with the observation that by Corollary 1.39 it is enough (for general g) to show the claimed isomorphism after pullback to the connected components of X', we easily reduce to the case that X' is connected. Then g factors over a connected component of X, so by a similar argument we can also assume that X is connected. Thus from now on we are in the situation that X and X' are the spectra of aic valuation rings.

Let $Z := \pi_0(Y) \times X$, so that we get the diagram of cartesian squares

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f_1'}\downarrow & & \downarrow{\scriptstyle f_1} \\ Z' & \longrightarrow & Z \\ {\scriptstyle f_2'}\downarrow & & \downarrow{\scriptstyle f_2} \\ X' & \xrightarrow{g} & X \end{array}$$

It is enough to show that the claimed base-change holds in both squares of the diagram. In the top square we note that the map $\pi_0(Y) \to \pi_0(Z)$ induced by f_1 is a homeomorphism. To prove base-change in this diagram we can as before reduce to the case that the bottom spaces are aic valuation rings, which now even implies that the upper right space is an aic valuation ring. Factoring the right vertical map over its scheme-theoretic image we can further reduce to the cases that the right vertical map is either flat or a closed immersion. We are thus left with the following cases:

(a) X, X' and Y are spectra of aic valuation rings and f is flat.
(b) X, X' and Y are spectra of aic valuation rings and f is a closed immersion.
(c) X and X' are spectra of aic valuation rings and $Y = S \times X$ for some profinite set S.

To prove the desired equivalence of (derived) sheaves in these cases, it is enough to show that they agree after applying $R\Gamma(U', -)$ for all $U' \in X'^{\mathrm{aff}}_{\mathrm{unr}}$. Fix such a U' and consider the diagram

$$\begin{array}{ccccc} V' & \xrightarrow{j'} & Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U' & \xrightarrow{j} & X' & \xrightarrow{g} & X \end{array}$$

where both squares (and hence also the whole rectangle) are cartesian. We need to show that the natural map

$$R\Gamma(U', j^* g^* Rf_* \mathscr{F}) \xrightarrow{\sim} R\Gamma(U', j^* Rf'_* g'^* \mathscr{F})$$

is an isomorphism. By [Sta14, 0D6G] we have base-change in the left square, so that the right-hand side of the desired equivalence becomes $R\Gamma(U', Rf''_* j'^* g'^* \mathscr{F})$. Thus the claimed equivalence boils down to showing that the base-change along the big rectangle holds after applying $R\Gamma(U', -)$. Note that by Lemma 4.1 (and Lemma 3.3) we can write U' as a finite disjoint union of spectra of aic valuation rings, which easily reduces us to the case that U' is a spectrum of an aic valuation ring. Thus, by replacing X' by U', Y' by V', g by $j \circ g$ and g' by $j' \circ g'$ we reduce to showing that the claimed base-change holds after applying global sections, i.e. we are left with showing that the natural map

$$R\Gamma(X', g^* Rf_* \mathscr{F}) \xrightarrow{\sim} R\Gamma(X', Rf'_* g'^* \mathscr{F})$$

is an isomorphism (while we maintained the assumptions in cases (a), (b) and (c) above). Let $U \subset X$ be the scheme-theoretic image of X. Then $U \hookrightarrow X$ is pro-unramified and thus U is again the spectrum of an aic valuation ring (cf. Lemma 4.1) and g factors over U. As discussed above we know that base-change along pro-unramified maps holds, so we are free to replace X by U and thus assume that g is flat. We can now further replace X by the set-theoretic image of g (which is pro-open and thus affine) to reduce to the case that g is faithfully flat.

So far all our arguments were rather formal. In all of the above cases (a), (b) and (c) we are further allowed to assume that g is faithfully flat and we only need to show the desired isomorphism of sheaves after applying $R\Gamma(X', -)$. In what follows, we will reduce this claim to an étale base-change result by Huber.

For any scheme S let $\mu = \mu_S \colon S_{\mathrm{unr}} \to S_{\text{ét}}$ be the projection of sites from Section 1.7 and for any map $h\colon S \to T$ let us denote $h_{\text{ét}}\colon S_{\text{ét}} \to T_{\text{ét}}$ the associated morphism on étale sites (while $h\colon S_{\mathrm{unr}} \to T_{\mathrm{unr}}$ denotes the morphism of unramified sites). Note that all of X, X', Y and Y' are geometrically unibranch and straight: For the first three spaces this is easy to see (cf. Lemma 3.11.(v)); for the last one it follows from Corollary 3.6 and Proposition 3.8 in case (a) and is easy in case (b) (where Y' is the spectrum of an aic valuation ring) and in case (c) (where $Y' = S \times X'$). In particular on all of these spaces μ_* is exact by Theorem 1.40 and since g and g' are flat it follows from Lemma 4.4 that μ_* commutes with pullback along these maps (in case (c) one easily passes to connected components in order to reduce to the setting of Lemma 4.4). Clearly $\mu_* = R\mu_*$ also commutes with (derived) pushforward along f and f', so we can commute μ_* past all the functors in the claimed base-change isomorphism. More concretely, denoting $\mathscr{F}_{\text{ét}} := \mu_{Y,*}\mathscr{F}$ we get

$$\begin{aligned} R\Gamma(X', g^* Rf_* \mathscr{F}) &= R\Gamma(X'_{\text{ét}}, \mu_{X',*} g^* Rf_* \mathscr{F}) = R\Gamma(X'_{\text{ét}}, g^*_{\text{ét}} Rf_{\text{ét},*} \mu_{Y,*} \mathscr{F}) \\ &= R\Gamma(X'_{\text{ét}}, g^*_{\text{ét}} Rf_{\text{ét},*} \mathscr{F}_{\text{ét}}), \\ R\Gamma(X', Rf'_* g'^* \mathscr{F}) &= R\Gamma(X'_{\text{ét}}, \mu_{X',*} Rf'_* g'^* \mathscr{F}) = R\Gamma(X'_{\text{ét}}, Rf'_{\text{ét},*} g'^*_{\text{ét}} \mu_{Y,*} \mathscr{F}) \\ &= R\Gamma(Y'_{\text{ét}}, g'^*_{\text{ét}} \mathscr{F}_{\text{ét}}). \end{aligned}$$

Thus, proving the desired isomorphism of the above two complexes leaves us with an étale base-change in the situation at hand. But this was proved by Huber in [Hub96, Corollary 4.2.7]. □

Remark 4.3. In the proof of Theorem 4.2 we reduced to the case that g is surjective by replacing X be the image $U \subset X$ of g. This is possible because $U \to X$ is pro-unramified and we work with unramified sheaves. This is the place where it is crucial that we work with unramified sheaves and not with étale sheaves!

Lemma 4.4. *Let S be the spectrum of an aic valuation ring. Let $g\colon T \to S$ be a flat map, with T qcqs, geometrically unibranch and straight. Then the following hold:*

(i) $g^*_{\mathrm{proet}} \mu_{S*} \cong \mu_{T*} g^*_{\mathrm{pu}}$ as functors $S^\sim_{\mathrm{pu}} \to T^\sim_{\mathrm{proet}}$,
(ii) $g^*_{\text{ét}} \mu_{S*} = \mu_{T*} g^*_{\mathrm{unr}}$ as functors $S^\sim_{\mathrm{unr}} \to T^\sim_{\text{ét}}$.

Proof. (ii) reduces to (i) by fully faithfullness of the functors $\nu^*_{X,\tau}\colon X_{p\tau} \to X_\tau$ for $X \in \{S, T\}$ and $\tau \in \{\text{ét}, \mathrm{unr}\}$, and the fact that $\nu^*_{X\mathrm{proet}} \mu_{X*} \cong \mu_{X*} \nu^*_{X\mathrm{pu}}$, which are immediate from [BS15, Lemma 5.1.2] and Lemma 1.37. To prove (i), let $F \in S^\sim_{\mathrm{pu}}$. It suffices to check that $g^*_{\mathrm{proet}} \mu_{S*} F(T') = \mu_{T*} g^*_{\mathrm{pu}} F(T')$ for any w-contractible $T' \in T_{\mathrm{proet}}$. As T' is w-contractible (and as $\mu_{S,*}$ is just restriction of sheaves), one immediately checks that $(g^*_{\mathrm{proet}} \mu_{S*} F)(T') = F(\lambda_{S^\circ}(T'))$, where λ_{S° is the Henselization over S. On the other hand, T' is straight by Lemma 1.43 and our assumptions. Thus T' is a contractible object of T_{pu} by Theorem 1.33(2), and just as above one immediately gets $(\mu_{T*} g^*_{\mathrm{pu}} F)(T') = F(\lambda_{S\mathrm{pu}\circ}(T'))$. It remains to show that $\lambda_{S^\circ}(T') = \lambda_{S\mathrm{pu}\circ}(T')$. For $\tau \in \pi_0(T')$, let $T'_\tau \subseteq T'$ denote the corresponding component, and let $\lambda_{S\mathrm{pu}\circ}(T')_\tau$ be the corresponding connected component of $\lambda_{S\mathrm{pu}\circ}(T')$. As T' is w-contractible and straight, Lemma **??** shows that for each $\tau \in \pi_0(T')$, $\lambda_{S\mathrm{pu}\circ}(T')_\tau$ is simply the image of $T'_\tau \to T' \to S$. As $T' \in T_{\mathrm{proet}}$ and $T \to S$ is flat, $T' \to S$ is flat, and hence $\mathrm{im}(T'_\tau \to T' \to S)$ is a quasi-compact pro-open subscheme of S, and in particular, pro-étale over S. Thus $\lambda_{S^\circ}(T') = \lambda_{S\mathrm{pu}\circ}(T')$ holds componentwise, and hence also globally. □

Theorem 4.2 implies the following analogue of [Sch18, Theorem 16.1].

Corollary 4.5. *Let $f\colon Y' \to Y$ be a map of combs. Consider the diagram of sites*

$$\begin{array}{ccc} Y'_v & \xrightarrow{\lambda_{Y'}} & Y'_{\mathrm{pu}} \\ \downarrow f_v & & \downarrow f_{\mathrm{pu}} \\ Y_v & \xrightarrow{\lambda_Y} & Y_{\mathrm{pu}} \end{array}$$

Let $\mathscr{F}$ be a small torsion sheaf of abelian groups with torsion prime to the characteristic. Then the base change morphism

$$\lambda_Y^* R^i f_{\mathrm{pu}*}\mathscr{F} \xrightarrow{\sim} R^i f_{v*}\lambda_{Y'}^*\mathscr{F}$$

is an isomorphism. Moreover, if $\widetilde{Y} \in Y_v$ is a comb, then the natual map

$$H^i((\lambda_{Y^\circ}(\widetilde{Y}) \times_Y Y')_{\mathrm{pu}}, \mathscr{F}) \to H^i((\widetilde{Y} \times_Y Y')_v, \mathscr{F})$$

is an isomorphism.

Proof. As in the beginning of the proof of [Sch18, Theorem 16.1] it suffices to show the last claim and moreover, for this it suffices to show that whenever $\widetilde{X} \xrightarrow{f} X \xleftarrow{g} X'$ are maps of combs with $\lambda_{X^\circ\mathrm{pu}}(\widetilde{X}) = X$ and $\mathscr{F}$ is an abelian torsion sheaf on X'_{unr} (satisfying the assumption in the corollary), then the natural map $H^i(X'_{\mathrm{unr}}, \mathscr{F}) \to H^i((\widetilde{X} \times_X X')_{\mathrm{unr}}, \mathscr{F})$ is an isomorphism. Write $f'\colon \widetilde{X} \times_X X' \to X'$ and $\widetilde{g}\colon \widetilde{X} \times_X X' \to \widetilde{X}$ for the base changed maps. In the following computation all maps are between the unramified sites. Using Theorem 4.2 for the first equality we have

$$Rf_*R\widetilde{g}_*f'^*\mathscr{F} = Rf_*f^*Rg_*\mathscr{F} = Rg_*\mathscr{F},$$

where the last equality holds by assumption on f and as higher (unramified) cohomology on the comb $\widetilde{X}$ vanishes. This implies the claim by taking global sections. □

4.2. **Category of unramified sheaves on a scheme.** Fix a ring Λ. By Proposition 3.26(iii), for any comb X, the functor λ_X^* is exact on the level of abelian categories; we write $L\lambda_X^* = \lambda_X^*\colon D(X_{\mathrm{pu}}, \Lambda) \to D(X_v, \Lambda)$ for the derived functor. We start with the analogues of [Sch18, Propositions 14.10 and 14.11].

Lemma 4.6. *Let X be a comb. The categories $D(X_{\mathrm{pu}}, \Lambda)$, $D(X_{\mathrm{unr}}, \Lambda)$ are left-complete and the pullback functors*

$$\nu_X^*\colon D(X_{\mathrm{unr}}, \Lambda) \to D(X_{\mathrm{pu}}, \Lambda), \quad \lambda_X^*\nu_X^*\colon D(X_{\mathrm{unr}}, \Lambda) \to D(X_v, \Lambda)$$

are fully faithful. The same statements hold for D^+.

Proof. This follows from Proposition 3.26 and Lemmas 1.37, 1.38 in the same way as [Sch18, Propositions 14.10 and 14.11]. To get the result in the unbounded case, we need to ensure that X_{pu} (resp. X_{unr}) have basis for topology consisting of such U such that $H^i(U, \mathscr{F}) = 0$ for all $i > 0$ and all sheaves $\mathscr{F}$ on X_{pu} (resp. X_{unr}). For X_{pu} we can, by Theorem 1.33, take the w-contractible straight $U \in X_{\mathrm{pu}}$. For X_{unr} we can, by Lemma 3.19, take all objects in $X_{\mathrm{unr}}^{\mathrm{aff}}$. □

From Corollary 4.5 and Lemma 4.6, the analogue of [Sch18, Theorem 14.12] follows with the same proof:

Proposition 4.7. *Let $f\colon Y' \to Y$ be a v-cover of combs. Suppose $A \in D(Y_{\mathrm{pu}}, \Lambda)$ or $A \in D(Y_v, \Lambda)$. If $f^*A \in D(Y'_{\mathrm{unr}}, \Lambda)$ (resp. $f^*A \in D^+(Y'_{\mathrm{unr}}, \Lambda)$), then $A \in D(Y_{\mathrm{unr}}, \Lambda)$ (resp. $f^*A \in D^+(Y_{\mathrm{unr}}, \Lambda)$).*

Definition 4.8. Let Y be a small v-stack. We define $D_{\mathrm{unr}}(Y,\Lambda) \subseteq D(Y_v,\Lambda)$ as the full subcategory consisting of all objects $A \in D(Y_v,\Lambda)$ such that $f^*A \in D(Y_{\mathrm{unr}},\Lambda)$ for all $f\colon X \to Y$ with X comb.

By Proposition 4.7 this definition makes sense, we have $D_{\mathrm{unr}}(Y,\Lambda) = D(Y_{\mathrm{unr}},\Lambda)$ if Y is a comb, and it suffices to test containement in D_{unr} after pullback to a single v-cover by a comb. Also note that Lemma 4.6 implies that for any stack Y, $D_{\mathrm{unr}}(Y,\Lambda)$ is left-complete as isomorphisms may be tested after a v-cover.

Question 4.9. For a general scheme X we have the categories $D_{\mathrm{unr}}(X,\Lambda)$ and $D(X_{\mathrm{unr}},\Lambda)$. The latter seems not to be well-behaved. Still, one might ask about the relation of these categories: is $D_{\mathrm{unr}}(X,\Lambda)$ the left-completion of $D(X_{\mathrm{unr}},\Lambda)$ and can this be realized inside $D(X_{\mathrm{pu}},\Lambda)$? With other words, do the analogues of [BS15, 5.3.2] and [Sch18, 14.15] hold?

Proposition 4.10. *Let Y be a small v-stack and $A \in D(Y_v,\Lambda)$. Then $A \in D_{\mathrm{unr}}(Y,\Lambda)$ if and only for each $i \in \mathbb{Z}$, the v-sheaf $\mathscr{H}^i(A)[0]$ lies in $D_{\mathrm{unr}}(Y,\Lambda)$.*

Proof. Same as the proof of [Sch18, Proposition 14.16]. □

Lemma 4.11. *Let $f\colon Y' \to Y$ be a map of combs. Assume that $n\Lambda = 0$ for some n prime to p. Then for any $A \in D(Y'_{\mathrm{unr}},\Lambda) \subseteq D(Y'_{\mathrm{pu}},\Lambda)$, the base change map $\lambda_Y^* Rf_{\mathrm{pu}*}A \to Rf_{v*}\lambda_{Y'}^*A$ in $D(Y_v,\Lambda)$ is an isomorphism.*

Proof. The proof of [Sch18, Corollary 16.4] applies. □

Just as in [Sch18, 16.6 and 16.7] we can relate unramified and pro-unramified pushforward on combs.

Lemma 4.12. *Let $f\colon Y' \to Y$ be a qcqs morphism of combs.*

(i) Let $\mathscr{F}$ be a sheaf of abelian groups on Y_{pu}. Then the base change map $\nu_Y^* R^i f_{\mathrm{unr}*}A \to Rf^i_{\mathrm{pu}*}\nu_{Y'}^*A$ is an isomorphism for each $i \geq 0$.

(ii) For any $A \in D(Y'_{\mathrm{unr}},\Lambda)$, the base change map $\nu_Y^* Rf_{\mathrm{unr}*}A \to Rf_{\mathrm{pu}*}\nu_{Y'}^*A$ in $D(Y_{\mathrm{pu}},\Lambda)$ is an isomorphism.

4.3. **Four functors on unramified sheaves.** Let $f\colon Y' \to Y$ be a map of v-stacks. As in [Sch18, §17], there is a pullback functor $f_v^*\colon D(Y_v,\Lambda) \to D(Y'_v,\Lambda)$, inducing by restriction a functor

$$f^*\colon D_{\mathrm{unr}}(Y,\Lambda) \to D_{\mathrm{unr}}(Y',\Lambda).$$

By Lurie's adjoint functor theorem, both admit right adjoints, which we denote by Rf_{v*}, Rf_*. If Y, Y' are combs, then f^* and Rf_* identify with the functors f^*_{unr} and $Rf_{\mathrm{unr}*}$ between $D(Y_{\mathrm{unr}},\Lambda)$ and $D(Y'_{\mathrm{unr}},\Lambda)$.

Proposition 4.13 (Analogue of [Sch18], Proposition 17.6)**.** *Let $f\colon Y' \to Y$ be a qcqs map of small v-stacks. Then for any $A \in D^+_{\mathrm{unr}}(Y',\Lambda)$, one has $Rf_{v*}A \in D^+_{\mathrm{unr}}(Y,\Lambda)$ and therefore $Rf_*A = Rf_{v*}A$. Moreover, for such A, the formation of Rf_*A commutes with arbitrary base changes.*

If Rf_* has finite cohomological dimension, then the above claims hold with D^+ replaced by D.

Proof. We may assume that Y is a comb and choose a hypercover $X'_\bullet \to Y'$ (over Y) such that each X_n is a comb. Let $g'_\bullet\colon X'_\bullet \to Y$ denote the resulting map. Let $A \in D^+_{\mathrm{unr}}(Y',\Lambda)$. Then $Rf_{v*}A$ is the limit of the simplicial object $Rg'_{\bullet v*}A|'_{X_\bullet}$. By Lemmas 4.11 and 4.12, for each $i \geq 0$, $Rg'_{iv*}A|'_{X_i} \in D^+(Y_{\mathrm{unr}},\Lambda)$ and there is some $n \geq 0$, such that all of them lie in $D^{\geq -n}(Y_{\mathrm{unr}},\Lambda)$. Thus also their derived limit, $Rf_{v*}A$, lies in $D^+(Y_{\mathrm{unr}},\Lambda)$. The claim $Rf_*A = Rf_{v*}A$ follows from it (cf. [Sch18, §17]). Commutation with base change follows formally as v-pushforward commutes with v-slices.

The last statement follows from the previous ones exploiting left-completeness and Proposition 4.10 in the same way as in the proof of [Sch18, Proposition 17.6]. □

Lemma 4.14. *Let Y be a v-stack.*

(i) The tensor product $-\otimes^{\mathbb{L}}_{\Lambda}-\colon D(Y_v,\Lambda)\times D(Y_v,\Lambda)\to D(Y_v,\Lambda)$ restricts to a functor

$$-\otimes^{\mathbb{L}}_{\Lambda}-\colon D_{\mathrm{unr}}(Y,\Lambda)\times D_{\mathrm{unr}}(Y,\Lambda)\to D_{\mathrm{unr}}(Y,\Lambda)$$

(ii) For any $A\in D_{\mathrm{unr}}(Y,\Lambda)$, the functor $-\otimes^{\mathbb{L}}_{\Lambda}A\colon D_{\mathrm{unr}}(Y,\Lambda)\to D_{\mathrm{unr}}(Y,\Lambda)$ admits a right adjoint $R\,\underline{\mathrm{Hom}}(A,-)\colon D_{\mathrm{unr}}(Y,\Lambda)\to D_{\mathrm{unr}}(Y,\Lambda)$. For varying A, this defines a functor

$$R\,\underline{\mathrm{Hom}}_{\Lambda}(-,-)\colon D_{\mathrm{unr}}(Y,\Lambda)^{\mathrm{op}}\times D_{\mathrm{unr}}(Y,\Lambda)\to D_{\mathrm{unr}}(Y,\Lambda).$$

Proof. (i): Note that when Y is a comb, $D_{\mathrm{unr}}(Y,\Lambda)$ is equipped with a natural tensor product, commuting with pullback along $Y_v\to Y_{\mathrm{unr}}$. Then the claim follows as the containment in $D_{\mathrm{unr}}(Y,\Lambda)$ can be checked v-locally. (ii): The proof is formal and the same as in [Sch18, Lemma 17.8]. □

Note that, just as in [Sch18], the inner hom is not the same as the restriction of the inner hom of $D(Y_v,\Lambda)$. Purely formal we have the following property.

Lemma 4.15. *Let $Y'\to Y$ be a map of small v-stacks. Then there is a natural equivalence*

$$Rf_*R\,\underline{\mathrm{Hom}}_{\Lambda}(f^*A,B)\cong R\,\underline{\mathrm{Hom}}_{\Lambda}(A,Rf_*B)$$

of functors $D_{\mathrm{unr}}(Y,\Lambda)^{\mathrm{op}}\times D_{\mathrm{unr}}(Y,\Lambda)\to D_{\mathrm{unr}}(Y,\Lambda)$

Proof. The same as in [Sch18, Corollary 17.9]. □

Question 4.16. Suppose that $n\Lambda=0$ for some n coprime to p. Does the assignment $X\mapsto D_{\mathrm{unr}}(X,\Lambda)$ define a 6-functor formalism on v-stacks, staisfying the usual properties?

To prove this, one would have to establish the proper base change for unramified cohomology. (Conversely: would PBC suffice?)

4.4. **Relation of etale and unramified cohomology.** Suppose that $n\Lambda=0$ for some n coprime to p. For any v-stack X, $D_{\acute{e}t}(X,\Lambda)$ and $D_{\mathrm{unr}}(X,\Lambda)$ are both full subcategories of $D(X_v,\Lambda)$ consisting of complexes, which a v-cover by a(ny) comb become étale resp. unramified. Note that $D_{\acute{e}t}(X,\Lambda)$ is contained in $D_{\mathrm{unr}}(X,\Lambda)$, as this is obviously true for any comb. As both are by definition full subcategories of $D(X_v,\Lambda)$, the embedding

$$\mu_X\colon D_{\acute{e}t}(X,\Lambda)\to D_{\mathrm{unr}}(X,\Lambda)$$

is fully faithfur. With other words, the canonical map $\mathrm{id}\to R\mu_{X*}\mu_X^*\colon D_{\acute{e}t}(X,\Lambda)\to D_{\acute{e}t}(X,\Lambda)$ is an equivalence.

We deduce that unramified cohomology agrees with étale cohomology on étale complexes.

Lemma 4.17. *Let X be any v-stack. For any $A\in D_{\acute{e}t}(X,\Lambda)$, we have $R\Gamma_{\acute{e}t}(X,A)\cong R\Gamma_{\mathrm{unr}}(X,\mu_X^*A)$.*

Proof. We have $R\Gamma_{\mathrm{unr}}(X,\mu^*A)=\mathrm{RHom}(\Lambda,\mu_X^*A)=\mathrm{RHom}(\Lambda,A)=R\Gamma_{\acute{e}t}(X,A)$, where we use that $D_{\acute{e}t}(X,\Lambda)\subseteq D_{\mathrm{unr}}(X,\Lambda)$ is a fully faithful embedding. □

5. Solid Sheaves on Schemes

In the previous section we have constructed the pro-unramified site X_{pu} associated to a scheme X and established many of its properties. We will now study the category of abelian sheaves on X_{pu} more closely, which will eventually lead to the definition of solid sheaves on X_{pu}. We will later use these results mostly in the case that X is a comb, but all the results of this section hold (with essentially the same proofs) on more general schemes. We therefore state everything in its natural generality.

This section is structured as follows.

5.1. **Cohomology and Colimits.** In the following we will often let τ be either the étale site $X_{\text{ét}}$ or the unramified site X_{unr} of a scheme X. In either case we denote by $p\tau$ the associated pro-site, i.e. X_{proet} or X_{pu} respectively. In [BS15, Corollary 5.1.6] it was shown that for a pro-étale $U = \varprojlim_i U_i \in X_{\mathrm{proet}}^{\mathrm{aff}}$ and any étale sheaf $\mathscr{F}$ on X we have $H^k(U, \mathscr{F}) = \varinjlim_i H^k(U_i, \mathscr{F})$. In the following we will generalize this result by allowing more general limits for $U = \varprojlim_i U_i$ and by showing the same result also for the unramified site. We start with the following observation:

Like in the pro-étale setting we get a derived version of Lemma 1.37. The following analog of [Sta14, Theorem 09YQ] holds (cf. [Sta14, Definition 0EZL] for the notion of systems of sheaves on $(X_i)_i$).

Proposition 5.1. *Let $(X_i)_{i\in I}$ be a cofiltered inverse system of qcqs schemes with affine transition maps and let $\tau \in \{\text{ét}, \mathrm{unr}\}$. Let $(\mathscr{F}_i)_i$ be a system of τ-sheaves on $(X_i)_i$. Denote $X = \varprojlim_i X_i$ and $\mathscr{F} = \varinjlim_i f_i^* \mathscr{F}_i$, where $f_i \colon X \to X_i$ is the projection. Then the natural map*

$$\varinjlim_i H^n(X_{i,\tau}, \mathscr{F}_i) \xrightarrow{\sim} H^n(X_\tau, \mathscr{F})$$

is an isomorphism for $n = 0$, resp. $n = 0, 1$, resp. all n, if the $\mathscr{F}_i$ are sheaves of sets, resp. sheaves of groups, resp. sheaves of abelian groups.

Proof. We only prove the abelian case in detail. This proof is the same as that of [Sta14, Theorem 09YQ]: For any qcqs scheme Y let $Y_{\tau,\mathrm{qcqs}}$ denote the subsite of qcqs schemes over Y. Then all cohomologies can be computed on the sites $X_{\tau,\mathrm{qcqs}}$ and $X_{i,\tau,\mathrm{qcqs}}$. By [Sta14, Lemma 09YP] the claim reduces to showing that the natural functor of sites

$$\varinjlim_i X_{i,\tau,\mathrm{qcqs}} \xrightarrow{\sim} X_{\tau,\mathrm{qcqs}}$$

is an equivalence. By [Sta14, Lemma 09YL] the colimit of sites on the left-hand side is computed as follows: The underlying category is the colimit of categories and the site is the coarsest site such that all the induced maps $X_{i,\tau,\mathrm{qcqs}} \to X_{\tau,\mathrm{qcqs}}$ are continuous (in the sense of [Sta14, Definition 00WV]). To show the claimed equivalence, let us first show that it is true on underlying categories. By [Sta14, Lemma 01ZM] this reduces to showing the following claim: Pick some $i_0 \in I$ and some qcqs map $U_{i_0} \to X_{i_0}$ with pullbacks $U \to X$ and $U_i \to X_i$ for $i \geq i_0$ and assume that $U \in X_\tau$; then there is some $i \geq i_0$ such that $U_i \in X_{i,\tau}$. In the case $\tau = \text{ét}$ this follows immediately from [Sta14, Lemma 07RP]. In the case $\tau = \mathrm{unr}$ we employ Lemma 1.35 to get a G-unramified v-cover $V \twoheadrightarrow U$ such that the composition $V \to X$ is G-unramified. Then by [Sta14, Lemma 01ZM] the map $V \to U$ comes via pullback from a map $V_i \to U_i$ for some $i \geq i_0$, which by [Ryd10, Theorem 6.4] becomes a v-cover after possibly enlarging i and by [Sta14, Lemma 0C4W] becomes G-unramified after enlarging i. By [Sta14, Lemma 0C4W] again, we can further assume that the composition $V_i \to U_i \to X$ is G-unramified, because $V \to X$ is so. But then Lemma 1.35 implies that $U_i \to X_i$ is in $X_{i,\mathrm{unr},\mathrm{qcqs}}$, as desired. This finishes the proof that the above claimed equivalence holds on the underlying categories. It follows easily from [Sta14, Lemma 07RR] (in the case $\tau = \text{ét}$) and [Ryd10, Theorem 6.4] (in the case $\tau = \text{unr}$) that the equivalence also holds on the level of sites. □

For later use we also record the following relative version of Proposition 5.1, in analogy with [Sta14, Lemma 0EYM]:

Corollary 5.2. *Let $(f_i\colon X_i \to S_i)_{i\in I}$ be a cofiltered diagram of qcqs maps of schemes with all transition maps being affine. Let $f\colon X \to S$ denote the limit of the diagram and for all $i \in I$ let $g_i\colon X \to X_i$ and $h_i\colon S \to S_i$ denote the projections. For fixed $\tau \in \{\text{ét}, \text{unr}\}$ let $(\mathscr{F}_i)_i$ be a system of τ-sheaves on $(X_i)_i$ with colimit $\mathscr{F} = \varinjlim_i g_i^* \mathscr{F}_i$. Then the natural map*

$$\varinjlim_i h_i^* R^n f_{i,*}\mathscr{F}_i \xrightarrow{\sim} R^n f_* \mathscr{F}.$$

is an isomorphism of sheaves on S_τ for $n = 0$, resp. $n = 0, 1$, resp. all n, if the $\mathscr{F}_i$ are sheaves of sets, resp. sheaves of groups, resp. sheaves of abelian groups.

Proof. We will only treat the case of abelian sheaves in detail. One can directly implement the proof of [Sta14, Lemma 0EYM], but we prefer the following alternative route: It follows formally (by inducting over finite (co)limits in derived ∞-categories and countable colimits over truncations) that Proposition 5.1 holds more generally in the case that $(\mathscr{F}_i)_i$ is a system of sheaves in $\mathcal{D}(\text{Ab}(X_{i,\text{unr}}))$ which are uniformly left-bounded in the sense that $H^k(\mathscr{F}_i) = 0$ for $k \ll 0$ independent of i. Applying this to the system $(Rf_{i,*}\mathscr{F}_i)_i$ we deduce that

$$\begin{aligned} R\Gamma(S, \varinjlim_i h_i^* Rf_{i,*}\mathscr{F}_i) &= \varinjlim_i R\Gamma(S_i, Rf_{i,*}\mathscr{F}_i) = \varinjlim_i R\Gamma(X_i, \mathscr{F}_i) = R\Gamma(X, \mathscr{F}) \\ &= R\Gamma(S, Rf_*\mathscr{F}), \end{aligned}$$

i.e. the claimed isomorphism of (derived) sheaves holds on global sections. But given any $U \in S_\tau$, we know (e.g. by the proof of Proposition 5.1) that U comes via base-change from some $U_i \in S_{i,\tau}$. Note that the claim is stable under the base-change along $U_i \to S_i$, so we can apply the same argument to U in place of S to also obtain the equivalence of sections on U. □

We now prove a different incarnation of Proposition 5.1 where all the X_i's are replaced by *arbitrary* qcqs sheaves on the pro-étale (resp. pro-unramified) site of some base scheme. The idea is to reduce this result to the case that all X_i's are representable by schemes by constructing a functorial resolution of the qcqs sheaves in terms of qcqs representable sheaves. This is enabled by the following lemmas.

Lemma 5.3. *Let X be a qcqs scheme, $\tau \in \{\text{ét}, \text{unr}\}$ and let $(U_i)_{i\in I}$ be a (small) diagram of qcqs sheaves on $X_{p\tau}$. Then $U := \varprojlim_i U_i$ is qcqs.*

Proof. Every limit can be constructed from fiber products and products, so it is enough to handle these two cases separately. For fiber products the claim is true in any algebraic topos (see [AGV71, VI.2.2]). We are left with the case of a product $U = \prod_i U_i$. Choose a well-ordering on I and replace U_i by $U_i' := \prod_{j<i} U_j$. Arguing by transfinite induction (and using that finite products of qcqs objects are qcqs by the fiber product case; here we need that X is qcqs) we reduce the claim to showing that a cofiltered limit $\varprojlim_{i\in I} U_i$ of qcqs sheaves is qcqs.,, Then by transfinite induction one can easily construct a diagram $(V_i \to U_i)_i$ of sheaves on $X_{p\tau}$ such that each V_i is representable by an object in $X_{p\tau}^{\text{aff}}$ and for every $i \in I$ the map

$$V_i \twoheadrightarrow \varprojlim_{j<i} V_j \times_{(\varprojlim_{j<i} U_j)} U_i$$

is surjective. It follows easily that the map

$$V := \varprojlim_i V_i \to U$$

is surjective (surjectivity of sheaves on $X_{p\tau}$ can be checked on weakly contractible objects, so the claim reduces to a surjectivity of maps of sets, which is easily verified). On the other hand we have $V \in X_{p\tau}^{\mathrm{aff}}$ so that V is qcqs. It follows that U is quasicompact. One argues similarly to see that $V \times_U V = \varprojlim_i V_i \times_{U_i} V_i$ is quasicompact, proving that U is quasiseparated. □

Lemma 5.4. *Let X be a scheme, $\tau \in \{\text{ét}, \text{unr}\}$, I a directed set and $(U_i)_{i \in I}$ a diagram of qcqs sheaves on $X_{p\tau}$. Then $U := \varprojlim_i U_i$ is qcqs and there is a cofinal subset $I' \subset I$ and a diagram $(V_i \to U_i)_{i \in I'}$ of sheaves on $X_{p\tau}$ with the following properties:*

(i) Each V_i is representable by a weakly contractible object in $X_{p\tau}^{\mathrm{aff}}$.
(ii) All the maps $V_i \to U_i$ and the map $V := \varprojlim_i V_i \to U$ are surjective.

Proof. We can assume that I has a final object $0 \in I$. Pick any cover $X' \twoheadrightarrow U_0$ by some qcqs scheme $X' \in X_{p\tau}$ and replace U_i by $U_i' := U_i \times_{U_0} X'$. It is enough to prove the claim for the system U_i' of pro-τ sheaves on X': Then the surjectivity of $U' := \varprojlim_i U_i' = U \times_{U_0} X' \to U$ implies that U is quasicompact and by the same argument applied to $U_i' \times_{U_i} U_i'$ one obtains that $U' \times_U U'$ is quasicompact, hence U is qcqs. Moreover the diagram $(V_i \to U_i')_i$ induces the diagram $(V_i \to U_i)_i$. Thus we can replace X by X' to assume that X is qcqs from now on.

Any limit of qcqs sheaves on $X_{p\tau}$ is qcqs by Lemma 5.3, so in particular U is qcqs. Let $\mathscr{X}$ be the set of pairs $(J, (V_j \to U_j)_{j \in J})$ where $J \subset I$ is a subset and $(V_j \to U_j)_{j \in J}$ is a diagram of maps of sheaves on $X_{p\tau}$ such that the following properties are satisfied:

(a) Each V_j is representable by a weakly contractible object in $X_{p\tau}^{\mathrm{aff}}$.
(b) For every pair $J'' \subset J' \subset J$ of subsets of J the map

$$V_{J'} \twoheadrightarrow V_{J''} \times_{U_{J''}} U_{J'}$$

is surjective.

We put a partial order on $\mathscr{X}$ by letting $(J, (V_j \to U_j)_{j \in J}) \le (J', (V_j' \to U_j)_{j \in J'})$ if $J \subset J'$, no element of $J' \setminus J$ is less than an element of J, and $V_j = V_j'$ for all $j \in J$.

Let $S \subset \mathscr{X}$ be a totally ordered subset. We claim that S has an upper bound in $\mathscr{X}$. Indeed, the underlying full subcategory $J \subset I$ of that upper bound will be the union of the full subcategories $J_s \subset I$ of the objects $s \in S$. There is then an obvious (and unique) choice of the diagram $(V_j \to U_j)_{j \in J}$ extending the diagrams on all $s \in S$. It is clear that the obtained pair $(J, (V_j \to U_j)_{j \in J})$ satisfies condition (a). To verify condition (b) we can pass to a cofinal subset of S and hence assume that S is well-ordered. Let $J'' \subset J' \subset J$ be given. By possibly enlarging J'' we can assume that no element of $J' \setminus J''$ is less than an element of J''. For $s \in S$ let $J_s' := J' \cap J_s$ and $J_{<s}' := J' \cap \bigcup_{t<s} J_t$ and similarly for J_s'' and $J_{<s}''$. By definition of $s > t$, no element of $J_s \setminus J_t$ is less than an element of J_t and by the assumption on J'' and J' no element of $J_t' \setminus J_t''$ is less than an element of J_s''; together this implies that there are no relations between elements of $J_{<s}' \setminus J_{<s}''$ and elements of $J_s'' \setminus J_{<s}''$ and hence

$$V_{J_{<s}' \cup J_s''} = V_{J_{<s}'} \times_{V_{J_{<s}''}} V_{J_s''}.$$

Now use the surjectivity of

$$V_{J_s'} \twoheadrightarrow V_{J_{<s}' \cup J_s''} \times_{(U_{J_{<s}' \cup J_s''})} U_{J_s'}$$

to inductively prove the desired surjectivity of $V_{J'} \twoheadrightarrow V_{J''} \times_{U_{J''}} U_{J'}$.

By Zorn's lemma there is a maximal element $(I', (V_i \to U_i)_{i \in I'})$ in $\mathscr{X}$. We claim that $I' \subset I$ is cofinal. If this is not the case then there is some $i_0 \in I \setminus I'$ which is not smaller than any element of I'. Consider the subsets $J := \{i_0\} \cup I' \subset I$ and $J' := \{i \in I' \mid i \leq i_0\}$. Let $V_{i_0} \in X_{p\tau}^{\text{aff}}$ be any weakly contractible cover of $U_{i_0} \times_{U_{J'}} V_{J'}$ (this fiber product is qcqs by Lemma 5.3). We claim that the thus produced pair $(J, (V_j \to U_j)_{j \in J})$ satisfies conditions (a) and (b) above and hence contradicts the maximality of I'. It is clear that the pair satisfies (a). Condition (b) follows easily from the fact that surjectivity is preserved by base-change (note that e.g. $U_J = U_{I'} \times_{U_{J'}} U_{i_0}$). □

We are finally in the position to prove the promised commutation of cohomology and limits of qcqs sheaves:

Proposition 5.5. *Let X be a scheme, $\tau \in \{\text{ét}, \text{unr}\}$, $(U_i)_{i \in I}$ a cofiltered system of qcqs sheaves on $X_{p\tau}$ and $\mathscr{F}$ a sheaf on X_τ (viewed as a sheaf on $X_{p\tau}$ via the fully faithful embedding). Then $U := \varprojlim_i U_i$ is qcqs and the natural map*

$$\varinjlim_i H^n(U_i, \mathscr{F}) \xrightarrow{\sim} H^n(\varprojlim_i U_i, \mathscr{F})$$

is an isomorphism for $n = 0$, resp. $n = 0, 1$, resp. all n, if $\mathscr{F}$ is a sheaf of sets, resp. a sheaf of groups, resp. a sheaf of abelian groups.

Proof. We only handle the case that $\mathscr{F}$ is a sheaf of abelian groups; the other cases are similar (but easier). By Lemma 5.4 U is qcqs and we can inductively find a cofinal subset I' and a diagram $(V_{i,\bullet} \to U_i)_{i \in I'}$ such that each $V_{i,\bullet} \to U_i$ is a truncated hypercover by weakly contractible objects $V_{i,k} \in X_{p\tau}^{\text{aff}}$ and such that $V_\bullet := \varprojlim_i V_{i,\bullet} \to U$ is still a truncated hypercover. Then $\mathscr{F}$ is acyclic on each $V_{i,k}$ and hence also on each V_k by Proposition 5.1. Thus by the Čech-to-sheaf-cohomology spectral sequence (see e.g. [Sta14, Lemma 01GY]) the cohomologies $H^n(U_i, \mathscr{F})$ and $H^n(U, \mathscr{F})$ are computed as the Čech cohomology with respect to the hypercovers $V_{i,\bullet} \to U_i$ and $V_\bullet \to U$ (for n small enough, depending on the truncation of our hypercovers). Thus the claimed isomorphism follows from the fact that filtered colimits are exact. □

5.2. **Relative Pontrjagin Duality.** The Pontrjagin duality functor is the functor

$$A \mapsto \mathbb{D}(A) := \underline{\mathrm{Hom}}(A, \mathbb{R}/\mathbb{Z})$$

on the category of locally compact abelian groups A, where $\underline{\mathrm{Hom}}$ denotes continuous group homomorphisms equipped with the compact-open topology. It is a classical result that $\mathbb{D}$ is contravariant autoduality, so in particular the natural map $A \to \mathbb{D}(\mathbb{D}(A))$ is an isomorphism for all locally compact A. Via restriction this gives an equivalence of the category of discrete abelian groups to the opposite of the category of compact abelian groups. We will now formulate and prove a version of this result in the relative setting, i.e. for groups over some fixed compact Hausdorff space X.

In the following we will often speak of the étale site $X_{\text{ét}}$ and the pro-étale site X_{proet} of a profinite set X. There are obvious definitions for that (e.g. X_{proet} consists of pro-étale sets over X with covers being finite jointly surjective families of maps). One can also view $X_{\text{ét}}$ and X_{proet} as the corresponding sites of the scheme $\operatorname{Spec} \mathcal{C}(X, k)$ for any algebraically closed field k, which immediately implies that all the results for the pro-étale site of schemes are valid for X_{proet} as well.

Definition 5.6. Let X be a compact Hausdorff space, viewed as a qcqs condensed set (cf. [Sch19, Theorem 2.16.(i)]). A *condensed abelian group over X* is a commutative group object A in the category of condensed sets over X, i.e. it is a condensed set A with a map $A \to X$ together with a multiplication map $A \times_X A \to A$, an inversion map $A \to A$ and a neutral element map $X \to A$ (all maps are over X).

Definition 5.7. Let X be a compact Hausdorff space and $A \to X$ a condensed abelian group over X.

(a) A is called *compact (over X)* if A is qcqs as a condensed set (equivalently the map $A \to X$ is qcqs).
(b) A is called *discrete over* X if the pullback of A to every profinite set $Y \to X$ is an étale sheaf on Y (i.e. lies in the image of the functor $\mathrm{Shv}(Y_{\text{ét}}) \hookrightarrow \mathrm{Shv}(Y_{\text{proet}})$).

Definition 5.8. Let X be a compact Hausdorff space and $A \to X$ a condensed abelian group over X. The *relative Pontrjagin dual of* A is the condensed abelian group over X defined as

$$\mathbb{D}_X(A) := \underline{\mathrm{Hom}}_X(A, \mathbb{R}/\mathbb{Z})$$

Here, $\mathbb{R}/\mathbb{Z}$ is implicitly viewed as the relative condensed abelian group $\mathbb{R}/\mathbb{Z} \times X \to X$ and the $\underline{\mathrm{Hom}}_X$ denotes the internal Hom in the category of abelian sheaves on the slice topos $(*_{\text{proet}})_{/X}$.

Lemma 5.9. *Let X be a compact Hausdorff space and $A \to X$ a compact abelian group over X. Then*

$$R\,\underline{\mathrm{Hom}}_X(A, \mathbb{R}) = 0$$

Proof. We argue as in the proof of [Sch19, Theorem 4.3.(ii)]. Namely, by the Breen-Deligne resolution [Sch19, Theorem 4.5] there is a resolution $F(A)_\bullet \to A \to 0$ of A of the form $F(A)_i = \bigoplus_{j=1}^{n_i} \mathbb{Z}_X[A^{r_{i,j}}]$, where for any map $j\colon Y \to X$ of sheaves we denote $\mathbb{Z}_X[Y] := j_!\mathbb{Z}$ and for any integer $r \geq 0$, A^r denotes the r-fold fiber product of A over X. By the vanishing of $\mathbb{R}$-cohomology on compact Hausdorff spaces (see [Sch19, Theorem 3.3]), for every compact Hausdorff space $X' \to X$, $R\,\underline{\mathrm{Hom}}_X(A, \mathbb{R})(X')$ is computed by the complex

$$0 \to \bigoplus_{j=0}^{n_0} \mathcal{C}(A^{r_{0,j}} \times_X X', \mathbb{R}) \to \bigoplus_{j=0}^{n_1} \mathcal{C}(A^{r_{1,j}} \times_X X', \mathbb{R}) \to \dots$$

Since all $A^{r_{0,j}} \times_X X'$ are qcqs (and hence compact Hausdorff) the same argument as in the proof of [Sch19, Theorem 4.3.(ii)] shows that the above complex of Banach spaces is acyclic, as desired. □

Lemma 5.10. *Let X be a compact Hausdorff space and $A \to X$ a condensed abelian group over X.*

(i) If A is discrete over X then $\mathbb{D}_X(A)$ is compact over X.
(ii) If A is compact over X then $\mathbb{D}_X(A)$ is discrete over X.

Proof. Pulling back to any cover $X' \twoheadrightarrow X$ by some profinite set X', we can assume that X itself is profinite.

We first prove (i), so assume that A is discrete over X, i.e. given by a sheaf on $X_{\text{ét}}$. The site $X_{\text{ét}}$ is generated by maps $j\colon U \to X$ such that U is a disjoint union of clopen subsets of X. Write $\mathbb{Z}[U] := j_! Z$; then the shape of U shows that $\mathbb{Z}[U]$ is a direct sum of direct summands of the sheaf $\mathbb{Z}$ on $X_{\text{ét}}$. By general properties of sites there is a resolution

$$\bigoplus_{j \in J} \mathbb{Z}[V_j] \to \bigoplus_{i \in I} \mathbb{Z}[U_i] \to A \to 0$$

of sheaves on $X_{\text{ét}}$, with V_j and U_i of the same shape as U. Thus we get an exact sequence

$$0 \to \mathbb{D}_X(A) \to \prod_{i \in I} \mathbb{D}_X(\mathbb{Z}[U_i]) \to \prod_{j \in J} \mathbb{D}_X(\mathbb{Z}[V_j]).$$

Letting $j_i\colon U_i \to X$ denote the structure map we get by adjunction $\mathbb{D}_X(\mathbb{Z}[U_i]) = j_{i*}\mathbb{R}/\mathbb{Z}$, which as above is a direct sum of direct summands of the sheaf $\mathbb{R}/\mathbb{Z}$ on X_{proet} and hence is qcqs. It follows that $\mathbb{D}_X(A)$ is a limit of qcqs sheaves on X_{proet} and hence qcqs by Lemma 5.3, as desired.

We now prove (ii), so assume that A is qcqs. Applying $R\,\underline{\mathrm{Hom}}_X(A,-)$ to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ on X_{proet} and using Lemma 5.9 we get an isomorphism

$$\mathbb{D}_X(A) = \underline{\mathrm{Hom}}_X(A, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} \underline{\mathrm{Ext}}^1_X(A, \mathbb{Z}).$$

For every $U \in X_{\mathrm{proet}}$, the Breen-Deligne resolution (cf. [Sch19, Theorem 4.5]) produces a sequence

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times_X U, \mathbb{Z}) \Rightarrow \underline{\mathrm{Ext}}^{p+q}(A, \mathbb{Z})(U),$$

where A^r denotes the r-fold fiber product of A over X. To see that $\underline{\mathrm{Ext}}^1(A,\mathbb{Z})$ is étale we need to verify that for every $U = \varprojlim_i U_i \in X_{\mathrm{proet}}$ with all $U_i \in X_{\text{ét}}$ the canonical map $\varinjlim_i \underline{\mathrm{Ext}}^1(A,\mathbb{Z})(U_i) \xrightarrow{\sim} \underline{\mathrm{Ext}}^1(A,\mathbb{Z})(U)$ is an isomorphism. But by the above spectral sequence this amounts to showing that the map $\varinjlim_i H^q(A^r \times_X U_i, \mathbb{Z}) \xrightarrow{\sim} H^q(A^r \times_X U, \mathbb{Z})$ is an isomorphism. But this is Proposition 5.5. □

Lemma 5.11. *Let X be a compact Hausdorff space and let $f\colon A \to A'$ be a map of condensed abelian groups over X. Suppose that A and A' are both discrete or both compact and for every point $x \in X$ the fiber $f_x\colon A_x \to A'_x$ is an isomorphism. Then f is an isomorphism.*

Proof. We can w.l.o.g. assume that X is profinite. If A and A' are compact then they are representable by compact Hausdorff spaces and f is a continuous bijection, hence a homeomorphism. If A and A' are discrete and hence étale sheaves on X then A_x and A'_x are the stalks of A and A' at x (e.g. write $x = \varprojlim_i U_i$ for a system of open neighbourhoods U_i of x) and the claim is also clear. □

Theorem 5.12. *Let X be a compact Hausdorff space. Then for every compact or discrete abelian group A over X the canonical map*

$$A \xrightarrow{\sim} \mathbb{D}_X(\mathbb{D}_X(A))$$

is an isomorphism. In particular $\mathbb{D}_X$ is an equivalence of categories

$$\{\textit{discrete abelian groups over } X\} \xleftrightarrow{\sim} \{\textit{compact abelian groups over } X\}^{\mathrm{op}}$$

Proof. Given a discrete or compact group A over X, by Lemma 5.10 the map $f\colon A \to \mathbb{D}_X(\mathbb{D}_X(A))$ is a map of both discrete or both compact abelian groups over X. Thus by Lemma 5.11, to see that f is an isomorphism it is enough to check this on all fibers f_x. But then the result follows from standard Pontrjagin duality (see e.g. [Sch19, Theorem 4.1.(iii)]). To get the equivalence of categories, it is only left to check full faithfulness of $\mathbb{D}_X$; but this is formal using the adjointness of tensor and $\underline{\mathrm{Hom}}$:

$$\begin{aligned}
\mathrm{Hom}_X(\mathbb{D}_X(A), \mathbb{D}_X(A')) &= \mathrm{Hom}_X(\mathbb{D}_X(A), \underline{\mathrm{Hom}}_X(A', \mathbb{R}/\mathbb{Z})) \\
&= \mathrm{Hom}_X(A', \underline{\mathrm{Hom}}_X(\mathbb{D}_X(A), \mathbb{R}/\mathbb{Z})) \\
&= \mathrm{Hom}_X(A', \mathbb{D}_X(\mathbb{D}_X(A))) \\
&= \mathrm{Hom}_X(A', A).
\end{aligned}$$

□

5.3. **Limits and Colimits of Abelian Sheaves.** The previous subsections culminate in the following results about the behavior of abelian sheaves on the pro-étale and the pro-unramified site. These results can be seen as generalizations of similar results in [Sch19] and are thus crucial for setting up the theory of solid sheaves.

We start with the vanishing of higher inverse limits of qcqs sheaves on the pro-étale and the pro-unramified site:

Proposition 5.13. *Let X be a qcqs scheme and let $(\mathscr{K}_i)_{i\in I}$ be a cofiltered inverse system of qcqs pro-étale sheaves of abelian groups on X. Then for all $n > 0$ the higher inverse limit*

$$R^n \varprojlim_i \mathscr{K}_i = 0,$$

in the category of pro-étale sheaves on X, vanishes. If X is straight and geometrically unibranch then the same holds for cofiltered inverse limits of qcqs pro-unramified sheaves.

Remark 5.14. In [FS, Proposition VII.1.6] the analogous result for diamonds only stated for étale constructible sheaves $\mathcal{K}_i$. This is enough to set-up the theory of solid sheaves, but with a little more effort (namely the relative Pontrjagin duality, Theorem 5.12) we get the result for all qcqs sheaves.

Proof of Proposition 5.13. We first reduce the case of pro-unramified sheaves to the case of pro-étale sheaves. By Theorem 1.33 it suffices to show that for any w-contractible straight $U \in X_{\mathrm{pu}}^{\mathrm{aff}}$, we have $\Gamma(U_{\mathrm{pu}}, R^n \varprojlim_i \mathscr{K}_i) = 0$. We have $R\Gamma(U_{\mathrm{proet}}, -) = \Gamma(U_{\mathrm{proet}}, -)$ and (by Theorem 1.33) also $R\Gamma(U_{\mathrm{pu}}, -) = \Gamma(U_{\mathrm{pu}}, -)$. By Theorem 1.40 we also have $R\mu_* = \mu_*$. As $R\varprojlim$ commutes with $R\mu_*$, we thus have

$$\Gamma(U_{\mathrm{pu}}, R\varprojlim_i \mathscr{K}_i) = \Gamma(U_{\mathrm{proet}}, \mu_* R\varprojlim_i \mathscr{K}_i) = \Gamma(U_{\mathrm{proet}}, R\varprojlim_i \mu_*\mathscr{K}_i).$$

Now, $\mu_*\mathscr{K}_i$ is again qcqs by Lemma 1.47 and so the claim about pro-unramified sheaves reduces to the claim about pro-étale sheaves. In the case of pro-étale sheaves, the proof goes as in [FS, VII.1.6]. It is enough to show that for any w-contractible $U \in X_{\mathrm{proet}}$ we have $\Gamma(U_{\mathrm{proet}}, R^n \varprojlim_i \mathscr{K}_i) = 0$. Since $\Gamma(U, -)$ is limit-preserving (always) and exact (as U is w-contractible), we can pull out $R\varprojlim_i$, reducing the claim to showing that

$$R^n \lim_i \mathscr{K}_i(U) = 0.$$

Now we have $\mathscr{K}_i(U) = \mathscr{K}_i(U^c)$ by Proposition 2.19(ii). Replacing X by U^c we are reduced to the case that X is the spectrum of an absolutely flat ring. Now $|X|$ is extremally disconnected set. In particular any qc open of $|X|$ is clopen, hence still extremally disconnected. It follows that every qcqs étale map $V \to X$ is a Zariski localization and thus the site X_{proet} is equivalent to the site $|X|_{\mathrm{proet}}$ of profinite sets over $|X|$. Thus, we are reduced to the case of a profinite set $|X|$.

By Theorem 5.12 we have $\mathscr{K}_i = \mathbb{D}_X(\mathscr{F}_i)$ for some discrete abelian groups $\mathscr{F}_i = \mathbb{D}_X(\mathscr{K}_i)$ over X, i.e., $\mathscr{F}_i$ are sheaves on X_{et}, or equivalently, on $|X|$. Now $\mathbb{R}/\mathbb{Z}$ is an injective sheaf on X by [FS, VII.1.7], so in particular, $\mathscr{K}_i = \mathrm{RHom}(\mathscr{F}_i, \mathbb{R}/\mathbb{Z})$, and then

$$\begin{aligned} R\varprojlim_i \mathscr{K}_i(X) &= R\varprojlim_i \mathrm{RHom}(\mathscr{F}_i, \mathbb{R}/\mathbb{Z}) = \mathrm{RHom}(\varinjlim_i \mathscr{F}_i, \mathbb{R}/\mathbb{Z}) \\ &= \mathrm{Hom}(\varinjlim_i \mathscr{F}_i, \mathbb{R}/\mathbb{Z}). \end{aligned}$$

Thus, the higher cohomologies of $R\varprojlim_i \mathscr{K}_i(X)$ vanish, as desired. ∎

Proposition 5.15. *Let X be a qcqs scheme.*

(i) *Let $(\mathscr{K}_i)_{i\in I}$ a cofiltered system of qcqs sheaves of abelian groups on X_{proet} and $\mathscr{F}$ an étale sheaf of abelan groups on X. Then the natural map*
$$\varinjlim_i R\,\underline{\mathrm{Hom}}(\mathscr{K}_i,\mathscr{F}) \xrightarrow{\sim} R\,\underline{\mathrm{Hom}}(\varprojlim_i \mathscr{K}_i,\mathscr{F})$$
is an isomorphism.

(ii) *Let $\mathscr{K}$ be a qcqs sheaf of abelian groups on X_{proet} and $(\mathscr{F}_i)_{i\in I}$ a filtered system of pro-étale sheaves on X. Then the natural map*
$$\varinjlim_i R\,\underline{\mathrm{Hom}}(\mathscr{K},\mathscr{F}_i) \xrightarrow{\sim} R\,\underline{\mathrm{Hom}}(\mathscr{K},\varinjlim_i \mathscr{F}_i)$$
is an isomorphism

Proof. By the Breen-Deligne resolution (cf. [Sch19, Corollary 4.8], which can be directly adapted to any site, as everything is functorial), for any sheaves of abelian groups $\mathcal{M}$ and $\mathcal{N}$ on X_{proet} and every $U \in X_{\mathrm{proet}}$ there is a functorial spectral sequence
$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(\mathcal{M}^{r_{p,j}} \times_X U, \mathcal{N}) \implies \underline{\mathrm{Ext}}^{p+q}(\mathcal{M},\mathcal{N})(U).$$
Thus (i) reduces to
$$H^k(\varprojlim_i \mathcal{K}_i^r \times_X U, \mathcal{F}) = \varinjlim_i H^k(\mathcal{K}_i^r \times_X U, \mathcal{F}),$$
for all $k, r \geq 0$ and qcqs U. This is Proposition 5.5. Similarly (ii) reduces to
$$H^k(\mathcal{K}^r \times_X U, \varinjlim_i \mathcal{F}_i) = \varinjlim_i H^k(\mathcal{K}^r \times_X U, \mathcal{F}_i),$$
which is [Sta14, Lemma 0739]. □

5.4. **Solid sheaves.** Now we define solid sheaves first on combs and then on arbitrary v-stacks; then we consider functors between the respective sheaf categories. Using our previous results, we may follow [FS, §VII.1-2] in a quite formal way.

Let X be a comb. For $j\colon U \to X$ in X_{pu}, we write $\widehat{\mathbb{Z}}[U] = j_!\mathbb{Z}$. If one can write U as a cofiltered inverse system of qcqs objects $j_i\colon U_i \to X$ in X_{unr}, then we put $\widehat{\mathbb{Z}}_\blacksquare[U] = \varprojlim_i \mathbb{Z}[U_i]$. This is independent of the presentation $U = \varprojlim_i U_i$, hence well-defined. We have a map $\widehat{\mathbb{Z}}[U] \to \varprojlim_i \widehat{\mathbb{Z}}[U_i] = \widehat{\mathbb{Z}}_\blacksquare[U]$.

Definition 5.16. Let X be a comb. A sheaf $\mathscr{F} \in \mathrm{Shv}(X_{\mathrm{pu}}, \widehat{\mathbb{Z}})$ is called *solid*, if for all $U \in X_{\mathrm{pu}}^{\mathrm{aff}}$ as above, the natural map $\mathrm{Hom}(\widehat{\mathbb{Z}}_\blacksquare[U], \mathscr{F}) \to \mathscr{F}(U)$ is an isomorphism.

As in [FS, VII.1.2], Proposition 5.1 implies that unramified torsion sheaves are solid:

Corollary 5.17. *For any $n \geq 1$ and any $\mathscr{F} \in \mathrm{Shv}(X_{\mathrm{unr}}, \mathbb{Z}/n\mathbb{Z})$, $\nu^*\mathscr{F}$ is solid, where $\nu\colon X_{\mathrm{pu}} \to X_{\mathrm{unr}}$ is the natural map of sites.*

As in [FS, Theorem VII.1.3], the category of solid sheaves satisfies various nice properties:

Proposition 5.18. *Let X be a comb. The category of solid $\hat{\mathbb{Z}}$-sheaves on X is an abelian subcategory of $\mathrm{Shv}(X_{\mathrm{pu}}, \widehat{\mathbb{Z}})$, stable under all limits, colimits and extensions. It is generated by the finitely presented objects $\widehat{\mathbb{Z}}_\blacksquare[U]$ for $U \in X_{\mathrm{pu}}^{\mathrm{aff}}$, and the inclusion into $\mathrm{Shv}(X_{\mathrm{pu}}, \widehat{\mathbb{Z}})$ admits a left adjoint $\mathscr{F} \to \mathscr{F}^\blacksquare$, that commutes with all colimits. Let $\mathscr{F} \in \mathrm{Shv}(X_{\mathrm{pu}}, \widehat{\mathbb{Z}})$. The following statements are equivalent.*

(i) *The $\widehat{\mathbb{Z}}$-sheaf $\mathscr{F}$ is finitely presented in* $\mathrm{Shv}(X_{\mathrm{pu}}, \widehat{\mathbb{Z}})$*, and is solid.*
(ii) *The $\widehat{\mathbb{Z}}$-sheaf $\mathscr{F}$ is solid and is finitely presented in the category of solid $\widehat{\mathbb{Z}}$-sheaves on X.*
(iii) *The $\widehat{\mathbb{Z}}$-sheaf $\mathscr{F}$ can be written as a cofiltered inverse limit of torsion constructible unramified sheaves.*

Let X be a comb. Consider the category of torsion *constructible* unramified sheaves on X, which is the smallest full subcategory of $\mathrm{Shv}(X_{\mathrm{unr}}, \widehat{\mathbb{Z}})$, which contains $j_{\mathrm{unr}!}\mathbb{Z}/n\mathbb{Z}$ for all $n \in \mathbb{Z}_{\geq 1}$ and all $(j : U \to X) \in X_{\mathrm{unr}}^{\mathrm{aff}}$, and is closed under finite limits and colimits.

Lemma 5.19. *Any torsion constructible unramified sheaf on the comb X is representable by a comb.*

Proof. This is true for the sheaves $j_{\mathrm{unr}!}\mathbb{Z}/n\mathbb{Z}$ ... □

Lemma 5.20. *Let $(Y_i)_{i\in I}$ be a cofiltered system of combs. Then $Y = \varprojlim_i Y_i$ is a comb and the natural map $|Y| \to \varprojlim_i |Y_i|$ is a homeomorphism.*

Proof. The last claim in [Sta14, 0CUF]. For the first claim, note that any connected component Z of Y is the inverse limit of the connected components $Z_i \subseteq Y_i$ to which it maps. Now the claim follows from the fact that a filtered colimit of valuation rings is a valuation ring. □

Proof of Proposition 5.18. This follows from Proposition 5.1, Proposition 5.13, Corollary 5.17, Lemma 5.19, Lemma 5.20 in the same way as [FS, Theorem VII.1.3]. □

Lemma 5.21. *Let $f : Y \to X$ be a map of combs. Then f_{pu}^* preserves solid sheaves and commutes with solidification. If f is a v-cover and $\mathscr{F} \in \mathrm{Shv}(X_{\mathrm{pu}}, \widehat{\mathbb{Z}})$ such that $f^*\mathscr{F}$ is solid, then $\mathscr{F}$ is solid.*

Proof. The proof is the same as in [FS, Proposition VII.1.8], where we use Proposition 3.26(i),(iv) instead of [Sch18, Proposition 14.7]. □

We now define (derived) solid sheaves on arbitrary small v-stacks.

Definition 5.22. Let X be a v-stack and let $\mathscr{F}$ be a sheaf of $\widehat{\mathbb{Z}}$-modules on X. Then $\mathscr{F}$ is called *solid* if for all maps $f : Y \to X$ from a comb X, the pullback $f^*\mathscr{F}$ is solid. We let $D_{\blacksquare}(X, \hat{\mathbb{Z}}) \subseteq D(X_v, \widehat{\mathbb{Z}})$ be the full subcategory of all $A \in D(X_v, \widehat{\mathbb{Z}})$ such that for each $i \in \mathbb{Z}$ the cohomology sheaf $\mathscr{H}^i(A)$ is solid.

By Lemma 5.21 it suffices to check solidness of a sheaf after pullback to a single v-cover by a comb. As the category of solid sheaves is stable under (co)kernels and extensions, $D_{\blacksquare}(X, \hat{\mathbb{Z}})$ is a triangulated subcategory of $D_v(X, \widehat{\mathbb{Z}})$. We have the solid version of Proposition 3.26(i),(iv):

Lemma 5.23. *Let X be a comb, and let $\mathscr{F}$ be a solid sheaf of $\widehat{\mathbb{Z}}$-modules on X_{pu}. Let $\lambda : X_v \to X_{\mathrm{pu}}$ be the natural map of sites. Then $\mathscr{F} \to R\lambda_*\lambda^*\mathscr{F}$ is an isomorphism.*

Proof. Again, the proof is the same as for [FS, Proposition VII.1.11], where we use Proposition 3.26(i,iii,iv) instead of the analogous results from [Sch18]. □

Lemma 5.24. *Let X be a comb. For all $A \in D_{\blacksquare}(X, \widehat{\mathbb{Z}})$ and all $j : U \to X$ in $X_{\mathrm{pu}}^{\mathrm{aff}}$, the map $R\,\underline{\mathrm{Hom}}(\widehat{\mathbb{Z}}_{\blacksquare}[U], A) \to Rj_*A|_U$ is an isomorphism.*

Proof. As [FS, Proposition VII.1.12], this first reduces to the case that A is concentrated in degree 0 by a Postnikov limit argument. Then, by using Breen's resolution, one reduces to the case that A is finitely presented, in which case it is a limit of constructible unramified sheaves by Proposition 5.18. For such sheaves, the result now follows from Proposition 5.1. □

As in [FS, Proposition VII.1.13-14] this gives the derived solidification and the solid tensor product:

Corollary 5.25. *Let X be a comb. The inclusion $D_{\blacksquare}(X, \widehat{\mathbb{Z}}) \subseteq D(X_{\mathrm{pu}}, \widehat{\mathbb{Z}})$ admits a left adjoint*

$$A \mapsto A^{\blacksquare} \colon D(X_{\mathrm{pu}}, \widehat{\mathbb{Z}}) \to D_{\blacksquare}(X, \widehat{\mathbb{Z}})$$

commuting with any base change. Moreover, $D_{\blacksquare}(X, \widehat{\mathbb{Z}})$ identifies with the derived category of solid $\widehat{\mathbb{Z}}$-sheaves on X and $A \mapsto A^{\blacksquare}$ with the left derived functor of $\mathscr{F} \to \mathscr{F}^{\blacksquare}$.

Furthermore, the kernel of $A \mapsto A^{\blacksquare}$ is a tensor ideal; there is a unique symmetric monoidal structure $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} -$ on $D_{\blacksquare}(X, \widehat{\mathbb{Z}})$, making $A \mapsto A^{\blacksquare}$ symmetric monoidal; $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} -$ commutes with all colimits and pullbacks and is the left derived of functor of the induced symmetric monoidal structure on solid $\widehat{\mathbb{Z}}$-sheaves.

From now on we let Λ be any solid $\widehat{\mathbb{Z}}^p$-algebra on the pro-unramified (equivalently, pro-étale) site of a point. We denote its pullback to any small v-stack X again by Λ.

Definition 5.26. Let X be any small v-stack. We denote by $D_{\blacksquare}(X, \Lambda) \subseteq D_v(X, \widehat{\mathbb{Z}})$ be the full subcategory of all $A \in D_v(X, \Lambda)$, whose image in $D_v(X, \widehat{\mathbb{Z}})$ is solid.

Let $Y \to X$ be a map of small v-stacks, so that we have the pullback functor $f^* \colon D_{\blacksquare}(X, \Lambda) \to D_{\blacksquare}(Y, \Lambda)$. It admits a right adjoint Rf_*, which coincides with the restriction of Rf_{v*}, generalizing Proposition 4.13.

Lemma 5.27. *Let $Y \to X$ be a map of small v-stacks. Let $A \in D_{\blacksquare}(Y, \Lambda) \subseteq D(Y, \Lambda)$. Then $Rf_{v*}A \in D(X_v, \Lambda)$ lies in $D_{\blacksquare}(X, \Lambda)$. In particular, $Rf_{v*} \colon D(Y_v, \Lambda) \to D(X_v, \Lambda)$ restricts to the right adjoint $Rf_* \colon D_{\blacksquare}(Y, \Lambda) \to D_{\blacksquare}(X, \Lambda)$.*

Proof. One formally reduces to the case that $\Lambda = \widehat{\mathbb{Z}}^p$. We may follow the proof of [FS, Proposition VII.2.1]. By Lemma 5.21 we may reduce to the case that X is a comb. Then, taking simplicial resolution of Y, we may assume that Y is a comb, too. Next, we may reduce to the case that A is a constructible unramified sheaf sitting in degree 0, where the result follows from Proposition 4.13. □

Proposition 5.28. *Let X be a small v-stack. The inclusion $D_{\blacksquare}(X, \Lambda) \subseteq D(X_v, \Lambda)$ admits a left adjoint*

$$A \mapsto A^{\blacksquare} \colon D(X_v, \Lambda) \to D_{\blacksquare}(X, \Lambda)$$

commuting with any base change. Moreover, the kernel of $A \mapsto A^{\blacksquare}$ is a tensor ideal. In particular, there is a unique symmetric monoidal structure $-\overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda}-$ on $D_{\blacksquare}(X, \widehat{\mathbb{Z}})$, making $A \mapsto A^{\blacksquare}$ symmetric monoidal; the functor $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} -$ commutes with all colimits and pullbacks.

Proof. For the existence of the left adjoint, we may reduce to the case that X is a comb, as being solid can be checked v-locally. Then, for a comb X, the proof of [FS, Proposition VII.1.15] carries over to our situation, where we use Lemma 3.21 instead of [Sch18, Lemma 14.5] and (the proof of) Proposition 3.26(iii) instead of [Sch18, Lemma 14.4].

Now, commutation with any base change is a consequence of Lemma 5.27, and the property of being a $\otimes$-ideal follows from the existence of the left adjoint for f_v^* (cf. [FS, proof of Proposition VII.2.2]). □

We also have the analogue of [FS, Proposition VII.2.3] (with the same proof).

Proposition 5.29. *Let X be a small v-stack and let $A, B \in D_{\blacksquare}(X, \widehat{\mathbb{Z}}^p)$ be concentrated in degree 0. Then $A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} B$ sits in cohomological degrees -1 and 0. If X is a comb, and $A = \varprojlim_i A_i$, $B = \varprojlim_i B_i$ are finitely presented solid $\widehat{\mathbb{Z}}^p$-sheaves written as limits of constructible unramified sheaves killed by some integer coprime to p, then the natural map $A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}} B \to R\varprojlim_{i,j} A_i \otimes^{\mathbb{L}} B_j$ is an isomorphism.*

The existence of the solid tensor product $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} -$ leads as usual to an internal Hom on solid sheaves, i.e., there is a (partial) right adjoint to $- \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} -$,

$$R\,\underline{\mathrm{Hom}}_{\Lambda}(-,-)\colon D_{\blacksquare}(X,\Lambda)^{\mathrm{op}} \times D_{\blacksquare}(X,\Lambda) \to D_{\blacksquare}(X,\Lambda),$$

which equals the restriction of the $R\,\underline{\mathrm{Hom}}_{\Lambda}(-,-)$ on $D(X_v, \Lambda)$.

Just as in [FS, VII.2.4] we have the "unrestricted basechange" for solid sheaves.

Proposition 5.30. *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

be a cartesian diagram of small v-stacks. For all $A \in D_{\blacksquare}(Y, \Lambda)$ the base change map $g^* Rf_* A \to Rf'_* g'^* A$ in $D_{\blacksquare}(X', \Lambda)$ is an isomorphism. For any map $f\colon Y \to X$ of small v-stacks and all $A, B \in D_{\blacksquare}(X, \Lambda)$, the map

$$f^* R\,\underline{\mathrm{Hom}}_{\Lambda}(A, B) \to R\,\underline{\mathrm{Hom}}_{\Lambda}(f^*A, f^*B)$$

is an isomorphism.

Proof. This follows from Lemma 5.27 as pushforward on the v-site commutes with arbitrary base change. The claim about $R\,\underline{\mathrm{Hom}}$ follows from the fact that solid $R\,\underline{\mathrm{Hom}}$ agrees with the restriction of the $R\,\underline{\mathrm{Hom}}$ on the v-site, which satisfies the claimed formula. □

Finally, we can introduce the left adjoint $f_\natural$ of f^* on solid sheaves (homology functor) with the same properties as in [FS, VII.3].

Proposition 5.31. *Let $Y \to X$ be a map of small v-stacks.*

(1) The functor $f^*\colon D_{\blacksquare}(X,\Lambda) \to D_{\blacksquare}(Y,\Lambda)$ admits a left adjoint

$$f_\natural\colon D_{\blacksquare}(Y,\Lambda) \to D_{\blacksquare}(X,\Lambda).$$

The natural maps

$$f_\natural(A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} f^*B) \to f_\natural A \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda} B$$
$$R\,\underline{\mathrm{Hom}}_{\Lambda}(f_\natural A, B) \to Rf_* R\,\underline{\mathrm{Hom}}_{\Lambda}(A, f^*B)$$

are isomorphisms for all $A \in D_{\blacksquare}(Y,\Lambda)$, $B \in D_{\blacksquare}(X,\Lambda)$

(2) Formation of $f_\natural$ is functorial in maps $\Lambda \to \Lambda'$.

(3) For any cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

of small v-stacks and any $A \in D_\blacksquare(Y,\Lambda)$, the natural map

$$f'_\natural g'^* A \to g^* f_\natural A$$

is an isomorphism.

Proof. This follows from Proposition 5.28, Proposition 5.30, Proposition 5.28 in the same formal way as [FS, Proposition VII.3.1]. □

Note that $f_\natural$ equals the composition

$$D_\blacksquare(Y,\Lambda) \subseteq D(Y_v,\Lambda) \xrightarrow{f_v^*} D(X_v,\Lambda) \to D_\blacksquare(X,\Lambda)$$

where the first functor is the fully faithful embedding and the last functor is solidification from Proposition 5.28.

5.5. **Homology of the affine space.** If $n < \infty$, then $f_n \colon \mathbb{A}^n \to \operatorname{Spec} \overline{\mathbb{F}}_q$ is smooth, so that $f_{n\natural} = f_{n!}[2n]$, so that $f_{n\natural}\Lambda = \Lambda[0]$. We prove the following general result, covering the infinite dimensional affine space.

Theorem 5.32. *Let $n \leq \infty$ and let $f \colon X \to Y$ be a morphism, which is v-locally on Y a trivial $\mathbb{A}^n$-bundle. Then the natural transformations of functors*

$$f_\natural f^* \to \mathrm{id} \to f_* f^* \colon D_\blacksquare(Y,\Lambda) \to D_\blacksquare(Y,\Lambda)$$

are equivalences. In particular, $f_\natural\Lambda \cong \Lambda[0] \cong f_*\lambda$.

Proof. As being an isomorphism can be checked v-locally, we may assume that $f \colon X = \mathbb{A}^n_Y \to Y$ is the n-dimensional affine space over a comb Y. Both equivalences are equivalent to f^* being fully faithful, so it suffices to prove that $f_* f^* K \cong K$ is an equivalence for any $K \in D_\blacksquare(Y,\Lambda)$. As $D_\blacksquare(Y,\Lambda)$ is left-complete (cite), we have $K \cong R\lim_m \tau^{\geq -m} K$. As both f_* and f^* commute with limits, it suffices to prove the claim when $K \in D^+_\blacksquare(Y,\Lambda)$. For such K, we have $K \cong \varinjlim \tau_{\leq n} K$ and f_* being a functor on D^+, commutes with this colimit [Sta14, 0739]. Thus, it suffices to show the claim for K bounded, and even for K concentrated in one degree. Any such K can be written as a quotient $\operatorname{coker}(\alpha \colon F_1 \to F_2)$ with $F_i = \varinjlim_j j_\natural\Lambda$ being filtered colimit of solid generators. Now f_* will not in general commute with cokernels, but we may do the following. Let $F_0 = \ker(\alpha)$, so that $K \simeq \operatorname{cone}(F_0[1] \to \operatorname{cone}(\alpha))$. Now, as $\Lambda = \mathbb{Z}_\ell$ it follows from [FS, Theorem VII.1.3] that F_0 is again a filtered colimit of solid generators. Now, by [Sta14, 0739], f_* commutes with filtered colimits of objects concentrated in one degree. As f_* also commutes with arbitrary limits, we are reduced to show the result for abelian sheaves on Y_{unr}. Then by Proposition 5.1 we are reduced to the case that $n < \infty$. But in this case f is smooth of dimension n and $f^!\Lambda = \Lambda[2n]$, so that $f_\natural \cong f_![2n]$, and we are done. □

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