# INNER PRODUCTS OF DEEP LEVEL DELIGNE–LUSZTIG REPRESENTATIONS OF COXETER TYPE

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Abstract. In this article we prove orthogonality relations for deep level Deligne–Lusztig varieties of Coxeter type, attached to a reductive group over a local non-archimedean field, which splits over an unramified extension. This extends results of [DI24], where the quasi-split case was handled. This allows to construct new irreducible representations of parahoric subgroups of $p$-adic groups.

## 1. Introduction

Let $k$ be a non-Archimedean local field with residue characteristic $p > 0$, integers $\mathcal{O}_k$, uniformizer $\varpi$ and residue field $\mathbb{F}_q$. Let $\breve{k}$ be the completion of the maximal unramified extension of $k$, let $\mathcal{O}_{\breve{k}}$ denote the integers of $\breve{k}$. Let $F$ denote the Frobenius automorphism of $\breve{k}$ over $k$.

Let $G$ be a reductive group over $k$, which splits over $\breve{k}$. Let $T \subseteq B$ be a maximal torus and a Borel subgroup of $G$, such that $T$ splits and $B$ becomes rational over $\breve{k}$. Denote by $U$ the unipotent radical of $B$ and assume that $(T, U)$ is a Coxeter pair (see §2.1). In particular, $T$ is elliptic and the apartment of $T$ in the reduced Bruhat–Tits building of $G$ consists of one point. Bruhat–Tits theory attaches to this point a (connected) parahoric $\mathcal{O}_k$-model $\mathcal{G}$ of $G$. For any $r \leq \infty$ we can regard $\mathcal{G}(\mathcal{O}_{\breve{k}}/\varpi^r) = \mathbb{G}_r(\overline{\mathbb{F}}_q)$ as the geometric points of a perfect $\overline{\mathbb{F}}_q$-scheme $\mathbb{G}_r$. This is done via the (truncated, if $r < \infty$) positive loop functor, see e.g. [Zhu17, §1.1] (or [DI24, §2]) for details.

We fix now some $r < \infty$ and write $\mathbb{G} = \mathbb{G}_r$. If $\mathcal{G}$ is defined over $\mathcal{O}_k$, then $\mathbb{G}$ is defined over $\mathbb{F}_q$; in this case we also denote by $F$ the geometric Frobenius of $\mathbb{G}$, so that $\mathbb{G}^F = \mathbb{G}(\mathbb{F}_q)$. Moreover, if $H \subseteq G$ is a subscheme, then we denote by $\mathcal{H}$ its closure in $\mathcal{G}$ and by $\mathbb{H} \subseteq \mathbb{G}$ the corresponding subscheme of $\mathbb{G}$. Consider the closed subscheme

$$X = X^{\mathcal{G}}_{T,U,r} = \{x \in \mathbb{G} \colon x^{-1}F(x) \in F\mathbb{U}\} \tag{1.1}$$

of $\mathbb{G}$. The group $\mathbb{G}^F \times \mathbb{T}^F$ acts on it by $(g, t) \colon x \mapsto gxt$. Let $\ell \neq p$ be a prime. For any smooth character $\chi \colon \mathbb{T}^F \to \overline{\mathbb{Q}}_\ell^\times$ we have the virtual $\mathbb{G}^F$-module

$$H^*_c(X)[\chi] = \sum_{i\in\mathbb{Z}} (-1)^i H^i(X_{T,U}, \overline{\mathbb{Q}}_\ell^\times)[\chi],$$

where for any $\mathbb{T}^F$-module $M$, $M[\chi]$ denotes the $\chi$-isotypic subspace. The following is our main result.

**Theorem 1.1.** *Suppose that $q$ satisfies condition (2.1) (this is always true when $q > 5$). Then there exists a Coxeter pair $(T, U)$ such that*

$$\dim_{\overline{\mathbb{Q}}_\ell} \mathrm{Hom}_{\mathbb{G}^F}(H_c^*(X)[\chi], H_c^*(X)[\chi']) = \sharp \left\{w \in W_e^F; w(\chi) = \chi'\right\}$$

*for any two smooth characters $\chi, \chi'$ of $\mathbb{T}^F$, where $W_e$ denotes the Weyl group of the special fiber of $\mathcal{T}$ in the reductive quotient of the special fiber of $\mathcal{G}$.*

**Remark 1.2.** Recently, under a mild condition on $p$, Chan [Cha24] shows that the inner product formula in Theorem 1.1 holds if $(T, \theta)$ is split-generic, using a different approach.

In particular, if $\{w \in W_e^F \colon w(\chi) = \chi\} = \{1\}$, then $H_c^*(X)[\chi]$ is up to sign an irreducible $\mathbb{G}^F$-representation. Note that Theorem 1.1 generalizes [DI24, Theorem 3.2.3] and [CI23, Theorem 4.1].

Now we comment on Theorem 1.1. First, we explain why "it suffices" to establish the theorem for a single Coxeter pair $(T, U)$. Ultimately, we are interested in smooth representations of the $p$-adic group $G(k)$. One has the $p$-adic Deligne–Lusztig space $X_{T,U} \cong \dot{X}_{\dot{c}}(b)$ from [Iva23a, §7.2 and §11.2] equipped with an action of $G(k) \times T(k)$. By [Nie24, Iva23b], one has

$$X_{T,U} = \coprod_{\gamma \in G(k)/\mathcal{G}(\mathcal{O}_k)} \gamma X^{\mathcal{G}}_{T,U,\infty}.$$

Thus for any smooth character $\theta \colon T(k) \to \overline{\mathbb{Q}}_\ell^\times$, the $\theta$-isotypic part of the cohomology of $X_{T,U}$ equals

$$R_{T,U}(\theta) = \text{c-Ind}_{\mathcal{G}(\mathcal{O}_k)Z(k)}^{G(k)} H_c^*(X)[\theta|_{\mathcal{T}(\mathcal{O}_k)}],$$

where $Z \subseteq G$ is the center and $H_c^*(X)[\theta|_{\mathcal{T}(\mathcal{O}_k)}]$ is extended to a representation of $\mathcal{G}(\mathcal{O}_k)Z(k)$ in the unique way such that its central character is $\theta|_{Z(k)}$. Now, the point is that by [Iva23a, Corollary 7.25, Lemma 11.3], $X_{T,U}$ are all mutually $G(k) \times T(k)$-equivariantly isomorphic, when $(T, U)$ varies through all Coxeter pairs $(T, U)$ with a fixed $T$. Thus, the $G(k)$-representation $R_{T,U}(\theta)$ is independent of the choice of $U$, as long as $(T, U)$ remains Coxeter. So, it suffices to know the statement of the theorem for at least one Coxeter pair. In fact, our proof shows that for many groups $G$ Theorem 1.1 holds for all pairs $(T, U)$, see Remark 2.4.

Next, we explain why the condition on $q$ in the theorem is very mild, so that Theorem 1.1 even allows to construct new irreducible representations. Recall that by the work of Yu and Kaletha [Yu01, Kal19], one can attach a supercuspidal irreducible $G(k)$-representation $\pi_{(S,\theta)}$ to any regular elliptic pair $(S, \theta)$ consisting of a maximal elliptic torus $S \subseteq G$ and a sufficiently nice smooth character $\theta \colon S(k) \to \overline{\mathbb{Q}}_\ell^\times$. A crucial point for this to work is the existence of a Howe factorization of $\theta$, cf. [Kal19, §3.6]. However, not all characters admit a Howe factorization, when the residue characteristic $p$ is small and $G$ is not an inner form of $\mathrm{GL}_n$.

For instance, if $p \in \{2,3,5\}$, there exist many examples of pairs $(T,\theta)$ with $T$ unramified Coxeter (hence covered by our main result when $q$ satisfied condition (2.1) – in particular, whenever $q > 5$) such that $\mathrm{Stab}_{W_e^F}(\theta) = \{1\}$, but $\theta$ does not admit a Howe factorization. For examples of $(T,\theta)$ not admitting a Howe factorization we refer to the forthcoming work of Fintzen–Schwein [FS], where an algebraic approach to the extension of Yu's construction is pursued. As $\mathrm{Stab}_{W_e^F}(\theta) = \{1\}$, one should expect an irreducible supercuspidal $G(k)$-representation attached to $(T,\theta)$, but Yu's construction does not apply as there is no Howe factorization. The point is now that our cohomological construction does not require any condition on $p$, but only a mild one on $q$. In particular, there are many examples of $k, G, T, \theta$ such that $\pm H_c^*(X)[\theta|_{\mathcal{T}(\mathcal{O}_k)}]$ is an irreducible $\mathcal{G}(\mathcal{O}_k)$-representation, which does not appear in Yu's construction.

However, let us also note that our method is not yet complete, in the sense that it does not prove that the resulting $G(k)$-representation $R_{T,U}(\theta)$ is irreducible supercuspidal. So far there is no purely Deligne–Lusztig theoretic proof of this fact; the closest purely Deligne–Lusztig theoretic result states (in the case of inner forms of $\mathrm{GL}_n$) that $R_{T,U}(\theta)$ is admissible and hence a direct sum of finitely many irreducible supercuspidal representations, see [CI23, Theorem 6.1].

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## 2. Preparations

2.1. **More notation.** We use the notation from the introduction. Moreover, we denote by $Z$ the center of $G$, $N_G(T)$ the normalizer of $T_{\breve{k}}$ in $G_{\breve{k}}$, by $W = N_G(T)/T$ the Weyl group of $T$, by $X^*(T)$ (resp. $X_*(T)$) the group of characters (resp. cocharacters) of $T_{\breve{k}}$ and by $\langle\cdot,\cdot\rangle\colon X^*(T)\times X_*(T)\to\mathbb{Z}$ the natural pairing. We write $\Phi$ for the root system of $T_{\breve{k}}$ in $G_{\breve{k}}$, $\Phi^+$ for the subset of positive roots determined by $B$, and $\Delta \subseteq \Phi^+$ for the subset of positive simple roots. We write $S \subseteq W$ for the corresponding set of simple reflections.

Let $c \in W$ be the unique element such that $FB = {}^cB$. Then for any lift $\dot{c}$ of $c$, $\mathrm{Ad}(\dot{c})^{-1} \circ F : G(\breve{k}) \to G(\breve{k})$ fixes the pinning $(T,B)$, hence defines automorphisms, denoted by $\sigma$, of the based root system $\Delta \subseteq \Phi$ and of the Coxeter system $(W,S)$. Note that $\sigma$ does not depend on the choice of the lift $\dot{c}$. We call $(T,B)$ (or $(T,U)$) a *Coxeter pair* if $c$ is a $\sigma$-Coxeter element in the Coxeter triple $(W,S,\sigma)$, that is, if a(ny) reduced expression of $c$ contains precisely one element from each $\sigma$-orbit on $S$. *Moreover, we* *assume throughout the article that $c$ is $\sigma$-Coxeter, and hence $(T, U)$ is a Coxeter pair.*

Except for $G$, $\mathcal{G}$ and their subgroups (which are defined over $k, \breve{k}$ resp. $\mathcal{O}_k, \mathcal{O}_{\breve{k}}$), all schemes appearing below are perfect schemes perfectly of finite presentation and perfectly smooth over $\overline{\mathbb{F}}_q$. For a review of perfect geometry we refer to [Zhu17, Appendix A]. We freely make use of the 6-functor formalism of étale cohomology for such schemes with $\overline{\mathbb{Q}}_\ell$-coefficients. Moreover, for a perfect $\overline{\mathbb{F}}_q$-scheme we denote by $H_*(Y) = H_c^*(Y, \overline{\mathbb{Q}}_\ell)$ its $\ell$-adic étale cohomology with compact support.

2.2. **Pinning.** We may express the action of the Frobenius $F$ on $X_*(T)_{\mathbb{Q}}$ as $F = \mu c\sigma \colon x \mapsto \mu + c\sigma(x)$ for some $\mu \in X_*(T)$. There is a unique point $e \in \mathbb{Q}\Phi^\vee$ such that $F(e) \in e + X_*(Z)_{\mathbb{Q}}$, or equivalently, $\mu + c\sigma(e) - e \in X_*(Z)_{\mathbb{Q}}$. Let

$$\Phi_e = \{\alpha \in \Phi; \langle \alpha, e \rangle \in \mathbb{Z}\}.$$

We denote by $\Delta_e$ the set of simple roots of $\Phi_e^+ = \Phi_e \cap \Phi^+$. Let $W_e \subseteq W$ be the Weyl group of $\Phi_e$. Note that $\mathcal{G}$ from the introduction is the parahoric model attached to the image of $e$ in the reduced building of $G$, and that $\Phi_e$ (resp. $W_e$) is the root system (resp. Weyl group) of the reductive quotient of the special fiber of $\mathcal{G}$.

Also, note that the action of $F$ on $W$ agrees with $\mathrm{Ad}(c) \circ \sigma$; we denote it by $F = c\sigma \colon W \to W$. This action stabilizes $W_e \subseteq W$. Finally, for an element $w \in W_e$ we denote by $\dot{w} \in \mathbb{G}(\overline{\mathbb{F}}_q)$ an arbitrary (fixed) lift of $w$.

2.3. **A condition on $q$.** Let $\omega_\alpha$ denotes the fundamental coweight of $\alpha \in \Delta$. For a $\sigma$-orbit $\mathcal{O} \subseteq \Delta$ of simple roots, we set $\omega_{\mathcal{O}}^\vee = \sum_{\alpha \in \mathcal{O}} \omega_\alpha^\vee$, where $\omega_\alpha^\vee$ denotes the fundamental coweight of $\alpha \in \Delta$. We prove our main result under the following condition on $q$:

$$q > M = \max\{\langle \gamma, \omega_{\mathcal{O}}^\vee \rangle; \gamma \in \Phi^+, \mathcal{O} \in \Delta/\langle\sigma\rangle\}. \tag{2.1}$$

Note that $M$ only depends on the (relative) Dynkin diagram $\Delta$ of the quasi-split inner form of $G$ over $k$. If $\Delta$ is connected then $M$ takes the following values: $M = 1$ for type $A_n$; $M = 2$ for types $B_n, C_n, D_n, {}^2A_n, {}^2D_n$; $M = 3$ for types $G_2, E_6, {}^3D_4$; $M = 4$ for types $F_4, E_7, {}^2E_6$; $M = 6$ for type $E_8$. If the quasi-split inner form of $G$ is split, then $M$ is the same as in [DI24, §2.7], and it differs otherwise. Just as in [DI24, §2.7], for arbitrary $G$ the constant $M$ equals the maximum of the values of $M$ over all connected components of the Dynkin diagram of $G_{\breve{k}}$ (equipped with the smallest power of $\sigma$ fixing the connected component). In particular, (2.1) holds whenever $q > 5$.

2.4. **A Coxeter element in $W_e$.** It turns out that $c$ determines a (twisted) Coxeter element of $W_e$. Write $c = s_{\alpha_1} \cdots s_{\alpha_r}$, where $\{\alpha_1, \dots, \alpha_r\} \subseteq \Delta$ is a set of representatives of $\sigma$-orbits of $\Delta$.

Let $I = (i_1 < i_2 < \cdots < i_m)$ be a subsequence of $[r] := (1 < 2 < \cdots < r)$, and let $I' = (j_1 < j_r < \cdots < j_{r-m})$ be the complement sequence of $I$ in $[r]$.

We define

$$\sigma_{I,c} = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_m}} \sigma;$$
$$c_I = s_{\beta_{j_1}} s_{\beta_{j_2}} \cdots s_{\beta_{j_{r-m}}};$$
$$\Delta_{I,c} = \{\beta_{j_l}; 1 \leqslant l \leqslant r-m\}$$

where $\beta_{j_l} = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_t}}(\alpha_{j_l})$ with $1 \leqslant t \leqslant m-1$ such that $i_t < j_l < i_{t+1}$. By definition, $c\sigma = c_I \sigma_{I,c}$.

**Theorem 2.1.** *Let $c$, $\mu$ and $e = e_{\mu,c}$ be as in §2.2. Then there exist a sequence $I = I_{\mu,c}$ of $1 < 2 < \cdots < r$ such that*
*(1) $\sigma_{I,c}(\Delta_e) = \Delta_e$;*
*(2) $\Delta_{I,c} \subseteq \Delta_e$ is a representative set of $\sigma_{I,c}$-orbits of $\Delta_e$;*
*(3) $\sigma_{I,c}^i = 1$ if and only if $\sigma_{I,c}^i$ fixes each root of $\Delta_e$.*
*In particular, $c_I$ is a $\sigma_{I,c}$-Coxeter element of $W_e$.*

This theorem is proven in §4.

2.5. **Support.** For $\alpha \in \Phi$ we denote by $\mathrm{supp}(\alpha) \subseteq \Delta$ the minimal subset whose linear span contains $\alpha$. For a subset $C \subseteq \Phi$ we set $\mathrm{supp}(C) = \cup_{\alpha\in C} \mathrm{supp}(\alpha)$. For $w \in W$ we denote by $\mathrm{supp}(w)$ the set of simple reflections which appear in some/any reduced expression of $w$.

**Lemma 2.2.** *Let $C \subseteq \Phi$ be a $c\sigma$-orbit. Then $\mathrm{supp}(C)$ is $\sigma$-stable.*

*Proof.* Let $c = s_{\alpha_1} \cdots s_{\alpha_r}$ be as in §2.4. Let $\alpha \in \mathrm{supp}(\gamma)$ for some $\gamma \in C$. It suffices to show that the $\sigma$-orbit $\mathcal{O}$ of $\alpha$ is contained in $\mathrm{supp}(C)$. Set $\delta = \sharp\mathcal{O}$. Let $1 \leqslant j \leqslant r$ be the unique integer such that $\alpha_j \in \mathcal{O}$. Let $0 \leqslant i_0 \leqslant \delta - 1$ such that

$$\alpha, \sigma^{-1}(\alpha), \ldots, \sigma^{1-i_0}(\alpha) \neq \alpha_j \text{ and } \sigma^{-i_0}(\alpha) = \alpha_j.$$

Then one checks that $(c\sigma)^{-i_0} = \sigma^{-i_0} w$ for some $w \in W$ such that $\mathrm{supp}(w) \in S - \{s_\alpha\}$. Hence $\alpha \in \mathrm{supp}(w(\gamma))$ and $\alpha_j = \sigma^{-i_0}(\alpha) \in \mathrm{supp}(\sigma^{-i_0} w(\gamma)) = \mathrm{supp}((c\sigma)^{-i_0}(\gamma))$. So we can assume that $\alpha = \alpha_j$. Let $0 \leqslant i \leqslant \delta - 1$. Note that $(c\sigma)^i = u_i \sigma^i$ for some $u_i \in W$ with $\mathrm{supp}(u_i) \in S - \{\sigma^i(\alpha_j)\}$. It follows that $\sigma^i(\alpha) \in \mathrm{supp}((c\sigma)^i(\gamma))$. So the statement follows. □

**Proposition 2.3.** *Let $C$ be a $c\sigma$-orbit of $\Phi$. Then $\mathrm{supp}(C) = \cup_{i\in\mathbb{Z}} \sigma^i(H)$, where $H$ is a connected component of $\Delta$.*

*Proof.* Without loss of generality we may assume that $\Delta = \cup_{i\in\mathbb{Z}} \sigma^i(H)$. We argue by induction on $\sharp\Delta$. Assume the statement is false. Let $c = s_{\alpha_1} \cdots s_{\alpha_r}$ be as in §2.4. By Lemma 2.2 there exists $1 \leqslant j \leqslant r$ such that $C \subseteq \Phi_K$, where $K = \Delta - \mathcal{O}$ and $\mathcal{O}$ is the $\sigma$-orbit of $\alpha_j$. By replacing $c$ with its $W_K$-$\sigma$-conjugate $s_{\alpha_j} \cdots s_{\alpha_r} \sigma(s_{\alpha_1} \cdots s_{\alpha_{j-1}})$, we can assume that $j = 1$ and $\alpha_1 \in \mathcal{O}$. Let $c' = s_{\alpha_1} c$, which is a $\sigma$-Coxeter element of $W_K$. As $C \subseteq \Phi_K$, $C$ is also a $c'\sigma$-orbit of $\Phi_K$. By induction hypothesis we have $\mathrm{supp}(C) = \cup_{i\in\mathbb{Z}} \sigma^i(D)$, where $D$ is a connected component of $H - \{\alpha_1\}$. As $H$ is connected, there exists $\gamma \in C$ and $\beta \in \mathrm{supp}(\gamma)$ such that

$$0 > \langle \alpha_1, \beta^\vee \rangle \geqslant \langle \alpha_1, \gamma^\vee \rangle.$$

Then we have $\sigma^{-1}(\alpha_1) \in \mathrm{supp}((c\sigma)^{-1}(\gamma))$, contradicting that $C \subseteq \Phi_K$. The proof is finished. □

2.6. **A condition on the $\sigma$-Coxeter element.** Let $c$, $\mu$, $e = e_{\mu,c}$, $I = I_{\mu,c}$, $c_I$, $\sigma_I = \sigma_{I,c}$ and $\Delta_I = \Delta_{I,c} \subseteq \Delta_e$ be as in Theorem 2.1. Denote by $\ell \colon W \to \mathbb{Z}_{\geq 0}$ (resp. $\ell_e : W_e \to \mathbb{Z}_{\geqslant 0}$) the length function associated to the set $\Delta$ (resp. $\Delta_e$) of simple roots. Let $w_0$ and $w_e$ be the longest elements of $W$ and $W_e$ respectively. We consider the following condition on $c$, or, equivalently, on the pair $(T, U)$:

$(*)$ There exists $N \in \mathbb{Z}_{\geqslant 1}$ such that $(c\sigma)^N = w_0\sigma^N, \quad N\ell(c) = \ell(w_0)$.

**Remark 2.4.** If $\Delta$ is connected, then there always exists a $\sigma$-Coxeter element $c \in W$ satisfying $(*)$, see [Bou68, Chap. V, Prop. 6.2]. Moreover, if the Coxeter number of $G$ is even, then any $c$ satisfies this condition.

**Lemma 2.5.** *Suppose $c$ satisfies condition $(*)$. Then $(c_I\sigma_I)^N = w_e\sigma_I^N$ and $N\ell_e(c_I) = \ell_e(w_e)$.*

*Proof.* By Theorem 2.1, $c_I\sigma_I = c\sigma$ and $\sigma_I(\Delta_e) = \Delta_e$. As $(c_I\sigma_I)^N = w_0\sigma^N$, it follows that $(c_I\sigma_I)^N$ sends $\Phi_e^+$ to $-\Phi_e^+$, that is, $(c_I\sigma_I)^N = w_e\sigma_I^N$.

It remains to show $\ell_e((c_I\sigma_I)^{i+1}) = \ell_e((c_I\sigma_I)^i) + \ell_e(c_I\sigma_I)$ for $1 \leqslant i \leqslant N-1$. Indeed, this is equivalent to that for any $\alpha \in \Phi_e^+$ with $(c_I\sigma_I)^{-1}(\alpha) < 0$ we have $(c_I\sigma_I)^i(\alpha) > 0$. This statement follows from that $c_I\sigma_I = c\sigma$ and $\ell((c\sigma)^{i+1}) = \ell((c\sigma)^i) + \ell(c\sigma)$ for $1 \leqslant i \leqslant N-1$. □

For $w \in W$ we denote by $\mathrm{supp}(w)$ the set of simple reflections in $\Delta$ that appears in some/any reduced expression of $w$. For $u \in W_e$, we can define $\mathrm{supp}_{\Delta_e}(u) \subseteq \Delta_e$ in a similar way.

**Corollary 2.6.** *Suppose $c$ satisfies condition $(*)$. Let $K \subsetneq \Delta_e$ be a proper $\sigma_I$-stable subset. Then there exists a proper $\sigma$-stable subset $J \subsetneq \Delta$ such that $\sigma_I \in W_J\sigma$ and $w_eW_K \subseteq w_0W_J$.*

*Proof.* Let notation be as in §2.4. As $\Delta_I = \{\beta_j; j \in I'\}$ with $I' = [r] - I$ is a representative set of $\Delta_e$, there exists $i \in I'$ such that $\beta_i \notin K$. Let $J = \Delta - \mathcal{O}_i$, where $\mathcal{O}_i$ is the $\sigma$-orbit of $\alpha_i$. By construction, $\mathrm{supp}(s) \subseteq J$ for $s \in K$ and $\mathrm{supp}(\sigma_I\sigma^{-1}) \subseteq J$. By Lemma 2.5 we have

$$w_e = (c_I\sigma_I)^N\sigma_I^{-N} = (c\sigma)^N\sigma_I^{-N} = w_0\sigma^N\sigma_I^{-N} \subseteq w_0W_J.$$

Thus $w_eW_K \subseteq w_0W_J$ as desired. □

**Lemma 2.7.** *Let $K_1, K_2 \subseteq \Delta_e$ be two $\sigma_I$-stable subsets. Let $c_1$ and $c_2$ be two $\sigma_I$-Coxeter elements of $W_{K_1}$ and $W_{K_2}$ respectively. Let $w \in W_e$ such that $c_1\sigma_I(w) = wc_2$. Then there exists $x \in {}^{K_1}W_e{}^{K_2}$ such that ${}^xK_2 = K_1$ and $w \in xW_{K_2}$.*

*Proof.* By symmetry we may assume $\sharp K_1 \leqslant \sharp K_2$. Let $x \in {}^{K_1}W_e$ such that $w \in W_{K_1}x$. Then there exists $c_2' \leq c_2$ such that $xc_2 \in W_{K_1}xc_2'$ and $xc_2' \in {}^{K_1}W_e$. Hence we have $\sigma_I(x) = xc_2'$. Note that $c_2'$ is a partial $\sigma_I$-Coxeter element, which is of minimal length (in the sense of $\ell_e$) in its $\sigma_I$-conjugacy class. Thus $c_2' = 1$, $x = \sigma_I(x)$ and $x(\mathrm{supp}_{\Delta_e}(c_2)) \subseteq K_1$, which implies that $x(K_2) \subseteq K_1$. Hence $x(K_2) = K_1$ since $\sharp K_1 \leq \sharp K_2$. Thus $x \in {}^{K_1}W_e{}^{K_2}$ as desired. □

## 3. Cohomology of $X$

Recall the scheme $X$ from (1.1) equipped with $\mathbb{G}^F \times \mathbb{T}^F$-action.

3.1. **The schemes $\Sigma^i$.** Let $i \in \mathbb{Z}$. We define

$$\Sigma^i = \{(x, x', y) \in \mathbb{U} \times F^{i+1}\mathbb{U} \times \mathbb{G}; xF(y) = yx'\}.$$

Let $\mathrm{pr}_3 : \Sigma^i \to \mathbb{G}$ be the natural projection. There is a locally closed decomposition

$$\Sigma^i = \bigsqcup_{w \in W_e} \Sigma^i_w,$$

where $\Sigma^i_w = \mathrm{pr}_3^{-1}(\mathbb{U}w\mathbb{T}\mathbb{G}^1 F^i\mathbb{U})$.

The group $\mathbb{T}^F \times \mathbb{T}^F$ acts on $\Sigma^i$ and on each of the pieces $\Sigma^i_w$ by

$$(t, t') : (x, x', y) \longmapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1}).$$

As in [DL76, p.137] there is a $\mathbb{T}^F \times \mathbb{T}^F$-equivariant isomorphism $X \times X/\mathbb{G}^F \xrightarrow{\sim} \Sigma^0$, and for characters $\chi, \chi'$ of $\mathbb{T}^F$ we have

$$\dim_{\overline{\mathbb{Q}}_\ell} \mathrm{Hom}_{\mathbb{G}^F}(H_*(X)[\chi'], H_*(X)[\chi]) = \dim H_*(\Sigma^0)_{\chi', \chi^{-1}},$$

where $H_*(\Sigma^0)_{\chi', \chi}$ is the corresponding isotropic subspace of $H_c^*(\Sigma^0)$.

Let $Z \subseteq G$ denote the centre of $G$ and consider the embedding $z \mapsto (z, z^{-1}) \colon Z \to T \times T$. Then the above $\mathbb{T}^F \times \mathbb{T}^F$-action on $\Sigma^i$ factors through an action of the quotient $\mathbb{T}^F \times^{\mathbb{Z}^F} \mathbb{T}^F$. This latter action extends to the action of $\mathbb{T}^F \times^{\mathbb{Z}^F} \mathbb{T}^F \subseteq (\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$ on $\Sigma^i$ (and $\Sigma^i_w$ for $w \in W_e$) given by the same formula. By the discussion in [DI24, §4.2] which applies in our more general setting, Theorem 1.1 follows from the next result.

**Theorem 3.1.** *Suppose that $q$ satisfies condition (2.1). Then there exists a Coxeter pair $(T, U)$ such that*

$$\dim_{\overline{\mathbb{Q}}_\ell} H_*(\Sigma^0_w) = \begin{cases} H_*((\dot{w}\mathbb{T})^{c\sigma}) & \text{if } w \in W_e^{c\sigma}, \\ 0 & \text{otherwise.} \end{cases}$$

*as virtual $(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$-modules.*

As a first step towards the proof of Theorem 3.1 we observe that the whole discussion of [DI24, §4.3] applies *mutatis mutandis* in our setting. Thus it suffices to prove Theorem 3.1 in the case that $\Delta$ is connected. In particular, there exists some $c$ satisfying condition (∗), cf. Remark 2.4. Now Theorem 3.1 follows from Corollary 3.7 and Proposition 3.12 below.

3.2. **An extension of action.** Let $w \in W_e$. We set $K_{w,i} = {}^{w^{-1}}U^- \cap F^i U^-$. Define

$$\hat{\Sigma}_w^i = \{(\tilde{x}, \tilde{x}', y_1, \tau, z, y_2) \in F\mathbb{U} \times F^{i+1}\mathbb{U} \times \mathbb{U} \times \dot{w}\mathbb{T} \times \mathbb{K}_{w,i}^1 \times F^i\mathbb{U}; \tilde{x}F(\tau z) = y_1 \tau z y_2 \tilde{x}'\}.$$

We define an action of $\mathbb{T}^F \times \mathbb{T}^F$ on $\Sigma_w^i$ by

$$(t, t') : (\tilde{x}, \tilde{x}', y_1, \tau, z, y_2) \longmapsto (t\tilde{x}t^{-1}, t'\tilde{x}'t'^{-1}, ty_1t^{-1}, t\tau t'^{-1}, t'zt'^{-1}, t'y_2t'^{-1}).$$

Then there is an $\mathbb{T}^F \times \mathbb{T}^F$-equivariant affine space bundle

$$\pi_w^i : \hat{\Sigma}_w^i \longrightarrow \Sigma_w^i, \quad (\tilde{x}, \tilde{x}', y_1, \tau, z, y_2) \longmapsto (\tilde{x}F(y_1)^{-1}, \tilde{x}'F(y_2), y_1 \tau z y_2).$$

Let $\chi \in X_*(T)$ which centralizes $K_{w,i}$. Define

$$H_{w,\chi} = \{(t, t') \in \mathbb{T}_h \times \mathbb{T}_h; w^{-1}t^{-1}F(t)w = t'^{-1}F(t') \in \mathrm{Im}(\chi)\}.$$

Then $H_{w,\chi}$ acts on $\hat{\Sigma}_w^i$ by

$$(t, t') : (\tilde{x}, \tilde{x}', y_1, \tau, z, y_2) \longmapsto ({}^{F(t)}\tilde{x}, {}^{F(t')}\tilde{x}', {}^{F(t)}y_1, t\tau t'^{-1}, {}^{t'}z, {}^{F(t')}y_2).$$

**Lemma 3.2.** *Let $w \in W_e \setminus W_e^{c\sigma}$ such that $\Sigma_w^i \neq \emptyset$. Then there exists a proper subset $K = \sigma_I(K) \subsetneq \Delta_e$ such that $w(c_I\sigma_I)^i \sigma_I^{-i} \in w_e W_K$.*

*Proof.* Let $w_i = w(c_I\sigma_I)^i\sigma_I^{-i} \in W_e$. By assumption we have

$$c\sigma \mathbb{B} w(c\sigma)^i \mathbb{B}\mathbb{G}^1 (c\sigma)^{-i-1} \cap \mathbb{B}w(c\sigma)^i\mathbb{B}\mathbb{G}^1 c\sigma \mathbb{B}(c\sigma)^{-i-1} \neq \emptyset.$$

As $c_I\sigma_I = c\sigma$, this implies that

$$c_I\sigma_I\mathbb{B}_1 w(c_I\sigma_I)^i\mathbb{B}_1 \cap \mathbb{B}_1 w(c_I\sigma_I)^i \mathbb{B}_1 c_I \sigma_I \neq \emptyset,$$

that is,

$$c_I \mathbb{B}_1 \sigma_I(w_i)\mathbb{B}_1 \cap \mathbb{B}_1 w_i \mathbb{B}_1 (\sigma_I)^i(c_I) \neq \emptyset.$$

In particular there are $\sigma_I$-Coxeter elements $v' \leq_e c_I$ and $v \leq_e (\sigma_I)^i(c_I)$ of some $\sigma_I$-stable subsets $K'$ and $K$ of $\Delta_e$ respectively (one of which is a proper subset of $\Delta_e$ since $w \in W_e \setminus W_e^{c\sigma}$) such that $v'\sigma_I(w_i) = w_i v$ and

(a) $$\mathbb{B}_1 w_i \mathbb{B}_1 (\sigma_I)^i(c_I) \cap \mathbb{B}_1 w_i v \mathbb{B}_1 \neq \emptyset.$$

Applying Lemma 2.7, there exist $x = \sigma(x) \in {}^{K'}W_e{}^K$ such that $K' = {}^xK$ and $w_i \in xW_K$. Moreover, it follows from (a) that for any simple reflection $s \in \mathrm{supp}_{\Delta_e}(\sigma_I^i(c_I)) \setminus K$ we have $xs \in W_{K'}x$ or $xs \leq_e x$. The former is impossible since $s \notin W_K = xW_{K'}x^{-1}$. So we have $xs \leq_e x$. Moreover, as $xsx^{-1} \notin W_{K'}$ we have $w_{K'}xs \leq_e w_{K'}x = xw_K$, where $w_K$ and $w_{K'}$ are the maximal elements of $W_K$ and $w_{K'}$ respectively. As $xw_K$ is $\sigma_I$-stable, we have $xw_K s \leq_e xw_K$ for all $s \in \Delta_e$, that is, $xw_K = w_e$. Hence $w_i \in w_e W_K$. □

Let $N_0 \in \mathbb{Z}_{\geqslant 0}$ be the order of $c\sigma \in W \rtimes \langle\sigma\rangle$. Define

$$N_F^{F^{N_0}} : \mathbb{T}_h \longrightarrow \mathbb{T}_h, \quad t \longmapsto tF(t)\cdots F^{N_0-1}(t).$$

**Lemma 3.3.** *Let $\chi \in X_*(T)$ and let $C$ be a $c\sigma$-orbit of $\Phi$. Assume $\chi$ is non-central on $C$ and $|\langle\chi, \beta\rangle| < q$ for $\beta \in C$. Then $\sum_{i=0}^{N_0-1} q^i \langle\gamma, (c\sigma)^i(\chi)\rangle \neq 0$ for $\gamma \in C$. In particular, the action of $\mathbb{G}_m$ on $\mathbb{U}_\gamma$ for $\gamma \in C$, via the morphism $N_F^{F^{N_0}} \circ \chi$, is nontrivial.*

*Proof.* By assumption, $|\langle\gamma, (c\sigma)^i(\chi)\rangle| = |\langle(c\sigma)^{-i}(\gamma), \chi\rangle| < q$ for $0 \leqslant i \leqslant N_0 - 1$, and there exists $0 \leqslant i_0 \leqslant N_0 - 1$ such that $\langle(c\sigma)^{-i_0}(\gamma), \chi\rangle \neq 0$. Hence the statement follows. □

Let $\mathbb{G}_m \subseteq \mathcal{O}_{\breve{k}}^\times$ be the Teichmüller lift of the quotient map $\mathcal{O}_{\breve{k}}^\times \to \overline{\mathbb{F}}_q^\times$. Assume that $r \in \mathbb{Z}_{\geqslant 1}$.

**Lemma 3.4.** *Consider the homomorphism*

$$f_{w,\chi} : \mathbb{G}_m \longrightarrow \mathbb{T} \times \mathbb{T}, \quad x \longmapsto (N_F^{F^{N_0}}({}^w\chi(x)), N_F^{F^{N_0}}(\chi(x))).$$

*Then* $\mathrm{Im}(f_{w,\chi}) \subseteq H_{w,\chi}^\circ$.

*Proof.* By definition. $F^{N_0}(\lambda(x)) = \lambda(x^{q^{N_0}})$ for $x \in \breve{k}$. Hence

$$N_F(\chi(x))^{-1}F(N_F(\chi(x))) = \chi(x)^{-1}F^{N_0}(\chi(x)) = \chi(x^{-1}\sigma^{N_0}(x)).$$

So the statement follows. □

3.3. **Handling $\Sigma_w^0$ for $w \in W_e \setminus W_e^{c\sigma}$.** Let $i \in \mathbb{Z}$. Following [DI24, §5] we define an isomorphism of varieties

$$\alpha_i : \Sigma^i \longrightarrow \Sigma^{i+1}, \quad (x, x', y) \longmapsto (x, F(x'), yx').$$

For $w, u \in W_e$ we define

$$Y_{w,u}^i = \Sigma_w^i \cap (\alpha_i)^{-1}(\Sigma_u^{i+1});$$
$$Z_{w,u}^{i+1} = \alpha_i(\Sigma_w^i) \cap \Sigma_u^{i+1} = \alpha_i(Y_{w,u}^i).$$

Let $\hat{Y}_{w,u}^i = (\pi_w^i)^{-1}(Y_{w,u}^i)$ and $\hat{Y}_{w,u}^i = (\pi_u^{i+1})^{-1}(Z_{w,u}^{i+1})$.

**Lemma 3.5.** *Let $w, u \in W_e$. Let $\chi, \mu \in X_*(T)$ which centralizes $\mathbb{K}_{w,i}$ and $\mathbb{K}_{u,i+1}$ respectively. Then $H_{w,\chi}$ and $H_{u,\mu}$ preserve $\hat{Y}_{w,u}^i$ and $\hat{Z}_{w,u}^{i+1}$ respectively.*

*Proof.* This is proved in [DI24, §5]. □

**Proposition 3.6.** *Suppose that condition* (∗) *holds and that $q$ satisfies condition* (2.1)*. Let $i \in \mathbb{Z}$. Then*

$$H_*(\hat{Y}_{w,u}^i) = H_*(Y_{w,u}^i) = H_*(Z_{w,u}^{i+1}) = H_*(\hat{Z}_{w,u}^{i+1}) = 0$$

*if $w$ or $u$ belongs to $W_e \setminus W_e^{c\sigma}$.*

*Proof.* Without loss of generality we can assume that $w \in W_e \setminus W_e^{c\sigma}$ and $\hat{Y}_{w,u}^i \neq \emptyset$. In particular, $\Sigma_w^i \neq \emptyset$. By Lemma 3.2 and Corollary 2.6, there are subsets $K = \sigma_I(K) \subsetneq \Delta_e$ and $J = \sigma(J) \subsetneq \Delta$ such that

$$w(c\sigma)^i \in w_e W_K (\sigma_I)^i \subseteq w_e W_J \sigma^i = w_0 W_J \sigma^i.$$

Thus

$$K_{w,i} \subseteq {}^{w^{-1}}(U^- \cap {}^{w(c\sigma)^i}U^-) \subseteq {}^{w^{-1}w_0}M_J,$$

where $M_J$ is the Levi subgroup generated by $T$ and $U_\gamma$ for $\gamma \in \Phi_J$. Let $\mathcal{O} \in \Delta \backslash J$ be a $\sigma$-orbit. Then $W_J$ fixes $\omega_{\mathcal{O}}^\vee$, and $K_w \subseteq {}^{w^{-1}w_0}M_J$ is centralized by

$$\chi := w^{-1}w_0(\omega_{\mathcal{O}}^\vee) = w^{-1}w(c\sigma)^i\sigma^{-i}(\omega_{\mathcal{O}}^\vee) = (c\sigma)^i(\omega_{\mathcal{O}}^\vee).$$

Moreover, $w(\chi) = w_0\sigma^N(\omega_{\mathcal{O}}^\vee) = (c\sigma)^N(\omega_{\mathcal{O}}^\vee)$.

Let $f_{w,\chi} : \mathbb{G}_m \to H_{w,\chi}$ be the as in Lemma 3.4. In view of Lemma 3.5, via $f_{w,\chi}$ the action of $H_{w,\chi}$ on $\hat{Y}_{w,u}^i$ induces an action of $\mathbb{G}_m$ on $\hat{Y}_{w,u}^i$, which commutes with action of $\mathbb{T}^F \times \mathbb{T}^F$. Hence

$$H_c^*(Y_{w,u}) = H_c^*(\hat{Y}_{w,u}^i) = H_c^*((\hat{Y}_{w,u}^i)^{\mathbb{G}_m}),$$

it suffices to show $(\hat{Y}_{w,u}^i)^{\mathbb{G}_m} = \emptyset$. To this end, we can assume that $\Delta = \cup_{i\in\mathbb{Z}}\sigma^i(H)$ for some/any connected component $H$ of $\Delta$. Then by Proposition 2.3, $\chi, w(\chi) \in \{(c\sigma)^i(\omega_{\mathcal{O}}^\vee); i \in \mathbb{Z}\}$ are non-central on each $c\sigma$-orbit of $\Phi$. As $q > M$, it follows from Lemma 3.3 that

$$(\hat{\Sigma}_{w,u}^i)^{\mathbb{G}_m} \subseteq \{1\} \times \{1\} \times \{1\} \times \mathbb{T} \times \{1\} \times \{1\}.$$

As $w \in W_e \setminus W_e^{c\sigma}$, we deduce that $(\hat{Y}_{w,u}^i)^{\mathbb{G}_m} = \emptyset$ as desired. □

**Corollary 3.7.** *Let $i \in \mathbb{Z}$ and $w \in W_e$. If $w \in W_e \setminus W_e^{c\sigma}$ then $H_*(\Sigma_w^i) = 0$. Otherwise,*

$$H_*(\Sigma_w^i) = \sum_{u\in W_e^{c\sigma}} H_*(Y_{w,u}^i) = \sum_{u\in W_e^{c\sigma}} H_*(Z_{u,w}^i) = \sum_{u\in W_e^{c\sigma}} H_*(Y_{u,w}^{i-1}).$$

*Proof.* Note that $\Sigma_w^i = \sqcup_{u\in W_e} Y_{w,u}^i = \sqcup_{u\in W_e} Z_{u,w}^i$ and $Z_{u,w}^i \cong Y_{u,w}^{i-1}$. Then the statement follows from Proposition 3.6. □

### 3.4. **Handling $\Sigma_w^0$ for $w \in W_e^{c\sigma}$.**

**Lemma 3.8.** *Suppose that Condition (∗) holds. Let $i \in \mathbb{Z}$ and $w, u \in W_e^{c\sigma}$ such that $Y_{w,u}^i \neq \emptyset$. Then $w = u$ if either $\sigma_I \neq 1$ or $\sigma_I = 1$ and $wc_I^i \neq w_e$.*

*Proof.* By assumption, we have

$$\mathbb{B}_1 w(c_I\sigma_I)^i \mathbb{B}_1 c_I\sigma_I \mathbb{B}_1 (c_I\sigma_I)^{-i-1} \cap \mathbb{B}_1 u(c_I\sigma_I)^{i+1}\mathbb{B}_1(c_I\sigma_I)^{-i-1} \neq \emptyset,$$

that is, $\mathbb{B}_1 w(c_I\sigma_I)^i\mathbb{B}_1 c_I\sigma_I\mathbb{B}_1 \cap \mathbb{B}_1 u(c_I\sigma_I)^{i+1}\mathbb{B}_1 \neq \emptyset$. Thus there exists $v \leq_e c_I$ such that $w(c_I\sigma_I)^i v\sigma_I = u(c_I\sigma_I)^{i+1}$. Note that $w, u \in W_e^{c\sigma} \subseteq \langle c_I\sigma_I\rangle$. We have

$$v\sigma_I = (c_I\sigma_I)^{-i} w^{-1} u (c_I\sigma_I)^{i+1} = w^{-1}u(c_I\sigma_I) \in \langle c_I\sigma_I\rangle.$$

In particular, it follows from Lemma 2.5 that $\ell_e(v)$ is divided by $\ell_e(c_I)$.

Assume that either $\sigma_I \neq 1$ or $\sigma_I = 1$ and $wc_I^i \neq w_e$. If $v \neq 1$, then $\ell_e(v) = \ell_e(c_I)$ since $1 \neq v \leq c_I$. Hence we have $v = c_I$ and $w = u$ as desired. Suppose $v = 1$. Then $c_I = u^{-1}w \in W_e^{c\sigma}$, which means that $\sigma_I(c_I) = c_I$. Hence $\sigma_I = 1$ by Theorem 2.1 (3). By assumption we have $\sigma_I = 1$ and $w\sigma_I^i \neq w_e$. As $v = 1$, we have $wc_I^i s < wc_I^i$ for all $s \in \mathrm{supp}_{\Delta_e}(c_I) = \Delta_e$, that is, $wc_I^i = w_e$, a contradiction. □

**Theorem 3.9** ([IN24], Theorem 3.1)**.** *The map $(u_1, u_2) \mapsto u_1^{-1}u_2F(u_1)$ gives an isomorphism*

$$\phi : (F\mathbb{U} \cap \mathbb{U}) \times (F\mathbb{U} \cap \mathbb{U}^-) \cong F\mathbb{U}.$$

*In particular, $\phi$ restricts to an isomorphism*

$$(F\mathbb{U}^1 \cap \mathbb{U}) \times (F\mathbb{U} \cap \mathbb{U}^-) \cong \mathbb{U}^1(F\mathbb{U} \cap \mathbb{U}^-).$$

For $i \in \mathbb{Z}$ and $w \in W_e$ we define

$$^\flat\Sigma_w^i = \{(x, x', y) \in (F\mathbb{U} \cap \mathbb{U}^-) \times F^i(F\mathbb{U} \cap \mathbb{U}^-) \times \mathbb{G}; xF(y) = yx'\}.$$

**Lemma 3.10.** *The map $(x, x', y) \mapsto (x_2, x_2', x_1 y F^i(x_1')^{-1}, x_1, x_1')$ gives an $\mathbb{T}^F \times \mathbb{T}^F$-equivariant isomorphism*

$$\Sigma_w^i \cong {}^\flat\Sigma_w^i \times (F\mathbb{U} \cap \mathbb{U}) \times (F\mathbb{U} \cap \mathbb{U}),$$

*where $(x_1, x_2) = \phi^{-1}(x)$ and $(x_1', x_2') = \phi^{-1}(x')$. In particular, $H_c^*(\Sigma_w^i) \cong H_c^*({}^\flat\Sigma_w^i)$.*

*Proof.* It follows by definition and Theorem 3.9. □

**Lemma 3.11.** *Suppose that $c$ satisfies condition $(*)$. Let $w = (c\sigma)^m \in W_e^{c\sigma}$ for some $m \in \mathbb{Z}$. Then we have $H_*(\Sigma_w^{N-m}) = H_*(\dot{w}\mathbb{T}^F) = H_*({}^\flat\Sigma_w^{2N-m}) = H_*(\Sigma_w^{2N-m})$.*

*Proof.* The first statement is proved in [DI24]. We show the second one. Let $(x, x', y) \in {}^\flat\Sigma_w^{-m}$. By definition,

$$y \in \mathbb{G}^1\mathbb{B}\dot{w}(c\sigma)^{2N-m}\mathbb{B}(c\sigma)^{-m+2N} = \mathbb{U}\mathbb{T}\mathbb{U}^{-,1}\dot{w}.$$

So we may write $y = y_1\tau y_2 w$ uniquely with $y_1 \in \mathbb{U}$, $\tau \in \mathbb{T}$ and $y_2 \in \mathbb{U}^{-,1}$.

Then the equality $xF(y) = yx'$ is equivalent to

$$\tau^{-1}y_1^{-1}xF(y_1)F(\tau) = y_2\dot{w}x'\dot{w}^{-1}F(y_2^{-1}) = y_2x''F(y_2^{-1}),$$

where $x'' = {}^{\dot{w}}x' \in F\mathbb{U} \cap \mathbb{U}^-$ since $w = (c\sigma)^m$.

By Theorem 3.9, the map $(g_1, g_2) \mapsto g_1^{-1}g_2F(g_1)$ gives isomorphisms

$$\mathbb{U} \times (F\mathbb{U} \cap \mathbb{U}^-) \cong \mathbb{U}(F\mathbb{U} \cap \mathbb{U});$$
$$\mathbb{U}^{-,1} \times (F\mathbb{U} \cap \mathbb{U}^-) \cong (F\mathbb{U} \cap \mathbb{U}^-)F\mathbb{U}^{-,1}.$$

So we can make changes of variables $(x, x'', y_1, y_2) \mapsto (z_1, z_2, z_3, z_4)$, where

$$(z_1, z_2, z_3, z_4) \in \mathbb{U} \times F\mathbb{U} \cap \mathbb{U}^- \times \mathbb{U}^{-,1}(F\mathbb{U} \cap \mathbb{U}^-) \times F\mathbb{U}^{-,1} \cap \mathbb{U}$$

such that $y_1^{-1}xF(y_1) = z_1z_2$ and $y_2x''F(y_2)^{-1} = z_3z_4$. Then we have

$$\tau^{-1}z_1z_2F(\tau) = {}^{\tau^{-1}}z_1L(\tau)^{F(\tau)^{-1}}z_2 = z_3z_4,$$

where $L(\tau) = \tau^{-1}F(\tau)$. As $z_4 \in \mathbb{U}^1$ we can have

$${}^{F(\tau)^{-1}}z_2z_4^{-1} = h_+h_0h_- \in \mathbb{U}\mathbb{T}\mathbb{U}^-,$$

where $h_+ \in \mathbb{U}^1$, $h_0 \in \mathbb{T}^1$ and $h_- \in (F\mathbb{U} \cap \mathbb{U}^-)\mathbb{U}^{-,1} = F(\mathbb{U}\mathbb{U}^{-,1}) \cap \mathbb{U}^-$. Hence

$${}^{\tau^{-1}}z_1{}^{L(\tau)}h_+L(\tau)h_0h_- = z_3.$$

It follows that $z_1 = {}^{F(\tau)}h_+$, $L(\tau) = h_0^{-1}$ and $z_3 = h_-$. Therefor,

$$\Sigma_w^i = \{(\tau, z_2, z_4) \in \mathbb{T} \times (F\mathbb{U} \cap \mathbb{U}^-) \times (F\mathbb{U}^{-,1} \cap \mathbb{U}); L(\tau) = \mathrm{pr}_0({}^{F(\tau)^{-1}}z_2z_4)\},$$

where $\mathrm{pr}_0 : \mathbb{U}^1\mathbb{T}\mathbb{U}^- \to \mathbb{T}$ is the natural projection.

Note that $(t, t') \in \mathbb{T}^F \times \mathbb{T}^F$ acts on ${}^\flat\Sigma_w^i$ by $(\tau, z_2, z_4) \mapsto (t\tau w(t')^{-1}, {}^tz_2, {}^{w(t')}z_4)$. Now we define and action of $s \in \mathbb{T}$ on ${}^\flat\Sigma_w^i$ by $(\tau, z_2, z_4) \mapsto (\tau, {}^sz_2, {}^sz_4)$. Then the actions of $\mathbb{T}$ and $\mathbb{T}^F \times \mathbb{T}^F$ commutes with each other. Thus, by Lemma 3.10 we have

$$H_*(\Sigma_w^{2N-m}) = H_*({}^\flat\Sigma_w^{2N-m}) \cong H_*((\Sigma_w^i)^{\mathbb{T}}) = H_*(\dot{w}\mathbb{T}^F)$$

as desired. □

**Proposition 3.12.** *Suppose that Condition (∗) holds and that $\Delta$ is connected. Then $H_c^*(\Sigma_w^0) = H_c^*(\dot{w}\mathbb{T}^F)$ for $w \in W_e^{c\sigma}$.*

*Proof.* Let $w \in W_e^{c\sigma}$. As $\Delta$ is connected, we may write $w = (c\sigma)^m$ for some $m \in \mathbb{Z}$. By Corollary 3.7 we have

(a) $$H_*(\Sigma_w^i) = \sum_{u \in W_e^{c\sigma}} H_*(Y_{w,u}^i), \quad H_*(\Sigma_w^{i+1}) = \sum_{u \in W_e^{c\sigma}} H_*(Y_{u,w}^i).$$

First we assume $\sigma_I \neq 1$. By Lemma 3.8 for any $w', u' \in W_e^{c\sigma}$ we have $Y_{w',u'}^i \neq \emptyset$ if and only if $w' = u'$. It follows by (a) that

$$H_*(\Sigma_w^i) = H_*(Y_{w,w}^i) = H_*(\Sigma_w^{i+1}).$$

By Lemma 3.11 we have $H_*(\Sigma_w^0) = H_*(\Sigma_w^{N-m}) = H_*(w\mathbb{T}^F)$ as desired.

Now we assume $\sigma_I = 1$. Let notation be as in Lemma 2.5. We can assume that $w = c_I^m$ with $0 \leqslant m \leqslant 2N-1$. If $0 \leqslant m \leqslant N$, it follows from (a), Lemma 3.8 and Lemma 3.11 that

$$H_*(\Sigma_w^0) = H_*(\Sigma_w^1) = \cdots = H_*(\Sigma_w^{N-m}) = H_*(\dot{w}\mathbb{T}^F).$$

If $N+1 \leqslant m \leqslant 2N-1$, similarly we have

$$H_*(\Sigma_w^0) = H_*(\Sigma_w^1) = \cdots = H_*(\Sigma_w^{2N-m}) = H_*(\dot{w}\mathbb{T}^F).$$

So the statement follows. □

## 4. Proof of Theorem 2.1

In this section, we fill in the proof for Theorem 2.1. First we show that it suffices to consider one particular Coxeter element.

**Lemma 4.1.** *Let $\alpha \in \{\alpha_1, \sigma^{-1}(\alpha_r)\}$ such that $c' = s_\alpha c\sigma(s_\alpha)$. Suppose Theorem 2.1 holds for $(\mu, c)$. Then it also holds for $(s_\alpha(\mu), c')$.*

*Proof.* Let $\mu, e, I$ be as in Theorem 2.1. Let $e' = e_{s_\alpha(\mu),c'} = s_\alpha(e)$ and $\Phi_{e'} = s_\alpha(\Phi_e)$. Assume that $I = (i_1 < \cdots < i_m)$. Without loss of generality we can assume $\alpha = \sigma^{-1}(\alpha_r)$ and $c' = s_{\alpha_1'} s_{\alpha_2'} \cdots s_{\alpha_r'}$ with $\alpha_1' = \alpha_r$ and $\alpha_i' = \alpha_{i-1}$ for $2 \leqslant i \leqslant r$.

First we assume that $r \in I$. Then $r = i_m$ and $\sigma_{I,c}(\alpha) < 0$, which means that $\alpha \notin \Delta_e = \sigma_{I,c}(\Delta_e)$. Thus $\Phi_{e'}^+ = s_\alpha(\Phi_e^+)$ since $\alpha \in \Delta$ is a simple root. In particular, $\Delta_{e'} = s_\alpha(\Delta_e)$. We take

$$I' = (1 < i_1 + 1 < i_2 + 1 < \cdots < i_{m-1} + 1).$$

Then $\sigma_{I',c'} = s_\alpha \sigma_{I,c} s_\alpha$, $c'_{I'} = s_\alpha c_I s_\alpha$ and the statement follows.

Now we assume that $r \notin I$. Then $\sigma_{I,c}(\alpha) \in \Delta_{I,c} \subseteq \Delta_e = \sigma_{I,c}(\Delta_e)$. Thus $\alpha \in \Delta_e$ and $\Delta_{e'} = \Delta_e$. We take

$$I' = (i_1 + 1 < i_2 + 1 < \cdots < i_m + 1).$$

Then $\sigma_{I',c'} = \sigma_{I,c}$, $c'_{I'} = s_\alpha c_I \sigma_{I,c}(s_\alpha)$ and the statement also follows. □

To finish the proof, we will take a particular $\sigma$-Coxeter element $c$ such that and verify the statement directly. Moreover, we can assume $\Delta$ is connected.

Let $P$ be the coweight be the coweight lattice of $\Phi$. If $\mu = 0 \in P/(1 - c\sigma)P$, then $\Delta_e = \Delta$ and the statement is trivial. So we may assume that $X_*(T)/(1-c)X_*(T) \neq \{0\}$, which excludes the types ${}^2A_{n-1}$ ($n$ even), ${}^3D_4$, $E_8$, ${}^2E_6$, $F_4$, $G_2$. Then we will take a case-by-case analysis for the remaining types.

We adopt the labeling of Dynkin diagrams by positive integers as in [Hum72]. For $i \in \mathbb{Z}_{\geqslant 1}$. let $s_i$ and $\omega_i^\vee$ denote the corresponding simple reflection and fundamental coweight, respectively.

Case (1): $\Delta$ is of type $A_{n-1}$. Take $c = s_1 s_2 \cdots s_{n-1}$. Then we have $P/(1 - c\sigma)P = \{0, \omega_1^\vee, \omega_2^\vee, \ldots \omega_{n-1}^\vee\}$. Assume $\mu = \omega_k^\vee$ with $k \in \mathbb{Z}$. Let $m = \gcd(k, n) \in \mathbb{Z}_{\geqslant 1}$. Then we take $I$ to be the complement of the sequence $I^c = (n/m, 2n/m, \cdots, (m-1)n/m)$.

Case (2): $\Delta$ is of type ${}^2A_{n-1}$ with $n \geqslant 4$ even. Take $c = s_1 s_2 \cdots s_{n/2}$. Then $P/(1 - c\sigma)P = \{0, \omega_1^\vee\}$. Assume $\mu = \omega_1^\vee$. Then we take $I = (n/2)$.

Case (3): $\Delta$ is of type $B_n$ with $n \geqslant 2$. Take $c = s_1 s_2 \cdots s_n$. Then $P/(1 - c\sigma)P = \{0, \omega_1^\vee\}$. Assume $\mu = \omega_1^\vee$. Then we take $I = (n)$.

Case (4): $\Delta$ is of type $C_n$ with $n \geqslant 3$. Take $c = s_1 s_2 \cdots s_n$. Then $P/(1 - c\sigma)P = \{0, \omega_n^\vee\}$. Assume $\mu = \omega_n^\vee$. Then we take $I = (1, 3, \ldots, n - \frac{(-1)^n+1}{2})$.

Case (5): $\Delta$ is of type $D_n$ with $n \geqslant 4$. Take $c = s_1 s_2 \cdots s_n$. Then $P/(1 - c\sigma)P = \{0, \omega_1^\vee, \omega_{n-1}^\vee, \omega_n^\vee\}$. If $\mu = \omega_1^\vee$, take $I = (n-1, n)$. It remains to handle the case $\mu = \omega_{n-1}^\vee$ by symmetry. If $n$ is even, take $I = (1, 3, \ldots, n-3, 4)$ if $4 \mid n$ and $I = (1, 3, \ldots, n-3, n-1)$ if $4 \nmid n$. If $n$ is odd, take $I = (1, 3, \ldots, n-3, n-4, \ldots, n-2, n)$.

Case (6): $\Delta$ is of type $E_6$. Take $c = s_1 s_3 s_4 s_2 s_5 s_6$. Then $P/(1 - c\sigma)P = \{0, \omega_1^\vee, \omega_6^\vee\}$. By symmetry we can assume $\mu = \omega_1^\vee$. Then take $I = (1, 3, 5, 6)$.

Case (7): $\Delta$ is of type $E_6$. Take $c = s_7 s_6 s_5 s_4 s_2 s_3 s_1$. Then $P/(1 - c\sigma)P = \{0, \omega_7^\vee\}$. If $\mu = \omega_7^\vee$, take $I = (7, 5, 2)$.

## References


[Bou68] Nicolas Bourbaki, *Groupes et algèbres de Lie, Chap. 4,5 et 6*, Hermann, Paris, 1968.

[Cha24] Charlotte Chan, *The scalar product formula for parahoric Deligne-Lusztig induction*, preprint (2024), arXiv:2405.00671.

[CI23] Charlotte Chan and Alexander B. Ivanov, *On loop Deligne–Lusztig varieties of Coxeter type for $GL_n$*, Camb. J. Math. **11** (2023), no. 2, 441–505.

[DI24] Olivier Dudas and Alexander Ivanov, *Orthogonality relations for deep level Deligne–Lusztig schemes of Coxeter type*, Forum Math. Sigma **12** (2024), no. e66, 1–27.

[DL76] Pierre Deligne and George Lusztig, *Representations of reductive groups over finite fields*, Ann. Math. **103** (1976), no. 1, 103–161.

[FS] Jessica Fintzen and David Schwein, *In preparation.*

[Hum72] James Humphreys, *Introduction to lie algebras and representation theory*, 1st edition ed., Graduate Texts in Mathematics, vol. 9, Springer Verlag, 1972.

[IN24] Alexander B. Ivanov and Sian Nie, *The cohomology of $p$-adic Deligne–Lusztig schemes of Coxeter type*, preprint (2024), arXiv:2402.09017.

[Iva23a] Alexander B. Ivanov, *Arc-descent for the perfect loop functor and $p$-adic Deligne–Lusztig spaces*, J. reine angew. Math. (Crelle) **2023** (2023), no. 794, 1–54.

[Iva23b] ______, *On a decomposition of $p$-adic Coxeter orbits*, Épijournal de Géom. Alg. **7** (2023).

[Kal19] Tasho Kaletha, *Regular supercuspidal representations*, J. Amer. Math. Soc. **32** (2019), 1071–1170.

[Nie24] Sian Nie, *Steinberg's cross-section of Newton strata*, Math. Ann. (2024), https://doi.org/10.1007/s00208-024-02872-2.

[Yu01] Jiu-Kang Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), 579–622.

[Zhu17] Xinwen Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Ann. Math. **185** (2017), no. 2, 403–492.



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