Habilitationsschrift

zur Erlangung der Venia legendi für das Fach Mathematik
an der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Dr. rer. nat. Alexander Boris Ivanov

Bonn 2020

For my parents, Evgenia and Boris

For Medea and Golda

Table of Contents

There are two levels of page numbering: a global one throughout the whole Habilitationsschrift (also used in the table of contents above), as well as the page numberings in the single articles.

List of publications and preprints included into this Habilitationsschrift

The publications and preprints are listed here and included below in a content-related (not in a chronological) order.

1. *On ind-representability of loop Deligne–Lusztig sheaves*, preprint (2020), arXiv:2003.04399, 28 pages.

2. (with Charlotte Chan) *Cohomological representations of parahoric subgroups*, preprint (2019), arXiv:1903.06153, 24 pages, submitted.

3. (with Charlotte Chan) *Affine Deligne–Lusztig varieties at infinite level*, preprint (2018), arXiv:1811.11204, 68 pages, submitted.

4. (with Charlotte Chan) *On loop Deligne–Lusztig varieties of Coxeter-type for inner forms of* GL_n, preprint (2019), arXiv:1911.03412, 35 pages, submitted.

5. (with Charlotte Chan) *The Drinfeld stratification for* GL_n, preprint (2020), arxiv:2001.06600, 40 pages, submitted.

6. *Affine Deligne–Lusztig varieties of higher level and the local Langlands correspondence for* GL_2, Advances in Mathematics **299** (2016), 640-686.

7. *Ramified automorphic induction and zero-dimensional affine Deligne–Lusztig varieties*, Mathematische Zeitschrift **288** (2018), 439-490.

8. *Ordinary* $\mathrm{GL}_2(F)$*-representations in characteristic two via affine Deligne–Lusztig constructions*, to appear in Mathematical Research Letters (2019+), arXiv:1911.03412, 35 pages.

Zusammenfassung

Der Leitgedanke, an dem sich die vorliegende Habilitationsschrift orientiert, ist es, ein Analogon der klassischen Deligne–Lusztig Theorie über p-adischen Körpern zu entwickeln. Somit gilt das Hauptinteresse einerseits (glatten ℓ-adischen) Darstellungen von p-adischen reduktiven Gruppen und andererseits geometrischen Objekten, deren Kohomologie diese Darstellungen realisiert. Ein Ziel dieser Entwicklung ist es, die dabei entstehenden p-adischen Analoga der Deligne–Lusztig Varietäten zu beschreiben. Ein weiteres Ziel besteht darin, die Darstellungstheorie von p-adischen reduktiven Gruppen besser und von einem neuen – geometrischen, rein lokalen und expliziten – Standpunkt aus zu verstehen. Außerdem erhofft man einen Erkenntnissgewinn über lokale automorphe Induktion, sowie Langlands und Jacquet–Langlands Korrespondenzen.

Die vorliegende Habilitationsschrift ist kumulativ. Sie umfasst acht Arbeiten beziehungsweise Vordrucke. In fünf dieser Arbeiten wird eine direkte Verallgemeinerung von klassischen Deligne–Lusztig Varietäten untersucht: eine Konstruktion, deren erste Inkarnation auf Lusztig zurückgeht. In [Iva20][1] wird eine neue Definition für p-adische Deligne–Lusztig Garben $X_w(b)$ gegeben. Es werden einige ihrer Grundeigenschaften studiert und es wird gezeigt, dass sie oft durch Ind-Schemata darstellbar sind. Das gibt eine partielle Antwort auf die entsprechende Frage von Boyarchenko. In [CI19a] wird eine ganzzahlige Version der Konstruktion vor allem vom darstellungstheoretischen Standpunkt studiert. Insbesondere wird Lusztig's Mackey-Formel vom reduktiven auf den parahorischen Fall verallgemeinert.

Die drei Arbeiten [CI18, CI19b, CI20] nehmen einen Spezialfall dieser Konstruktion genau unter die Lupe: p-adische Deligne–Lusztig Varietäten vom Coxeter-Typ, die zu inneren Formen $\mathbf{G}$ von $\mathbf{GL}_n$ und elliptischen unverzweigten Tori $\mathbf{T} \subseteq \mathbf{G}$ assoziiert sind. In ihrer Kohomologie wird p-adische Deligne–Lusztig Induktion $\theta \mapsto \pm R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ nachgewiesen, die glatten Charakteren θ von $\mathbf{T}(k)$ (hier bezeichnet k den p-adischen Körper) glatte irreduzible superkuspidale Darstellungen von $\mathbf{G}(k)$ zuordnet. Dieses Prozedere bietet (im gegebenen Spezialfall) eine exakte Verallgemeinerung der klassischen Theorie und realisiert darüberhinaus einige Instanzen der lokalen Langlands bzw. Jacquet–Langlands Korrespondenzen (bis auf den rektifizierenden Charakter, der in diesem Fall aber sehr einfach ist).

Die drei übrigen Arbeiten [Iva16, Iva18, Iva19] untersuchen eine verwandte, aber dennoch unterschiedliche Konstruktion, die durch eine Verallgemeinerung von affinen Deligne–Lusztig Varietäten entsteht. Mittels dieser wird erstmals auch p-adische Deligne–Lusztig Induktion für verzweigte elliptische Tori realisiert. Genauer gesagt wird dies im Spezialfall $\mathbf{G} = \mathbf{GL}_2$ für verzweigte (auch sehr wild verzweigte) Tori und Charaktere von beliebig tiefem Level gezeigt. Zumindest im beschriebenen Spezialfall steht diese Induktion im engen Verhältnis mit der Theorie von kuspidalen Typen von Bushnell–Kutzko (beziehungsweise allgemeiner, von Yu). Man kann vermuten, dass dieses Verhältnis für beliebige reduktive Gruppen bestehen bleibt.

[1] Hier und im Folgenden beziehen sich die Referenzen auf die Bibliographie der weiter unten folgenden englischsprachigen Einleitung.

Introduction

The main themes of this work are p-adic[2] analoga of classical Deligne–Lusztig theory [DL76]. Thus the main objects of our investigations are (smooth ℓ-adic) representations of p-adic reductive groups and the geometry of objects whose cohomology realizes these representations. The objects we consider live over the residue field of the p-adic field, and carry actions of the (locally compact) p-adic reductive group and its tori. They generalize classical Deligne–Lusztig varieties, which are designed to realize representations of finite reductive groups.

This habilitation thesis is cumulative and consists of eight articles resp. preprints[3]. Three of them [Iva16, Iva18, Iva19] deal with constructions based on affine Deligne–Lusztig varieties. The remaining five deal with a different generalization of classical Deligne–Lusztig varieties, first incarnations of which appeared in Lusztig's work [Lus79, Lus04]. Roughly, [Iva20] deals with geometric aspects of the general construction, [CI19a] investigates some representation-theoretic aspects, whereas [CI18, CI19b, CI20] contain a detailed study of the important special case related to elliptic tori in GL_n and its inner forms.

The structure of the rest of this introduction is as follows:

- In Section 1 we recall some basic constructions and some of the main results of the classical Deligne–Lusztig theory [DL76].
- After Section 1 the notation used in the rest of the introduction is explained.
- In Section 2 we give a small survey on p-adic Deligne–Lusztig theory.
- In Section 3 we explain the most recent work [Iva20], where the so far most natural definition for a p-adic Deligne–Lusztig object is given, and some representability results are shown.
- In Section 4 we review results of Lusztig [Lus04] on representations of reductive groups over the integers of a p-adic field and their generalization [CI19a] to parahoric group schemes.
- In Section 5 we discuss parts of [CI19b] (and [CI18]) where the construction from [Iva20] in the special case of Coxeter-type p-adic Deligne–Lusztig schemes attached to inner forms of GL_n is carried out.
- Section 6 is also devoted to the Coxeter-type schemes for inner forms of GL_n, but here we investigate the representations realized in the cohomology of these schemes and compare them with special cases of local Langlands and Jacquet–Langlands correspondences. This is related to the content of [CI19b, CI20].
- In Section 7 we explain the "other" construction, via extended affine Deligne–Lusztig varieties, where in particular p-adic Deligne–Lusztig induction for ramified tori is realized in some cases (in contrast to the above, where only unramified tori appear). This is based on [Iva16, Iva18, Iva19].
- In Section 8 some open questions resp. directions of further development are listed.

[2]The theory works well over all local non-archimedean fields. For simplicity we only speak about p-adic fields here.

[3]Four of the eight articles were written in joint work with Charlotte Chan: [CI18, CI19a, CI19b, CI20]. According to the regulations of the Habilitationsordnung, the author hereby confirms that a significant part of the content of each article is due to him.

1. Review of classical Deligne–Lusztig theory

The following review is brief and by no means complete. Let $\mathbf{G}$ be a connected reductive group over the finite field $\mathbb{F}_q$ with q elements and characteristic p, and put $G = \mathbf{G}(\mathbb{F}_q)$. Fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ and let σ denote the Frobenius automorphism of $\overline{\mathbb{F}}_q/\mathbb{F}_q$. In 1968 MacDonald conjectured that there should be a natural map

$$\begin{aligned} \{(\mathbf{T},\theta)\}/\sim \quad &\to \quad \{\text{complex irreducible representations of } G\}/\cong \\ (\mathbf{T},\theta) \quad &\mapsto \quad \pi_{(\mathbf{T},\theta)}, \end{aligned}$$

where on the left side $\mathbf{T}$ goes through $\mathbb{F}_q$-rational maximal tori of $\mathbf{G}$, θ through sufficiently generic characters $\theta\colon \mathbf{T}(\mathbb{F}_q) \to \mathbb{C}^\times$, and the equivalence relation is induced by G-conjugacy. Moreover, if $\mathbf{T}$ is elliptic (i.e., not contained in any $\mathbb{F}_q$-rational proper parabolic subgroup of $\mathbf{G}$), then $\pi_{(\mathbf{T},\theta)}$ should be cuspidal. This conjecture was based on the character tables of $\mathbf{GL}_n(\mathbb{F}_q)$, $\mathbf{Sp}_4(\mathbb{F}_q)$ known at that time. Deligne–Lusztig theory [DL76] developed in 1976 not only resolved this conjecture completely, but gave on top a natural cohomological realization of *all* irreducible representations of G. Even today it remains the only tool allowing a uniform construction of all such representations. The starting point for [DL76] was a computation by Drinfeld, who found cuspidal representations of $\mathbf{SL}_2(\mathbb{F}_q)$ in the ℓ-adic cohomology of a very particular curve over $\mathbb{F}_q$.

Fix an $\mathbb{F}_q$-rational maximal torus of $\mathbf{G}$, contained in an $\mathbb{F}_q$-rational Borel subgroup: $\mathbf{T}_0 \subseteq \mathbf{B}_0 \subseteq \mathbf{G}$. Let W be the Weyl group of $\mathbf{T}_0$. The Frobenius σ acts on W. Up to G-conjugacy the $\mathbb{F}_q$-rational maximal tori of $\mathbf{G}$ are parametrized by the σ-conjugacy classes of W. For $w \in W$, let $\mathbf{T}_w \subseteq \mathbf{G}$ be a representative of the corresponding conjugacy class (for example, for $w = 1$ we recover $\mathbf{T}_0$, and Coxeter elements of W give elliptic tori).

The projective $\mathbb{F}_q$-scheme $\mathbf{G}/\mathbf{B}_0$ is isomorphic to the variety of all Borel subgroups of $\mathbf{G}$. By the Bruhat decomposition the orbits for the action of $\mathbf{G}$ on $(\mathbf{G}/\mathbf{B}_0)^2$ by left multiplication in each factor are parametrized by W, i. e., we have the locally closed decomposition $(\mathbf{G}/\mathbf{B}_0)^2 = \coprod_{w\in W} \mathcal{O}(w)$, where $\mathcal{O}(w)$ is the orbit corresponding to $w \in W$.

Definition 1.1. The *Deligne–Lusztig variety* X_w attached to $w \in W$ is the locally closed subset of $\mathbf{G}/\mathbf{B}_0$ of all elements $g \in \mathbf{G}/\mathbf{B}_0$ such that $(g, \sigma(g)) \in \mathcal{O}(w)$. With other words, X_w is defined by the Cartesian diagram,

$$\begin{array}{ccc} X_w & \longrightarrow & \mathcal{O}(w) \\ \downarrow & & \downarrow \\ \mathbf{G}/\mathbf{B}_0 & \xrightarrow{(\mathrm{id},\sigma)} & \mathbf{G}/\mathbf{B}_0 \times \mathbf{G}/\mathbf{B}_0, \end{array}$$

where the lower map is the graph of the Frobenius morphism.

Then X_w is a smooth quasi-projective $\overline{\mathbb{F}}_q$-scheme. Moreover, for any lift $\dot{w} \in \mathbf{G}(\overline{\mathbb{F}}_q)$ of $w \in W$, X_w has a natural finite Galois covering $\dot{X}_{\dot{w}} \to X_w$ with Galois group $T_w := \mathbf{T}_w(\mathbb{F}_q)$ (changing the lift $\dot{w}$ gives an isomorphic covering). The group G acts on X_w, $\dot{X}_{\dot{w}}$, and the actions of G and T_w on $\dot{X}_{\dot{w}}$ commute. Let $\ell \neq p$ be a prime and let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of the ℓ-adic numbers. Fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$. For a character $\theta\colon T_w \to \overline{\mathbb{Q}}_\ell^\times$ put

$$R_w(\theta) := \sum_{i\in\mathbb{Z}} (-1)^i H_c^i(\dot{X}_{\dot{w}}, \overline{\mathbb{Q}}_\ell)_\theta,$$

where the subscript θ denotes the θ-isotypic component, and the alternating sum is necessarily finite and makes sense as an element of the Grothendieck group of $\overline{\mathbb{Q}}_\ell$-representations of G. Some of the main results of [DL76] can be summarized in the following theorem.

Theorem 1.2 (Corollary 4.3, Theorem 6.8, Theorem 8.3 and Corollary 7.7 of [DL76]).

(i) The virtual G-representation $R_w(\theta)$ depends only on the G-conjugacy class of $(\mathbf{T}_w, \theta)$, but not on the choice of w.
(ii) If θ is generic, then $\pm R_w(\theta)$ is an irreducible G-representation.
(iii) If θ is generic and $\mathbf{T}_w$ elliptic, then $\pm R_w(\theta)$ is cuspidal.
(iv) Every irreducible G-representation occurs in the support (in the Grothendieck group) of $R_w(\theta)$ for some $(\mathbf{T}_w, \theta)$.

Let us mention two important technical tools from [DL76].

Theorem 1.3 (Mackey formula; Theorem 6.8 of [DL76]). *Let $(\mathbf{T}_w, \theta)$, $(\mathbf{T}_{w'}, \theta')$ be two pairs as above. Then*

$$\langle R_w(\theta), R_{w'}(\theta')\rangle_G = \#\{w \in W(\mathbf{T}_w, \mathbf{T}_{w'})^\sigma \colon \mathrm{Ad}(w)(\theta) = \theta'\},$$

where $W(\mathbf{T}_w, \mathbf{T}_{w'}) = \{x \in \mathbf{G} \colon x\mathbf{T}_w x^{-1} = \mathbf{T}_{w'}\}/\mathbf{T}_w$.

This is a vast generalization of the original Mackey formula for representations induced from a Borel subgroup (which corresponds to the special case $w = 1$). Theorem 1.2(ii) is an immediate consequence of Theorem 1.3. An other important tool is the character formula [DL76, Theorem 4.2], which expresses the traces of elements $g \in G$ in $R_w(\theta)$ inductively in terms of Deligne–Lusztig representations attached to subgroups of $\mathbf{G}$ of smaller rank. The main ingredient in its proof is the following result.

Theorem 1.4 (Deligne–Lusztig fixed point formula; Theorem 3.2 of [DL76]). *Let X be a separated scheme of finite type over $\overline{\mathbb{F}}_q$ and let $g \colon X \to X$ be an automorphism of finite order. Decompose $g = s \cdot u$, where s,u are the powers of g respectively of order prime to p and a power of p. If X^s is the subscheme of fixed points of s on X, then*

$$\mathrm{Tr}\left(g; \sum_{i\in\mathbb{Z}} (-1)^i H^i_c(X, \overline{\mathbb{Q}}_\ell)\right) = \mathrm{Tr}\left(u; \sum_{i\in\mathbb{Z}} (-1)^i H^i_c(X^s, \overline{\mathbb{Q}}_\ell)\right).$$

Notation for the rest of this introduction. We fix a non-archimedean local field k, and let $\breve{k}$ be the completion of a maximal unramified extension of k, $\overline{\mathbb{F}}_q/\mathbb{F}_q$ the corresponding extension of residue fields, σ the Frobenius automorphism of $\breve{k}/k$ (and of $\overline{\mathbb{F}}_q/\mathbb{F}_q$). We write $\mathcal{O}_k$, $\mathfrak{p}_k$ resp. $\mathcal{O}_{\breve{k}}$, $\mathfrak{p}_{\breve{k}}$ for the integers and maximal ideal of k resp. $\breve{k}$, ϖ for a fixed uniformizer of k, ord for the valuation of $\breve{k}$, normalized such that $\mathrm{ord}(\varpi) = 1$. Denote by k^{alg} an algebraic closure of k containing $\breve{k}$.

Further, $\mathbf{G}$ always denotes a reductive group over k and $G = \mathbf{G}(k)$. We denote the base change of $\mathbf{G}$ to $\breve{k}$ again by $\mathbf{G}$. In general, by bold letters we denote algebro-geometric objects over k (or $\breve{k}$), by usual letters their k-points, and by calligraphic letters their $\mathcal{O}_k$-models, e.g. $\mathcal{G}$ will often be a smooth affine $\mathcal{O}_k$-model of $\mathbf{G}$.

Most of the time we work with *perfect* (ind-)schemes. To simplify notation, we will write "(ind-)scheme" to mean a "perfect (ind-)scheme". Also, $\mathbb{A}^n_{\overline{\mathbb{F}}_q}$ will denote the perfection of the usual affine n-space over $\overline{\mathbb{F}}_q$.

We fix $\overline{\mathbb{Q}}_\ell$ as in Section 1. All cohomology groups are ℓ-adic étale cohomology groups with compact support. We sometimes write $H_c^*(X, \overline{\mathbb{Q}}_\ell)$ for $\sum_{i\in\mathbb{Z}}(-1)^i H_c^i(X, \overline{\mathbb{Q}}_\ell)$. Purely inseparable morphisms induce isomorphisms on ℓ-adic cohomology, so if X is of perfectly finite type over $\overline{\mathbb{F}}_q$ – say, X is the perfection of some X_0, which is of finite type over $\overline{\mathbb{F}}_q$ – then $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ makes sense and is naturally isomorphic to $H_c^i(X_0, \overline{\mathbb{Q}}_\ell)$.

2. Deligne–Lusztig theory over local fields: a survey

This section contains a brief and not strictly chronological survey. The main focus lies on topics related to the present work.

(2.1) The story of Deligne–Lusztig theory for p-adic reductive groups began 1979 with Lusztig's brilliant article [Lus79]. To an unramified k-rational maximal torus $\mathbf{T} \subseteq \mathbf{G}$ and a (only $\breve{k}$-rational) Borel subgroup $\mathbf{B} = \mathbf{TU}$ containing $\mathbf{T}$ and with unipotent radical $\mathbf{U}$, he attached a set,

$$X_{\mathrm{Lus}} = \{g \in \mathbf{G}(\breve{k})\colon g^{-1}\sigma(g) \in \mathbf{U}(\breve{k})\}/\mathbf{U}(\breve{k}) \cap (\sigma^{-1}\mathbf{U})(\breve{k})$$

carrying the commuting actions of $G = \mathbf{G}(k)$ and $T = \mathbf{T}(k)$ by left and right multiplication, of which he conjectured that it has a natural structure of an (infinite-dimensional) scheme over the residue field $\overline{\mathbb{F}}_q$ of $\breve{k}$. Using an *ad hoc* method he verified this conjecture in the case that $\mathbf{G}$ is an anisotropic form of $\mathbf{SL}_n$, and proved that the alternating sum of ℓ-adic homology groups of the resulting scheme (makes sense and) realizes many supercuspidal representations of G.

(2.2) For a reductive group $\mathcal{G}$ over the integers $\mathcal{O}_k$, one can write down the analog X_h of X_{Lus} attached to the base change $\mathcal{G} \otimes_{\mathcal{O}_k} \mathcal{O}_k/\mathfrak{p}^h$ ($h \geq 1$) and make sense of it as a (perfect) scheme. Moreover, it is of (perfectly) finite type over $\overline{\mathbb{F}}_q$. In [Lus04], Lusztig extended his techniques from [Lus79] to compute the alternating sum of the ℓ-adic cohomology groups of X_h, however with some unpleasant restriction, which will be explained later (Section 4.2). Lusztig formulated his results only in the case $\operatorname{char} k > 0$; the quite similar case $\operatorname{char} k = 0$ was treated later by Stasinski [Sta09]. More recently, Chan and the author generalized these results to the case that $\mathcal{G}$ is just a parahoric group scheme over $\mathcal{O}_k$, and computed the traces of certain *very regular* elliptic elements in $\mathcal{G}(\mathcal{O}_k)$ in the cohomology of the resulting varieties [CI19a].

(2.3) Using Lusztig's *ad hoc* scheme structure, the scheme X_{Lus} was extensively studied in the "division algebra case" (i.e., $\mathbf{G}$ is an inner form of $\mathbf{GL}_n$, such that G is the group of units of a division algebra with Hasse invariant s/n, $\gcd(s, n) = 1$) by Boyarchenko [Boy12] and Chan [Cha16, Cha18, Cha19]. In particular, Boyarchenko developed technical tools to study the cohomology groups of X_{Lus} in single degrees, which then were further improved by Chan. An important observation is that certain subschemes $Z_h \subseteq X_h$ strongly related to $\varprojlim_h X_h \subseteq X_{\mathrm{Lus}}$ turn out to be (perfections of) *maximal varieties* over $\mathbb{F}_{q^n}$, i.e., the number $\#Z_h(\mathbb{F}_{q^n})$ attains the Weil–Deligne bound with respect to the ℓ-adic Betti numbers of Z_h. For example, this allows to completely determine the action induced by the Frobenius σ^n in their cohomology.

(2.4) In [Iva16] the author suggested a different approach, noticing that certain covers of Rapoport's affine Deligne–Lusztig varieties [Rap05] could serve as a good alternative to Lusztig's set X_{Lus}. If $\mathcal{J}$ is a (nice) $\mathcal{O}_k$-model of $\mathbf{G}$, then the quotient fpqc-sheaf $L\mathbf{G}/L^+\mathcal{J}$ (here $L\mathbf{G}$ is the loop group of $\mathbf{G}$, and $L^+\mathcal{J}$ is the group of positive loops of $\mathcal{J}$) is representable by an ind-scheme, and one can define subsets $X_{\dot{w}}^{\mathcal{J}}(1) \subseteq (L\mathbf{G}/L^+\mathcal{J})(\overline{\mathbb{F}}_q)$ (with $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)(\breve{k})$ for a split maximal torus $\mathbf{T}_0 \subseteq \mathbf{G}$) of them, generalizing the construction from [Rap05] (where $\mathcal{J}$ is a parahoric model)[4]. Shrinking $\mathcal{J}$ one gets a tower $\{X_{\dot{w}}^{\mathcal{J}}(1)\}_{\mathcal{J}}$ of subsets of $(L\mathbf{G}/L^+\mathcal{J})(\overline{\mathbb{F}}_q)$, each with the actions of G and the k-points of an unramified maximal torus of $\mathbf{G}$, depending on $\dot{w}$. An advantage is that there is a natural candidate for the (ind-)scheme-structure: one only has to show that $X_{\dot{w}}^{\mathcal{J}}(1)$ is locally closed in $L\mathbf{G}/L^+\mathcal{J}$. Already for $\mathbf{G} = \mathbf{GL}_2$ and an elliptic unramified torus, the scheme-structure on X_{Lus} was not known, whereas $\{X_{\dot{w}}^{\mathcal{J}}(1)\}_{\mathcal{J}}$ gave reasonable schemes, and in [Iva16] it was shown that in this case the expected representations appear in the cohomology of the tower. A disadvantage is the higher complexity (e. g., the choices of $\mathcal{J}, \dot{w}$).

(2.5) The two apparently different approaches [Lus79] and [Iva16] were brought together in the case that $\mathbf{G}$ is an inner form of $\mathbf{GL}_n$ and $\mathbf{T}$ an elliptic unramified torus. More precisely, Chan and the author showed in [CI18] that in this case there is an equivariant isomorphism between the *set* X_{Lus} and the set of $\overline{\mathbb{F}}_q$-points of the $\overline{\mathbb{F}}_q$-scheme $\varprojlim_{\mathcal{J}} X_{\dot{w}}^{\mathcal{J}}(b)$ (at least for appropriate choices of $\mathcal{J}, \dot{w}$). This *a posteriori* endows X_{Lus} with a scheme structure. Combining and improving now techniques from [Lus79, Lus04] and [Iva16], it was then shown that the cohomology of X_{Lus} parametrizes certain supercuspidal representations π of G (those π which satisfy (i) the L-parameter of π factors through ${}^L\mathbf{T} \to {}^L\mathbf{G}$ and (ii) π is "*très cuspidal*" in the sense of Carayol [Car84])[5].

(2.6) In two follow-up articles [CI19b, CI20], which still deal with elliptic unramified tori in inner forms of $\mathbf{GL}_n$, Chan and the author extended the results of [CI18], by considerably simplifying the proof of representability of X_{Lus}, which does not require anymore the bypass through affine Deligne–Lusztig varieties. Moreover, we extended the results on cohomology by (almost) removing the assumption "*très cuspidale*" on π mentioned above. This cohomological study requires a serious improvement of most of the above-mentioned techniques.

(2.7) Until now only analoga over k of classical Deligne–Lusztig varieties *of Coxeter type* were considered (except for Section (2.2) above, but there it was over $\mathcal{O}_k$ only). If one wants achieve a natural generalization of classical Deligne–Lusztig theory, then one should also cover those which are not of Coxeter type. The expectation that such make sense as ind-schemes was first formulated by Boyarchenko [Boy12, Problem 1]. In the recent work [Iva20], the author gave – following a suggestion by Scholze – a more natural definition of Deligne–Lusztig spaces $X_w(b)$ as arc-sheaves on perfect schemes over $\mathbb{F}_q$, and proved their ind-representability in many cases.

(2.8) All Deligne–Lusztig spaces discussed above are attached to unramified maximal tori $\mathbf{T} \subseteq \mathbf{G}$. A very natural question [Boy12, Problem 3] to ask is, whether there are similar constructions corresponding to *ramified* tori of $\mathbf{G}$. The first try in this direction was undertaken by Stasinski [Sta11], who suggested a construction of so called *extended Deligne–Lusztig varieties* (only for a group $\mathcal{G}/\mathcal{O}_k$ as in (2.2) above), and computed an example attached to $\mathrm{SL}_2(\mathcal{O}_k/\mathfrak{p}^2)$. Unfortunately, this does not yet lead to a realization of supercuspidal representations. In [Iva18] the author then defined *extended affine Deligne–Lusztig varieties*, which combine certain features of both, Stasinski's construction, as well as the affine Deligne–Lusztig varieties of higher

[4]It was shown by He [He14] that the cohomology of Iwahori-level affine Deligne–Lusztig varieties does not contain supercuspidal representations of positive level; this fits well into the general picture: only shrinking the level subgroup $\mathcal{J}$, one gets representations of higher level.

[5]If $\mathbf{G} \neq \mathbf{GL}_n$, we encountered here the problem that methods from [Lus04] had to be generalized to non-reductive – but still parahoric – group schemes $\mathcal{G}/\mathcal{O}_k$. This was the motivation for our work [CI19a] (see Section (2.2) above).

level from [Iva16]. For ramified tori in $\mathbf{GL}_2$ these schemes (which in that case are simply disjoint unions of points, endowed with quite complicated group actions) allow a cohomological realization of Bushnell–Kutzko types, even in very wild situations.

(2.9) Finally, we discuss two further related developments. The work of Boyarchenko–Weinstein [BW16] deals with certain formal models of special affinoids in the Lubin–Tate space, centered around points with CM by an unramified extension of the base field k. The cohomology of the reductions of these affinoids realizes the local Langlands and Jacquet–Langlands correspondences for supercuspidal representations satisfying conditions (i) and (ii) in (2.5) above. These reductions seem to be (not exactly equal but) related to Lusztig's schemes X_h.

A generalization of [Lus04] going roughly in a similar direction as some of the abovementioned articles is the work of Stasinski–Chen [CS17] and Chen [Che18], who related Lusztig's representations to those constructed by Gérardin in purely algebraic terms [Gé75].

3. Deligne–Lusztig arc-sheaves and ind-representability

This section explains the most recent article [Iva20](cf. Section (2.7) above), which partially resolves the following problem.

Problem [Boy12, Problem 1] *Formalize Lusztig's construction from Section (2.1) for an arbitrary reductive group* $\mathbf{G}$*, that is, define* X_{Lus} *as an ind-scheme over* $\overline{\mathbb{F}}_q$ *and define its homology groups* $H_i(X_{\mathrm{Lus}}, \overline{\mathbb{Q}}_\ell)$*, in such a way that the action of* $G \times T$ *on* X_{Lus} *yields smooth representations of* $G \times T$ *in* $H_i(X_{\mathrm{Lus}}, \overline{\mathbb{Q}}_\ell)$ *for all* $i \geq 1$*.*

In [Iva20] a formal definition of Deligne–Lusztig spaces $X_w(b)$ is given in terms of sheaves on perfect $\overline{\mathbb{F}}_q$-algebras, and it is shown that these sheaves are ind-representable in many cases (but nothing is said about homology). We now explain this definition. First, Definition 1.1 can not be carried over literally. Indeed, in the classical case the geometric Frobenius $\mathbf{G}(\overline{\mathbb{F}}_q) \to \mathbf{G}(\overline{\mathbb{F}}_q)$, $x \mapsto \sigma(x)$ (induced by the $\mathbb{F}_q$-rational structure of $\mathbf{G}$) is a honest morphism of $\overline{\mathbb{F}}_q$-schemes. Over k, there is still a natural Frobenius map $\mathbf{G}(\breve{k}) \to \mathbf{G}(\breve{k})$, but it does not make sense as a morphism of $\breve{k}$-schemes. The problem is resolved by artificially making Frobenius a scheme morphism, which requires the loop functor construction.

3.1. **Loop functor.** Let $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ denote the category of perfect $\overline{\mathbb{F}}_q$-algebras. For $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$, let

$$\mathbb{W}(R) = \begin{cases} W_{\mathcal{O}_{\breve{k}}}(R) & \text{if } \operatorname{char} k = 0, \\ R[\![\varpi]\!] & \text{if } \operatorname{char} k > 0, \end{cases}$$

where $W_{\mathcal{O}_{\breve{k}}}(R)$ are the ramified (p-typical) Witt-vectors relative to $\mathcal{O}_{\breve{k}}$ (as in [FF18, 1.2]). In particular, $\mathbb{W}(\overline{\mathbb{F}}_q)[1/\varpi] = \breve{k}$. For a scheme $\mathbf{X}/\breve{k}$, we can define the functor $L\mathbf{X}$ on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ by setting

$$L\mathbf{X}(R) = \mathbf{X}(\mathbb{W}(R)[1/\varpi]).$$

If $\mathbf{X}$ is an affine scheme of finite type over $\breve{k}$, then $L\mathbf{X}$ is an ind-scheme[6] over $\overline{\mathbb{F}}_q$. Let us illustrate this for $\mathbf{X} = \mathbb{A}^1_{\breve{k}}$. For $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$, any element of $\mathbb{W}(R)[1/\varpi]$ has a unique expression as a convergent sum $\sum_{i \gg -\infty} [a_i]\varpi^i$, where $[\cdot]\colon R \to \mathbb{W}(R)$ denotes either the Teichmüller lift ($\operatorname{char} k = 0$) or the natural inclusion ($\operatorname{char} k > 0$). Hence $L\mathbb{A}^1_{\breve{k}}(R) = \mathbb{W}(R)[1/\varpi] \cong \varinjlim_{N \to -\infty} \prod_{i=N}^{+\infty} R$ (as

[6] recall our convention that "ind-scheme" in fact means "perfect ind-scheme".

sets), and hence $L\mathbb{A}^1_{\breve{k}} \cong \varinjlim_{N\to-\infty} \prod_{i=N}^{+\infty} \mathbb{A}^1_{\overline{\mathbb{F}}_q}$, which is an ind-scheme over $\overline{\mathbb{F}}_q$. If $\mathbf{X}$ is only quasi-projective over $\breve{k}$, then $L\mathbf{X}$ seems to be not ind-representable in general. At least it was known to be an fpqc-sheaf. The following improvement was suggested by Scholze.

Theorem 3.1 (Theorem A of [Iva20])**.** *Let $\mathbf{X}$ be a quasi-projective $\breve{k}$-scheme. Then $L\mathbf{X}$ is an arc-sheaf.*

Here by an arc-sheaf we mean a sheaf for the arc-topology on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ in the sense of Bhatt–Mathew [BM18]. Roughly a map $R \to R'$ in $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ is an arc-cover if any immediate specialization in $\mathrm{Spec}\, R$ lifts to $\mathrm{Spec}\, R'$. The proof of Theorem 3.1 is based on two inputs:

(i) A precise analysis of what $\mathrm{Spec}\, \mathbb{W}(\cdot)[1/\varpi]$ does to arc-covers. In particular, it is shown that if $R \to R'$ is an arc-cover, then the map $\mathrm{Spec}\, \mathbb{W}(R')[1/\varpi] \to \mathrm{Spec}\, \mathbb{W}(R)[1/\varpi]$ is dominant and its image contains all closed points of the target. This is achieved by studying the adic spectrum $\mathrm{Spa}(\mathbb{W}(R)[1/\varpi], \mathbb{W}(R))$ and its relation to $\mathrm{Spa}(R, R)$.

(ii) Descent (along arc-covers in $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$) for vector bundles on $\mathrm{Spa}\, \mathbb{W}(R)[1/\varpi]$. This uses perfectoid techniques of Scholze [Sch18, SW20].

With regard to our original problem of defining Deligne–Lusztig spaces, the most important property of the loop functor is that if $\mathbf{X}/\breve{k}$ is the base change of a k-scheme, then the sheaf $L\mathbf{X}$ is endowed with a geometric Frobenius automorphism $\sigma = \sigma_{L\mathbf{X}} \colon L\mathbf{X} \to L\mathbf{X}$. For example, in the case $\mathbf{X} = \mathbb{A}^1_{\breve{k}}$ discussed above (with the obvious k-rational structure), if we have coordinates $\prod_{i=N}^{+\infty} \mathbb{A}^1_{\overline{\mathbb{F}}_q} \cong \mathrm{Spec}\, \overline{\mathbb{F}}_q[x_N^{1/p^\infty}, x_{N+1}^{1/p^\infty}, \dots]$, then σ is the $\overline{\mathbb{F}}_q$-morphism given by $x_i \mapsto x_i^q$.

3.2. **Loop Deligne–Lusztig spaces.** After the above discussion, it is clear what the right definition must be: take the Cartesian square in Definition 1.1 and replace all entries (except the upper left one) by corresponding loop functors. To fix ideas, assume that $\mathbf{G}$ is an *unramified* reductive group over k, and fix a k-rational maximal torus contained in a k-rational Borel subgroup: $\mathbf{T}_0 \subseteq \mathbf{B}_0 \subseteq \mathbf{G}$. Let W be the Weyl group of $\mathbf{T}_0$ in $\mathbf{G}$. In contrast to the classical case, now only *stable conjugacy classes*[7] of unramified k-rational maximal tori in $\mathbf{G}$ are parametrized by σ-conjugacy classes of W [DeB06, Lemma 4.3.1], whereas the parametrization of G-conjugacy classes of those tori is more complicated (see [DeB06]). The following (probably most elegant) definition was suggested by P. Scholze.

Definition 3.2 (Definition 7.2 of [Iva20])**.** Let $b \in \mathbf{G}(\breve{k})$, $w \in W$. Define $X_w(b)$ by the following Cartesian diagram of functors on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$:

$$\begin{array}{ccc} X_w(b) & \longrightarrow & L\mathcal{O}(w) \\ \downarrow & & \downarrow \\ L(\mathbf{G}/\mathbf{B}_0) & \xrightarrow{(\mathrm{id}, b\sigma)} & L(\mathbf{G}/\mathbf{B}_0) \times L(\mathbf{G}/\mathbf{B}_0) \end{array}$$

Few remarks are in order after this definition:

- By Theorem 3.1 all $X_w(b)$ are arc-sheaves.
- Following [RZ96, 1.12], for $b \in \mathbf{G}(\breve{k})$, let $\mathbf{G}_b$ be the functor on k-algebras defined by

$$\mathbf{G}_b(A) = \{g \in \mathbf{G}(A \otimes_k \breve{k}) \colon g(b\sigma) = (b\sigma)g\}. \tag{3.1}$$

[7] maximal k-tori $\mathbf{T}_1, \mathbf{T}_2$ of $\mathbf{G}$ are *stably conjugate*, if there exists some $g \in \mathbf{G}(k^{\mathrm{alg}})$ such that $\mathbf{T}_1(k) = g\mathbf{T}_2(k)g^{-1}$.

It is representable by a smooth affine k-group scheme, which is an inner form of a Levi subgroup of $\mathbf{G}$. In particular, $\mathbf{G}_b(k)$ is a locally profinite group. Denote by $\underline{\mathbf{G}_b(k)}$ the corresponding constant group scheme[8]. Then $\underline{\mathbf{G}_b(k)}$ acts on $X_w(b)$.

- Essentially, $X_w(b)$ only depends on the σ-conjugacy class $[b]_{\mathbf{G}} = \{g^{-1}b\sigma(g) \colon g \in \mathbf{G}(\breve{k})\}$ of b. This also holds in the classical Deligne–Lusztig theory, and due to Lang's theorem, the parameter b is obsolete there, as it always can be replaced by $b = 1$ (which we did in Definition 1.1). However, here the set of σ-conjugacy classes – i.e., Kottwitz's set $B(\mathbf{G})$, see [Kot85] – is in general more complicated.
- Any reductive k-group $\mathbf{G}'$, which splits over $\breve{k}$ is (up to a z-extension) an inner form of an unramified group $\mathbf{G}$, i.e., there is a (basic) $b \in \mathbf{G}(\breve{k})$ and an isomorphism $\mathbf{G}' \cong \mathbf{G}_b$, so that the Deligne–Lusztig varieties "belonging to $\mathbf{G}'$" are covered by the above definition for $\mathbf{G}$. Hence the assumption "$\mathbf{G}$ unramified" is not a severe restriction.
- Replacing $\mathbf{G}/\mathbf{B}_0$ by $\mathbf{G}/\mathbf{U}_0$ (with $\mathbf{U}_0$ = unipotent radical of $\mathbf{B}_0$) and $w \in W$ by $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)(\breve{k})$, one can define sheaves $\dot{X}_{\dot{w}}(b)$ over $X_w(b)$, endowed with an action of the constant group scheme $\underline{\mathbf{T}_w(k)}$, where $\mathbf{T}_w$ is a representative of the stable conjugacy class of tori corresponding to w. In contrast to the classical theory, $\dot{X}_{\dot{w}}(b)$ really depends on the lift $\dot{w}$[9], e. g., it might happen that $\dot{X}_{\dot{w}_1}(b) \neq \varnothing$, whereas $\dot{X}_{\dot{w}_2}(b) = \varnothing$ for $\dot{w}_1, \dot{w}_2$ with the same image in W [CI19b, Lemma 2.5(iii)].
- Although $\dot{X}_{\dot{w}}(b)$ and Lusztig's sets X_{Lus} from Section (2.1) look differently, the sheaf version of X_{Lus} can be expressed as some $\dot{X}_{\dot{w}}(b)$, and when b is basic plus some condition holds, then also the converse is possible. (This is something which already appears in the classical theory: there are two slightly different ways to write down the same torsor $\dot{X}_{\dot{w}}$ over a Deligne–Lusztig variety X_w.)

Next, we investigate the basic properties of the sheaves $X_w(b)$. Let $b \in \mathbf{G}(\breve{k})$ with σ-conjugacy class $[b]_{\mathbf{G}} \subseteq \mathbf{G}(\breve{k})$. If $\mathbf{B}_0 \subseteq \mathbf{P} \subseteq \mathbf{G}$ is a k-rational parabolic subgroup of $\mathbf{G}$, then $[b]_{\mathbf{G}} \cap \mathbf{P}(\breve{k})$ decomposes as a finite disjoint union $[b]_{\mathbf{G}} = \dot{\bigcup}_{i=1}^{r} [b_i]_{\mathbf{P}}$ of σ-conjugacy classes of $\mathbf{P}(\breve{k})$. If $w \in W$, there is a smallest k-rational parabolic subgroup $\mathbf{P}$ (containing $\mathbf{B}$) such that w is represented by an element of $\mathbf{P}(\breve{k})$. Let $\mathbf{M}$ be the unique Levi factor of $\mathbf{P}$, which contains $\mathbf{T}$. We have the images of $b_i \in \mathbf{P}(\breve{k})$ in $\mathbf{M}(\breve{k})$, again denoted b_i. Then

$$X_w^{\mathbf{G}}(b) \cong \coprod_{i=1}^{r} \underline{\mathbf{G}_{b_i}(k)/\mathbf{P}_{b_i}(k)} \times X_w^{\mathbf{M}}(b_i), \tag{3.2}$$

where $\underline{H}$ denotes the constant sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ attached to a profinite set H, and the upper indices $\mathbf{G}, \mathbf{M}$ denote the group to which $X_w(b)$ is attached. This is proven in [Iva20, Theorem B]. The proof is more complicated than the proof of the analogous fact in the classical case [Lus76b, 3].

Corollary 3.3 (Corollary 7.11 of [Iva20])**.** *Let $b, w, \mathbf{P}$ be as above. If $[b]_{\mathbf{G}} \cap \mathbf{P}(\breve{k}) = \varnothing$, then $X_w(b) = \varnothing$.*

This generalizes an observation by Viehmann, that $X_1(b) = \varnothing$ for b superbasic. Further, one has the technique of the *Frobenius cyclic shift* [Iva20, §7.4], which establishes an isomorphism

[8] If $H \subseteq \mathbf{G}_b(k)$ is an open compact subgroup, then $\underline{H} = \mathrm{Spec}\,\mathrm{Cont}(H, \overline{\mathbb{F}}_q)$, where $\overline{\mathbb{F}}_q$ is equipped with the discrete topology, and $\underline{\mathbf{G}_b(k)}$ is (as a scheme) disjoint union of copies of $\underline{H}$.

[9] We suspect that this might also be related to the difference between rational and stable conjugacy classes of tori.

$X_w(b) \cong X_{w'}(b)$ if $w = w_1 w_2$, $w' = w_2\sigma(w_1)$ such that $\ell(w) = \ell(w_1) + \ell(w_2) = \ell(w')$. This generalizes the classical situation [DL76, Proof of Theorem 1.4] with essentially the same proof.

3.3. **Ind-representabilty.** Finally, in [Iva20] the ind-representability of $X_w(b)$ is proven for many w and all b. One can quite closely follow the strategy of Bonnafé–Rouquier [BR08], who give a new proof of a theorem of Orlik–Rapoport [OR08, §5] and He [He08, Theorem 1.3] claiming affineness of certain classical Deligne–Lusztig varieties. The *Braid monoid* is the monoid with presentation

$$B^+ = \left\langle (\underline{x})_{x\in W} : \forall x, x' \in W, \ell(xx') = \ell(x) + \ell(x) \Rightarrow \underline{xx'} = \underline{x}\,\underline{x'} \right\rangle.$$

For any set I of simple reflections of W, let w_I denote the longest element in the parabolic subgroup $W_I \subseteq W$ corresponding to I.

Theorem 3.4 (Theorem 8.1 of [Iva20]). *If I is a σ-stable set of simple reflections and $w \in W_I$ is such that there exists an integer $d > 0$ and $a \in B^+$ with $\underline{w}\sigma(\underline{w})\dots\sigma^{d-1}(\underline{w}) = \underline{w}_I a$, then for all $b \in \mathbf{G}(\breve{k})$ and all $\dot{w}$ lifting w, the arc-sheaves $X_w(b)$, $\dot{X}_{\dot{w}}(b)$ are representable by ind-schemes.*

The main idea of the proof – following [BR08] – is that certain generalization $\mathcal{O}(w_1,\dots,w_r) \subseteq (\mathbf{G}/\mathbf{B}_0)^{r+1}$ of the $\mathbf{G}$-orbits $\mathcal{O}(w) \subseteq (\mathbf{G}/\mathbf{B}_0)^2$ becomes an affine $\breve{k}$-scheme under some condition on the w_i's naturally expressed in B^+ (whereas $\mathcal{O}(w)$ itself is affine if and only if w is the longest element of W). Then one can write $X_w(b)$ as a pull-back of $L\mathcal{O}(w_1,\dots,w_r) \hookrightarrow L(\mathbf{G}/\mathbf{B}_0)^{r+1}$ along a morphism $L(\mathbf{G}/\mathbf{B}_0) \hookrightarrow L(\mathbf{G}/\mathbf{B}_0)^{r+1}$ which is represented by closed immersions. As the loop functor transforms affine schemes of finite type into ind-schemes, $X_w(b)$ is becomes expressed as a closed sub-ind-scheme of an ind-scheme, hence itself is an ind-scheme. Using now some known results about the combinatorics of the Weyl group, one deduces – again, similar as in [BR08] – the following corollary.

Corollary 3.5 (Theorem C of [Iva20]). *Let $w \in W$ be of minimal length in its σ-conjugacy class. Then for all $b \in \mathbf{G}(\breve{k})$, and all lifts $\dot{w}$ of w, the arc-sheaves $X_w(b)$, $\dot{X}_{\dot{w}}(b)$ are representable by ind-schemes.*

The proofs of Theorem 3.4 and Corollary 3.5 require also (3.2) and the Frobenius-cyclic shift. Lusztig's original conjecture goes beyond ind-representability.

Conjecture 3.6 (Lusztig [Lus79]). *If w is a Coxeter element, then $X_w(b)$ is representable by a scheme.*

There is some examples-based evidence for this conjecture. Mainly it is given by the well-understood case of inner forms of $\mathbf{GL}_n$, discussed below. Scholze conjectured moreover that in fact *all* $X_w(b)$ are representable by schemes. At least, there is no example known, where $X_w(b)$ is an ind-scheme, but not a scheme.

3.4. **Ramified tori: some remarks.** All said above only concerns *unramified* tori in $\mathbf{G}$, in the sense that the torus $\mathbf{T}_w$, such that $\underline{\mathbf{T}_w(k)}$ acts on $\dot{X}_{\dot{w}}(b)$ (over $X_w(b)$), splits over $\breve{k}$. An important question, already formulated by Boyarchenko [Boy12, Problem 3], is to extend the above construction to cover also ramified tori. It is not clear yet how this can be achieved in a satisfactory manner. Few observations in this directions are:

- A naive generalization does not work. Cf. Remark 7.15.

- There is a construction via *extended affine Deligne–Lusztig varieties*, which indeed produces irreducible supercuspidals attached to ramified tori. See Section 7.4.
- Although properly realizing irreducible supercuspidals, these extended affine Deligne–Lusztig varieties show however quite degenerated behavior. For example, for ramified tori in $\mathbf{GL}_2$ the corresponding varieties are zero-dimensional and reduced, so just infinite disjoint unions of geometric points with quite complicated group actions. See Theorem 7.14 and Conclusion 7.16.

What one would rather hope for, are varieties similar to those appearing in the work of Weinstein [Wei16] (and further articles by Imai, Takamatsu and others) on the reduction of special affinoids around points with CM by a ramified extension in the perfectoid Lubin–Tate curve. Indeed, for unramified CM points, there is certainly *some* (still not quite clear) connection between Deligne–Lusztig constructions and the reduction of special affinoids, see [BW16]. Heuristically, this means the following: say $\mathbf{G} = \mathbf{GL}_2$. The curve with equation $y^{q+1} = x^q + x$ (resp. higher-dimensional analoga) appears in the reduction of special affinoids around unramified CM-points, as well as in some $\dot{X}_{\dot{w}}(b)$ considered above. Now, the curve $y^2 = x^q + x$ (resp. its analoga) appears in the reduction of special affinoids around ramified CM-points, and the hope is that there exists a natural generalization of $\dot{X}_{\dot{w}}(b)$ such that this curve appears.

4. Lusztig's "deeper level" Mackey formula and its generalization

In this section Lusztig's work [Lus04] and its generalization [CI19a] due to Chan and the author will be explained. Before the results of [Iva20] explained above, there was no general way to endow X_{Lus} with a scheme structure, but it was possible to do so with "truncated integral versions" X_h of X_{Lus} (as in Section (2.2)) attached to an affine smooth group $\mathcal{G}$ over $\mathcal{O}_k$.

4.1. **Positive loops and its truncations.** We need an integral version of the loop functor $L\mathbf{X}$ from Section 3.1. For an $\mathcal{O}_{\breve{k}}$-scheme $\mathcal{X}$, we have the functors $L^+\mathcal{X}$ and $L_h^+\mathcal{X}$ ($h \geq 1$) on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$,

$$R \mapsto L^+\mathcal{X}(R) = \mathcal{X}(\mathbb{W}(R)) \quad \text{and} \quad R \mapsto L_h^+\mathcal{X}(R) = \mathcal{X}(\mathbb{W}_h(R)),$$

where $\mathbb{W}_h(R) = \mathbb{W}(R)/(\varpi^h)$. The first is called the functor of positive loops of $\mathcal{X}$ and the second is its h-truncated version. If $\mathcal{X}$ is an affine scheme of finite type over $\mathcal{O}_{\breve{k}}$, then $L^+\mathcal{X}$, $L_h^+\mathcal{X}$ are representable by $\overline{\mathbb{F}}_q$-schemes, and the latter is of perfectly finite type over $\mathrm{Spec}\,\overline{\mathbb{F}}_q$ [PR08, Zhu17].

Similar as in Section 3.1 we have the Frobenius morphisms $\sigma\colon L^+\mathcal{X} \to L^+\mathcal{X}$, $\sigma\colon L_h^+\mathcal{X} \to L_h^+\mathcal{X}$, whenever $\mathcal{X}$ is defined over $\mathcal{O}_k$. In this case let us write $X_h := L_h^+\mathcal{X}(\mathbb{F}_q)$ and if $\mathcal{X}$ is a group and $1 \leq r \leq h$, $X_h^r := \ker(X_h \to X_r)$. In particular, we will apply this to $\mathcal{X} = \mathcal{G}$ and its subgroups.

4.2. **Lusztig's deep level Deligne–Lusztig schemes.** Let $\mathcal{G}$ be a *reductive* group over $\mathcal{O}_k$. Let $\mathcal{T}$ be an $\mathcal{O}_k$-rational $\mathcal{O}_{\breve{k}}$-split maximal torus in $\mathcal{G}$. Let $\mathcal{B} = \mathcal{T}\mathcal{U}$ be a $\mathcal{O}_{\breve{k}}$-rational Borel subgroup containing it, with unipotent radical $\mathcal{U}$.

Definition 4.1 (§2 of [Lus04])**.** Define the $\overline{\mathbb{F}}_q$-scheme $X_{\mathcal{T},\mathcal{U},h}$ by the Cartesian diagram,

$$\begin{array}{ccc} X_{\mathcal{T},\mathcal{U},h} & \longrightarrow & L_h^+\mathcal{U} \\ \big\downarrow & & \big\downarrow \\ L_h^+\mathcal{G} & \xrightarrow{\mathrm{Lang}_{\mathscr{G}}} & L_h^+\mathcal{G} \end{array}$$

where $\mathrm{Lang}_{\mathscr{G}}\colon L_h^+\mathcal{G} \to L_h^+\mathcal{G}$ is the Lang map, given by $g \mapsto g^{-1}\sigma(g)$.

Left and right multiplication induce an action of $\mathcal{G}(\mathcal{O}_k) \times \mathcal{T}(\mathcal{O}_k) = L^+\mathcal{G}(\mathbb{F}_q) \times L^+\mathcal{T}(\mathbb{F}_q)$ on $X_{\mathcal{T},\mathcal{U},h}$, which factors through an action of the h-truncation $G_h \times T_h$.

Remark 4.2. Lusztig's original definition in [Lus04] was not formulated in terms of the loop functors, instead the $\overline{\mathbb{F}}_q$-*variety* $X_{\mathcal{T},\mathcal{U},h}$ was rather defined by describing its $\overline{\mathbb{F}}_q$-points. Moreover, Lusztig handled only the case $\operatorname{char} k > 0$. The mixed characteristic case was worked out later by Stasinski [Sta09]. It requires technically a little more care (e.g., it really becomes important to work with *perfect* schemes).

Remark 4.3. Let us briefly sketch the relation of the "Borel-level version"

$$X_{\mathcal{T},\mathcal{B},h} := L_h^+\mathcal{B} \times_{L_h^+\mathcal{G},\mathrm{Lang}_\mathcal{G}} L_h^+\mathcal{G}$$

of $X_{\mathcal{T},\mathcal{U},h}$ with the spaces $X_w(b)$ from Section 3. Let $\mathcal{T}_0 \subseteq \mathcal{G}$ be an $\mathcal{O}_k$-rational maximal torus, which is contained in an $\mathcal{O}_k$-rational Borel subgroup $\mathcal{B}_0$. Let W_0 denote the Weyl group of $\mathcal{T}_0 \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$ in $\mathcal{G} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$. Then there is an element $w_1 \in W$,[10] such that the space $X_{w_1}(1)$ produced out of $\mathcal{G}, \mathcal{T}_0, w_1$ via the obvious version of Definition 3.2 (with $L(\cdot)$ replaced by $L_h^+(\cdot)$) is G_h-equivariantly isomorphic to $X_{\mathcal{T},\mathcal{B},h}$.

Just as in the classical Deligne–Lusztig theory, to a character $\theta\colon T_h \to \overline{\mathbb{Q}}_\ell^\times$ we can attach the G_h-representation

$$R_{T_h,U_h}^{G_h}(\theta) = \sum_{i\in\mathbb{Z}} (-1)^i H_c^i(X_h, \overline{\mathbb{Q}}_\ell)_\theta.$$

We also regard it as a $\mathcal{G}(\mathcal{O}_k)$-representation by inflation via $\mathcal{G}(\mathcal{O}_k) \to G_h$. In general, it depends on h [CI19a, §4.2][11] The main result of [Lus04] is a version of the Mackey formula (Theorem 1.3) for $R_{T_h,U_h}^{G_h}(\theta)$, which we now explain. There is one strong restriction on the involved characters θ.

Definition 4.4 (Rough version of §1.5 of [Lus04])**.** Let $h \geq 2$. A character $\theta\colon T_h \to \overline{\mathbb{Q}}_\ell^\times$ is called *primitive*[12] if it is very far from being trivial on T_h^{h-1}.

The precise definition is slightly technical. We only remark that it also depends on the roots of $\mathcal{T} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$ in $\mathcal{G} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$.

Theorem 4.5 (Proposition 2.3 of [Lus04])**.** *Let $(\mathcal{T},\mathcal{U},\theta)$, $(\mathcal{T}',\mathcal{U}',\theta')$ be two triples as above. If $h \geq 2$, assume that θ or θ' is primitive. Then*

$$\left\langle R_{T_h,U_h}^{G_h}(\theta), R_{T_h',U_h'}^{G_h}(\theta') \right\rangle_{G_h} = \#\left\{v \in W(\mathcal{T},\mathcal{T}')^\sigma \colon \theta' = \theta \circ \mathrm{Ad}(v)\right\},$$

where $W(\mathcal{T},\mathcal{T}')$ is the transporter from $\mathcal{T} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$ to $\mathcal{T}' \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$ in $\mathcal{G} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$ modulo $\mathcal{T} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$.

Corollary 4.6 (Corollary 2.4 of [Lus04])**.** *Let $(\mathcal{T},\mathcal{U},\theta)$ be as in Theorem 4.5, with θ primitive if $h \geq 2$. Then $R_{T_h,U_h}^{G_h}(\theta)$ is independent of the choice of $\mathcal{U}$, and if additionally there are no $1 \neq v \in W(\mathcal{T},\mathcal{T})^\sigma$ with $\theta = \theta \circ \mathrm{Ad}(v)$, then $\pm R_{T_h,U_h}^{G_h}(\theta)$ is an irreducible G_h-representation.*

[10] w_1 is determined by the element w in the Weyl group of $\mathcal{T} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$ in $\mathcal{G} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$, such that $\sigma(\mathcal{B} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q) = w(\mathcal{B} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q)w^{-1}$.

[11] This dependence is expected to disappear for elliptic tori.

[12] In [Lus04] the term *regular* was used. We prefer *primitive*, which was also used in [BW16]. May be *very regular* would be a better name (cf. Section 4.5).

4.3. **Lusztig's proof of Theorem 4.5.** In fact, all geometric proofs of Mackey formulas in various "Deligne–Lusztig" contexts are all based on similar ideas. For deeper level Deligne–Lusztig varieties, this first appeared in Lusztig [Lus04]. As the tools used in the proof, as well as the method of proof itself play an important role in what follows, we give more details on both. The two important general principles, which are used in the proof are the following.

Theorem 4.7 (10.15 of [DM91], Corollary 6.5 of [DL76])**.**

(i) Let H be a torus acting on a scheme X, separated and of finite type over $\overline{\mathbb{F}}_q$, and (for simplicity) affine. Then $\dim H^*_c(X, \overline{\mathbb{Q}}_\ell) = \dim H^*_c(X^H, \overline{\mathbb{Q}}_\ell)$.

(ii) Let H be a connected algebraic group acting on a scheme X, separated and of finite type over $\overline{\mathbb{F}}_q$. Then for any $h \in H$, the action of h in $H^*_c(X, \overline{\mathbb{Q}}_\ell)$ is trivial.

Part (i) is a consequence of the Deligne–Lusztig fixed point formula (Theorem 1.4). Lusztig's strategy of proof of Theorem 4.5 can be divided into the following steps:

(1) First, express $\left\langle R^{G_h}_{T_h,U_h}(\theta), R^{G_h}_{T'_h,U'_h}(\theta')\right\rangle_{G_h}$ as the $\theta \otimes \theta'$-isotypic part of the ℓ-adic Euler characteristic of the $\overline{\mathbb{F}}_q$-scheme $\Sigma = G_h \backslash (X_{\mathcal{T},\mathcal{U},h} \times X_{\mathcal{T}',\mathcal{U}',h})$ on which $T_h \times T'_h$ acts.

(2) There is a locally closed partition $\Sigma = \coprod_{v \in W(\mathcal{T},\mathcal{T}')} \Sigma_v$, and for each v a fibration $\widehat{\Sigma}_v \to \Sigma_v$ with all fibers being isomorphic to a fixed affine space, such that $T_h \times T'_h$ compatibly acts on $\widehat{\Sigma}_v$.

(3) There is some (highly non-trivial) locally closed decomposition $\widehat{\Sigma}_v = (\coprod_{\lambda \in J} \widehat{\Sigma}'_{v,\lambda}) \dot{\cup} \widehat{\Sigma}''_v$ into finitely many pieces, such that $T_h \times T'_h$ still acts on $\widehat{\Sigma}'_{v,\lambda}, \widehat{\Sigma}''_v$.

(4) The action of $T_h \times T'_h$ on the closed part Σ''_v extends to an action of a big subgroup scheme $H \subseteq L^+_h\mathcal{T} \times L^+_h\mathcal{T}'$, with reductive part $H_{\rm red}$, whose connected component $H^\circ_{\rm red}$ is a (still sufficiently big) torus. Then one gets

$$\dim H^*_c(\widehat{\Sigma}''_v, \overline{\mathbb{Q}}_\ell)_{\theta\otimes\theta'} = \dim H^*_c((\widehat{\Sigma}''_v)^{H^\circ_{\rm red}}, \overline{\mathbb{Q}}_\ell)_{\theta\otimes\theta'} = \begin{cases} 1 & \text{if } v \in W(\mathcal{T},\mathcal{T}')^\sigma \text{ and } \theta' = \theta \circ \mathrm{Ad}(v), \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

where the first equality is by Theorem 4.7(i), and the second is easy, as $(\widehat{\Sigma}''_v)^{H^\circ_{\rm red}}$ is a (very simple) finite set of points.

(5) Wlog we can assume that θ is primitive. The action of $T^{h-1}_h \times \{1\} (\subseteq T_h \times T_h)$ on each $\widehat{\Sigma}'_{v,\lambda}$ extends in a highly non-trivial way to an action of a positive-dimensional group $\mathscr{T} \subseteq \ker(L^+_h\mathcal{T} \to L^+_{h-1}\mathcal{T})$ (depending on λ). By Theorem 4.7(ii), $\mathscr{T}^\circ \cap (T_h \times \{1\})$ acts trivially in $H^*_c(\widehat{\Sigma}'_{v,\lambda}, \overline{\mathbb{Q}}_\ell)$. On the other side, as θ is primitive, θ is non-trivial on $\mathscr{T}^\circ \cap (T^{h-1}_h \times \{1\})$. Thus we deduce $H^*_c(\widehat{\Sigma}'_{v,\lambda}, \overline{\mathbb{Q}}_\ell)_{\theta\otimes\theta'} \subseteq H^*_c(\widehat{\Sigma}'_{v,\lambda}, \overline{\mathbb{Q}}_\ell)_\theta = 0$.

4.4. **A generalization to parahoric models.** In [CI19a] Chan and the author extended Lusztig's definition to groups $\mathcal{G}/\mathcal{O}_k$, which are not necessarily reductive, but are *parahoric models* of reductive k-groups. This was motivated by the special case of inner forms of $\mathbf{GL}_n$ (see Sections 5 and 6 below), where the proofs of [Lus04] do not apply without modification. An investigation of that case (inner forms of $\mathbf{GL}_n$) was already performed in [CI18, §8]. The application of results [CI19a] is however not limited to non-quasi-split forms of unramified k-groups. Even for split groups (like for example Sp_4) over k there exist non-reductive parahoric $\mathcal{O}_k$-models, and the corresponding generalization of Lusztig's schemes naturally arises.

To be more precise, let the notation be as in Section 4.2, with the assumption on $\mathcal{G}$ relaxed, i.e., we assume only that $\mathcal{G}$ is the parahoric model of a reductive k-group $\mathbf{G}$. We also let $\mathcal{T}$ (resp. $\mathcal{B}$, resp. $\mathcal{U}$) be the closure in $\mathcal{G}$ of a k-rational $\breve{k}$-split maximal torus of $\mathbf{G}$ (resp. a $\breve{k}$-rational Borel containing it, resp. the unipotent radical of the latter[13]). Then Definition 4.1 carries over literally, giving $\overline{\mathbb{F}}_q$-schemes $X_{\mathcal{T},\mathcal{U},h}$. We also have corresponding G_h-representations $R_{T_h,U_h}^{G_h}(\theta)$.

Theorem 4.8 (Theorem 1.1 and Corollary 4.7 of [CI19a])**.** *Statements analogous to Theorem 4.5 and Corollary 4.6 hold for representations $R_{T_h,U_h}^{G_h}(\theta)$ arising from parahoric $\mathcal{O}_k$-models of reductive k-groups.*

The strategy of the proof of Theorem 4.8 is the same as that of Theorem 4.5, sketched above. The only, but quite subtle difference concerns the most complicated step (5). Elaboration of the corresponding details occupies a big part (Sections 2 and 3) of [CI19a]. It also becomes clear that the proof does not work for all affine smooth models (which is also not surprising).

4.5. **Traces of very regular elements.** Let the setup be as in Section 4.4. In [CI18] and [CI19a] the traces of certain elements inside $T_h \subseteq G_h$ in the representation $R_{T_h,U_h}^{G_h}(\theta)$ are computed. Roughly, an element $g \in L_h^+\mathcal{T}(\overline{\mathbb{F}}_q)$ is *very regular* if its image in T_1 does not lie in any singular subtorus (a precise definition is given in [CI19a, Definition 5.1], generalizing [Hen92, §2.3]).

Theorem 4.9 (rough form of Theorem 1.2 of [CI19a], Theorem 11.2 of [CI18])**.** *Let $g \in T_h \subseteq G_h$ be a very regular element. For any character $\theta\colon T_h \to \overline{\mathbb{Q}}_\ell^\times$,*

$$\mathrm{Tr}(g; R_{T_h,U_h}^{G_h}(\theta)) = \sum_{w \in W_{\mathcal{G}}(\mathcal{T})^\sigma} (\theta \circ \mathrm{Ad}(w))(g).$$

where $W_{\mathcal{G}}(\mathcal{T})$ denotes the Weyl group of the special fiber of $\mathcal{T}$ in the reductive quotient of the special fiber of $\mathcal{G}$.

This is a generalization to deep level Deligne–Lusztig varieties of a special case of the classical trace formula in [DL76, Theorem 4.2]. Currently this seems to be the only case where such a formula can be proven. The reason for the severe restriction to very regular elements is that a proof requires at some point the use of Deligne–Lusztig fixed point formula (Theorem 1.4), which is useless for elements in the p-subgroup G_h^1 of G_h. Note that on the other side, there is no restriction on θ.

Remark 4.10. Although Theorem 4.9 gives the traces for a relatively small class of elements of $\mathcal{G}(\mathcal{O}_k)$ in $R_{T_h,U_h}^{G_h}(\theta)$ (only very regular elements of $\mathcal{T}(\mathcal{O}_k)$), at least in the case of $\mathbf{G}$ = inner form of $\mathbf{GL}_n$ this (+ some further properties of $R_{T_h,U_h}^{G_h}(\theta)$) is still sufficient to uniquely pin down this representation. This uses a result of Henniart [Hen92], see Section 6.1 below.

5. Coxeter varieties for inner forms of GL_n

Similar as in the classical theory, if $w \in W$ is a Coxeter element, the ind-scheme $X_w(b)$ considered in Section 3 is called *of Coxeter type* and has some particular properties, making it sometimes more accessible. In this section I discuss a geometric description of the (covers of) Coxeter varieties $\dot{X}_{\dot{w}}(b)$ for $\mathbf{GL}_n$ and inner forms obtained in joint work with Ch. Chan [CI18, CI19b], and I explain the *ad hoc* construction (going back to [Lus79]) of smooth representations in their cohomology.

[13]Caution: the special fiber $\mathcal{B} \otimes_{\mathcal{O}_{\breve{k}}} \overline{\mathbb{F}}_q$ of $\mathcal{B}$ needs not to be a Borel subgroup of $\mathcal{G} \otimes_{\mathcal{O}_{\breve{k}}} \overline{\mathbb{F}}_q$

5.1. **The setup.** Let $\mathbf{G}$ be an inner form of $\mathbf{GL}_n$ over k ($n \geq 2$). Then there exists a unique integer $0 \leq \kappa \leq n-1$ such that putting $n' = \gcd(n, \kappa)$, $n = n'n_0$, $\kappa = n'\kappa_0$, we have $G \cong \mathbf{GL}_{n'}(D_{\kappa_0/n_0})$, where D_{κ_0/n_0} is the central division algebra over k with Hasse-invariant κ_0/n_0. The group $\mathbf{G}$ possesses a k-rational unramified elliptic maximal torus $\mathbf{T} \subseteq \mathbf{G}$, which is unique up to conjugation; we write $T = \mathbf{T}(k)$.

We can express $\mathbf{G}$ in terms of (3.1): let $b \in \mathbf{GL}_n(\breve{k})$ be any *basic* element with $\mathrm{ord} \circ \det(b) = \kappa$[14]. Then $\mathbf{G} \cong \mathbf{GL}_{n,b}$. Fix a maximal split torus $\mathbf{T}_0 \subseteq \mathbf{GL}_n$ and let W be the Weyl group of $\mathbf{T}_0$ in $\mathbf{GL}_n$. We are thus interested in the corresponding ind-schemes $X_w(b)$ with $w \in W$ Coxeter and b basic element as above, and even more in their covers $\dot{X}_{\dot{w}}(b)$ with $\dot{w} \in \mathbf{GL}_n(\breve{k})$ lifting w. On $\dot{X}_{\dot{w}}(b)$ we have the action of $G \times T$. The situation is essentially unique (*i. e.*, independent of the choices of $b, w, \dot{w}$), indeed:

(i) all basic b with $\mathrm{ord} \circ \det(b) = \kappa$ are mutually σ-conjugate,
(ii) all Coxeter elements $w \in W$ are connected by a finite sequence of Frobenius-cyclic shifts.

Thus all $X_w(b)$ with w, b are above are mutually isomorphic, and for fixed b, w we have

(iii) if $\dot{w}$ varies through the lifts of w, then $\dot{X}_{\dot{w}}(b) = \varnothing$, unless $\mathrm{ord} \circ \det(\dot{w}) = \kappa$, and if this holds, then $\dot{X}_{\dot{w}}(b)$ is essentially independent of the lift $\dot{w}$ of w.

5.2. **An explicit description.** We now describe $\dot{X}_{\dot{w}}(b)$ more in the style of Section 4, i. e., intrinsically in terms of $\mathbf{G}, \mathbf{T}$ and without involving $\mathbf{GL}_n, b, \dot{w}$. There is a (unique up to conjugation) maximal parahoric $\mathcal{O}_k$-model $\mathcal{G}$ of $\mathbf{G}$, and $G_{\mathcal{O}} := \mathcal{G}(\mathcal{O}_k) = \mathbf{GL}_{n'}(\mathcal{O}_{D_{\kappa_0/n_0}})$, where $\mathcal{O}_{D_{\kappa_0/n_0}} \subseteq D_{\kappa_0/n_0}$ are the integers of D_{κ_0/n_0}. Let $\mathbf{B} = \mathbf{TU} \subseteq \mathbf{G}$ be a $\breve{k}$-rational Borel subgroup containing $\mathbf{T}$ with unipotent radical $\mathbf{U}$, and let $\mathbf{U}^-$ be the unipotent radical of the opposite Borel subgroup. We have the closure $\mathcal{T}$ resp. $\mathcal{U}^{\pm}$ of $\mathbf{T}$ resp. $\mathbf{U}^{\pm}$ in $\mathcal{G}$, and we put $T_{\mathcal{O}} = \mathcal{T}(\mathcal{O}_k)$. The Frobenius σ of $\breve{k}/k$ acts on the roots of $\mathbf{T}$ in $\mathbf{G}$, hence the subgroups $\sigma\mathbf{U}^{\pm}$ (generated by σ-translates of the root subgroups generating $\mathbf{U}^{\pm}$) make sense. The following theorem generalizes the classical example [DL76, §2.2].

Theorem 5.1 (Proposition 2.6 of [CI19b], Theorem 6.4 and §7.2 of [CI18])**.** *Let $b, \dot{w}$ be as in Section 5.1 with* $\mathrm{ord} \circ \det(b) = \mathrm{ord} \circ \det(\dot{w})$. *We have $G \times T$-equivariantly, $\dot{X}_{\dot{w}}(b) \cong \coprod_{g \in G/G_{\mathcal{O}}} g.X_{\mathcal{O}}$, where $X_{\mathcal{O}}$ is the $\overline{\mathbb{F}}_q$-scheme, defined by the Cartesian diagram*

$$\begin{array}{ccc} X_{\mathcal{O}} & \longrightarrow & L^+(\sigma\mathcal{U} \cap \mathcal{U}^-) \\ \big\downarrow & & \big\downarrow \\ L^+\mathcal{G} & \xrightarrow{g \mapsto g^{-1}\sigma(g)} & L^+\mathcal{G} \end{array} \tag{5.1}$$

The $\overline{\mathbb{F}}_q$-points of $X_{\mathcal{O}}$ can be described explicitly: For simplicity, assume we that are in the split case $\mathbf{G} = \mathbf{GL}_n$ (that is, $\kappa = 0$). Then

$$X_{\mathcal{O}}(\overline{\mathbb{F}}_q) = \left\{ v \in \mathcal{O}_{\breve{k}}^{\oplus n} \colon \det g(v) \in \mathcal{O}_k^{\times} \right\},$$

where $g(v) = (v; \sigma(v); \sigma^2(v); \ldots; \sigma^{n-1}(v)) \in M_n(\mathcal{O}_{\breve{k}})$ is the matrix whose i-th column is $\sigma^{i-1}(v)$. The scheme $X_w(b)$ admits a similar description.

Following remarks concern Theorem 5.1 and its proof:

[14] "b basic" means that its Newton point ν_b factors through the center of $\mathbf{GL}_n$, cf. [Kot85]. For example, we can take $b = \begin{pmatrix} 0 & \varpi^{\kappa} \\ \mathbf{1}_{n-1} & 0 \end{pmatrix}$.

- It is crucial to work with perfect (hence reduced!) ind-schemes in the theorem. Otherwise the proof does not apply, and the statement is most probably wrong, even if $\operatorname{char} k > 0$.
- The main ingredient in the proof of Theorem 5.1 (along with Corollary 3.5 or any ad hoc argument proving ind-representability) is a statement for isocrystals, which is a precise analog of the well-known fact, that the valuations of the coefficients of a (monic) polynomial $f \in \breve{k}[T]$ lie over the Newton-polygon of f (cf. the proof of [CI19b, Proposition 2.6][15]).

5.3. **The ℓ-adic representations in the cohomology of $\dot{X}_{\dot{w}}(b)$.** We have the $\overline{\mathbb{F}}_q$-scheme $\dot{X}_{\dot{w}}(b) = \coprod_{G/G_{\mathcal{O}}} gX_{\mathcal{O}}$ from Theorem 5.1. We can write $X_{\mathcal{O}}$ as an inverse limit $X_{\mathcal{O}} = \varprojlim_h X_h$ of schemes X_h (as in Section 4.4[16]) perfectly of finite presentation over $\overline{\mathbb{F}}_q$, such that each X_h admits a compatible action of the subquotient $G_h \times T_h$ of $G_{\mathcal{O}} \times T_{\mathcal{O}}$.

Example 5.2. Let $\mathbf{G} = \mathbf{GL}_n$. Then

$$X_h(\overline{\mathbb{F}}_q) = \{v \in \left(\mathcal{O}_{\breve{k}}/\mathfrak{p}_{\breve{k}}^h\right)^{\oplus n} \,:\, \det(v; \sigma(v); \dots; \sigma^{n-1}(v)) \in (\mathcal{O}_k/\mathfrak{p}_k^h)^\times\}.$$

Let $\theta\colon T \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character. Then for $h \geq 1$ big enough, θ is trivial on $\ker(T_{\mathcal{O}} \to T_h)$, so induces a character of T_h, again denoted θ. As in Section 4, we have the virtual G_h-representation $R_{T_h,U_h}^{G_h}(\theta)$. By Section 5.1 our "Coxeter choices" are essentially unique, so $R_{T_h,U_h}^{G_h}(\theta)$ is independent of U_h. Thus we simply write $R_{T_h}^{G_h}(\theta)$ instead of $R_{T_h,U_h}^{G_h}(\theta)$. Using that the fibers of $X_h/\ker(T_h \to T_{h-1}) \to X_{h-1}$ are isomorphic to $\mathbb{A}^{n-1}_{\overline{\mathbb{F}}_q}$ [CI18, Proposition 7.6][17], one shows that $R_{T_h}^{G_h}(\theta)$ regarded by inflation as a virtual $G_{\mathcal{O}}$-representation is independent of h.

Let $Z \subseteq G$ be the center. Compactly inducing and taking care of the center, we can – in accordance with the disjoint decomposition in Theorem 5.1– make sense of the "θ-isotypic component of the ℓ-adic Euler characteristic with compact support of $\dot{X}_{\dot{w}}(b)$":

$$R_T^G(\theta) := \operatorname{cInd}_{ZG_{\mathcal{O}}}^G R_{T_h}^{G_h}(\theta),$$

where $R_{T_h}^{G_h}(\theta)$ is extended to $ZG_{\mathcal{O}}$ by letting $z \in Z \subseteq T$ act by $\theta(z)$.

6. Deligne–Lusztig induction for elliptic unramified tori in inner forms of GL_n

This section is based on joint work with Chan [CI19b, CI20, CI18] (special cases were independently obtained in [Iva16, Cha19]). The results on representations obtained by p-adic Deligne–Lusztig induction for inner forms of $\mathbf{GL}_n$ in Section 5.3, and their relation with local Langlands (LL) and Jacquet–Langlands (JL) correspondences will be explained. We use notation from Section 5. In particular, we have the locally compact groups $G = \mathrm{GL}_{n'}(D_{\kappa_0/n_0}), T$, and for a smooth character $\theta\colon T \to \overline{\mathbb{Q}}_\ell^\times$, we have the virtual smooth G-representation $R_T^G(\theta)$, which is attached to the essentially unique (non-empty) Coxeter-type Deligne–Lusztig variety corresponding to $T \subseteq G$. We use notation from Section 4.1.

[15]The original proof in [CI18] of representability in Theorem 5.1 was more complicated and used the comparison with affine Deligne–Lusztig varieties, as well as results of Viehmann on the moduli of p-divisible groups [Vie08].
[16]In Definition 4.1 we used $\mathcal{U}$, whereas in (5.1) we use $\sigma\mathcal{U} \cap \mathcal{U}^-$. This discrepancy is unessential and can be ignored.
[17]In the division algebra case, this crucial observation goes back to Lusztig [Lus79]

6.1. **Main result.** The brief upshot is:

(i) One can explicitly attach to θ a representation σ_θ of the Weil group $\mathcal{W}_k$ of k, such that $\sigma_\theta \mapsfrom \theta \mapsto \pm R_T^G(\theta)$ coincides with LL. The *rectifier* of Bushnell–Henniart [BH10, Definition 1] is encoded in $\theta \mapsto \sigma_\theta$ and is not "seen" by $R_T^G(\cdot)$.[18]

(ii) Fixing $\mathbf{T}$ and varying $\mathbf{G}$, $\theta \mapsto \pm R_T^G(\theta)$ induces JL.

To state the results we need more notation. Let L/k be the unramified degree n extension, such that $T \cong L^\times$ (determined up to an element in $\mathrm{Gal}(L/k)$). We have $T_\mathcal{O} \cong U_L$ and $T_h \cong U_L/U_L^h$, where U_L resp. U_L^h are the units resp. h-units of L. Denote by ε a character of $k^\times$ with $\ker(\varepsilon) = \mathrm{N}_{L/k}(L^\times)$, the image of the norm map of L/k. Also let

- $\mathscr{X}$ be the set of smooth characters of $L^\times$ *in general position*, i.e., with trivial stabilizer in $\mathrm{Gal}(L/k)$,
- $\mathscr{G}_k^\varepsilon(n)$ be the set of isomorphism classes of smooth n-dimensional representations σ of $\mathcal{W}_k$ of k satisfying $\sigma \cong \sigma \otimes (\varepsilon \circ \mathrm{rec}_k)$,
- $\mathscr{A}_k^\varepsilon(n,\kappa)$ be the set of smooth irreducible supercuspidal representations π of G (where $\kappa = \kappa_0 n'$ determines G) such that $\pi \cong \pi \otimes (\varepsilon \circ \mathrm{Nrd}_G)$.

Then there are natural bijections of sets

$$\begin{array}{ccccccc} \mathscr{X}\,/\,\mathrm{Gal}(L/k) & \longrightarrow & \mathscr{G}_k^\varepsilon(n) & \xrightarrow{\mathrm{LL}} & \mathscr{A}_k^\varepsilon(n,0) & \xrightarrow{\mathrm{JL}} & \mathscr{A}_k^\varepsilon(n,\kappa) \\ \theta & \longmapsto & \sigma_\theta & \longmapsto & \mathrm{LL}(\sigma_\theta) & \longmapsto & \mathrm{JL}(\mathrm{LL}(\sigma_\theta)) =: \pi_\theta. \end{array}$$

Here $\sigma_\theta := \mathrm{Ind}_{\mathcal{W}_L}^{\mathcal{W}_k}(\theta \cdot \mu)$ is the induction to $\mathcal{W}_k$ of the character $\mathcal{W}_L \to \mathcal{W}_L^{\mathrm{ab}} \overset{\mathrm{rec}_L}{\to} L^\times \overset{\theta\cdot\mu}{\to} \overline{\mathbb{Q}}_\ell^\times$, where μ is the *rectifier*, that is the character of $L^\times$ defined by $\mu(\varpi) = (-1)^{n-1}$ and $\mu(U_L) = 1$.

Theorem 6.1 (Theorem A of [CI19b])**.** *Assume that $p > n$. Let $\theta \colon T \cong L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character such that $\theta|_{U_L^1}$ has trivial stabilizer in* $\mathrm{Gal}(L/k)$. *Then $\pm R_T^G(\theta)$ is a genuine G-representation and*

$$\pm R_T^G(\theta) \cong \pi_\theta.$$

In particular, $\pm R_T^G(\theta)$ is irreducible supercuspidal and $\sigma_\theta \leftrightarrow \pm R_T^G(\theta)$ is a realization of the local Langlands and Jacquet–Langlands correspondences.

In the division algebra case Theorem A gets easier and essentially follows (for all p, n and all θ with trivial $\mathrm{Gal}(L/k)$-stabilizer) from Lusztig's original work [Lus79] along with a result of Henniart [Hen92, 3.1 Théorème], see [Cha19]. Special cases of Theorem 6.1 were proven in [CI18, Theorem 12.3, Theorem 12.6], [Iva16, Theorem 4.3], [Cha19, Corollary to Main Theorem 1]. We expect that the assumption $p > n$ in Theorem 6.1 is redundant. For a conjectural generalization of Theorem 6.1 to all θ in general position see Conjecture 6.12.

The proof of Theorem 6.1 proceeds in five steps:

(1) Prove a "geometric" Mackey formula for the G_h-representations $\pm R_{T_h}^{G_h}(\theta)$. It follows that when θ is sufficiently generic, $\pm R_{T_h}^{G_h}(\theta)$ is irreducible.

[18] The sign $\pm$ such that $\pm R_T^G(\theta)$ is a genuine G-representation – an important invariant – can be determined explicitly from θ by Theorems 6.11, 6.6. This gives an analog of the classical result [DL76, Theorem 8.3]. In particular, $\pm R_T^G(\cdot)$ "sees" the sign appearing in the characterization of JL by matching traces in (ii), and $\theta \mapsto \sigma_\theta$ in (i) is does not depend on which inner form $\mathbf{G}$ of $\mathbf{GL}_n$ we start with.

(2) There is a special closed $G_h \times T_h$-stable subscheme $X_{h,n'} \subseteq X_h$, which is a *maximal variety* over $\mathbb{F}_{q^n}$ (see Example 6.5 and Section 6.4 below). Prove a version of the Mackey formula for $\pm H_c^*(X_{h,n'}, \overline{\mathbb{Q}}_\ell)_\theta$ and show $\pm H_c^*(X_{h,n'}, \overline{\mathbb{Q}}_\ell)_\theta \cong \pm R_{T_h}^{G_h}(\theta)$ for sufficiently generic θ.
(3) Using (1) and further geometric input show that $\pm R_T^G(\theta) = \mathrm{cInd}_{ZG_{\mathcal{O}}}^G(\pm R_{T_h}^{G_h}(\theta))$ is admissible (equivalently, a finite direct sum of irreducible supercuspidals).
(4) Using that $X_{h,n'}$ is a maximal variety, compute the eigenvalues of q^n-Frobenius in the single cohomology groups $H_c^*(X_{h,n'}, \overline{\mathbb{Q}}_\ell)$. Use this to compute the integer $\dim_{\overline{\mathbb{Q}}_\ell} H_c^*(X_{h,n'}, \overline{\mathbb{Q}}_\ell)_\theta$, which by (2) is also equal to $\dim_{\overline{\mathbb{Q}}_\ell} R_{T_h}^{G_h}(\theta)$.
(5) Conclude as follows: Use (3) together with the traces of $R_T^G(\theta)$ (Theorem 4.9) and of π_θ on very regular elements in $T \subseteq G$ to show (using a linear independence of characters argument due to Henniart [Hen92]) to show that π_θ is one of the irreducible constituents of $R_T^G(\theta)$. Matching $\dim_{\overline{\mathbb{Q}}_\ell} R_{T_h}^{G_h}(\theta)$ from (4) with the explicitly known formal degree of π_θ [CMS90] finishes the proof.

Below we discuss results related to steps (1) and (2) in Section 6.2, those related to step (3) in Section 6.3, and those related to step (4) in Section 6.4. Step (5) is explained in [CI19b, §7.2].

6.2. **The "integral" representation $R_{T_h}^{G_h}(\theta)$.** In our special case (inner forms of $\mathbf{GL}_n$, elliptic tori) we significantly improve the Mackey formula (Theorem 4.8), getting rid of the primitivity assumption on θ. Let $W_{\mathcal{O}}$ be the Weyl group of $\mathcal{T} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$ in $\mathcal{G} \otimes_{\mathcal{O}_k} \overline{\mathbb{F}}_q$.[19]

Theorem 6.2 (Theorem 3.1 of [CI19b])**.** *Let $\theta, \theta' \colon T_h \cong U_L/U_L^h \to \overline{\mathbb{Q}}_\ell^\times$ be two characters. Then*

$$\left\langle R_{T_h}^{G_h}(\theta), R_{T_h}^{G_h}(\theta') \right\rangle_{G_h} = \# \left\{ v \in W_{\mathcal{O}}^\sigma \colon \theta' = \theta \circ \mathrm{Ad}(v) \right\}.$$

We have $W_{\mathcal{O}}^\sigma \cong \mathrm{Gal}(L/k)[n']$ (the n'-torsion), such that $T \cong L^\times$ is equivariant with respect to the action of $W_{\mathcal{O}}^\sigma$ on T and of $\mathrm{Gal}(L/k)[n']$ on $L^\times$. The following corollary is immediate.

Corollary 6.3 (Corollary 3.3 of [CI19b])**.** *Let $\theta \colon T_h \to \overline{\mathbb{Q}}_\ell^\times$ be a character, whose stabilizer in $\mathrm{Gal}(L/k)[n']$ is trivial. Then $\pm R_{T_h}^{G_h}(\theta)$ is irreducible G_h-representation. In particular, the map*

$$\left\{\text{characters } \theta \colon T_h \to \overline{\mathbb{Q}}_\ell^\times \text{ in general position}\right\}/W_{\mathcal{O}}^\sigma \to \left\{\text{irreducible } G_h\text{-representations}\right\}$$
$$\theta \mapsto \pm R_{T_h}^{G_h}(\theta)$$

is injective.

In the present context, θ is primitive (Definition 4.4) if and only if $\theta|_{U_L^{h-1}/U_L^h}$ has trivial stabilizer in $\mathrm{Gal}(L/k)[n']$.

6.2.1. *Strategy of the proof of Theorem 6.2.* Without the primitivity assumption on θ or θ', Lusztig's proof of Theorem 4.5 (Section 4.3) collapses: the crucial step (5) simply does not work. But precisely in our special case[20], the following quite miraculous workaround works (we use notation as in Section 4.3):

- There is a interesting trichotomy: either $v = 1$, or $\Sigma_v = \varnothing$, or $\mathbf{U}^- \cap v^{-1}\mathbf{U}^- v$ is contained in a proper (k-rational) Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ (containing $\mathbf{T}$).

[19]The reductive quotient of the special fiber $\mathcal{G} \otimes_{\mathcal{O}} \mathbb{F}_q$ of $\mathcal{G}$ is isomorphic to $\mathrm{Res}_{\mathbb{F}_{q^{n_0}}/\mathbb{F}_q} \mathrm{GL}_{n', \mathbb{F}_{q^{n_0}}}$, and $W_{\mathcal{O}}$ is isomorphic to n_0 copies of the symmetric group $S_{n'}$ and carries an action of σ permuting these.

[20]In fact, only for a very special choice of a Coxeter element; observe that making such a choice, we do not loose generality by Section 5.1.

- If $\Sigma_v = \varnothing$, nothing must be done. If for $v \in W_{\mathcal{O}}$ a Levi $\mathbf{L}$ as above exists, an improved version of step (4) of Section 4.3 works for the whole $\widehat{\Sigma}_v$ instead of $\widehat{\Sigma}''_v$: using the center of $\mathbf{L}$ one can produce an extension of action of $T_h \times T_h$ to a subgroup which contains a (much smaller than in Section 4.3, but still regular enough) one-dimensional subtorus of $L_1^+\mathbf{T}_{\mathcal{O}} \times L_1^+\mathbf{T}_{\mathcal{O}}$. From this one deduces (4.1) for $\widehat{\Sigma}_v$.
- The remaining case $v = 1$ can either be done by an ad hoc extension of action argument, or may be deduced from a twisted version of Lusztig's strategy [CI19b, §3.4], where Σ_v's are generalized to allow more flexibility (but the core idea – to extend the action to a *torus*, and then conclude as in step (4) of Section 4.3 – remains the same).

Remark 6.4. Computations using GAP, recently performed by O. Dudas, show that it is not clear whether and how this workaround extends outside this special case. Indeed, neither for *other* (than those mentioned in the footnote 20 above) Coxeter elements in type A_{n-1}, nor for a single Coxeter element in type B_2[21], we could make this strategy work.

6.2.2. *The special subvariety $X_{h,n'}$.* The variety X_h has a very interesting closed subvariety $X_{h,n'}$: the induced action of σ^n in the cohomology of $X_{h,n'}$ is just multiplication with a scalar (see Theorem 6.11). At the end, it is $X_{h,n'}$ whose cohomology contributes the supercuspidal representations appearing in Theorem 6.1 (see also Conjecture 6.12).

Example 6.5. (i) Let $\mathbf{G} = \mathbf{GL}_n$, that is $n' = n$. If $\bar{v}$ denotes the reduction of v modulo $\mathfrak{p}_{\breve{k}}$, then $X_{h,n}$ is the closed subvariety of X_h (see Example 5.2) defined by the condition $\bar{v} = (-1)^{n-1}\bar{v}$.

(ii) In the division algebra case (i.e., $n' = 1$), we have simply $X_{h,1} = X_h$.

Theorem 6.6. *Let $\theta \colon T_h \to \overline{\mathbb{Q}}_\ell^\times$ be a character. Assume that $p > n$, and that $\theta|_{T_h^1}$ has trivial stabilizer in $W_{\mathcal{O}}^\sigma$. Then*

$$\left\langle R_{T_h}^{G_h}(\theta), H_c^*(X_{h,n'}, \overline{\mathbb{Q}}_\ell)_\theta \right\rangle_{G_h} = \left\langle H_c^*(X_{h,n'}, \overline{\mathbb{Q}}_\ell)_\theta, H_c^*(X_{h,n'}, \overline{\mathbb{Q}}_\ell)_\theta \right\rangle_{G_h} = 1.$$

In particular, $H_c^*(X_{h,n'}, \overline{\mathbb{Q}}_\ell)_\theta \cong R_{T_h}^{G_h}(\theta)$ is up to sign an irreducible representation of G_h.

This is a slightly weaker version of the Mackey formula (Theorem 6.2) for $X_{h,n'}$. The strategy of the proof remains roughly the same as in Section 6.2.1. The significant difference, is that we only can extend the action of $T_h^1 \times T_h^1$ to an action of a positive-dimensional subgroup of $H \subseteq \ker(L_h^+\mathbf{T}_{\mathcal{O}} \to L_1^+\mathbf{T}_{\mathcal{O}}) \times \ker(L_h^+\mathbf{T}_{\mathcal{O}} \to L_1^+\mathbf{T}_{\mathcal{O}})$, so that

- as H is *unipotent*, one only can apply Theorem 4.7(ii) (and not Theorem 4.7(i)), which does not give such good quantitative results, and forces the assumption on θ.
- one has to determine the connected component $H^\circ \subseteq H$, which is quite subtle and forces us to impose the assumption $p > n$.

6.3. **Admissibility of $R_T^G(\theta)$.** Thanks to Section 6.2 $R_{T_h}^{G_h}(\theta)$ is now understood sufficiently well. Building on this we have the following result for $R_T^G(\theta)$.

[21] We observe here that specifically for the group Sp_4 of type $C_2 = B_2$ all non-trivial characters θ of the elliptic torus $\mathbf{T}_w$ (w Coxeter) are primitive, so that in this particular case, Theorem 6.2 would follow from Theorem 4.5 along with some (relatively easy) claim about the fibers of $X_h \to X_{h-1}$. The next non-trivial case is type B_3, but here it also seems that the method of proof of Theorem 6.2 does not generalize straightforwardly.

Theorem 6.7 (Theorem 5.1 of [CI19b]). *Let $\theta\colon T \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character in general position. Then $R_T^G(\theta)$ is a finite sum of irreducible supercuspidal representations.*

Theorem 6.7 follows by standard arguments from Corollary 6.3 and the next proposition.

Proposition 6.8 (Proposition 5.2 of [CI19b]). *Let N be the group of k-points of the unipotent radical of a proper k-rational parabolic subgroup of $\mathbf{G}$. Then*

$$R_{T_h}^{G_h}(\theta)^{N\cap G_{\mathcal{O}}} = 0.$$

This proposition (as well as the theorem above) should be regarded as a generalization of the cuspidality result in the classical setup, i.e., Theorem 1.2(iii). It can be deduced by an "extension of action to a torus"-argument (i.e., Theorem 4.7(i)) from the following result, which is the technically deep one of this section. For simplicity we only explain the case $\mathbf{G} = \mathbf{GL}_n$. For $r \geq 1$ and $x \in (L_h^+\mathbb{G}_a)^{\oplus r}$, let $g_r(x)$ be the $r\times r$-matrix with i-th column $\sigma^{i-1}(x)$, and put

$$Y_{r,h} = \left\{x \in (L_h^+\mathbb{G}_a)^{\oplus r}\colon \det g_r(x) \in L_h^+\mathbb{G}_m\right\}$$

(regarded as a scheme of perfectly finite type over $\overline{\mathbb{F}}_q$).

Proposition 6.9 (Lemma 5.6 of [CI19b]). *If $N = \left\{\left(\begin{smallmatrix} 1 & B \\ 0 & 1\end{smallmatrix}\right) : B \in M_{i_0\times(n-i_0)}(k)\right\}$ for some $1 \leq i_0 \leq n-1$,*[22] *then*

$$N_h\backslash X_h \cong \left\{(m,x') \in Y_{i_0,h}\times Y_{n-i_0,h}\colon \frac{|g_{i_0}(m)|}{|g_{n-i_0}(x')|^{\sum_{j=1}^{i_0-1}\sigma^j}} \in (L_h^+\mathbb{G}_m)(\mathbb{F}_q)\right\},$$

induced by $x = (x_i)_{i=1}^n \mapsto ((m_i(x))_{i=1}^{i_0}, (x_i)_{i=i_0+1}^n)$, where $m_i(x)$ is the $(n-i_0+1)\times(n-i_0+1)$-minor of $g_n(x)$ given by

$$m_i(x) := \det\begin{pmatrix} x_i & \sigma(x_i) & \dots & \sigma^{n-i_0}(x_i) \\ x_{i_0+1} & \sigma(x_{i_0+1}) & \dots & \sigma^{n-i_0}(x_{i_0+1}) \\ x_{i_0+2} & \sigma(x_{i_0+2}) & \dots & \sigma^{n-i_0}(x_{i_0+2}) \\ \dots & \dots & \dots & \dots \\ x_n & \sigma(x_n) & \dots & \sigma^{n-i_0}(x_n)\end{pmatrix}$$

The proof is based (among other things) on technical calculations with classical determinantal identities, going back to the work of Turnbull [Tur09] from 1909 (see also [Lec93] for a more modern presentation of Turnbull's results). What makes Proposition 6.9 difficult, is that it is a statement about the "covering" $\dot{X}_{\dot{w}}(b)$. A corresponding statement for $X_w(b)$ would be easier to prove, and should follow along the lines of [Lus76a, (2.10)]. In particular, for $X_w(b)$ one would have a purely group-theoretical proof, which immediately should generalize to all groups $\mathbf{G}$, whereas it is not clear a priori how to generalize Proposition 6.9 to other groups.

6.4. **Maximal varieties and single cohomology groups of $X_{h,n'}$.** In the classical Deligne–Lusztig theory, one important question is to study single cohomology groups of $\dot{X}_{\dot{w}}$ and X_w. In general, it is a very hard question, but for w Coxeter quite complete results on X_w have been achieved by Lusztig [Lus76a].

One might ask for similar results on single cohomology groups in the case of $\dot{X}_{\dot{w}}(b)$. General techniques allowing such results were first studied by Boyarchenko [Boy12]. They were applied to

[22] i.e., N is the set of k-points of the unipotent radical of a maximal proper k-rational parabolic subgroup of $\mathbf{G}$. Some caution is in order: in Sections 6.2 and 6.3 different presentations of $X_{\mathcal{O}}$ are used, cf. [CI19b, Remark 2.1].

obtain particular results in the division algebra case, and to study special affinoids in the Lubin–Tate tower [BW16]. Then there was an improvement by Chan [Cha19] (see also [Cha16,Cha18]), finalizing the division algebra case.

6.4.1. *Maximal varieties.* We recall the following definition from [BW16]. Let $\mathbb{F}_Q$ be a finite field with Q elements of characteristic p. Let X be a separated scheme of finite type over $\mathbb{F}_Q$, and let $X_{\overline{\mathbb{F}}_Q} = X \otimes_{\mathbb{F}_Q} \overline{\mathbb{F}}_Q$. By [Del80, Theorem 3.3.1], for each i and each eigenvalue α of Fr_Q (the geometric Frobenius relative to $\mathbb{F}_Q$) in $H^i_c(X_{\overline{\mathbb{F}}_Q}, \overline{\mathbb{Q}}_\ell)$, there exists an integer $m \leq i$, such that all complex conjugates of α have the absolute value $Q^{m/2}$. Thus the Grothendieck–Lefschetz trace formula implies the upper bound

$$\#X(\mathbb{F}_Q) \leq \sum_{i\in\mathbb{Z}} Q^{i/2} \dim H^i_c(X_{\overline{\mathbb{F}}_Q}, \overline{\mathbb{Q}}_\ell).$$

on the number of $\mathbb{F}_Q$-points of X.

Definition 6.10. The $\mathbb{F}_Q$-scheme X is called *maximal* if this upper bound is achieved. Equivalently, X is maximal if and only if for each i, Fr_Q acts in $H^i_c(X_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$ by $(-1)^i Q^{i/2}$.

6.4.2. *Howe decomposition.* Let θ be a character of $T \cong L^\times$. There is a unique tower of fields $L = L_t \supsetneq L_{t-1} \supsetneq \cdots \supsetneq L_1 \supsetneq L_0 = k$ and (not necessarily uniquely determined) characters $\phi_0, \phi_1, \ldots, \phi_t$ of $k^\times$, $L_1^\times, \ldots, L_t^\times$ respectively, such that $\theta = (\phi_0 \circ \mathrm{N}_{L/K})(\phi_1 \circ \mathrm{N}_{L/L_1}) \ldots (\phi_t)$, where N_{L/L_i} are the respective norm maps. This is a Howe decomposition of θ, introduced by Howe [How77]. Similarly, one can define a Howe decomposition of characters of T_h and T_h^1.

6.4.3. *Single cohomology groups of $X_{h,n'}$.* Recall the closed subvariety $X_{h,n'}$ of X_h from Section 6.2.2. Generalizing the results of [Cha19] in the division algebra case, we have the following theorem, describing the single cohomology groups of $X_{h,n'}$ for all inner forms of $\mathbf{GL}_n$. There is a disjoint decomposition $X_{h,n'} = \coprod_{g\in G_1} g.X^1_{h,n'}$ into connected components, and $G_h^1 \times T_h^1$ acts on $X^1_{h,n'}$.

Theorem 6.11 (Theorem 6.1.1 of [CI20])**.** *Let $\chi\colon T_h^1 \to \overline{\mathbb{Q}}_\ell^\times$ be any character. There is an integer r_χ explicitly determined by the Howe decomposition χ, such that*

$$H^i_c(X^1_{h,n'}, \overline{\mathbb{Q}}_\ell)_\chi = \begin{cases} \textit{irreducible } G_h^1\textit{-representation} & \textit{if } i = r_\chi, \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, there exists an $\mathbb{F}_{q^n}$-rational structure on $X_{h,n'}$ making it into a maximal variety over $\mathbb{F}_{q^n}$, i.e., for each $i \geq 0$ the action induced by σ^n in $H^i_c(X_{h,n'}, \overline{\mathbb{Q}}_\ell)$ is given by multiplication with the scalar $(-1)^i q^{ni/2}$.

The method of proof of this result is based on several technical lemmas from [Boy12, Cha19], and is roughly a complicated induction procedure on partial products of factors in the Howe decomposition of χ.

For θ as in Theorem 6.1 (trivial on U_L^h) and $\chi := \theta|_{T_h^1}$, the explicitly determined integer r_χ plays an important role in step (5) of the proof Theorem 6.1. We also note that in the special case $\mathbf{G} = \mathbf{GL}_2$, Theorem 6.11 was essentially shown in [Iva16, Theorem 3.5, Corollary 3.13] by other methods (there the $\mathbb{F}_{q^2}$-rational structure was slightly different, hence the Frobenius acted by slightly different scalars).

6.4.4. *Drinfeld stratification.* The closed subscheme $X_{h,n'}$ of X_h can be put into a more general context: in fact, in [CI20, §4.4] Chan and the author defined a stratification of X_h into locally closed parts, which in analogy with a stratification on the Drinfeld half-space were called the Drinfeld stratification. This stratification generalizes to other groups [CI20, §3] and seems to be an important refinement of the "integral part" of $\dot{X}_{\dot{w}}(b)$.

For example, for inner forms of $\mathbf{GL}_n$ the strata are parametrized by divisors of n': $X_h = \bigcup_{r|n'} X_{h,r}$ with $X_{h,n'}$ being the closed stratum. Extending Theorems 6.1, 6.11 and the well-understood situation for the closed stratum, we make a precise conjecture predicting the stratum in which specific supercuspidal representations should be expected.

Conjecture 6.12 (cf. Conjecture 7.2.1 of [CI20])**.** *Let $r \mid n'$ and let $\theta\colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a character with trivial* $\mathrm{Gal}(L/k)$*-stabilizer. Suppose that the stabilizer of $\theta|_{U_L^1}$ in* $\mathrm{Gal}(L/k)$ *is equal to the unique subgroup of index $n_0 r$. Then*

$$R_{T_h}^{G_h}(\theta) \cong H_c^*(X_{h,r}, \overline{\mathbb{Q}}_\ell)_\theta \cong \pi_\theta.$$

where π_θ is as in Section 6.1. Moreover, $H_c^i(X_{h,r}, \overline{\mathbb{Q}}_\ell)_\theta = 0$, unless $i = r_\theta$, an integer explicitly determined by the Howe decomposition of θ.

Only the last part of this conjecture requires a proof (by extending the methods from [Cha19, CI20]); the first part then follows from this and results explained in Sections 6.2, 6.3.

7. Extended affine Deligne–Lusztig varieties

In this section we discuss results from [Iva16, Iva18, Iva19] (and [CI18]). We begin with affine Deligne–Lusztig varieties, which were introduced by Rapoport in [Rap05] to study the reduction mod p of some Shimura varieties. The name is justified by the fact that they arise by exactly the same procedure as their classical analoga, but inside the *affine* flag manifold.

Before [Iva20] (see Section 3) or Theorem 5.1, there was the hope to use covers of affine Deligne–Lusztig varieties introduced in [Iva16] instead of X_{Lus} – which was lacking a scheme-structure – to realize p-adic Deligne–Lusztig induction, or alternatively to use them to provide X_{Lus} with a scheme structure [CI18]. Both ideas work in several cases, but they seem to be less efficient than the more recently developed techniques of [Iva20, CI19b].

Nevertheless, there is a third important point where the construction [Iva16] has an independent value: it concerns the generalization in [Iva18] to the case of *ramified* tori in k-groups. Indeed, all constructions and results in previous sections only concern tori which split over $\breve{k}$, and it is not clear yet how one could uniformly cover all tori. At least, no *ad hoc* generalization of X_{Lus} seems to produce ramified supercuspidals[23] via $\theta \mapsto R_T^G(\theta)$. Instead, the construction of higher level affine Deligne–Lusztig varieties has a quite natural generalization which indeed realizes many ramified supercuspidals at least for $\mathbf{G} = \mathbf{GL}_2$.

Conventions. For simplicity we make two additional assumptions: (i) $\mathrm{char}\, k > 0$, that is $k = \mathbb{F}_q((\varpi))$, $\breve{k} = \overline{\mathbb{F}}_q((\varpi))$ (in fact, in [Iva16, Iva18] only this case was handled; the other case works similarly). (ii) We assume that $\mathbf{G}$ is split (everything below works for arbitrary reductive $\mathbf{G}$).

[23]Here and below, by ramified supercuspidals we will mean those supercuspidal G-representations whose L-parameter does not factor through ${}^L\mathbf{T} \to {}^L\mathbf{G}$ for an unramified maximal torus $\mathbf{T} \subseteq \mathbf{G}$.

7.1. **Affine Deligne–Lusztig varieties.** Let $\mathbf{T}_0$ be a split maximal torus of $\mathbf{G}$. We have the Bruhat–Tits buildings $\mathscr{B}_k$ and $\mathscr{B}_{\breve{k}}$ of the adjoint group $\mathbf{G}_{ad}$ over k resp. $\breve{k}$. The Frobenius σ of $\breve{k}/k$ acts on $\mathscr{B}_{\breve{k}}$ and $\mathscr{B}_{\breve{k}}^{\langle\sigma\rangle} = \mathscr{B}_k$ (as simplicial complexes). Let $I \subseteq \mathbf{G}(\breve{k})$ be an Iwahori subgroup corresponding to a σ-stable alcove of $\mathscr{B}_{\breve{k}}$ contained in the apartment of $\mathbf{T}_0$. By Bruhat–Tits theory, there is an $\mathcal{O}_k$-model $\mathcal{G}$ of $\mathbf{G}$ such that $\mathcal{G}(\mathcal{O}_{\breve{k}}) = I$. Attached to it we have the affine flag manifold, which is the fpqc-quotient $L\mathbf{G}/L^+\mathcal{G}$. It is represented by an ind-scheme, which is ind-proper of (perfectly) ind-finite type over $\overline{\mathbb{F}}_q$ [PR08, Theorem 1.4], and $(L\mathbf{G}/L^+\mathcal{G})(\overline{\mathbb{F}}_q) = \mathbf{G}(\breve{k})/I$. The extended affine Weyl group $\widetilde{W}$ of $\mathbf{T}_0$ in $\mathbf{G}$ sits in the short exact sequence

$$0 \to X_*(\mathbf{T}_0) \to \widetilde{W} \to W \to 1,$$

where $W = N_{\mathbf{G}}(\mathbf{T}_0)(\breve{k})/\mathbf{T}_0(\breve{k})$ is the finite Weyl group of $\mathbf{T}_0$. The Iwahori–Bruhat decomposition states now $\mathbf{G}(L) = \coprod_{x \in \widetilde{W}} IxI$, and an affine Deligne–Lusztig variety (of Iwahori level) attached to $x \in \widetilde{W}$, $b \in \mathbf{G}(\breve{k})$ is the locally closed subset

$$X_x^{\mathrm{aff}}(b) = \{gI \in \mathbf{G}(\breve{k})/I \colon g^{-1}b\sigma(g) \in IxI\}^{24}$$

of $L\mathbf{G}/L^+\mathcal{G}$, endowed with the structure of a reduced sub-ind-scheme. In fact, it is even a scheme, locally of perfectly finite type over $\overline{\mathbb{F}}_q$. The group $\mathbf{G}_b(k)$ from (3.1) acts on $X_x^{\mathrm{aff}}(b)$ by left multiplication.

Affine Deligne–Lusztig varieties are more complicated combinatorically, as well as scheme-theoretically, than classical Deligne–Lusztig varieties. The difference is highlighted by the following facts: whereas the classical variety X_w is non-empty and smooth of dimension $\ell(w)$, affine Deligne–Lusztig varieties $X_x^{\mathrm{aff}}(b)$ can be empty and non-equidimensional; moreover, the question when $X_x^{\mathrm{aff}}(b)$ is empty and which dimension it has (depending on x, b) is a very complicated one, and was finally essentially answered only by the work of many people (see, in particular, Görtz–Kottwitz–Haines–Reuman [GHKR10], Görtz–He [GH10]).

7.2. **Affine Deligne–Lusztig varieties of higher level.** Using the setup of Section 7.1, we sketch the construction from [Iva16]. The idea is very simple: replace the Iwahori-level model $\mathcal{G}$ with $\mathcal{G}(\mathcal{O}_{\breve{k}}) = I$ by models of higher level. Let $\Phi = \Phi(\mathbf{T}_0, \mathbf{G})$ be the root system of $\mathbf{T}_0$ in $\mathbf{G}$. For $a \in \Phi$, let $\mathbf{U}_a$ denote the corresponding root subgroup. Put $\mathbf{U}_0 = \mathbf{T}_0$. Fix a point $\mathbf{x}$ in $\mathscr{B}_k$ contained in the closure of the alcove corresponding to I. Then $\mathbf{x}$ determines by Bruhat–Tits theory [BT84, §6.2] a filtration $U_a(\breve{k})_r$ $(r \in \mathbb{R})$ on $\mathbf{U}_a(\breve{k})$ $(a \in \Phi)$. Moreover, we also have the Moy–Prasad filtration $\mathbf{U}_0(\breve{k})_r = \mathbf{T}(\breve{k})_r$ $(r \in \mathbb{R})$ on $\mathbf{T}_0(\breve{k})$. For a concave function $f \colon \Phi \cup \{0\} \to \mathbb{R}_{\geq 0}$, Bruhat–Tits theory gives a well-behaved $\mathcal{O}_{\breve{k}}$-model $\mathcal{G}_f$ of $\mathbf{G}$, such that $I^f := \mathcal{G}_f(\mathcal{O}_{\breve{k}})$ is the subgroup of $\mathbf{G}(\breve{k})$ generated by all $\mathbf{U}_a(\breve{k})_{f(a)}$ $(a \in \Phi \cup \{0\})$. We have $(L\mathbf{G}/L^+\mathcal{G}_f)(\overline{\mathbb{F}}_q) = G(\breve{k})/I^f$. If f_I is a concave function such that $I^{f_I} = I$ and $f \geq f_I$, then $I^f \subseteq I$ and there is a morphism $L^+\mathcal{G}_f \to L^+\mathcal{G}$ obtained by a series of dilatations along the unit section. This defines a map of fpqc-sheaves $L\mathbf{G}/L^+\mathcal{G}_f \to L\mathbf{G}/L^+\mathcal{G}$. Consider the set $D_f = I^f\backslash\mathbf{G}(\breve{k})/I^f$ of double cosets. There is a surjection $D_f \to \widetilde{W}$.

Definition 7.1. An *affine Deligne–Lusztig set of level* f attached to $b \in \mathbf{G}(\breve{k})$ and $\dot{w} \in D_f$ is

$$X_{\dot{w}}^{\mathrm{aff},f}(b) = \{gI^f \in G(\breve{k})/I^f \colon g^{-1}b\sigma(g) \in \dot{w}\}.$$

The following proposition justifies that we can call these sets varieties in many cases.

[24] We use the upper index "aff" to distinguish this construction from the $X_w(b)$ inside $L(G/B)$ in Section 3.

Proposition 7.2 (Proposition 2.4 of [Iva18]; Theorem 4.9 [CI18])**.** *Under a condition on $f, \dot{w}$ (which can be expressed in purely combinatorial terms), $X_{\dot{w}}^{\mathrm{aff},f}(b)$ is a locally closed subset of the ind-scheme $L\mathbf{G}/L^+\mathcal{G}_f$. Equipped with the reduced induced sub-ind-scheme structure, it is locally of perfectly finite type over $\overline{\mathbb{F}}_q$.*

The proof works by reduction to the well-known case of Iwahori or maximal compact level affine Deligne–Lusztig varieties [HV11, Corollary 6.5].

7.2.1. *Group actions.* The group $\mathbf{G}_b(k)$ acts on $X_{\dot{w}}^{\mathrm{aff},f}(b)$ by left multiplication. Suppose I_f is normal in I. Then I acts by left and right multiplication on D_f. Similarly as $\mathbf{T}_w(\mathbb{F}_q)$ acts on $\dot{X}_{\dot{w}}$ in Section 1, the σ-stabilizer of $\dot{w}$ in I,

$$I_{\dot{w},f} = \{i \in I \colon i^{-1}\dot{w}\sigma(i) = \dot{w}\}$$

acts by right multiplication on $X_{\dot{w}}^{\mathrm{aff},f}(b)$. This action extends to the action of $ZI_{\dot{w},f}$, where Z is the center of G, and factors through an action of $ZI_{\dot{w},f}/I_f$. Let w denote the image of $\dot{w}$ under $D_f \to \widetilde{W} \to W$. We have the maximal torus $\mathbf{T}_w \subseteq \mathbf{G}$ (determined only up to stable conjugacy). Varying f (and choosing $\dot{w}$ compatibly inside D_f), the group $ZI_{\dot{w},f}/I_f$ will contains various quotients of $\mathbf{T}_w(k)$, along with some unipotent part, whose action should not appear in the cohomology.

Example 7.3 (§3 of [Iva16])**.** Let $\mathbf{G} = \mathbf{GL}_2$, $1 \neq w \in W$, $\mathbf{T}_0$ the diagonal torus, $I = \begin{pmatrix} \mathcal{O}_{\breve{k}}^\times & \mathcal{O}_{\breve{k}} \\ \mathfrak{p}_{\breve{k}} & \mathcal{O}_{\breve{k}}^\times \end{pmatrix}$, $\dot{I}^h = \begin{pmatrix} 1+\mathfrak{p}_{\breve{k}}^{h+1} & \mathfrak{p}_{\breve{k}}^{h} \\ \mathfrak{p}_{\breve{k}}^{h+1} & 1+\mathfrak{p}_{\breve{k}}^{h+1} \end{pmatrix}$ (corresponding to an appropriate choice of $\mathbf{x} \in \mathscr{B}_k$ and functions $\dot{f}_h$, $h \geq 1$). Fix some $r \geq 1$ and let $\dot{w} = \begin{pmatrix} 0 & \varpi^{-r} \\ -\varpi^{r} & 0 \end{pmatrix} \in D_{\dot{f}_h}$, a compatible choice for all h. If L/k denotes unramified extension of degree 2 and U_L^{h+1} the $(h+1)$-units, then there is an exact sequence

$$0 \to L_{h+1}^+\mathbb{G}_a \to ZI_{\dot{w},h}/\dot{I}_h \to L^\times/U_L^{h+1} \to 1.$$

The action of $ZI_{\dot{w},h}/\dot{I}_h$ on the ℓ-adic cohomology of $X_{\dot{w}}^{\mathrm{aff},\dot{f}_h}(1)$ factors through an action of $L^\times/U_L^{h+1}$ [Iva16, Lemma 3.12].

7.3. **Examples in some Coxeter cases in type $\widetilde{A}_{n-1}$.** This is based on either [Iva16, §3] ($\mathbf{GL}_2$-case) or, more generally, [CI18, §6]. Consider the group $\mathbf{GL}_n$ ($n \geq 2$) and let

- I^m (with $m \geq 0$) be the preimage under $\mathrm{GL}_n(\mathcal{O}_{\breve{k}}) \twoheadrightarrow \mathrm{GL}_n(\mathcal{O}_{\breve{k}}/\varpi^{m+1}\mathcal{O}_{\breve{k}})$, of upper triangular matrices whose entries over the main diagonal lie in $\varpi^m\mathcal{O}_{\breve{k}}/\varpi^{m+1}\mathcal{O}_{\breve{k}}$.
- the basic element $b \in \mathbf{GL}_n(\breve{k})$ and $\kappa = \kappa_0 n'$, $n = n_0 n'$ be as in Section 5.1,
- for $r \geq 0$, $\dot{w}_r = \begin{pmatrix} 0 & \varpi \\ \mathbf{1}_{n_0-1} & 0 \end{pmatrix} \varpi^{(-r,\dots,-r,\kappa+(n-1)r)} \in \mathbf{GL}_n(\breve{k})$ (resp. the corresponding double I^m-coset),
- V be a fixed n'-dimensional $\overline{\mathbb{F}}_q$-vector space with a fixed $\mathbb{F}_{q^{n_0}}$-rational structure.

We have the $(n'-1)$-dimensional Drinfeld upper half-space over $\mathbb{F}_{q^{n_0}}$, the perfection of

$$\Omega_{\mathbb{F}_{q^{n_0}}}^{n'-1} := \mathbb{P}(V) \smallsetminus \bigcup_{\substack{H \subseteq V \\ \mathbb{F}_{q^{n_0}}-\text{rational hyperplane}}} H.$$

Theorem 7.4 (Theorem 6.14 of [CI18]). *Let m, b, $\dot{w}_r$ be as above. Assume $r > m \geq 0$. Then we have an isomorphism of $\overline{\mathbb{F}}_q$-schemes*

$$X^{\mathrm{aff},m}_{\dot{w}_r}(b) \cong \bigsqcup_{G/G_{\mathcal{O}}} \Omega^{n'-1}_{\mathbb{F}_{q^{n_0}}} \times \mathbb{A},$$

where $G, G_{\mathcal{O}}$ are as in Section 5, and $\mathbb{A}$ is the perfection of a finite dimensional affine space over $\overline{\mathbb{F}}_q$ (with dimension depending on r, m). In particular, all these schemes are perfections of smooth $\overline{\mathbb{F}}_q$-schemes.

The author learned from E. Viehmann about a method of proof of the disjoint decomposition in the theorem. Viehmann also proved related results at the maximal compact level [Vie08]. Another part of the proof of Theorem 7.4 is done by explicitly parametrizing some affine Schubert cells IvI/I (resp. their coverings IvI/I^m) and using an affine analog of the (very useful, when it comes down to computations) property of a classical Deligne–Lusztig variety attached to a Coxeter element, that it is entirely contained in the generic Schubert cell.

Remark 7.5. (i) Taking $m = 0$, we obtain a description of some affine Deligne–Lusztig varieties at Iwahori level (to our knowledge not appearing in the literature before).

(ii) Passing to the inverse limit over r and m (in some non-canonical way) gives an $\overline{\mathbb{F}}_q$-scheme, whose $\overline{\mathbb{F}}_q$-points are precisely $X_w(b)(\overline{\mathbb{F}}_q)$ for $w = \begin{pmatrix} 0 & 1 \\ \mathbf{1}_{n_0-1} & 0 \end{pmatrix}$. To obtain the $\overline{\mathbb{F}}_q$-points of the covering $\dot{X}_{\dot{w}}(b)$ from Theorem 5.1, one can apply the same procedure, after replacing I^m by the subgroup $\dot{I}^m$ of matrices whose entries on the main diagonal are congruent 1 modulo $\mathfrak{p}_{\breve{k}}^{m+1}$.

(iii) The passage to the inverse limit in (ii), was exactly the *modus operandi* in [CI18] to endow Lusztig's sets $X_w(b)(\overline{\mathbb{F}}_q)$, and $\dot{X}_{\dot{w}}(b)(\overline{\mathbb{F}}_q)$ with scheme structure. This was simplified by [CI19b, Proposition 2.6].

7.4. **Extended affine Deligne–Lusztig varieties.** The variety $X^{\mathrm{aff},f}_{\dot{w}}(b)$ is in a sense attached to the – necessarily unramified – torus $\mathbf{T}_w$, where w is the image of $\dot{w}$ in W. Following [Iva18] we will now extend this definition to cover other tori. Let E/k be a finite separable extension, such that $\breve{E} = E\breve{k}$ is the completion of a Galois extension of k. Let $\mathrm{Gal}(\breve{E}/k)$ denote the set of continuous automorphisms of $\breve{E}$ fixing k.

We have the Bruhat–Tits building $\mathscr{B}_{\breve{E}}$ of $\mathbf{G}$ over $\breve{E}$, and $\mathrm{Gal}(\breve{E}/k)$ acts on $\mathscr{B}_{\breve{E}}$. In general $\mathscr{B}_{\breve{E}}^{\mathrm{Gal}(\breve{E}/k)} \neq \mathscr{B}_k$ even as sets, which is due to the so called "barbs", which form a kind of a tubular neighborhood of $\mathscr{B}_k$ in $\mathscr{B}_{\breve{E}}$ and are pointwise fixed by the $\mathrm{Gal}(\breve{E}/k)$-action. Their size depends on how wild the ramification of $\breve{E}/k$ is, and if E/k is tamely ramified, then they vanish, i.e., $\mathscr{B}_{\breve{E}}^{\mathrm{Gal}(\breve{E}/k)} = \mathscr{B}_k$ at least as sets (however, still not as simplicial complexes, unless E/k is unramified), see [Rou77,Pra01]. Fix an alcove in the apartment of $\mathbf{T}_0$ in $\mathscr{B}_{\breve{E}}$ (it is automatically $\mathrm{Gal}(\breve{E}/k)$-stable, as $\mathbf{T}_0$ is split). Let $I \subseteq G(\breve{E})$ denote the corresponding Iwahori subgroup. Fix also a point of $\mathscr{B}_k$, in the closure of this alcove. As in Section 7.2 for a concave function $f\colon \Phi \cup \{0\} \to \mathbb{R}_{\geq 0}$ we have the corresponding integral model $\mathcal{G}_f$ of $\mathbf{G}$. Put $I^f = \mathcal{G}_f(\breve{E})$ (it is stable under $\mathrm{Gal}(\breve{E}/k)$-action, as $\mathbf{T}_0$ is split) and let $D_{\breve{E},f} = I^f \backslash G(\breve{E})/I^f$.

Definition 7.6 (Definition 2.1 of [Iva18]). Additionally to the above, fix the following data:

- A (finite) subset $\Sigma \subseteq \mathrm{Gal}(\breve{E}/k)$ such that $\breve{E}^{\Sigma} = k$
- An element $b \in G(\breve{k})$

- A map $\underline{w}\colon \Sigma \to D_{\breve{E},f}$.

The extended affine Deligne–Lusztig set attached to these data is

$$X_{\underline{w}}^{\mathrm{aff},\Sigma,f}(b) = \{gI^f \in G(\breve{E})/I^f : g^{-1}b\sigma(g) \in \underline{w}(\gamma)\,\forall \gamma \in \Sigma\}.$$

As in Section 7.2 we have the following result.

Proposition 7.7 (Proposition 2.4 of [Iva18])**.** *Under some conditions on f and $\underline{w}$, $X_{\underline{w}}^{\mathrm{aff},\Sigma,f}(b)$ is a locally closed subset of the ind-scheme $L\mathbf{G}_{\breve{E}}/L^+\mathcal{G}_f$. Equipped with the reduced induced sub-ind-scheme structure, it is locally of perfectly finite type over $\overline{\mathbb{F}}_q$.*

7.4.1. *Group actions.* The group $\mathbf{G}_b(k)$ acts on $X_{\underline{w}}^{\mathrm{aff},\Sigma,f}(b)$ by left multiplication. Further, assume I^f is normal in I, and let $\mathbf{Z}$ be the center of $\mathbf{G}$. The group $\mathbf{Z}(\breve{E})I/I^f$ acts (on the right) on the set of maps $\Sigma \to D_{\breve{E},f}$ by $\underline{w}.i = i^{-1}\underline{w}(\gamma)\gamma(i)$ for all $i \in \mathbf{Z}(\breve{E})I/I^f$ and $\gamma \in \Sigma$. Then the group

$$\widetilde{I}_{f,\underline{w}}/I^f = \{i \in \mathbf{Z}(\breve{E})I/I^f : \underline{w}.i = \underline{w}\}$$

acts on $X_{\underline{w}}^{\mathrm{aff},\Sigma,f}(b)$ by right multiplication. Similar as in Example 7.3, this group is related to a (stable) conjugacy class of maximal tori $\mathbf{T} \subset \mathbf{G}$ obtained as a twist of $\mathbf{T}_0$ by a Galois 1-cocycle, which is determined by the map $\underline{w}$ (if it determines one). See Proposition 7.10 for an example.

7.5. **The GL_2-case.** In this case the varieties $X_{\underline{w}}^{\mathrm{aff},\Sigma,f}(b)$ relevant for the construction of ramified supercuspidals are studied in detail in [Iva18] (tamely ramified case, i.e., $p \neq 2$) and in [Iva19] (wildly ramified case, i.e., $p = 2$). To simplify the presentation, we only consider the wildly ramified case here, as it is anyway the more interesting one. The tamely ramified case is easier and can be obtained by a few simple modifications (in particular, taking $d = 0$ everywhere below).

Remark 7.8. Note that in the wild case, there are infinitely many ramified elliptic maximal tori in $\mathbf{GL}_2$, which are non-isomorphic over k (just as by Artin–Schreier theory, for any p, there are infinitely many degree p Galois extensions of a local field of characteristic p).

Take $\mathbf{G} = \mathbf{GL}_2$, let $\mathbf{T}_0$ the (split) diagonal torus, and let

- E/k be a degree two ramified extension with discriminant of valuation d (positive odd integer),
- τ be the unique non-trivial element of $\mathrm{Gal}(\breve{E}/E)$, and σ the Frobenius element of $\breve{E}/E$,
- $\Sigma = \{\sigma, \tau\}$,
- $b = 1$, so that we obtain smooth $G = \mathbf{GL}_2(k)$-representations as output of our cohomological induction procedure,
- $\dot{I}_{\breve{E}}^m = \begin{pmatrix} 1+\mathfrak{p}_{\breve{E}}^{m+1} & \mathfrak{p}_{\breve{E}}^m \\ \mathfrak{p}_{\breve{E}}^{m+1} & 1+\mathfrak{p}_{\breve{E}}^{m+1} \end{pmatrix}$ be a level subgroup contained in the Iwahori subgroup $\begin{pmatrix} \mathcal{O}_{\breve{E}}^\times & \mathcal{O}_{\breve{E}} \\ \mathfrak{p}_{\breve{E}} & \mathcal{O}_{\breve{E}}^\times \end{pmatrix}$,
- π be a uniformizer of E, such that $\pi^2 + \Delta\pi + \varpi = 0$ with some $\Delta \in \mathcal{O}_k$ ($\mathrm{ord}(\Delta) = \frac{d+1}{2}$). Put $\varepsilon := \tau(\pi)\pi^{-1} \in U_E^d \smallsetminus U_E^{d+1}$,
- for $r \geq 1$, $w_r = \begin{pmatrix} 0 & \pi^r\varepsilon^{\lfloor \frac{r}{2} \rfloor} \\ \pi^{-r}\varepsilon^{\lfloor -\frac{r+1}{2} \rfloor} & 0 \end{pmatrix}$. Let $\underline{w}_r\colon \Sigma \to D_m$ be the function given by $\underline{w}_r(\sigma) = 1$, $\underline{w}_r(\tau) = w_r$.

We denote the varieties from Definition 7.6 corresponding to these choices by $X_{\underline{w}_r}^{\mathrm{aff},m}(1)$.

Proposition 7.9 (Proposition 2.5 and Section 3.2 of [Iva19]; Theorem 3.8 and Section 5.1 of [Iva18])**.** *With above choices, $X_{\underline{w}_r}^{\mathrm{aff},m}(1)$ is a discrete reduced scheme (a disjoint union of* *countably many points), in some sense naturally parametrized as*

$$X^{\mathrm{aff},m}_{\underline{w}_r}(1) \cong \coprod_{g\in G/I_k} g.\left\{\left(\mathfrak{p}_E/\mathfrak{p}_E^{r+d+m+1} \smallsetminus \mathfrak{p}_E^2/\mathfrak{p}_E^{r+d+m+1}\right) \times \left(U_E/U_E^{m+1}\right)\right\},$$

where I_k is an Iwahori subgroup of G. The group actions of $G \times \widetilde{I}_{m,\underline{w}_r}/I^m$ can be described by (complicated) explicit formulas.

Write $\widetilde{\Gamma} := \widetilde{I}_{m,\underline{w}_r}/I^m$. In contrast to the unramified case in Section 7.2, this group gets more complicated in the present situation. Nevertheless it is still related to the (conjugacy class) of tori in $\mathbf{GL}_2$, which are isomorphic to $\mathrm{Res}_{E/k}\,\mathbf{G}_m$.

Proposition 7.10 (Proposition 2.10 of [Iva19]; Lemma 3.5 of [Iva18])**.** *Suppose $r \geq \frac{m-d+1}{2}$ and $2r > d$. Let Γ' be the commutator of $\widetilde{\Gamma}$. Then there is a (non-split) short exact sequence of finite abelian groups*

$$0 \to E^\times/U_E^{m+1} \to \widetilde{\Gamma}/\Gamma' \to \mathcal{O}_E/\mathfrak{p}_E^d \to 0$$

The condition $2r > d$ in the proposition is only there to simplify the (still hard) computations in [Iva19], and should in principle be removable. The condition $r \geq \frac{m-d+1}{2}$ is much more interesting, as the following section will show.

7.6. **Ramified supercuspidals.** In Theorem 7.4 we made the assumption $r \geq m$, due to fact that the varieties $X^{\mathrm{aff},m}_{\underline{w}_r}(b)$ only behave well if this holds (or at least, our proof only applies in this case). The representations realized in the cohomology of $X^{\mathrm{aff},m}_{\underline{w}_r}(b)$ (more precisely, its covers of level $\dot{I}^m$), at the end do not depend on the specific choice of r (as long as $r > m$), as $\dot{X}^{\mathrm{aff},m}_{\underline{w}_r}(b)$ for fixed m but different r's only differ by an affine space.

Not so in the ramified situation of Section 7.5: fixing the parameter m and varying r will change the representations realized by $H^0_c(X^{\mathrm{aff},m}_{\underline{w}_r}(1), \overline{\mathbb{Q}}_\ell)$. We have the following observation.

Observation 7.11 (Section 2.6 of [Iva19])**.** *Estimating $\dim_{\overline{\mathbb{Q}}_\ell} H^0_c(X^{\mathrm{aff},m}_{\underline{w}_r}(1), \overline{\mathbb{Q}}_\ell)_{\tilde{\theta}}$ against known formal degrees of some ramified irreducible representations produced via Bushnell–Kutzko types, predicts that for varying r, m, the smooth G-representation $H^0_c(X^{\mathrm{aff},m}_{\underline{w}_r}(1), \overline{\mathbb{Q}}_\ell)_{\tilde{\theta}}$ should be irreducible precisely when $r = \frac{m-d+1}{2}$.*

In particular, for $r > \frac{m-d+1}{2}$, we cannot hope for $H^0_c(X^{\mathrm{aff},m}_{\underline{w}_r}(1), \overline{\mathbb{Q}}_\ell)_{\tilde{\theta}}$ to be irreducible G-representation for a smooth character $\tilde{\theta}\colon \widetilde{\Gamma} \to \overline{\mathbb{Q}}_\ell^\times$, and in general this also isn't true. For $r = \frac{m-d+1}{2}$, we indeed obtain supercuspidal representations, as the theorem below shows. To describe which supercuspidal representations we get, we observe the following slightly exotic isomorphism. Let $r > \frac{m-d+1}{2}$ and $2r > d$. Let

$$\Pi := E^\times/U_E^{m+1} \times_{\mathfrak{p}_E^{r+d}/\mathfrak{p}_E^{m+1}} \mathfrak{p}_E^r/\mathfrak{p}_E^{m+1}, \tag{7.1}$$

be the pushout along the natural embedding $\mathfrak{p}^{r+d}/\mathfrak{p}_E^{m+1} \hookrightarrow \mathfrak{p}_E^r/\mathfrak{p}_E^{m+1}$ and the map $\mathfrak{p}_E^{r+d}/\mathfrak{p}_E^{m+1} \hookrightarrow E^\times/U_E^{m+1}$, $x \mapsto 1 + x$.

Proposition 7.12 (Proposition 2.12 of [Iva19])**.** *Assume $r \geq \frac{m-d+1}{2}$ and $2r > d$. There is an isomorphism $\beta\colon \Pi \xrightarrow{\sim} \widetilde{\Gamma}/\Gamma'$ (depending on the choice of the uniformizer π modulo U_E^d).*

Remark 7.13 (Remark 2.13 of [Iva19])**.** The isomorphism β is slightly exotic. At least if $r \geq d$, one has a natural isomorphism

$$\widetilde{\Gamma}/\Gamma' \cong E^\times/U_E^{m+1} \times_{U_E^{r+d}/U_E^{m+1}} U_E^r/U_E^{m+1}.$$

Now $U_E^{r+d}/U_E^{m+1} \cong \mathfrak{p}_E^{r+d}/\mathfrak{p}_E^{m+1}$, via $1+x \mapsto x$. But U_E^r/U_E^{m+1} is obviously non-isomorphic to $\mathfrak{p}_E^r/\mathfrak{p}_E^{m+1}$ if $2r < m+1$ (the second group is killed by 2, the first has non-trivial 4-torsion).

The dual $\beta^\vee\colon (\widetilde{\Gamma}/\Gamma')^\vee \xrightarrow{\sim} \Pi^\vee$ of β transforms characters of $\widetilde{\Gamma}$ into characters of Π. But to give a character χ of Π is exactly equivalent (after an embedding $\iota\colon E^\times \hookrightarrow G$ is fixed) to give a *cuspidal type* in the sense of Bushnell–Henniart [BH06], that is, some amount of quite explicit algebraic data, which determine a supercuspidal representation BH_χ of G ([Iva19, Definition 3.13]). For $\mathbf{GL}_n$, the construction of these supercuspidals out of the algebraic data is precisely the content of the theory of Bushnell–Kutzko types [BK93] (for the $\mathbf{GL}_2$-case, see also [BH06]); for all reductive groups, but with a restriction to only "tamely ramified supercuspidals", the corresponding theory was developed by Yu [Yu01]. The following result concerns the comparison of these algebraic construction with the cohomology of our zero-dimensional extended affine Deligne–Lusztig varieties.

Theorem 7.14 (Theorems 1.1 and 1.2 of [Iva19])**.** *Let $\tilde{\theta}\colon \widetilde{\Gamma} \to \overline{\mathbb{Q}}_\ell^\times$ be generic. If $r = \frac{m-d+1}{2}$, the G-representation $H_c^0(X_{\underline{w}_r}^{\mathrm{aff},m}(1), \overline{\mathbb{Q}}_\ell)_{\tilde{\theta}}$ is irreducible, supercuspidal and is isomorphic to $\mathrm{BH}_{\beta^\vee(\tilde{\theta})}$. All smooth irreducible supercuspidal representations ρ of G, which are attached to $\iota(E^\times) \subseteq G$ via Bushnell–Kutzko types, whose level cannot be lowered by a central twist, and which satisfy $2\ell(\rho) \geq 3d$ are isomorphic to some $H_c^0(X_{\underline{w}_r}^{\mathrm{aff},m}(1), \overline{\mathbb{Q}}_\ell)_{\tilde{\theta}}$.*

Among the representations in the theorem also *exceptional* ones [BH06, 44.1 Definition] appear, i.e., those whose L-parameter does not factor through a torus. The proof of this theorem relies on a (quite technical) computation of traces of enough elements on both sides. The analogous result in the tamely ramified case is [Iva18, Theorem 1.2].

Remark 7.15. In Remark 7.5(ii),(iii) it was mentioned that forming an inverse limit over $r \geq m$ of affine Deligne–Lusztig varieties (in some non-canonical way!) gives the space $X_w(b)$. In the present situation,

- one *cannot* form a compatible system of $X_{\underline{w}_r}^{\mathrm{aff},m}(1)$ for $r = \frac{m+d-1}{2} \to \infty$ in a similar way.
- one can form a compatible system of $X_{\underline{w}_r}^{\mathrm{aff},m}(1)$ with $r \geq m+d-1$. This gives in the limit a version of $X_w(b)$ (that is, something inside $L(\mathbf{G}/\mathbf{B} \otimes_{\breve{k}} \breve{E})$) which is in a sense "attached to a ramified torus", but its cohomology does not realize supercuspidal representations (cf. Observation 7.11).

From Theorem 7.14 and Remark 7.15 we draw the following conclusion, summarizing the content of the last sections.

Conclusion 7.16. *Extended affine Deligne–Lusztig varieties are a useful tool to cohomologically realize the construction of supercuspidals associated with ramified tori via Bushnell–Kutzko's and Yu's cuspidal types. On the other side, their cohomology seems not to realize local Langlands correspondences. A naive parallel construction "inside $L(G/B)$" (in the spirit of Section 3) does not even properly realize supercuspidals.*

Question 7.17. It would be interesting to clarify the following:

- Compute the representations $H_c^0(X_{\underline{w}_r}^{\mathrm{aff},m}(1), \overline{\mathbb{Q}}_\ell)_\rho$, where ρ is an irreducible representation of $\widetilde{\Gamma}$ of dimension > 1 (in particular, ρ will not factor through $\widetilde{\Gamma}/\Gamma'$).
- What happens in the case $r < \frac{m-d+1}{2}$? Also, in the unramified situation (Section 7.2), determine $X_{w_r}^m(b)$ for $r < m$.

8. Further directions of study

In this final section some open questions resp. further directions of development are collected:

1. Investigate further representability properties of the arc-sheaves $X_w(b)$. In particular, prove an optimal result towards Conjecture 3.6.
2. Solve the second half of [Boy12, Problem 1], *i. e.*, make sense of the ℓ-adic (co)homology of $X_w(b)$, generalizing the *ad hoc* construction from Section 5.3.
3. Study the torsors $\dot{X}_{\dot{w}}(b)$ over $X_w(b)$ in situations beyond elliptic tori and $\mathbf{GL}_n$. Is there a relation to stable vs. rational conjugacy classes of tori?
4. For elliptic tori, generalize Theorem 6.2 (Mackey formula without restriction to primitive characters) and Theorem 6.7 (cuspidality of the induced representation) to all reductive groups $\mathbf{G}$. A purely group-theoretic generalization is at least not obvious, cf. Remark 6.4 and Section 6.3.
5. Obtain a p-adic version of [Lus76a], which (roughly) means to describe the single cohomology groups of $X_w(b)$ as well as the Frobenius action in them.
6. Find a natural generalization of Definition 3.2 to ramified tori.
7. Study the relation between extended affine Deligne–Lusztig varieties and (ramified) cuspidal types of Bushnell–Kutzko and Yu.
8. Develop the corresponding theory with $\overline{\mathbb{F}}_\ell$-coefficients instead of $\overline{\mathbb{Q}}_\ell$-coefficients.

This list is by no means complete, it just gives an impression of the authors opinion, which kind of questions related to the topics explained above are interesting and could probably be solved with today's state of technology.

Acknowledgements. During the period of time when the articles following below were written, I experienced a lot of love and support by my relatives, especially my parents Evgenia Ivanova and Boris Zeldin, my partner Medea Menzel and our daughter Golda. I would like to express my deepest gratitude to them for all the love and support they gave to me (in case of Golda, it was more love than support, but at the end this is what counts).

I especially would like to thank my long term coauthor Charlotte Chan from whom I learned many things, and with whom I really enjoyed working together. Also, I would like to express my special gratitude to Eva Viehmann, Laurent Fargues and Peter Scholze who helped me a lot in my mathematical development during the time after my Ph.D. I would also like to thank many people for uncountable many very helpful advices and enlightening discussions concerning various subjects related to the articles below: Johannes Anschütz, Jean-François Dat, Olivier Dudas, Laurent Fargues, Ulrich Görtz, Thomas Haines, Paul Hamacher, Guy Henniart, Tasho Kaletha, Gregor Kemper, Arthur-Cesar Le Bras, Christian Liedtke, Andreas Mihatsch, Stephan Neupert, Peter Scholze, Eva Viehmann, Jared Weinstein, and many others.

Finally I would like to express my gratitude to the various institutions and organizations I was affiliated to and supported by during the time when the articles contributing to this habilitation thesis were written: Technische Universität München, Institut de Mathématiques de Jussieu, Universität Bonn, ERC via the starting grant 277889 "Moduli spaces of local G-shtukas" of Eva Viehmann, DFG via a postdoctoral research grant and via the Leibniz Prize of Peter Scholze.

References

[BH06] C. J. Bushnell and G. Henniart. *The Local Langlands Correspondence for GL_2*. Springer, 2006.

[BH10] C.J. Bushnell and G. Henniart. The essentially tame local Langlands correspondence, III: the general case. *Proc. London Math. Soc.*, 101(3):497–553, 2010.

[BK93] C.J. Bushnell and P.C. Kutzko. *The Admissible Dual of $GL(N)$ Via Compact Open Subgroups*, volume 129 of *Annals of Mathematics Studies*. Princeton University Press, 1993.

[BM18] B. Bhatt and A. Mathew. The arc-topology. *Preprint*, 2018. arXiv:1807.04725.

[Boy12] M. Boyarchenko. Deligne-Lusztig constructions for unipotent and p-adic groups. *Preprint*, 2012. arXiv:1207.5876.

[BR08] C. Bonnafé and R. Rouquier. Affineness of Deligne–Lusztig varieties for minimal length elements. *J. Alg.*, 320:1200–1206, 2008.

[BT84] F. Bruhat and J. Tits. Groupes réductifs sur un corps local II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Inst. Hautes Études Sci. Publ. Math.*, 60:197–376, 1984.

[BW16] M. Boyarchenko and J. Weinstein. Maximal varieties and the local Langlands correspondence for $\mathrm{GL}(n)$. *J. Amer. Math. Soc.*, 29:177–236, 2016.

[Car84] H. Carayol. Représentations cuspidales du groupe linéaire. *Ann. Sci. École Norm. Sup.*, 17(4):191–226, 1984.

[Cha16] C. Chan. Deligne–Lusztig constructions for division algebras and the local Langlands correspondence. *Adv. Math.*, 294:332–383, 2016.

[Cha18] C. Chan. Deligne-Lusztig constructions for division algebras and the local Langlands correspondence, II. *Selecta Math.*, 24(4):3175–3216, 2018.

[Cha19] C. Chan. The cohomology of semi-infinite Deligne–Lusztig varieties. *J. Reine Angew. Math.*, to appear, 2019+. arXiv:1606.01795, v2 or later.

[Che18] Z. Chen. On the inner products of some Deligne–Lusztig-type representations. *Int. Math. Res. Not.*, 2018. rny195.

[CI18] C. Chan and A. B. Ivanov. Affine Deligne–Lusztig varieties at infinite level. *preprint*, 2018. arxiv:1811.11204.

[CI19a] C. Chan and A. B. Ivanov. Cohomological representations of parahoric subgroups. *preprint*, 2019. arxiv:1903.06153.

[CI19b] C. Chan and A. B. Ivanov. On loop Deligne–Lusztig varieties of Coxeter type for GL_n. *preprint*, 2019. arxiv:1911.03412.

[CI20] C. Chan and A. B. Ivanov. The Drinfeld stratification for GL_n. *Preprint*, 2020. arXiv:2001.06600.

[CMS90] L. Corwin, A. Moy, and P. Sally. Degrees and formal degrees for division algebras and GL_n over a p-adic field. *Pacific J. Math.*, 141:21–45, 1990.

[CS17] Z. Chen and A. Stasinksi. The algebraisation of higher Deligne–Lusztig representations. *Selecta Math.*, 23(4):2907–2926, 2017.

[DeB06] S. DeBacker. Parametrizing conjugacy classes of maximal unramified tori via Bruhat–Tits theory. *Michigan Math. J.*, 54:157–158, 2006.

[Del80] P. Deligne. La conjecture de Weil II. *Publ. Math. IHES*, 103(52):137–252, 1980.

[DL76] P. Deligne and G. Lusztig. Representations of reductive groups over finite fields. *Ann. of Math.*, 103(1):103–161, 1976.

[DM91] F. Digne and J. Michel. *Representations of finite groups of Lie type*. Cambridge University press, Cambridge, 1991.

[FF18] L. Fargues and J.-M. Fontaine. *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, volume 406 of *Astérisque*. 2018.

[GH10] U. Görtz and X. He. Dimensions of affine Deligne-Lusztig varieties in affine flag varieties. *Doc. Math.*, (15):1009–1028, 2010.

[GHKR10] U. Görtz, Th. J. Haines, R. E. Kottwitz, and D. C. Reuman. Affine Deligne–Lusztig varieties in affine flag varieties. *Comp. Math.*, 146:1339–1382, 2010.

[Gé75] P. Gérardin. *Construction des séries discrétes p-adiques*, volume 462 of *Lecture Notes in Mathematics*. Springer-Verlag, 1975.

[He08] X. He. On the affineness of Deligne-Lusztig varieties. *J. Alg.*, 320:1207–1219, 2008.

[He14] X. He. Geometric and homological properties of affine Deligne-Lusztig varieties. *Ann. Math.*, 179:367–404, 2014.

[Hen92] G. Henniart. Correspondance de Langlands–Kazhdan explicite dans le cas non ramifié. *Math. Nachr.*, 158(7-26), 1992.

[How77] R. Howe. Tamely ramified supercuspidal representations of GL_n. *Pacific J. Math.*, 73:437–460, 1977.

[HV11] U. Hartl and E. Viehmann. The Newton stratification on deformations of local G-shtuka. *J. reine und angew. Math.*, 656:87–129, 2011.

[Iva16] A. Ivanov. Affine Deligne-Lusztig varieties of higher level and the local Langlands correspondence for GL_2. *Adv. Math.*, 299:640–686, 2016.

[Iva18] A. Ivanov. Ramified automorphic induction and zero-dimensional affine Deligne-Lusztig varieties. *Math. Z.*, 288:439–490, 2018.

[Iva19] A. Ivanov. Ordinary $\mathrm{GL}_2(\mathrm{F})$-representations in characteristic two via affine Deligne-Lusztig constructions. *Math. Res. Lett.*, to appear, 2019+. arxiv:1802.02456.

[Iva20] A. Ivanov. On ind-representability of loop Deligne–Lusztig sheaves. *preprint*, 2020. arXiv:2003.04399.

[Kot85] R.E. Kottwitz. Isocrystals with additional structure. *Compos. Math.*, 56:201–220, 1985.

[Lec93] B. Leclerc. On identities satisfied by minors of a matrix. *Adv. Math.*, 100:101–132, 1993.

[Lus76a] G. Lusztig. Coxeter orbits and eigenspaces of Frobenius. *Invent. Math.*, 38:101–159, 1976.

[Lus76b] G. Lusztig. On the finiteness of the number of unipotent classes. *Invent. Math.*, 34:201–213, 1976.

[Lus79] G. Lusztig. Some remarks on the supercuspidal representations of p-adic semisimple groups. *In Automorphic forms, representations and L-functions, Proc. Symp. Pure Math.* ***33*** *Part 1 (Corvallis, Ore., 1977)*, pages 171–175, 1979.

[Lus04] G. Lusztig. Representations of reductive groups over finite rings. *Represent. Theory*, 8:1–14, 2004.

[OR08] S. Orlik and M. Rapoport. Deligne-Lusztig varieties and period domains over finite fields. *J. Alg.*, 320:1220–1234, 2008.

[PR08] G. Pappas and M. Rapoport. Twisted loop groups and their affine flag varieties. *Adv. in Math.*, 219:118–198, 2008.

[Pra01] G. Prasad. Galois-fixed points in the Bruhat-Tits building of a reductive group. *Bull. soc. math. France*, 129:169–174, 2001.

[Rap05] M. Rapoport. A guide to the reduction modulo p of Shimura varieties. *Astérisque*, 298:271–318, 2005.

[Rou77] G. Rousseau. Immeubles des groupes réductifs sur les corps locaux. Thése Université de Paris-Sud Orsay, 1977.

[RZ96] M. Rapoport and T. Zink. *Period spaces for p-divisible groups*. Annals of Mathematics Studies. Princeton University Press, 1996.

[Sch18] P. Scholze. Étale cohomology of diamonds. *Preprint*, 2018.

[Sta09] A. Stasinski. Unramified representations of reductive groups over finite rings. *Represent. Theory*, 13:636–656, 2009.

[Sta11] A. Stasinski. Extended Deligne-Lusztig varieties for general and special linear groups. *Adv. in Math.*, 226(3):2825–2853, 2011.

[SW20] P. Scholze and J. Weinstein. Berkeley lectures on p-adic geometry. *Preprint, to appear in Annals of Math Studies*, 2020+.

[Tur09] H. W. Turnbull. The irreducible concomitants of two quadratics in n variables. *Trans. Cambridge Philos. Soc.*, 21:197–240, 1909.

[Vie08] E. Viehmann. Moduli spaces of p-divisible groups. *J. Alg. Geom.*, 17:341–374, 2008.

[Wei16] J. Weinstein. Semistable models for modular curves of arbitrary level. *Invent. Math.*, 205(2):459–526, 2016.

[Yu01] J.-K. Yu. Construction of tame supercuspidal representations. *J. Amer. Math. Soc.*, 14:579–622, 2001.

[Zhu17] X. Zhu. Affine Grassmannians and the geometric Satake in mixed characteristic. *Ann. Math.*, 185(2):403–492, 2017.