# CONVEX ELEMENTS AND DEEP LEVEL DELIGNE-LUSZTIG VARIETIES

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Abstract. We essentially complete a program initiated by Boyarchenko–Weinstein to give a full description of the cohomology of deep level Deligne–Lusztig varieties for elliptic tori. We give several applications of our results: we show that the $\phi$-weight part of the cohomology is very often concentrated in a single degree, and is induced from a Yu-type subgroup. Also, we give applications to the work [Nie24] of the second author on decomposition of deep level Deligne–Lusztig representations, and to Feng's explicit construction of Fargues–Scholze parameters. Furthermore, a conjecture of Chan–Oi about the Drinfeld stratification follows as a special case from our results.

## 1. Introduction

In [BW16] Boyarchenko–Weinstein started a program toward a complete description of the cohomology of certain higher-dimensional varieties over $\overline{\mathbb{F}}_q$ equipped with interesting group actions. The varieties they considered came in two disguises: the first were related to special affinoids in Lubin–Tate spaces, and the second were very close to deep level Deligne–Lusztig varieties introduced in [Lus04, CI19]. In this article we introduce the notion of convex elements in a Weyl group, and essentially complete the program of Boyarchenko–Weinstein for deep level Deligne–Lusztig varieties associated to convex elements. We then give some applications. We note that various related/partial results in this direction were obtained in [Cha20, CI21, IN24] on deep level Deligne–Lusztig varieties of Coxeter type.

Let $k$ be a non-archimedean local field with residue field $\mathbb{F}_q$ of characteristic $p$. Let $\breve{k}$ be the completion of the maximal unramified extension of $k$. Let $F$ denote the Frobenius automorphism of $\breve{k}$ over $k$. Let $G$ be a reductive group over $k$, which splits over $\breve{k}$. Let $T$ be a $k$-rational $\breve{k}$-split elliptic maximal torus of $G$. Let $U$ be the unipotent radical of a $\breve{k}$-rational Borel subgroup of $G$ containing $T$.

Let $\mathbf{x}$ be a point in the Bruhat–Tits building of $G$ over $k$. Bruhat–Tits theory attaches to it a parahoric group $\mathcal{G}_{\mathbf{x}}$ over the integers $\mathcal{O}_k$ of $k$. By the work of Lusztig [Lus04] and Chan and the first author [CI19], one associates with $T, U, \mathbf{x}$ and any $r \geq 0$ a deep level Deligne–Lusztig variety

$$X_r = X_{T,U,\mathbf{x},r}$$

over $\overline{\mathbb{F}}_q$, equipped with an action of $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)\times\mathcal{T}_{\mathbf{x}}(\mathcal{O}_k)$, where $\mathcal{T}_{\mathbf{x}}$ is the closure of $T$ in $\mathcal{G}_{\mathbf{x}}$ (see §4 for definition).

For the rest of the introduction, we fix a prime number $\ell \neq p$ and a smooth character $\phi\colon \mathcal{T}_{\mathbf{x}}(\mathcal{O}_k) \to \overline{\mathbb{Q}}_\ell^\times$. The $\phi$-weight part $R^G_{T,U,r}(\phi) = \sum_{i\in\mathbb{Z}} H^i_c(X_r,\overline{\mathbb{Q}}_\ell)[\phi]$ of the (equivariant) $\ell$-adic Euler characteristic of $X_r$ is a virtual $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)$-representation.

Assume that $\phi$ admits a Howe factorization $(G^i, r_i, \phi_i)_{-1\leq i\leq d}$ in the sense of [Kal16, §3.6]. In [Nie24] the second author gave a very explicit decomposition of $R^G_{T,U,r}(\phi)$:

$$R^G_{T,U,r}(\phi) = \mathrm{ind}_{\mathcal{K}_\phi(\mathcal{O}_k)}^{\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)}\left(\kappa_\phi\otimes R^{G^0}_{T,U,0}(\phi_{-1})\right) \tag{1.1}$$

where $\mathcal{K}_\phi = \mathcal{K}_{\phi,\mathbf{x}}$ is a second $\mathcal{O}_k$-model of $G$ determined by the Howe datum $(G^i,r_i)_{-1\leqslant i\leqslant d}$, such that $\mathcal{K}_\phi(\mathcal{O}_k)\subseteq\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)$ is a "Yu-type subgroup", and $R^{G^0}_{T,U,0}(\phi_{-1})$ is a classical Deligne–Lusztig representation, regarded as a $\mathcal{K}_\phi(\mathcal{O}_k)$-representation by inflation. Here, $\kappa_\phi$ is a $\mathcal{K}_\phi(\mathcal{O}_k)$-representation, defined in cohomological terms, which is irreducible by [Nie24, Proposition 1.4]. As a comparison, there is the so-called Weil–Heisenberg representation $\kappa(\phi)$ appearing in J.-K. Yu's construction. One can expect that $\kappa_\phi$ and $\kappa(\phi)$ differ precisely by the quadratic character of Fintzen–Kaletha–Spice [FKS23, Theorem 4.1.13], see [Nie24, Remark 1.10].

There is also a second variety with a much simpler geometric structure,

$$Z_{\phi,U,r},$$

also equipped with the action of $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)\times\mathcal{T}_{\mathbf{x}}(\mathcal{O}_k)$ (see §6 for definition). It was first considered in special cases by Chen–Stasinski [CS17, CS] and plays also an important role in [Nie24]. Due to its simpler geometry, the cohomology of $Z_{\phi,U,r}$ is much easier to describe than that of $X_r$.

Our main technical result is the following (degreewise) comparison theorem. To state it we need the notion of *convex elements* of the Weyl group $W$ of $T$ in $G$, introduced in §3. Convex elements generalize Coxeter elements and share many properties with them. They have the advantage that any $\sigma$-conjugacy class of $W$ contains a convex element of minimal length. This and further properties of convex elements are shown in [NTY24].

**Theorem 1.1.** *Suppose the relative position of $U$ and $F(U)$ in the Weyl group of $T$ is a convex element of the Weyl group (cf. §3). Then there is a $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)$-equivariant isomorphism*

$$R\Gamma_c(X_r,\overline{\mathbb{Q}}_\ell)[\phi]\cong R\Gamma_c(Z_{\phi,U,r},\overline{\mathbb{Q}}_\ell)[\phi][2m]$$

*for some (explicit) shift $m\in\mathbb{Z}_{\geq 0}$.*

*If, moreover, $G^0$ is a standard Levi subgroup with respect to $U$ (such $U$ always exist by Proposition 3.5), then*

$$H^i_c(X_r,\overline{\mathbb{Q}}_\ell)[\phi]\cong \pm\mathrm{ind}_{\mathcal{K}_\phi(\mathcal{O}_k)}^{\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)}\left(\kappa_\phi\otimes H^{i-2n}_c(X^{G^0}_{T,U\cap G^0,\mathbf{x},\mathbf{0}},\overline{\mathbb{Q}}_\ell)[\phi_{-1}]\right),$$

*where* $X^{G^0}_{T,U,\mathbf{x},0}$ *is the classical Deligne-Lusztig variety for reductive quotient of* $(G^0)_{\mathbf{x}}$ *and* $n \in \mathbb{Z}$ *is some (explicit) shift. In particular, if* $\phi_{-1}$ *is non-singular in the sense of* [DL76, Definition 5.15]*, the cohomology groups* $H^i_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi]$ *concentrate at a single degree.*

The first part of Theorem 1.1 follows directly from Theorem 4.1 and Proposition 6.1. The second part is Theorem 6.6. The proof is of geometric flavor and relies on an analysis of the geometry of $X_r$ and $Z_{\phi,U,r}$. The methods used in the proof resemble those from [IN24]. Now we discuss some applications of Theorem 1.1. Most importantly, Theorem 1.1, along with further results from [Nie24], implies that $\phi$-weight part of cohomology of $X_r$ is often concentrated in a single cohomological degree.

**Corollary 1.2** (see Corollary 6.7 for a precise statement)**.** *Let* $T, U$ *be as in Theorem 1.1 and assume that* $G^0$ *is a standard Levi subgroup with respect to* $U$*. Suppose that* $\phi_{-1}$ *is non-singular for the special fiber of* $(\mathcal{G}^0)_{\mathbf{x}}$ *in the sense of* [DL76, Definition 5.15]*, then there exists a unique integer* $N_\phi$ *such that* $H^i_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi] \neq 0$ *if and only if* $i = N_\phi$*.*

We also have the following direct consequence.

**Corollary 1.3.** *Let* $T, U$ *be as in Theorem 1.1. Then the (derived)* $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)$*-representation* $R\Gamma_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi]$ *is induced from* $\mathcal{K}_\phi(\mathcal{O}_k)$*.*

*Proof.* This follows directly from Theorem 1.1 by (6.1). □

Moreover, Theorem 1.1 can be regarded as a stronger form of [CO23, Conjecture 6.5], which follows as a special case. For any (twisted) rational Levi subgroup $T \subseteq L \subseteq G$, there is a closed subvariety $X_r^{(L)} \subseteq X_r$, called a Drinfeld stratum, see [CI21, §3], [CO23, §6.2].

**Corollary 1.4.** *Let* $T \subseteq L \subseteq G$ *be a twisted rational Levi subgroup. If* $\phi$ *is such that* $G^0 \subseteq L$*, then* $R\Gamma_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi] = R\Gamma_c(X_r^{(L)}, \overline{\mathbb{Q}}_\ell)[\phi]$*. With other words,* [CO23, Conjecture 6.5] *holds true.*

*Proof.* This follows from Proposition 5.1 by proper base change. □

A fourth application concerns [Nie24]. Let $\mathcal{R}^G_{T,U,r}(\phi)$ denote the $\phi$-weight part of the equivariant $\ell$-adic Euler characteristic of $Z_{\phi,U,r}$. A major step in [Nie24] towards the proof of (1.1) was to show that

$$R^G_{T,U,r}(\phi) = \mathcal{R}^G_{T,U,r}(\phi) \tag{1.2}$$

as virtual $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)$-representations, see [Nie24, Theorem 5.7]. This relied in an essential way on inner product formulas for $R^G_{T,U,r}(\phi)$, proven by Chan [Cha24]. Now notice that (1.2) also follows directly from Theorem 1.1, giving a more geometric proof of [Nie24].

At the end of the introduction we give more applications of Theorem 1.1. Now we discuss the assumption on $U$ in Theorem 1.1, showing that it is not restrictive in the following sense. For any $T$, there is a choice of $U$, such that the relative position of $U$ with $F(U)$ satisfies this assumption and is, moreover, of minimal length in the (twisted) conjugacy class of the Weyl group containing it; this is shown by Tan, Yu and the second author in [NTY24], see §3. Note also that all the $p$-adic Deligne–Lusztig spaces $X_w(b)$ from [Iva23] (equipped with $G(k)$-action and closely related with $X_r$) attached to minimal length elements of a fixed twisted conjugacy class are equivariantly isomorphic by [Iva23, Corollary 7.25]; in particular, the specific choice of $U$ becomes irrelevant.

**Pro-unipotent Deligne–Lusztig varieties.** In the second part of the article we prove [IN24, Conjecture 1.2], thereby generalizing the [IN24, Theorem 1.1]. Let $\mathcal{G}_{\mathbf{x}}^+$ denote the pro-unipotent radical of $\mathcal{G}_{\mathbf{x}}$ and let $\mathcal{T}_{\mathbf{x}}^+$ be the closure of $T$ in $\mathcal{G}_{\mathbf{x}}^+$. Very similar to $X_r$, one can define a scheme $X^+$ over $\overline{\mathbb{F}}_q$ and its truncations $X_r^+$ (such that $X^+ = \varprojlim_r X_r^+$), equipped with natural $\mathcal{G}_{\mathbf{x}}^+(\mathcal{O}_k) \times \mathcal{T}_{\mathbf{x}}^+(\mathcal{O}_k)$-actions. See §7 for precise definition.

In *loc. cit.* we gave an essentially complete description of the homology of $X^+$ as a $(\mathbb{G}^+)^F \times (\mathbb{T}^+)^F$-module under some mild restrictions on $p$ and the condition that $T \subseteq G$ is of Coxeter type. Here, we generalize this in two ways: (1) we prove the result for all elliptic tori $T$ and (2) we relax the assumptions on $p$ (we only require $p$ not be a torsion prime for $G$). As in *loc. cit.*, we phrase our result in terms of the homology $f_\natural\overline{\mathbb{Q}}_\ell$ of the structure map $f\colon X^+ \to \operatorname{Spec}\mathbb{F}_q$ (whose $\phi$-weight part agrees up to a shift with the $\phi$-weight part of the compact support cohomology of $X_r^+$ for sufficiently big $r$). We refer to [IN24, §2.7] for a brief discussion of the homology functor. Let $N$ denote the order of $F$ as an automorphism of $\Phi$. The following generalizes [IN24, Theorem 1.1] and proves [IN24, Conjecture 1.2], except that in part (3) we have to assume convexity and in part (2) a different sign might appear.

**Theorem 1.5.** *Assume that $T$ is elliptic. Let $\chi\colon \mathcal{T}_{\mathbf{x}}^+(\mathcal{O}_k) \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character. Then the following hold.*

*(1) Assume that $p$ is not a torsion prime for $G$. The homology $f_\natural\overline{\mathbb{Q}}_\ell[\chi]$ is non-vanishing in precisely one degree $s_\chi \geq 0$.*

*(2) Assume that $p$ is not a torsion prime for $G$. The Frobenius $F^N$ acts in the space $H_{s_\chi}(X^+, \overline{\mathbb{Q}}_\ell)[\chi] := H^{-s_\chi} f_\natural\overline{\mathbb{Q}}_\ell[\chi]$ as multiplication by the scalar $(-1)^{s'_\chi} q^{s_\chi N/2}$ with some $s'_\chi \in \mathbb{Z}$. In particular, all Moy–Prasad quotients of $X^+$ are $\mathbb{F}_{q^N}$-maximal varieties.*

*(3) Assume the element $w\sigma \in W\sigma$ attached to $F$ in §2.4 is convex. For varying $\chi$, $H_{s_\chi}(X^+, \overline{\mathbb{Q}}_\ell)[\chi]$ runs through pairwise non-isomorphic irreducible smooth $\mathcal{G}^+(\mathcal{O}_k)$-representations.*

First, we remark that for part (3) the same proof as in [IN24, §7.1] applies, as for convex elements the (twisted) Steinberg cross-section map is an isomorphism by Theorem 3.3(2). It remains to prove parts (1) and (2) of Theorem 1.5. We do this in §7 by following the strategy of [IN24, §5].

**Further applications.** Exploiting concentration in one degree, our results apply to (1) T. Feng's explicit calculation of Fargues–Scholze parameters [Fen24] (we refer to [Fen24, §10] for the relevant setup) and (2) trace formulae in terms of $X_r$ for elements in $\mathbb{G}_r^F$ acting on $R\Gamma_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi]$.

**Corollary 1.6.** *Let $T, U$ be as in Theorem 1.1. Assume that $G^0 = T$ and that $p$ is not a torsion prime for $G$. Then the $R\Gamma_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi]$ is irreducible, concentrated in a single degree $s_{\phi,r} \in \mathbb{Z}$ and $F^N$ acts in $H_c^{s_{\phi,r}}(X_r, \overline{\mathbb{Q}}_\ell)$ by the scalar $(-1)^{s'_{\phi,r}} q^{N s_{\phi,r}/2}$ for some $s'_{\phi,r} \in \mathbb{Z}$. In particular, we get:*

*(1) If $\tilde{\phi}\colon T(k) \to \overline{\mathbb{Q}}_\ell^\times$ with $\tilde{\phi}|_{\mathcal{T}_{\mathbf{x}}(\mathcal{O}_k)} = \phi$, then* [Fen24, Corollary 10.4.2] *applies and provides an explicit description of the $L$-parameter of the smooth $G(k)$-representation $\pi_{T,U,\tilde{\phi}} := \text{c-ind}_{T(k)\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)}^{G(k)} R\Gamma_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi]$ (where we extend $R\Gamma_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi]$ to a $T(k)\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)$-representation via $\tilde{\phi}$).*

*(2) Let $g \in \mathcal{G}(\mathcal{O}_k)$. Then*

$$\operatorname{tr}(g, H_c^{s_{\phi,r}}(X_r, \overline{\mathbb{Q}}_\ell)[\phi]) = \frac{(-1)^{s_{\phi,r} - s'_{\phi,r}}}{\#\mathbb{T}_r^F \cdot q^{N s_{\phi,r}/2}} \sum_{t \in \mathbb{T}_r^F} \phi(t) \cdot \# S_{g,t},$$

*where $S_{g,t} = \{x \in X_r(\overline{\mathbb{F}}_q)\colon gF^N(x) = xt\}$.*

*Proof.* Indeed, by Corollary 1.4, $R\Gamma_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi] = R\Gamma_c(X_r^{(T)}, \overline{\mathbb{Q}}_\ell)[\phi]$. But $X_r^{(T)}$ is a disjoint union (indexed over $\mathbb{G}_0^F$) of copies of the scheme $X_r^+$ from §7 and by Theorem 1.5, $R\Gamma_c(X_r^+, \overline{\mathbb{Q}}_\ell)[\phi]$ is concentrated in one degree. This implies concentration in one degree. The assumption $G^0 = T$ and [Kal19, Lemma 3.6.5] imply $\mathrm{Stab}_{W^F}(\phi) = 1$. Then $R\Gamma_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi]$ is irreducible by (for example) [Nie24, Theorem 1.6]. Now, for claim (1) note that our techniques apply without change when the coefficient field $\overline{\mathbb{Q}}_\ell$ is replaced by any algebraically closed field of characteristic $\ell \neq p$ and claim (2) follows from [Boy12, Lemma 2.12]. □

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## 2. Notation and setup

2.1. **General notation.** We let $k \subseteq \breve{k}$ with integers $\mathcal{O}_k \subseteq \mathcal{O}$, residue field extension $\mathbb{F}_q \subseteq \overline{\mathbb{F}}_q$, and Frobenius $F$ be as in the introduction. We denote by $\varpi$ a uniformizer of $k$.

For a perfect $\mathbb{F}_q$-algebra $R$, put $\mathbb{W}(R) = R[\![\varpi]\!]$ if $\mathrm{char}(k) > 0$, resp. $\mathbb{W}(R) = W(R) \otimes_{\mathbb{Z}_p} \mathcal{O}_k$ if $\mathrm{char}(k) = 0$, where $W(R)$ denotes the ring of Witt vectors of $R$. In particular, we have $\mathbb{W}(\mathbb{F}_q) = \mathcal{O}_k$ and $\mathbb{W}(\overline{\mathbb{F}}_q) = \mathcal{O}$.

Let $[\cdot]\colon R \to \mathbb{W}(R)$ be the Teichmüller lift if $\mathrm{char}(k) = 0$, resp. $[x] = x$ if $\mathrm{char}(k) > 0$.

Let $\mathcal{X}$ be an $\mathcal{O}$-scheme, which is affine and of finite type over $\mathcal{O}$. Applying the (perfect) positive loop functor $L^+$ to $\mathcal{X}$ yields a perfect affine $\overline{\mathbb{F}}_q$-scheme

$$\mathbb{X} = L^+\mathcal{X} \quad \text{satisfying} \quad \mathbb{X}(R) = \mathcal{X}(\mathbb{W}(R))$$

for any perfect $\overline{\mathbb{F}}_q$-algebra $R$. If $\mathcal{X}$ is defined over $\mathcal{O}_k$, then $\mathbb{X}$ is naturally defined over $\mathbb{F}_q$, and we denote by $F$ the (geometric) Frobenius acting on $\mathbb{X}(\overline{\mathbb{F}}_q)$, so that $\mathbb{X}^F = \mathbb{X}(\mathbb{F}_q) = \mathcal{X}(\mathcal{O}_k)$.

2.2. **Groups.** We fix a reductive group $G$ defined over $k$ and split over $\breve{k}$. We write $Z(G)$ for the center of $G$, $G_{\mathrm{der}}$ for the derived group of $G$, and $G_{\mathrm{sc}}$ for the simply connected cover of $G_{\mathrm{der}}$; we write $T_{\mathrm{der}}, T_{\mathrm{sc}}$ for the preimage of $T$ in $G_{\mathrm{der}}, G_{\mathrm{sc}}$, respectively.

Let $\mathbf{x}$ be a point of the (reduced) Bruhat–Tits building of $G$ over $k$. By Bruhat–Tits theory there is an associated connected parahoric $\mathcal{O}_k$-model $\mathcal{G}_{\mathbf{x}}$ of $G$, equipped with filtration by the Moy–Prasad subgroups $\mathcal{G}_{\mathbf{x}}^r$ for $r \in \mathbb{R}_{\geq 0}$ ($\mathcal{G}_{\mathbf{x}}^r(\mathcal{O})$ contains exactly the affine roots $f$ with $f(\mathbf{x}) \geq r$. We let

$$J = \mathrm{Jumps}(\mathbf{x}, G) = \{r \in \mathbb{R}_{\geq 0} \colon \mathcal{G}_{\mathbf{x}}^r \neq \mathcal{G}_{\mathbf{x}}^{r'} \text{ for all } r' > r\},$$

This is a discrete subset and for $r \in J$ we denote by $r+ \in J$ (resp. $r-$) its descendant (resp. ascendant). Moreover, for $r \in \mathbb{R}_{\geq 0}$ such that $r_1 \leq r < r_1+$ with $r_1 \in J$, we put $r+ := r_1+$.

For any $s \leq r \in J$ we obtain the $\mathbb{F}_q$-rational perfectly smooth affine (Moy–Prasad) group scheme

$$\mathbb{G}_r^s := \mathbb{G}_{\mathbf{x}}^s / \mathbb{G}_{\mathbf{x}}^{r+},$$

where $\mathbb{G}_{\mathbf{x}}^s = L^+\mathcal{G}_{\mathbf{x}}^s$. If $s = 0$, we also write $\mathbb{G}_r$ for $\mathbb{G}_r^0$; if $r$ is fixed and clear from the context, we write $\mathbb{G}$, $\mathbb{G}^s$ for $\mathbb{G}_r$, $\mathbb{G}_r^s$. Note also that $(\mathbb{G}_r^s)^F$ is a finite Moy–Prasad subquotient of the $p$-adic reductive group $G(k)$.

If $H \subseteq G$ is a closed subgroup defined over $\breve{k}$, we may consider its closure $\mathcal{H}$ in $\mathcal{G}$, apply $L^+$ and pass to (sub)quotients to obtain a closed subgroup $\mathbb{H}_r^s \subseteq \mathbb{G}_r^s$ (see [CI19, §2.6]). If $H$ was $k$-rational, then $\mathbb{H}_r^s$ is $F$-stable.

2.3. **Pinning.** We fix a $k$-rational, $\breve{k}$-split maximal torus $T$ of $G$, we denote by $N_G(T)$ its normalizer. We identify its Weyl group $W = N_G(T)/T$ with the set of its $\breve{k}$-points; it is endowed with a natural action of $F$. We denote by $X_*(T)$, $X^*(T)$ the groups of (co)characters of $T_{\breve{k}}$, equipped with natural $F$-actions, and by $\langle , \rangle \colon X^*(T) \times X_*(T) \to \mathbb{Z}$ the natural $W$- and $F$-equivariant pairing.

We fix a Borel subgroup $T \subseteq B \subseteq G$ defined over $\breve{k}$, we denote by $U$ the unipotent radical of $B$, and by $\overline{U}$ the unipotent radical of the opposite Borel subgroup. We write $\Phi \subseteq X^*(T)$ for the set of roots of $T$ in $G$, and by $\Phi^+$ resp. $\Phi^-$ the subset of positive roots corresponding to $U$ resp. $\overline{U}$. For each $\alpha \in \Phi$, let $U_\alpha \colon \mathbb{G}_{a,\breve{k}} \to G$ denote a fixed parametrization of the root subgroup of $\alpha$. For $V \subseteq G$, we write $\Phi_V = \{\alpha \in \Phi \colon U_\alpha(\breve{k}) \subseteq \Phi_V\}$.

2.4. **Factorization of Frobenius.** There is a unique element $w \in W$, such that $FB = {}^wB$. Moreover, for any lift $\dot{w} \in N_G(T)(\breve{k})$, $\mathrm{Ad}(\dot{w})^{-1} \circ F \colon G(\breve{k}) \to G(\breve{k})$ fixes the pinning $(T, B)$ of $G$, and hence defines autormorphism $\sigma$ of the Coxeter system $(W, S)$, where $S$ is the set of simple reflections determined by $B$. Moreover, there is a unique automorphism of $X^*(T)$, again denoted by $\sigma$, such that the $F$-action on $X^*(T)$ is given by $qw\sigma$. This defines an action of $W \rtimes \langle \sigma \rangle$ on $X^*(T)$ satisfying $\sigma(\Phi^+) = \Phi^+$.

2.5. **Affine roots.** Denote by $\mathcal{T}$ the connected Néron model of $T$. Then $\mathcal{T}(\mathcal{O})$ is the maximal bounded subgroup of $T(\breve{k})$. Moreover, for $r \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{T}(\mathcal{O})^r = \{t \in \mathcal{T}(\mathcal{O}) \colon \operatorname{ord}_\varpi(\chi(t) - 1) \geq r \,\forall \chi \in X^*(T)\}$$

defines a descending separated filtration on $T(\breve{k})$, satisfying $\mathcal{T}(\mathcal{O})^0 = \mathcal{T}(\mathcal{O})$. For $r \geq 1$ one has an isomorphism

$$V := X_*(T) \otimes \overline{\mathbb{F}}_q \xrightarrow{\sim} \mathcal{T}(\mathcal{O})^r / \mathcal{T}(\mathcal{O})^{r+1}, \quad \lambda \otimes x \longmapsto \lambda(1 + [x]\varpi^r).$$

We denote by $\Phi_{\mathrm{aff}} \cong \Phi \times \mathbb{Z}$ the set of affine roots of $T$ in $G$ (with respect to a fixed point in the apartment of $T$ in the Bruhat–Tits building of $G$). For $f \in \Phi_{\mathrm{aff}}$, we write $\alpha_f \in \Phi$ for its vector part and $n_f \in \mathbb{Z}$ for the integer such that $f = (\alpha_f, n_f)$. We write $\widetilde{\Phi} = \Phi_{\mathrm{aff}} \sqcup \mathbb{Z}_{\geq 0}$ for the enlarged set of affine roots, with the affine root subgroup corresponding to $r \in \mathbb{Z}_{\geq 0}$ being the $r$-th slice of $T(\mathcal{O})$. There is a natural $F$-action on $\Phi_{\mathrm{aff}}$, and we extend it to an $F$-action on $\widetilde{\Phi}$ by letting $F$ act trivially on $\mathbb{Z}_{\geq 0}$.

## 3. Convex elements

We introduce convex elements in the Weyl group $W$. They behave like Coxeter elements in many respects, but they have the advantage that any $\sigma$-elliptic conjugacy class contains a minimal length element which is convex, as is proven by the Tan, Yu and the second author in [NTY24]. In later sections we will make use of the fact that higher level Deligne–Lusztig varieties attached to convex elements of $W$ can be studied by similar techniques as in the Coxeter case.

3.1. **Convex elements in the Weyl group.** Let $x \in W \rtimes \langle \sigma \rangle$. Set

$$\Delta_x = \Phi^+ \cap x(\Phi^-).$$

For $\alpha \in \Phi^\pm$ we define

$$n_x(\alpha) = \min\{i \in \mathbb{Z}_{\geqslant 1}; x^i(\alpha) \in \Phi^\mp\}.$$

**Definition 3.1.** We say an elliptic element $x \in W\sigma$ is *quasi-convex* if

$$n_x(\alpha + \beta) \leqslant \max\{n_x(\alpha), n_x(\beta)\}$$

for all $\alpha, \beta \in \Phi^\pm$ such that $\alpha + \beta \in \Phi$. Moreover, we say $x$ is *convex* if both $x$ and $x^{-1}$ are quasi-convex.

**Lemma 3.2.** *Let $x$ be a elliptic quasi-convex element. Let $\alpha, \beta \in \Phi^+$ and $i, j \in \mathbb{Z}_{\geqslant 1}$ such that $i\alpha + j\beta \in \Phi^+$. Then $n_x(i\alpha + j\beta) \leqslant \max\{n_x(\alpha), n_x(\beta)\}$.*

*Proof.* We always can find a sequence of roots $\gamma_0, \gamma_1, \dots, \gamma_t = i\alpha + j\beta$ with $\gamma_0$ and $\gamma_m - \gamma_{m-1}$ ($\forall 1 \leq m \leq t$) equal either $\alpha$ or $\beta$. Then the lemma follows by induction from the definition. □

Convex elements were studied in [NTY24], where the following was proven.

**Theorem 3.3** ([NTY24], Theorem 0.1)**.** *The following statements hold true.*

*(1) In each elliptic $W$-conjugacy class of $W\sigma$, there exists a minimal length element $x$ which is convex.*

*(2) (Steinberg cross-sections) The map $g, y \mapsto g^{-1}y\dot{x}\sigma(g)\dot{x}^{-1}\colon (U \cap {}^{x\sigma}U) \times (\overline{U} \cap {}^{x\sigma}U) \to {}^{x\sigma}U$ is an isomorphism.*

We will need further properties of convex elements.

**Lemma 3.4.** *Let $x \in W\sigma$ be convex. Let $\alpha, \beta \in \Phi$ such that $\beta - \alpha \in \mathbb{Z}_{\geqslant 0}\Delta_x$. Then*

*(1) if $\alpha \in \Phi^+$ then $n_{x^{-1}}(\beta) \leqslant n_{x^{-1}}(\alpha)$;*

*(2) If $\alpha, x^{-1}(\alpha) \in \Phi^-$ then either $\beta \in \Phi^+$ or $n_x(\beta) \leqslant n_x(\alpha)$.*

*Proof.* (1) By assumption, there exists a sequence of roots

$$\alpha = \gamma_0, \gamma_1, \dots, \gamma_m = \beta$$

such that $\gamma_i - \gamma_{i-1} \in \Delta_x$. Since $x^{-1}$ is quasi-convex and $n_{x^{-1}}(\Delta_x) = \{1\}$ we deduce that

$$n_{x^{-1}}(\alpha) = n_{x^{-1}}(\gamma_0) \geqslant n_{x^{-1}}(\gamma_1) \geqslant \cdots \geqslant n_{x^{-1}}(\gamma_m) = n_{x^{-1}}(\beta)$$

as desired.

(2) We can assume that $\alpha, x^{-1}(\alpha), \beta \in \Phi^-$. Since $x^{-1}(\Delta_x) \in \Phi^-$, we have $x^{-1}(\beta) \in \Phi^-$ and $n_x(x^{-1}(\Delta_x)) = \{1\}$. Note that

$$x^{-1}(\beta) - x^{-1}(\alpha) \in \mathbb{Z}_{\geqslant 0}x^{-1}(\Delta_x) = -\mathbb{Z}_{\geqslant 0}\Delta_{x^{-1}}.$$

Thus, by (1) we have

$$n_x(\beta) + 1 = n_x(x^{-1}(\beta)) \leqslant n_x(x^{-1}(\alpha)) = n_x(\alpha) + 1,$$

which implies that $n_x(\beta) \leqslant n_x(\alpha)$ as desired. □

3.2. **$M$-standard convex elements.** Let $M \subseteq G$ be an $F$-stable Levi subgroup containing fixed maximal torus $T$. We denote by $W_M \subseteq W$ the Weyl of $M$.

**Proposition 3.5.** *There exists a Borel subgroup $T \subseteq B$ such that*

*(1) $M$ is a standard Levi subgroup with respect to $B$;*

*(2) the relative position $x \in W\sigma$ of $B$ and $FB$ is a convex element with respect to the Coxeter system $(W, S)$ attached to $B$;*

*(3) $x$ is of minimal length in its $W_M$-conjugacy class.*

*Proof.* Let $V = \mathbb{R}\Phi$ be the Euclidean space together with an inner product preserved by $W \rtimes \langle\sigma\rangle$. Since $T$ is elliptic, there exists an orthogonal decomposition

$$V = \oplus_{i=1}^n V_i,$$

where each $V_i$ is an $F$-stable subspace of dimension $\leqslant 2$. Moreover, for each $i$ there exist $0 < \theta_i \leqslant \pi$ such that $F(v) + F^{-1}(v) = 2\cos\theta_i \cdot v$ for all $v \in V_i$.

Let $V_M \subseteq V$ be the subspace spanned by the roots of $M$. Denote by $V_M^\perp$ be the orthogonal complement of $V_M$. As $M$ is $F$-stable, $V_M^\perp$ is preserved by $w\sigma$. By reordering the subspaces $V_i$, we may assume that $V_M^\perp = \oplus_{i=1}^m V_i$ for some $0 \leqslant m \leqslant n$ and $\theta_{m+1} \leqslant \theta_{m+2} \leqslant \cdots \leqslant \theta_n$. By [HN12, Lemma 5.1], there exists a Weyl chamber $C \subseteq V$ for $\Phi$ such that for each $1 \leqslant i \leqslant n$ the Hausdorff closure $\overline{C}$ contains a $\Phi$-regular point of $\oplus_{j=1}^i V_i$. Here for any linear subspace $V' \subseteq V$ a point $v' \in V'$ is called a regular points of $V'$ if for each $\alpha \in \Phi$, $(\alpha, v') = 0$ implies that $(\alpha, V') = \{0\}$.

Let $T \subseteq B$ be the Borel subgroup associated to the Weyl chamber $C$. As $\overline{C}$ contains a regular point $V_M^\perp = \oplus_{i=1}^m V_i$, $M$ is a standard Levi subgroup with respect to $B$. Moreover, by [NTY24, Theorem 3.4], the relative position of $B$ and $FB$ is a convex element with respect to the Coxeter system $(W, S)$ attached to $B$. Moreover, as $V_M = \oplus_{j=m+1}^n V_j$ and $\theta_{m+1} \leqslant \theta_{m+2} \leqslant \cdots \leqslant \theta_n$, it follows from [HN12, Proposition 5.4] that $x$ is of minimal length in its $W_M$-conjugacy class. The proof is finished. □

3.3. **Action of convex elements on a Lie algebra.** Let $x \in W \rtimes \langle\sigma\rangle$. For $A \subseteq \Phi$ we consider the following $\overline{\mathbb{F}}_q$-vector spaces

$$H_A = \bigoplus_{\alpha\in A} \overline{\mathbb{F}}_q e_\alpha \subseteq H_\Phi = \bigoplus_{\alpha\in\Phi} \overline{\mathbb{F}}_q e_\alpha.$$

Assume that $A = x(A)$. Then we denote by $F = F_A$ the Frobenius map on $H_A$ given by $F(ce_\alpha) = c^q e_{x(\alpha)}$ for $c \in \overline{\mathbb{F}}_q$.

Let $B \subseteq -\Delta_x = \Phi^- \cap x(\Phi^+)$ such that for any $\alpha \in A$, $\beta \in B$ and $i \in \mathbb{Z}_{\geqslant 1}$ we have $\alpha + i\beta \in A$ if $\alpha + i\beta \in \Phi$. For $\beta \in B$ and $c \in \overline{\mathbb{F}}_q$ we define a linear map

$$\mathrm{Ad}_\beta(c) : H_A \longrightarrow H_A, \quad e_\alpha \longmapsto e_\alpha + \sum_{\substack{i\geqslant 1:\\ \alpha+i\beta\in\Phi}} c_{\alpha,\beta,i} c^i e_{\alpha+i\beta},$$

where $c_{\alpha,\beta,i} \in \overline{\mathbb{F}}_q$ are arbitrary but fixed constants.

Assume $B = \{\beta_1, \ldots, \beta_n\}$. Let $\phi = \mathrm{Ad}_{\beta_1}(c_1) \circ \cdots \circ \mathrm{Ad}_{\beta_n}(c_n)$, where $c_j \in \overline{\mathbb{F}}_q$ for $1 \leqslant j \leqslant n$ are arbitrary but fixed. For a fixed $z \in H_A$, let

$$V(\phi, x, z) := \{w \in H_A; \phi(w) - F(w) - z \in H_{A\cap -\Delta_x}\}.$$

This is a closed subvariety of $H_A$. In §5 we will use it to describe the fibers of a deep level Deligne–Lusztig variety over one of a shallower depth. Now we prove the following general structure result for $V(\phi, x, z)$.

**Proposition 3.6.** *Let notation be as above. Assume that $x$ is convex. Then the natural projection $H_A \to H_{A\cap\Delta_x}$ induces a homeomorphism*

$$V(\phi, x, z) \cong H_{A\cap\Delta_x}.$$

*Proof.* Write $w = \sum_{\alpha\in A} w_\alpha e_\alpha$, $z = \sum_{\alpha\in A} z_\alpha e_\alpha$ and $\phi(w) = \sum_{\alpha\in A} y_\alpha e_\alpha$ with $w_\alpha, z_\alpha, y_\alpha \in \overline{\mathbb{F}}_q$. Then the variety $V(\phi, x, z)$ is defined by the equations

$$(E_\alpha) \qquad y_\alpha - w^q_{x^{-1}(\alpha)} - z_\alpha = 0,$$

where $\alpha$ ranges over the roots in $A \smallsetminus (-\Delta_x)$.

For $\alpha \in A$, we set $\Gamma_\alpha = (\alpha + \mathbb{Z}_{\geqslant 0}B) \cap A$. As $B \subseteq -\Delta_x$, it follows from the definition of $\phi$ that

$$y_\alpha \in \sum_{\gamma\in\Gamma_\alpha} c_\alpha^\gamma w_\gamma,$$

where $c_\alpha^\gamma \in \overline{\mathbb{F}}_q$ are some constants such that $c_\alpha^\alpha = 1$. Hence the equation $(E_\alpha)$ is equivalent to

$$(E'_\alpha) \qquad w_\alpha - w^q_{x^{-1}(\alpha)} = z_\alpha - \sum_{\gamma\in\Gamma_\alpha\smallsetminus\{\alpha\}} c_\alpha^\gamma w_\gamma.$$

Now we show that given $z$ and $(w_\alpha)_{\alpha\in A\cap\Delta_x}$ there exists a unique tuple $(w_\alpha)_{\alpha\in A\smallsetminus\Delta_x}$ such that the equations $(E'_\alpha)$ hold for all $\alpha \in A \smallsetminus (-\Delta_x)$. To this end, for $\alpha \neq \beta \in \Phi^+$ we define $\beta \prec \alpha$ if either $n_{x^{-1}}(\beta) < n_{x^{-1}}(\alpha)$ or $n_{x^{-1}}(\beta) = n_{x^{-1}}(\alpha)$ and $\beta - \alpha$ is a sum of roots in $\Delta_x$.

First we claim that $w_\alpha$ is determined by the equation $(E'_\alpha)$ for $\alpha \in (A \cap \Phi^+) \smallsetminus \Delta_x$ (by which we mean that we may eliminate equation $(E'_\alpha)$ along with the variable $w_\alpha$). We use induction on the partial order $\preceq$ on $A \cap \Phi^+$. As $\alpha \in (A \cap \Phi^+) \smallsetminus \Delta_x$, we have $x^{-1}(\alpha) \in \Phi^+$ and hence $x^{-1}(\alpha) \prec \alpha$. Moreover, by Lemma 3.4 (1) we have $\gamma \prec \alpha$ for $\gamma \in \Gamma_\alpha \smallsetminus \{\alpha\}$. By induction hypothesis, $w_{x^{-1}(\alpha)}$ and $w_\gamma$ for $\gamma \in \Gamma_\alpha \smallsetminus \{\alpha\}$ are already determined. Hence $w_\alpha$ is determined by the equation $E'_\alpha$, and the claim is proved.

It remains to show that $w_\alpha$ is determined by the equation $(E'_{x(\alpha)})$ for $\alpha \in A \cap \Phi^-$. We argue by induction on $n_x(\alpha)$. In view of $(E'_{x(\alpha)})$, $w_\alpha$ is determined by $z_{x(\alpha)}$ and $w_\gamma$ for $\gamma \in \Gamma_{x(\alpha)}$. So it suffices to show $w_\gamma$ is already determined for $\gamma \in \Gamma_{x(\alpha)}$. Indeed, if $\gamma \in \Phi^+$, this follows from the previous claim. Now we assume $\gamma \in \Phi^-$ and hence $x(\alpha) \in \Phi^-$. By Lemma 3.4 (2), we have $n_x(\gamma) \leqslant n_x(x(\alpha)) < n_x(\alpha)$. Thus $w_\gamma$ is determined by the induction hypothesis. Thus $w_\alpha$ is determined by the equation $(E'_{x(\alpha)})$, and the proof is finished. □

**Proposition 3.7.** *Let notation be as above. Assume that $x$ is convex. Then the map $(x, y) \mapsto -\phi(x) + y - F(x)$ gives an isomorphism*

$$H_{A\cap x(\Phi^+)\cap\Phi^+} \times H_{A\cap -\Delta_x} \xrightarrow{\sim} H_{x(A\cap\Phi^+)}.$$

*Proof.* By Proposition 3.6, this map is injective. It suffices to show it is surjective, that is, for any $z \in H_{x(A\cap\Phi^+)}$, there exists $w \in H_{A\cap x(\Phi^+)\cap\Phi^+}$

such that $\phi(w) - F(w) - z \in H_{x(A \cap -\Delta_x)}$. This is equivalent to the following statement:

(a) For any $z \in H_{A\cap\Phi^+}$, there exists $w \in H_{A\cap x(\Phi^+)\cap\Phi^+}$ such that $-\varphi(w) + F^{-1}(w) - z \in H_{A\cap\Delta_{x^{-1}}}$. Here $\varphi = F^{-1} \circ \phi \circ F = \prod_{\gamma \in x^{-1}(B)\cap\Delta_{x^{-1}}} \mathrm{Ad}_\gamma(d_\gamma)$ for some $d_\gamma \in \overline{\mathbb{F}}_q$.

Now we prove (a). Let $z = \sum_\alpha c_\alpha e_\alpha \in H_{A\cap\Phi^+}$ for some $c_\alpha \in \overline{\mathbb{F}}_q$. Define

$$n_x(z) = \max\{n_x(\alpha); c_\alpha \neq 0\}.$$

We argue by induction on $n_x(z)$. If $n_x(z) = 1$, that is, $z \in H_{A\cap\Delta_{x^{-1}}}$ and we may take $w = 0$. Assume $n_x(z) \geqslant 2$. Let $z' = \sum_{\gamma, n_x(\gamma) = n_x(z)} c_\gamma e_\gamma \in H_{A\cap\Phi^+}$. Then $n_x(z - z') \leqslant n_x(z) - 1$ and $F(z') \in H_{A\cap x(\Phi^+)\cap\Phi^+}$. Moreover, as $x$ is convex and $n_x(\gamma) = 1$ for $\gamma \in \Delta_{x^{-1}}$, we have

$$n(\varphi(F(z'))) \leqslant n_x(F(z')) = n_x(z') - 1 = n_x(z) - 1.$$

Thus

$$n_x(\varphi(F(z')) - F^{-1}(F(z')) - z) = n_x(\varphi(F(z')) - (z - z')) \leqslant n_x(z) - 1.$$

Then the statement follows by induction hypothesis. The proof is finished. □

## 4. Deligne–Lusztig varieties

**4.1. Deligne–Lusztig varieties.** Recall the notation from §2.2. Fix $r \in J$. We have the $\overline{\mathbb{F}}_q$-group $\mathbb{G} = \mathbb{G}_r$ equipped with $\mathbb{F}_q$-Frobenius $F$ and its subgroups $\mathbb{T}, \mathbb{U}, \overline{\mathbb{U}}$. Consider the $\mathbb{F}_q$-varieties

$$X_r = \{g \in \mathbb{G} \colon g^{-1}F(g) \in \overline{\mathbb{U}} \cap F\mathbb{U}\}$$
$$Y_r = \{g \in \mathbb{G} \colon g^{-1}F(g) \in \mathbb{T}(\overline{\mathbb{U}} \cap F\mathbb{U})\}/\mathbb{T}.$$

There is an obvious map $h \colon X_r \to Y_r$, which is an étale $\mathbb{T}^F$-torsor with $\mathbb{T}^F$ acting by right multiplication. Hence

$$h_! \overline{\mathbb{Q}}_\ell = \bigoplus_\theta \theta \otimes \mathcal{E}_\theta,$$

where $\theta$ ranges over characters of $T_r^F$ and $\mathcal{E}_\theta$ is the associated local system on $Y_r$. For any $i \in \mathbb{Z}$ we have

$$H_c^i(X_r, \overline{\mathbb{Q}}_\ell)[\theta] \cong H_c^i(Y_r, \mathcal{E}_\theta).$$

**4.2. Howe strata.** Fix a character $\phi : T^F \to \overline{\mathbb{Q}}_\ell^\times$ of depth$(\phi) \leqslant r$. Then $\phi$ induces a character $\mathbb{T}^F \to \overline{\mathbb{Q}}_\ell^\times$, which we again denote by $\phi$. Assume that $\phi$ admits a *Howe factorization* in the sense of [Kal16, §3.6] and denote it by $(G^i, \phi_i, r_i)_{-1\leqslant i\leqslant d}$. That is,

$$T = G^{-1} \subseteq L := G^0 \subsetneq G^1 \subsetneq \dots G^{d-1} \subsetneq G^d$$

is a sequence of twisted Levi subgroups, $\phi_i$ $(0 \leq i \leq d)$ is a character of $(G^i)^F$, which is $(G^i : G^{i+1})$-generic for $i < d$, such that $\phi = \prod_{i=-1}^d \phi_i$.

Moreover, there is a sequence $0 = r_{-1} < r_0 < \cdots < r_{d-1} \leq r_d$ of integers such that $\phi_i$ has depth $r_i$ for $0 \leq i \leq d-1$; $\phi_d = 1$ if $r_{d-1} = r_d$ and $\phi_d$ has depth $r_d$ otherwise; $\phi_{-1} = 1$ if $G^0 = T$ and $\phi_{-1}$ has depth 0 otherwise. For $\alpha \in \Phi$ we denote by $i(\alpha)$ the unique integer $0 \leqslant i \leqslant d$ such that $\alpha \in \Phi(G^i, T) \setminus \Phi(G^{i-1}, T)$. Define $r(\alpha) = r_{i(\alpha)-1}$.

We define subgroups of $\mathbb{G}$ as follows.

$$\begin{aligned}
\mathbb{K}_\phi &= (\mathbb{G}_0)^0 (\mathbb{G}_1)^{r_0/2} \cdots (\mathbb{G}_d)^{r_{d-1}/2};\\
\mathbb{K}_\phi^+ &= (\mathbb{G}_0)^{0+} (\mathbb{G}_1)^{r_0/2+} \cdots (\mathbb{G}_d)^{r_{d-1}/2+};\\
\mathbb{H}_\phi &= (\mathbb{G}_0)^{0+} (\mathbb{G}_1)^{r_0/2} \cdots (\mathbb{G}_d)^{r_{d-1}/2};\\
\mathbb{E}_\phi &= (\mathbb{G}_0)^{0+} (\mathbb{G}_1^{\mathrm{der}})^{r_0/2+, r_0+} \cdots (\mathbb{G}_d^{\mathrm{der}})^{r_{d-1}/2+, r_{d-1}+}.
\end{aligned}$$

Here $(\mathbb{G}_i^{\mathrm{der}})^{r_{i-1}/2+, r_{i-1}+}$ is generated by $(\mathbb{G}_i^{\mathrm{der}})^{r_{i-1}+}$ and $\mathbb{U}_f$ for $f \in \widetilde{\Phi}_{\mathrm{aff}}^{r_{i-1}/2+}$ such that $\alpha_f \in R_i \setminus R_{i-1}$.

Furthermore, we let $\bar{\mathbb{K}}_\phi = \mathbb{K}_\phi / \mathbb{E}_\phi$ and let $\bar{\mathbb{H}}_\phi$, $\bar{\mathbb{L}}_\phi$, $\bar{\mathbb{T}}$, ... be the natural images of $\mathbb{H}_\phi$, $\mathbb{L}$, $\mathbb{T}$, ... in $\bar{\mathbb{K}}_\phi$ respectively.

The "discrete part" $(G^i, r_i)_{-1 \leqslant i \leqslant d}$ of the Howe datum of $\phi$ cuts out the following subvarieties of $X_r, Y_r$, which might therefore be called (closed) *Howe strata* of $X, Y$:

$$\begin{aligned}
X_r^\flat &= \{g \in \mathbb{G} \colon g^{-1}F(g) \in \mathbb{K}_\phi \cap \overline{\mathbb{U}} \cap F\mathbb{U}\},\\
Y_r^\flat &= \{g \in \mathbb{G} \colon g^{-1}F(g) \in \mathbb{T}(\mathbb{K}_\phi \cap \overline{\mathbb{U}}_r \cap F\mathbb{U}_r)\}/\mathbb{T}
\end{aligned}$$

The following is our first main result. It says the $\phi$-isotypic part of the cohomology of $X_r$ concentrates on the corresponding Howe stratum.

**Theorem 4.1.** *Suppose the element $w\sigma \in W\sigma$ attached to $F$ in §2.4 is convex. We have $R\Gamma_c(X_r \smallsetminus X_r^\flat, \overline{\mathbb{Q}}_\ell)[\phi] = R\Gamma_c(Y_r \smallsetminus Y_r^\flat, \mathcal{E}_\phi) = 0$.*

By proper base change, Theorem 4.1 follows from the vanishing of the cohomology of $\mathcal{E}_\phi$ on the fibers of $Y_r \smallsetminus Y_r^\flat \to Y_0$, which is Proposition 5.1 below.

## 5. Fibers over the classical Deligne–Lusztig variety

Here we complete the proof of Theorem 4.1. Let the notation be as in §4. In particular, we have a fixed character $\phi \colon T^F \to \overline{\mathbb{Q}}_\ell^\times$ of depth$(\phi) \leq r$, admitting a Howe factorization, and the corresponding groups $\mathbb{K}_\phi, \mathbb{K}_\phi^+, \dots$, as well the varieties $X_r \supseteq X_r^\flat$, $Y_r \supseteq Y_r^\flat$. We will denote the character induced by $\phi$ on the subquotient $\mathbb{T}_r^F$ of $T^F$ again by $\phi$. Recall the local system $\mathcal{E}_\phi$ on $Y_r$ attached to $\phi$. Let

$$\delta_r : Y_r \smallsetminus Y_r^\flat \hookrightarrow Y_r \longrightarrow Y_0$$

be the natural projection. Note that $Y_0$ is (essentially) a classical Deligne–Lusztig variety for $\mathbb{G}_0$.

**Proposition 5.1.** *Suppose the element* $w\sigma \in W\sigma$ *attached to* $F$ *in §2.4 is convex. Let* $\bar{g}_0 \in Y_0$. *Then we have* $R\Gamma_c(\delta_r^{-1}(\bar{g}_0), \mathcal{E}_\phi) = 0$.

After necessary preparations, we prove Proposition 5.1 at the end of §5. *Until the end of §5, we assume that the element* $w\sigma \in W\sigma$ *attached to* $F$ *is convex, so that results of §3 apply; we fix* $\bar{g}_0 \in Y_0 \subseteq \mathbb{G}_0/\mathbb{T}_0$ *and a lift* $g_0 \in \mathbb{G}_r$ *of* $\bar{g}_0$ *such that*

$$y_0 := g_0^{-1}F(g_0) \in \overline{\mathbb{U}}_r \cap F\mathbb{U}_r.$$

5.1. **Parametrization of Moy–Prasad quotients.** We set

$$\widetilde{\Phi}_r^0 = \{f \in \widetilde{\Phi}; 0 \leqslant f(\mathbf{x}) \leqslant r\};$$
$$\widetilde{\Phi}_r^+ = \{f \in \widetilde{\Phi}; 0 < f(\mathbf{x}) \leqslant r\};$$
$$\widetilde{\Delta}_r = \{f \in \widetilde{\Phi}_r^0; \alpha_f \in \Phi_U \cap F\Phi_{\overline{U}}\}.$$

Moreover, we set $\Phi_{\mathrm{aff},r}^0 = \{f \in \widetilde{\Phi}_r^0; \alpha_f \in \Phi\}$ and $\Phi_{\mathrm{aff},r}^+ = \{f \in \widetilde{\Phi}_r^+; \alpha_f \in \Phi\}$. For $f, f' \in \widetilde{\Phi}_r^0$ we write $f' < f$ if either $f'(\mathbf{x}) < f(\mathbf{x})$ or $f'(\mathbf{x}) = f(\mathbf{x})$ and $f' - f$ is a sum of affine roots in $\widetilde{\Delta}_r$. We extend this partial order to a total order on $\widetilde{\Phi}_r^0$, and still denote it by $\leqslant$. For $f \in \widetilde{\Phi}_r^0$, we write $\widetilde{\Phi}_r^f = \{f' \in \widetilde{\Phi}_r^0 \colon f' \geq f\}$.

Note that $\mathbb{T}_r \to \mathbb{T}_0$ admits a unique splitting, which we denote by $t \mapsto [t]$. Let $f \in \widetilde{\Phi}_r^+ \cup \{0\}$. Define

$$u_f \colon \mathbb{A}_f := \mathbb{A}^1 \longrightarrow \mathbb{T}_r\mathbb{G}_r^{0+}, \quad x \longmapsto U_{\alpha_f}([x]\varpi^{n_f}) \qquad \text{if} \quad f \in \widetilde{\Phi}_{\mathrm{aff}},$$
$$u_f \colon \mathbb{A}_f := \mathbb{T}_0 \longrightarrow \mathbb{T}_r\mathbb{G}_r^{0+}, \quad x \longmapsto [x] \qquad \text{if} \quad f = 0,$$
$$u_f \colon \mathbb{A}_f := X_*(T) \otimes \overline{\mathbb{F}}_q \longrightarrow \mathbb{T}_r\mathbb{G}_r^{0+}, \quad \lambda \otimes x \longmapsto \lambda(1 + [x]\varpi^{n_f}) \qquad \text{if } f \in \mathbb{Z}_{\geq 1},$$

where in the last line $\lambda \in X_*(T), x \in \overline{\mathbb{F}}_q$

Define an abelian group $\mathbb{A}[r] = \prod_{f \in \widetilde{\Phi}_r^+ \cup \{0\}} \mathbb{A}_f$. Then we have an isomorphism of varieties

$$u \colon \mathbb{A}[r] \longrightarrow \mathbb{T}_r\mathbb{G}_r^{0+}, \quad (x_f)_f \longmapsto \prod_f u_f(x_f), \tag{5.1}$$

where we the product is taken with respect to the order $\leq$ restricted to $\widetilde{\Phi}_r^+ \cup \{0\}$. Let $E \subseteq \widetilde{\Phi}_r^+ \cup \{0\}$. We define $\mathbb{A}_E = \prod_{f \in E} \mathbb{A}_f$ which is viewed as a subgroup of $\mathbb{A}[r]$ in the natural way. We denote by $p_E \colon \mathbb{A}[r] \to \mathbb{A}_E$ the natural projection. Define

$$\mathbb{G}_r^E = u(\mathbb{A}_E) \subseteq \mathbb{T}_r\mathbb{G}_r^{0+}.$$

Moreover, we denote by

$$\mathrm{pr}_E : \mathbb{T}_r\mathbb{G}_r^{0+} \cong \mathbb{A}[r] \longrightarrow \mathbb{A}_E \cong \mathbb{G}_r^E$$

the natural projection. If $E + E, \mathbb{Z}_{\geqslant 0} + E \subseteq E \cup \widetilde{\Phi}^{r+}$, then $\mathbb{G}_r^E$ is a subgroup of $G_r^{0+}$.

Let $g \in \mathbb{T}_r\mathbb{G}_r^{0+}$, $x \in \mathbb{A}[r]$ and $E \subseteq \widetilde{\Phi}_r^+ \cup \{0\}$. We set $g_E = \mathrm{pr}_E(g) \in u(\mathbb{A}_E)$, $x_E = p_E(x) \in \mathbb{A}_E$ and $\hat{x} = u(x) \in \mathbb{G}_r^+$. For $f \in \widetilde{\Phi}_r^+$ we will set $x_f = x_{\{f\}}$

and $x_{\geqslant f} = x_{\widetilde{\Phi}_r^f}$. We can define $g_f$ and $g_{\geqslant f} \in \mathbb{G}_r^+$ in a similar way. By abuse of notation, we will identify $g_f \in u(\mathbb{A}_f)$ with $u^{-1}(g_f) \in \mathbb{A}_f$ according to the context.

5.2. **Description of the fiber.** Define

$$\pi\colon \mathbb{G}_r \longrightarrow \mathbb{G}_r, \quad \pi(g) = g^{-1} y_0 F(g).$$

Let $Y_r(\bar{g}_0)$, $X_r(\bar{g}_0)$ be the preimages in $Y_r$, $X_r$ of $\bar{g}_0 \in Y_0$ under the natural projections. Then we have isomorphisms: $g \mapsto g g_0^{-1}$ induces an isomorphism

$$\begin{aligned} Y_r(\bar{g}_0) &\xrightarrow{\sim} \{g \in \mathbb{G}_r^{0+}\mathbb{T}_r \colon \pi(g) \in \mathbb{T}_r(\overline{\mathbb{U}}_r \cap F\mathbb{U}_r)\}/\mathbb{T}_r \\ &\xleftarrow{\sim} \{g \in \mathbb{G}_r^{\Phi^+_{\mathrm{aff},r}} \colon \pi(g) \in \mathbb{T}_r(\overline{\mathbb{U}}_r \cap F\mathbb{U}_r)\} \end{aligned} \tag{5.2}$$

where the first map is induced by $h \mapsto g_0^{-1} h$, and the second map is induced by $g \mapsto g\mathbb{T}_r$. Under these isomorphisms, $\delta_r^{-1}(\bar{g}_0)$ identifies with the subvariety of those $g$ for which $\pi(g) \in \mathbb{T}_r((\overline{\mathbb{U}}_r \cap F\mathbb{U}_r) \smallsetminus \mathbb{K}_\phi)$. Until the end of §5 we will identify $Y_r(\bar{g}_0)$, $\delta_r^{-1}(\bar{g}_0)$ with the models given by the last line of (5.2). This enables us to define a map

$$\pi_{\mathbb{T}_r}\colon Y_r(\bar{g}_0) \longrightarrow \mathbb{T}_r, \quad \pi_{\mathbb{T}_r}(g) = \pi(g)_{\mathbb{T}_r},$$

and we denote its restriction to $\delta_r^{-1}(\bar{g}_0)$ again by $\pi_{\mathbb{T}_r}$.

**Lemma 5.2.** *There is a cartesian diagram*

$$\begin{array}{ccc} X_r(\bar{g}_0) & \longrightarrow & \mathbb{T}_r \\ \downarrow & & \downarrow{\scriptstyle L_{\mathbb{T}_r}^{-1}} \\ Y_r(\bar{g}_0) & \xrightarrow{\pi_{\mathbb{T}_r}} & \mathbb{T}_r \end{array}$$

*where the left map is the natural projection, $L_{\mathbb{T}_r}$ is the Lang map of $\mathbb{T}_r$, and the upper map sends $h \in X_r(\bar{g}_0) \cong \{h \in \mathbb{G}_r^{0+}\mathbb{T}_r \colon \pi(h) \in \overline{\mathbb{U}}_r \cap F\mathbb{U}_r\}$ to $h_{\mathbb{T}_r}$.*

Note that in the diagram of the lemma both horizontal maps depend on the parametrization of $\mathbb{G}_r^{0+}\mathbb{T}_r$ fixed in §5.1.

*Proof.* As both vertical maps in the diagram are étale $\mathbb{T}_r^F$-torsors, it suffices to show that the diagram commutes. For this, let $h \in X_r(\bar{g}_0)$. Its image in $Y_r(\bar{g}_0)$ identifies (under the isomorphism from the previous paragraph) with $h_{\Phi^+_{\mathrm{aff},r}} = h h_{\mathbb{T}_r}^{-1}$. Its image under the lower map is then equal to

$$\begin{aligned} (h h_{\mathbb{T}_r}^{-1})^{-1} y_0 F(h h_{\mathbb{T}_r}^{-1}) &= (h_{\mathbb{T}_r} \cdot (h^{-1} y_0 F(h)) \cdot F(h_{\mathbb{T}_r})^{-1})_{\mathbb{T}_r} \\ &= h_{\mathbb{T}_r} F(h_{\mathbb{T}_r})^{-1} \\ &= L_{\mathbb{T}_r}(h_{\mathbb{T}_r})^{-1} \end{aligned}$$

where the second equality follows as $h \in X_r(\bar{g}_0)$. □

By proper base change theorem, Lemma 5.2 implies that

$$\mathcal{E}_\phi|_{Y_r(\bar{g}_0)} \cong \pi_{\mathbb{T}_r}^* \mathcal{L}_\phi,$$

where $\mathcal{L}_\phi$ denotes the multiplicative local system on $\mathbb{T}_r$ corresponding to $\phi$. Clearly, the same isomorphism holds after restricting to $\delta_r^{-1}(\bar{g}_0)$.

Now we prove that the cohomology of the fiber $\delta_r^{-1}(\bar{g}_0)$ with coefficients in $\mathcal{E}_\phi$ is independent of $r$ as long as $r \geq r_{d-1}$.

**Proposition 5.3.** *Let $r \in J$ satisfying $r \geqslant r_{d-1} > 0$. Then we have*

$$R\Gamma_c(\delta_r^{-1}(\bar{g}_0), \pi_{\mathbb{T}_r}^* \mathcal{L}_\phi) \cong R\Gamma_c(\delta_{r+}^{-1}(\bar{g}_0), \pi_{\mathbb{T}_{r+}}^* \mathcal{L}_\phi)[2m],$$

*where $m = \sharp(\widetilde{\Delta}_{r+} \smallsetminus \widetilde{\Delta}_r)$.*

*Proof.* In the setup of §3.3, let $A = \{\alpha_f \in \Phi\colon f(\mathbf{x}) = r\}$ and let $\phi\colon H_A \to H_A$ be the endomorphism determined by conjugation with $y_0 = \prod_{\alpha \in -\Delta} y_{0,\alpha}$. Note moreover, that (5.2) gives a section $s\colon Y_r(\bar{g}_0) \to Y_{r+}(\bar{g}_0)$ to the natural projection. The fiber of $Y_{r+}(\bar{g}_0) \to Y_r(\bar{g}_0)$ over $g$ is then given by $V(\phi, x, z(s(g)))$, where $z\colon Y_r(\bar{g}_0) \to \mathbb{G}_{r+}^{r+}/\mathbb{T}_{r+}^{r+} \cong H_A$ is some morphism. Then Proposition 3.6 gives an isomorphism $Y_{r+}(\bar{g}_0) \cong Y_r(\bar{g}_0) \times \mathbb{A}_{\widetilde{\Delta}_{r+} \smallsetminus \widetilde{\Delta}_r}$. By assumption on $r$, $\delta_{r+}^{-1}(\bar{g}_0)$ is the preimage in $Y_{r+}(\bar{g}_0)$ of $\delta_r^{-1}(\bar{g}_0) \subseteq Y_r(\bar{g}_0)$, so that we get an isomorphism

$$\delta_{r+}^{-1}(\bar{g}_0) \cong \delta_r^{-1}(\bar{g}_0) \times \mathbb{A}_{\widetilde{\Delta}_{r+} \smallsetminus \widetilde{\Delta}_r}.$$

Moreover, for $(x, y) \in \delta_r^{-1}(\bar{g}_0) \times \mathbb{A}_{\widetilde{\Delta}_{r+} \smallsetminus \widetilde{\Delta}_r}$ we have

$$\pi_{\mathbb{T}_{r+}}(x, y) - \pi_{\mathbb{T}_r}(x) \in \mathbb{T}_{\mathrm{der},r+}^{r+}.$$

Since the restriction of $\phi$ to $(\mathbb{T}_{\mathrm{der},r+}^{r+})^F$ is trivial, $\pi_{\mathbb{T}_{r+}}^* \mathcal{L}_\phi$ is isomorphic to the pullback of $\pi_{\mathbb{T}_r}^* \mathcal{L}_\phi$ under the natural projection $\mathbb{T}_{r+} \to \mathbb{T}_r$. Therefore, we have

$$\pi_{\mathbb{T}_{r+}}^* \mathcal{L}_\phi \cong \pi_{\mathbb{T}_r}^* \mathcal{L}_\phi \boxtimes \overline{\mathbb{Q}}_\ell$$

and the statement follows by the Künneth formula. □

5.3. **Handling jumps.** Let $(G^i, \phi_i, r_i)_{-1 \leqslant i \leqslant d}$ be the Howe factorization of $\phi$ from §4.2. Set

$$r = r_{d-1}, \qquad M = G^{d-1}, \qquad V = \overline{U} \cap FU.$$

We label the real numbers of $\{f(\mathbf{x}); f \in \widetilde{\Phi}_r^0 \smallsetminus \widetilde{\Phi}_M\} \subseteq J$ in the ascending order:

$$0 = s_0 < s_1 < \cdots < s_m = r.$$

Note that $s_i + s_{m-i} = r$ for $0 \leqslant i \leqslant m$. Set

$$\begin{aligned} C^0 &= \{f \in \Phi_{\mathrm{aff}}; f(\mathbf{x}) = 0\} \\ C^i &= \{f \in \widetilde{\Phi} \smallsetminus \widetilde{\Phi}_M; f(\mathbf{x}) = s_i\} \qquad \text{for } 0 < i < m \\ C^m &= \{f \in \widetilde{\Phi}; f(\mathbf{x}) = r\}. \end{aligned}$$

Put $C^{<i} = \bigcup_{j<i} C^j$ and define $C^{>i}$ similarly. Note that for for $j \geq \frac{m}{2} - 1$, $\mathbb{G}_r^{C^{>j}} \subseteq \mathbb{G}_r^{0+}$ is a subgroup normalized by $\mathbb{T}_r^{0+}$. For $0 \leqslant i, j \leqslant m$ with $j \geq \frac{m}{2} - 1$ we define

$$\begin{aligned} Y_{y_0,r}^{i,j} &:= \{g \in \mathbb{G}_r^{\Phi^+_{\mathrm{aff},r} \cap C^{\leq j}} : \pi(g) \in \mathbb{T}_r^{0+}\mathbb{G}_r^{C^{>j}}\mathbb{V}_r^{C^{\geqslant i}}\} \\ &\cong \{g \in \mathbb{G}_r^{0+} \colon \pi(g) \in \mathbb{T}_r^{0+}\mathbb{G}_r^{C^{>j}}\mathbb{V}_r^{C^{\geqslant i}}\}/\mathbb{T}_r^{0+}\mathbb{G}_r^{C^{>j}}, \end{aligned}$$

where

$$\mathbb{V}_r^{C^{\geqslant i}} := \{v \in \mathbb{V}_r \smallsetminus \mathbb{K}_{\phi,r}; v_{C^{<i} \smallsetminus \widetilde{\Phi}_M} = 0\}.$$

Note that $C^{>m} = C^{<0} = \varnothing$ and hence $Y_{y_0,r}^{0,m} = \delta_r^{-1}(\bar{g}_0)$. Moreover, if $0 < i$ then $Y_{y_0,r}^{i,j} \neq \varnothing$ if and only if $y_0 \in \mathbb{M}_r$.

**Lemma 5.4.** *Let $E \subseteq C^{>m/2}$ and $E' \subseteq \widetilde{\Phi}_r^+$ such that $E + (E' \smallsetminus \widetilde{\Phi}_M) \subseteq \widetilde{\Phi}^r$. Let $x \in \mathbb{A}_E$ and $y \in \mathbb{A}_{E'}$. Then*

$$(\hat{x}\hat{y})_{\mathbb{T}_r} = -\sum_f \alpha_f^\vee(1 + \varpi^r x_f y_{r-f}) + y_{\mathbb{T}_r} + x_{\mathbb{T}_r} \in \mathbb{T}_r,$$

*where $f$ ranges over $E$ such that $f > r - f$, and where we denote the group law in $\mathbb{T}_r$ by $+$.*

*Proof.* The proof is similar to [IN24, Lemma 5.13]. Write $\hat{x}\hat{y} = \hat{z}_{g_1} \dots \hat{z}_{g_n} \in \mathbb{G}_r^{0+}$ with $E'' := \{g_1 < \cdots < g_n\} \subseteq (\mathbb{Z}_{\geq 0}E + \mathbb{Z}_{\geq 0}E') \cap \widetilde{\Phi}_r^+$. Then each $z_{g_i} \in \mathbb{A}_{g_i}$ is a sum of $x_{g_i}$ (appears if $g_i \in E$), $y_{g_i}$ (appears if $g_i \in E'$) and possibly some iterated commutator terms arising from $x_f$,$y_{f'}$ with $f \in E, f' \in E'$.

As $E \subseteq C^{>m/2}$, we even have $E'' \smallsetminus (E \cup E') \subseteq E + \mathbb{Z}_{\geq 1}E'$. Let $1 \leq i \leq n$ be such that $g_i \in \mathbb{Z}_{\geq 1}$. Suppose that $g_i = f + \sum_{j=1}^s a_j f'_j$ with some $a_j \in \mathbb{Z}_{\geq 1}$, $f'_j \in E'$. As $M \subseteq G$ is a Levi subgroup, and $g_i \in \widetilde{\Phi}_M$, there must be some $j_0$ with $f'_{j_0} \in E' \smallsetminus \widetilde{\Phi}_M$. Then, by assumption, $f + f'_j \in \widetilde{\Phi}^r$, which forces $j_0 = s = 1$, $a_1 = 1$ and $g_i = f + f'_1 = r$.

Thus, if $g_i \in \mathbb{Z}_{\geq 1}$ and $g_i < r$, then $z_{g_i} = x_{g_i} + y_{g_i} = y_{g_i}$ (note that $E \cap \mathbb{Z}_{\geq 1} = \varnothing$, and thus $x_{g_i} = 0$). When $g_i = r$, then $(\hat{x}\hat{y})_r = -\sum_f \alpha_f^\vee(1 + \varpi^r x_f y_{r-f}) + y_r + x_r \in \mathbb{T}_r^r$, where the sum ranges of the same index set as in the lemma. As $z_{\mathbb{T}_r} = z_{g_{i_1}} \dots z_{g_{i_r}}$, where $g_{i_j} = j \in \mathbb{Z}_{\geq 1}$, this finishes the proof. □

**Lemma 5.5.** *Let $0 \leqslant i \leqslant m/2$ and $w \in \mathbb{A}_{C^{m-i}}$. For each $f \in C^i$ we have*

$$(y_0^{-1}\hat{w}y_0)_{r-f} = \sum_{f \leqslant f' \in C^i} c_{f,f'} w_{r-f'},$$

*where $c_{f,f'} \in \overline{\mathbb{F}}_q$ are some constants such that $c_{f,f} = 1$.*

*As a consequence, $(y_0^{-1}\hat{w}y_0)_{\mathbb{T}_r} = 0$ if $0 < i < m/2$ and $(y_0^{-1}\hat{w}y_0)_{\mathbb{T}_r} = \sum_{f \leqslant f' \in C^i} \alpha_f^\vee(1 + \varpi^r d_{f,f'}(y_0)_f w_{r-f'})$ if $i = 0$. Here $d_{f,f'} \in \overline{\mathbb{F}}_q$ are some constants such that $d_{f,f'} = 1$.*

*Proof.* Write $\hat{w} = \hat{w}_{f_1} \dots \hat{w}_{f_s}$ with $C^{m-i} = \{f_1 < \cdots < f_s\}$. It is clear that $y_0^{-1}\hat{w}y_0$ only depends on the image of $y_0$ in $\mathbb{G}_0$. By induction on the number of roots in $\widetilde{\Delta} \cap C^0$ needed to write $y_0$, we may assume that $y_0 = y_{0,g}$ with some $g \in \widetilde{\Delta} \cap C^0$. We compute

$$y_0^{-1}\hat{w}y_0 = \hat{w}_{f_1}[\hat{w}_{f_1}^{-1}, y_0^{-1}]\hat{w}_{f_2} \dots \hat{w}_{f_s}[\hat{w}_{f_s}^{-1}, y_0^{-1}]. \tag{5.3}$$

Moreover, $[\hat{w}_{f_j}^{-1}, y_0^{-1}] = \prod_a (c_a \hat{w}_{f_j})_{f_j + ag}$, where the product is taken over all $a \in \mathbb{Z}_{\geq 1}$ such that $f_j + ag \in \widetilde{\Phi}$ (and hence in $C^{m-i}$), and $c_a \in \overline{\mathbb{F}}_q$ is a constant depending on $a, y_0$. If $m - i > m/2$, then all terms $\hat{w}_{f_j}$, $[\hat{w}_{f_{j'}}^{-1}, y_0^{-1}]$ in (5.3) commute with each other (in $\mathbb{G}_r$) and the result follows. If $m - i = m/2$, then the terms in (5.3) commute up to $\mathbb{G}_r^r$, which may be ignored, as $r - f$ (from the statement of the lemma) lies in $C^{m/2}$. ☐

**Proposition 5.6.** *Let $0 \leqslant i \leqslant m/2$. The map $g \mapsto (g_{C^{<m-i}}, g_{C^{m-i} \cap \widetilde{\Delta}_r})$ induces an isomorphism*

$$Y_{y_0,r}^{i,m-i} \cong Y_{y_0,r}^{i,m-i-1} \times \mathbb{A}_{C^{m-i} \cap \widetilde{\Delta}_r}.$$

*Moreover, for $g = (g', z) \in Y_{y_0,r}^{i,m-i-1} \times \mathbb{A}_{C^{m-i} \cap \widetilde{\Delta}_r}$ we have*

- *if $0 \leqslant i < m/2$, then*
$$\pi(g)_{\mathbb{T}_r} = \sum_{f \leqslant f' \in C^i \cap -\widetilde{\Delta}_r} \alpha_f^\vee(1 + \varpi^r c_{f,f'} \pi(g')_f z_{r-f'}) + \pi(g')_{\mathbb{T}_r},$$
*where $c_{f,f'} \in \mathcal{O}_{\breve{k}}$ are some constants with $c_{f,f} = 1$;*
- *if $i = m/2$ and $g' \in \mathbb{M}_r \cap Y_{y_0,r}^{i,m-i-1}$, then*
$$\pi(g)_{\mathbb{T}_r} = \mu(z) + \pi(g')_{\mathbb{T}_r},$$
*where $\mu : \mathbb{A}_{C^{m/2} \cap \widetilde{\Delta}_r} \to \mathbb{T}_r$ is a certain morphism.*

*Proof.* By Proposition 3.6 we have an isomorphism

$$\psi : Y_{y_0,r}^{i,m-i} \xrightarrow{\sim} Y_{y_0,r}^{i,m-i-1} \times \mathbb{A}_{C^{m-i} \cap \widetilde{\Delta}_r},$$

and moreover, for $g = \psi^{-1}(g', z)$ with $(g', z) \in Y_{y_0,r}^{i,m-i-1} \times \mathbb{A}_{C^{m-i} \cap \widetilde{\Delta}_r}$ we have $g = g'\hat{w}$ for some $w \in \mathbb{A}_{C^{m-i}}$ such that

$$w_f = z_f \text{ for } f \in C^{m-i} \cap \widetilde{\Delta}_r. \tag{*}$$

We set $h = \pi(g') \in \mathbb{T}_r^{0+}\mathbb{G}_r^{C^{>m-1-i}}\mathbb{V}_r^{C^{\geq i}}$. By definition we have $y_0 = h_{C^0}$, $h_{C^{<i} \smallsetminus \widetilde{\Phi}_M} = 0$. Write $h = h_{C^0}h_+ = y_0 h_+$ with $h_+ = h_{\widetilde{\Phi}_r^{0+}}$. Then $h_{\mathbb{T}_r} = (h_+)_{\mathbb{T}_r}$. We have

$$\pi_{\mathbb{T}_r}(g) = (\hat{w}^{-1}hF(\hat{w}))_{\mathbb{T}_r} = (y_0 y_0^{-1}\hat{w}y_0 h_+ F(\hat{w}))_{\mathbb{T}_r} = (y_0^{-1}\hat{w}y_0 h_+ F(\hat{w}))_{\mathbb{T}_r}.$$

Assume $i = 0$. By Lemma 5.4 we have

$$\pi_{\mathbb{T}_r}(g) = ((y_0^{-1}\hat{w}y_0)h_+)_{\mathbb{T}_r} = (y_0^{-1}\hat{w}y_0)_{\mathbb{T}_r} + (h_+)_{\mathbb{T}_r}.$$

Hence the statement follows from Lemma 5.5 and (*).

Assume $0 < i < m/2$. Then $y_0^{-1}\hat{w}y_0 \in \mathbb{G}_r^{C^{\geqslant m-i}}$. Moreover, as $0 < i < m-i$, we have $h_+F(\hat{w}) \in \mathbb{M}_r^+\mathbb{G}_r^{C^{\geqslant i}}$ and $(h_+F(\hat{w}))_f = h_f = (h_+)_f$ for $f \in C^i$. Applying Lemma 5.4, Lemma 5.5 and (*) we deduce that

$$\begin{aligned}\pi_{\mathbb{T}_r}(g) &= ((y_0^{-1}\hat{w}y_0)(h_+F(\hat{w}))_{\mathbb{T}_r}\\ &= \sum_{f\in C^i} \alpha_f^\vee(1+\varpi^r(y_0^{-1}\hat{w}y_0)_{r-f}(h_+F(\hat{w}))_f) + (y_0^{-1}\hat{w}y_0)_{\mathbb{T}_r} + (h_+F(\hat{w}))_{\mathbb{T}_r}\\ &= \sum_{f\leqslant f'\in C^i\cap -\widetilde{\Delta}_r} \alpha_f^\vee(1+\varpi^r c_{f,f'}h_f w_{r-f'}) + (h_+)_{\mathbb{T}_r}\\ &= \sum_{f\leqslant f'\in C^i\cap -\widetilde{\Delta}_r} \alpha_f^\vee(1+\varpi^r c_{f,f'}\pi(g')_f z_{r-f'}) + \pi_{T,r}(g'),\end{aligned}$$

where the third equality follows from that $h_f = 0$ for $f \in C^i \smallsetminus -\widetilde{\Delta}_r$.

Finally, assume that $i = m/2 \in \mathbb{Z}$ and we may choose $g' \in \mathbb{M}_r$. Then $h \in \mathbb{M}_r$ and $h_{C^{\leqslant m/2}\smallsetminus\widetilde{\Phi}_M} = 0$. By Proposition 3.6, $w \in \mathbb{A}_{C^{m/2}}$ only depends on $z \in \mathbb{A}_{C^{m/2}\cap\widetilde{\Delta}_r}$ (and the fixed element $y_0$). We define $\mu(z) = (y_0^{-1}\hat{w}y_0F(\hat{w}))_{\mathbb{T}_r}$. Noticing that $y_0^{-1}\hat{w}y_0F(\hat{w}) \in \mathbb{G}_r^{C^{\geqslant m/2}}$, $h_+ \in \mathbb{M}_r^+$ and $[h_+^{-1}, F(\hat{w})^{-1}] \in \mathbb{G}_r^{C^{>m/2}\smallsetminus\widetilde{\Phi}_M}$, we deduce by Lemma 5.4 that

$$\begin{aligned}\pi_{\mathbb{T}_r}(g) &= (y_0^{-1}\hat{w}y_0F(\hat{w})h_+[h_+^{-1}, F(\hat{w})^{-1}]))_{\mathbb{T}_r}\\ &= (y_0^{-1}\hat{w}y_0F(\hat{w}))_{\mathbb{T}_r} + (h_+[h_+^{-1}, F(\hat{w})^{-1}])_{\mathbb{T}_r}\\ &= \mu(z) + (h_+)_{\mathbb{T}_r} + ([h_+^{-1}, F(\hat{w})^{-1}])_{\mathbb{T}_r}\\ &= \mu(z) + (h_+)_{\mathbb{T}_r}\\ &= \mu(z) + \pi_{\mathbb{T}_r}(g').\end{aligned}$$

The proof is finished. □

We have a decomposition

$$\mathbb{V}_r \smallsetminus \mathbb{K}_{\phi,r} = \mathbb{V}_r' \sqcup \mathbb{V}_r'',$$

where $\mathbb{V}_r'' = \{g \in \mathbb{V}_r \smallsetminus \mathbb{K}_{\phi,r}; g_{C^{<m/2}\smallsetminus\widetilde{\Phi}_M} = 0\}$ and $\mathbb{V}_r' = (\mathbb{V}_r \smallsetminus \mathbb{K}_{\phi,r}) \smallsetminus \mathbb{V}_r''$. This induces a natural decomposition $\delta_r^{-1}(\bar{g}_0) = \delta_r^{-1}(\bar{g}_0)' \sqcup \delta_r^{-1}(\bar{g}_0)''$.

**Proposition 5.7.** *Let $\pi : X \times \mathbb{G}_a \to \mathbb{T}_r$ be a morphism. Suppose that for each $x \in X$ the pull-back of $\mathcal{L}_\phi$ via the map $z \mapsto \pi(x,z)$ is isomorphic to a nontrivial multiplicative local system on $\mathbb{G}_a$. Then $R\Gamma_c(X \times \mathbb{G}_a, \pi^*\mathcal{L}_\phi) = 0$.*

*Proof.* Let $x\colon \operatorname{Spec}\overline{\mathbb{F}}_q \to X$ be a point, and let $x'\colon \mathbb{G}_a \to X \times \mathbb{G}_a$ be the base changed map. Denote by $f\colon X \times \mathbb{G}_a \to X$ the natural projection and by $f_x$ its pullback along $x$. Proper base change implies that $x^*f_!\pi^*\mathcal{L}_\phi \cong f_{x!}x'^*\pi^*\mathcal{L}_\phi$, which is zero by [Boy10, Lemma 9.4]. As this holds for any geometric point $x \in X$, we deduce $f_!\pi^*\mathcal{L}_\phi = 0$. Thus $R\Gamma_c(X \times \mathbb{G}_a, \pi^*\mathcal{L}_\phi) \cong R\Gamma_c(X, f_!\pi^*\mathcal{L}_\phi) = 0$. □

**Proposition 5.8.** *We have $R\Gamma_c(\delta_r^{-1}(\bar{g}_0)', \pi_{\mathbb{T}_r}^*\mathcal{L}_\phi) = 0$.*

*Proof.* Let $m/2 \leqslant j \leqslant m$, $0 \leqslant i < m/2$ and $f \in (C^{\leqslant i} \cap -\widetilde{\Delta}_r) \smallsetminus \widetilde{\Phi}_M$. We define

$$Y_{y_0,r}^{f,j} = \{g \in Y_{y_0,r}^{0,j}; \pi(g)_f \neq 0, \pi(g)_{C^{<f} \smallsetminus \widetilde{\Phi}_M} = 0\}.$$

Then $\delta_r^{-1}(\bar{g}_0)'$ is a disjoint union of locally closed subsets $Y_{y_0,r}^{f,m}$. It suffices to show $R\Gamma_c(Y_{y_0,r}^{f,m}, \pi_{\mathbb{T}_r}^* \mathcal{L}_\phi) = 0$.

By Proposition 5.6, we have

$$Y_{y_0,r}^{f,m} \cong Y_{y_0,r}^{f,m-1} \times \mathbb{A}_{C^m \cap \widetilde{\Delta}_r} \cong \cdots \cong Y_{y_0,r}^{f,m-i-1} \times \mathbb{A}_{C^{\geqslant m-i} \cap \widetilde{\Delta}_r}.$$

Moreover, for $g = (g', z) \in Y_{y_0,r}^{f,m-i-1} \times \mathbb{A}_{C^{\geqslant m-i} \cap \widetilde{\Delta}_r}$ we have $\pi(g')_{C^{<f} \smallsetminus \widetilde{\Phi}_M} = 0$ and hence

$$\pi(g)_{\mathbb{T}_r} \equiv \alpha_f^\vee(1 + \varpi^r \pi(g')_f z_{r-f}) + \pi(g')_{\mathbb{T}_r} \mod (\mathbb{T} \cap \mathbb{M}_{\mathrm{der}})_r^r.$$

Since the restriction of the character $\phi$ to $((\mathbb{T} \cap \mathbb{M}_{\mathrm{der}})_r^r)^F$ is trivial, it follows that the pull-back of $\mathcal{L}_\phi$ over $\mathbb{A}_{C^{\geqslant m-i} \cap \widetilde{\Delta}_r}$ under the morphism $z \mapsto \pi_{\mathbb{T}_r}(g', z)$ is isomorphic to the pull-back of $\mathcal{L}_\phi$ under the morphism $z \mapsto \alpha_f^\vee(1+\varpi^r \pi(g')_f z_{r-f})$, which is a nontrivial multiplicative local system. Thus the statement follows from Proposition 5.7. □

5.4. **Proof of Proposition 5.1.** We argue by induction on $d$ and the semisimple rank of $G$. If $d = 0$ or $G$ is a torus, then $V_r = \varnothing$ and the statement is trivial.

Suppose that $d \geqslant 1$ and hence $M = G^{d-1}$ is a proper Levi subgroup. In view of Proposition 5.8, it suffices to show $R\Gamma_c(\delta_r^{-1}(\bar{g}_0)'', \pi_{\mathbb{T}_r}^* \mathcal{L}_\phi) = 0$. For $m/2 - 1 \leqslant j \leqslant m$ we define

$$Y_{y_0,r}^{\prime\prime,j} = \{g \in Y_{y_0,r}^{0,j} : \pi(g)_{C^{<m/2} \smallsetminus \widetilde{\Phi}_M} = 0\}$$

Then $\delta_r^{-1}(\bar{g}_0)'' = Y_{y_0,r}^{\prime\prime,m}$. By Proposition 5.6, we have

$$Y_{y_0,r}^{\prime\prime,m} \cong Y_{y_0,r}^{\prime\prime,m/2-1} \times \mathbb{A}_{C^{\geqslant m/2} \cap \widetilde{\Delta}}.$$

Moreover, for $(g', z) \in Y_{y_0,r}^{\prime\prime,m/2-1} \times \mathbb{A}_{C^{\geqslant m/2} \cap \widetilde{\Delta}}$ we have $\pi_{\mathbb{T}_r}(g) = \mu(z) + \pi_{\mathbb{T}_r}(g')$. As $\mathcal{L}_\phi$ is multiplicative, by Künneth formula it suffices to show $R\Gamma_c(Y_{y_0,r}^{\prime\prime,m/2-1}, \pi_{\mathbb{T}_r}^* \mathcal{L}_\phi) = 0$.

Indeed, using the natural embedding $\mathbb{M}_{r-}^{0+}/\mathbb{T}_{r-}^{0+} \hookrightarrow \mathbb{G}_r^{0+}/\mathbb{T}_r^{0+}\mathbb{G}_r^{C^{\geqslant m/2}}$ we have

$$Y_{y_0,r}^{\prime\prime,m/2-1} = \sqcup_{g \in (\mathbb{G}_r^{0+})^{\mathrm{ad}(y_0)\circ F}/(\mathbb{M}_r^{0+}\mathbb{G}_r^{C^{\geqslant m/2}})^{\mathrm{ad}(y_0)\circ F}} g Y_{y_0,r-}^M,$$

where $Y_{y_0,r-}^M = \{g \in \mathbb{M}_{r-}^{0+}/\mathbb{T}_{r-}^{0+}; \pi(g) \in \mathbb{T}_{r-}^{0+}(\mathbb{M}_{r-} \cap \mathbb{V}_{r-})\}$. Now the statement follows by induction hypothesis that $R\Gamma_c(Y_{y_0,r-}^M, \pi_{T_{r-}}^* \mathcal{L}_\phi) = 0$. This proves Proposition 5.1 and hence Theorem 4.1.

## 6. Relation with the variety of Chen–Stasinski

We continue to work with notation from §4. We thus have a character $\phi\colon T^F \to \overline{\mathbb{Q}}_\ell^\times$ of depth$(\phi) \leq r$, which we assume to admit a Howe factorization with corresponding subgroups $\mathbb{K}_\phi, \mathbb{K}_\phi^+$, etc. As long as $r$ remains fixed, we sometimes omit it from notation and write $\mathbb{G}$ instead of $\mathbb{G}_r$, etc. We also write $\phi$ for the character of $\mathbb{T}^F$ induced by $\phi$.

Theorem 4.1 shows that $R\Gamma_c(X_r, \overline{\mathbb{Q}}_\ell)[\phi] = R\Gamma_c(X_r^\flat, \overline{\mathbb{Q}}_\ell)[\phi]$. Next, we relate the cohomology of $X_r^\flat$ with the cohomology of a different variety.

Define the subgroup

$$\mathbb{I}_{\phi,U} = (\mathbb{K}_\phi \cap \mathbb{U})(\mathbb{E}_\phi \cap \mathbb{T})(\mathbb{K}_\phi^+ \cap \mathbb{U}^-).$$

of $\mathbb{K}_\phi$ and the subvariety

$$Z_{\phi,U,r} = \{g \in \mathbb{G}\colon g^{-1}F(g) \in F\mathbb{I}_{\phi,U}\}$$

acted on by $\mathbb{G}^F \times \mathbb{T}^F$ by left and right multiplication. The variety $Z_{\phi,U,r}$ was first considered in a special case by Chen and Stasinski in [CS17], and later (in general) by the second author in [Nie24]. The following result gives a degreewise comparison of the cohomologies of $X_r^\flat$ and $Z_{\phi,U,r}$. This improves over [Nie24, Theorem 4.1], which only compares the ($\mathbb{G}^F$-equivariant) Euler characteristics.

**Proposition 6.1.** *We have a $\mathbb{G}^F$-equivariant isomorphism*

$$R\Gamma_c(X_r^\flat, \overline{\mathbb{Q}}_\ell)[\phi] \cong R\Gamma_c(Z_{\phi,U,r}, \overline{\mathbb{Q}}_\ell)[\phi][2m],$$

*where* $m = \dim(\overline{\mathbb{U}} \cap \mathbb{K}_\phi^+)(F\mathbb{U} \cap \mathbb{U} \cap \mathbb{K}_\phi) + \dim(\mathbb{T} \cap \mathbb{E}_\phi)$.

This follows directly from Lemma 6.2 and Proposition 6.3 below.

### 6.1. Proof of Proposition 6.1.

Consider

$$X_r^{\flat,\mathbb{K}} = X_r^\flat \cap \mathbb{K}_\phi$$
$$Z_{\phi,U,r}^{\mathbb{K}} = Z_{\phi,U,r} \cap \mathbb{K}_\phi,$$

both admitting $\mathbb{K}_\phi^F \times \mathbb{T}^F$-actions by left/right multiplication. It is immediate that $Z_{\phi,U,r} = \coprod_{\gamma \in \mathbb{G}^F/\mathbb{K}_\phi^F} \gamma Z_{\phi,U,r}^{\mathbb{K}}$, so that

$$R\Gamma_c(Z_{\phi,U,r}, \overline{\mathbb{Q}}_\ell)[\phi] = \mathrm{ind}_{\mathbb{K}_\phi^F}^{\mathbb{G}^F} R\Gamma_c(Z_{\phi,U,r}^{\mathbb{K}})[\phi], \tag{6.1}$$

and the same formulas hold for $X_r^\flat$. To prove Proposition 6.1 it thus suffices to show $R\Gamma_c(X_r^{\flat,\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi] \cong R\Gamma_c(Z_{\phi,U,r}^{\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi][2m]$.

Let $\mathbb{T}_\phi = \mathbb{E}_\phi \cap \mathbb{T}$. Define

$$X_r^{\natural,\mathbb{K}} = \{g \in \mathbb{K}_\phi; g^{-1}F(g) \in \mathbb{T}_\phi(F\mathbb{U} \cap \overline{\mathbb{U}} \cap \mathbb{K}_\phi)\}.$$

**Lemma 6.2.** *We have a natural $\mathbb{K}_\phi^F$-equivariant isomorphism $R\Gamma_c(X_r^{\flat,\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi] \cong R\Gamma_c(X_r^{\natural,\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi][2\dim \mathbb{T}_\phi]$.*

*Proof.* Since $\mathbb{T}_\phi$ is an affine space, the quotient map $X_r^{\natural,\mathbb{K}} \to X_r^{\natural,\mathbb{K}}/\mathbb{T}_\phi$ induces an isomorphism of $\mathbb{K}_\phi^F$-modules

$$R\Gamma_c(X_r^{\natural,\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi] \cong R\Gamma_c(X_r^{\natural,\mathbb{K}}/\mathbb{T}_\phi, \overline{\mathbb{Q}}_\ell)[\phi][2\dim \mathbb{T}_\phi].$$

On the other hand, there is a natural isomorphism $X_r^{\flat,\mathbb{K}}/\mathbb{T}_\phi^F \to X_r^{\natural,\mathbb{K}}/\mathbb{T}_\phi$. Thus we have natural isomorphisms $\mathbb{K}_\phi^F$-modules

$$\begin{aligned} R\Gamma_c(X_r^{\natural,\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi] &\cong R\Gamma_c(X_r^{\natural,\mathbb{K}}/\mathbb{T}_\phi, \overline{\mathbb{Q}}_\ell)[\phi][2\dim \mathbb{T}_\phi] \\ &\cong R\Gamma_c(X_r^{\flat,\mathbb{K}}, \overline{\mathbb{Q}}_\ell)^{\mathbb{T}_{\phi,r}^F}[\phi][2\dim \mathbb{T}_\phi] \\ &\cong R\Gamma_c(X_r^{\flat,\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi][2\dim \mathbb{T}_\phi], \end{aligned}$$

where the last isomorphism follows from that $\phi$ is trivial over $\mathbb{T}_\phi^F$. □

**Proposition 6.3.** *The map $(z, a) \mapsto za$ gives an isomorphism*

$$\varphi_r : X_r^{\natural,\mathbb{K}} \times (\overline{\mathbb{U}} \cap \mathbb{K}_\phi^+)(F\mathbb{U} \cap \mathbb{U} \cap \mathbb{K}_\phi) \xrightarrow{\sim} Z_{\phi,U}^{\mathbb{K}}.$$

*As a consequence, we have an isomorphism $R\Gamma_c(Z_{\phi,U}^{\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi][2m'] \cong R\Gamma_c(X_r^{\natural,\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi]$ as $\mathbb{K}_\phi^F$-modules, where $m' = \dim(\overline{\mathbb{U}} \cap \mathbb{K}_\phi^+)(F\mathbb{U} \cap \mathbb{U} \cap \mathbb{K}_\phi)$.*

*Proof.* First note that $\varphi_r$ is well-defined. Let $z \in Z_{\phi,U,r}^{\mathbb{K}}$. It suffices to show there exists a unique $a \in \mathcal{A}_r := (F\mathbb{U}_r \cap \mathbb{U}_r \cap \mathbb{K}_{\phi,r})(\overline{\mathbb{U}}_r \cap \mathbb{K}_{\phi,r}^+)$ such that $za \in X_r^{\natural,\mathbb{K}}$. We argue by induction on $r \in \mathbb{R}_{\geqslant 0}$. If $r = 0$, then $\mathcal{A}_r = F\mathbb{U}_r \cap \mathbb{U}_r \cap (\mathbb{G}^0)_r$, $F\mathcal{I}_{\phi,U,r} = F\mathbb{U}_r \cap (\mathbb{G}^0)_r$ and the statement follows from Proposition 3.7.

Suppose that the statement holds for $r-$. We show it also holds for $r > 0$. Indeed, by induction hypothesis, there exists $b \in \mathcal{A}_{r-}$ such that $zb \in X_{r-}^\natural$. Choose a lift of $b$ in $\mathcal{A}_r$ and still denote it by $b$. Then

$$(zb)^{-1}F(zb) \in \mathbb{T}_{\phi,r}(\mathbb{K}_{\phi,r} \cap F\mathbb{U}_r \cap \overline{\mathbb{U}}_r)\mathbb{H}_r$$

where $\mathbb{H}_r = (F\mathbb{U}_r \cap \mathbb{K}_{\phi,r} \cap \mathbb{G}_r^r)(F\overline{\mathbb{U}}_r \cap \mathbb{K}_{\phi,r}^+ \cap \mathbb{G}_r^r)$.

We assume that $r = r_{i-1}/2$ for some $1 \leqslant i \leqslant d$. The remaining case follows in a simpler way. Let $\Phi_j \subseteq \Phi$ be the root system of $G^j$ for $0 \leqslant j \leqslant d$. Then $\mathbb{H}_r = \mathbb{H}_r' \oplus \mathbb{H}_r''$, where $\mathbb{H}_r'$ (resp. $\mathbb{H}_r''$) is spanned by the (images) of affine root subgroups of $F(f)$ such that $f(\mathbf{x}) = r$ and $\alpha_f \in \Phi_i^+ \smallsetminus \Phi_{i-1}$ (resp. $\alpha_f \in \Phi_{i-1}$). Let $\mathcal{C}_r = \mathcal{A}_r \cap \mathbb{G}_r^r$. Then $\mathcal{C}_r = \mathcal{C}_r' \oplus \mathcal{C}_r''$, where $\mathcal{C}_r'$ (resp. $\mathcal{C}_r''$) is spanned by the (images) of affine root subgroups of $f$ such that $f(\mathbf{x}) = r$ and $f \in (F(\Phi_i^+) \cap \Phi_i^+) \smallsetminus \Phi_{i-1}$ (resp. $\alpha_f \in \Phi_{i-1} \smallsetminus (F(\Phi_{i-1}^+) \cap \Phi_{i-1}^-)$).

Applying Proposition 3.7 and Proposition 3.6, there exists $c \in \mathcal{C}_r$ such that

$$c^{-1}((zb)^{-1}F(zb))F(c) \in \mathbb{T}_{\phi,r}(\mathbb{K}_{\phi,r} \cap F\mathbb{U}_r \cap \overline{\mathbb{U}}_r),$$

that is, $za \in X_r^\natural$ with $a = bc \in \mathcal{A}_r$ as desired.

Let $a' \in \mathcal{A}_r$ be another element such that $za' \in X_r^{\natural,\mathbb{K}}$. By induction hypothesis, the images of $a$ and $a'$ in $\mathcal{A}_{r-}$ are the same. Hence we may assume $a' = ad$ for some $d \in \mathcal{C}_r$. Then it follows from the uniqueness in Proposition 3.6 that $d$ is trivial and hence $a = a'$ as desired.

The second statement follows from that $\varphi_r$ is $\mathbb{K}_{\phi,r}^F \times \mathbb{T}_r^F$-equivariant and that $(\overline{\mathbb{U}}_r \cap \mathbb{K}_{\phi,r}^+)(F\mathbb{U}_r \cap \mathbb{U}_r \cap \mathbb{K}_{\phi,r})$ is an affine space. □

6.2. **Cohomology of $Z_{\phi,U,r}^{\mathbb{K}}$.** We generalize the results of [Nie24, §5.2]. Let $(V, \overline{V})$ and $(U, \overline{U})$ be two pairs of opposite maximal unipotent subgroups of $G$ normalized by $T$. For a subset $R \subseteq \mathbb{K}_\phi$, write $\bar{R}$ for the image of $R$ under the natural projection $\mathbb{K}_\phi \to \mathbb{K}_\phi/\mathbb{E}_\phi$. As $\phi$ remains fixed, we omit it from notation and write $\mathbb{K}, \mathbb{H}, \mathbb{E}$ instead of $\mathbb{K}_\phi, \mathbb{H}_\phi, \mathbb{E}_\phi$. Write $L = G^0$. First note that

$$\bar{\mathbb{L}} = \sqcup_{w \in W_{\bar{\mathbb{L}}}(\bar{\mathbb{T}})} \bar{\mathbb{L}}_V \dot{w} \bar{\mathbb{T}} \bar{\mathbb{L}}_U,$$

where $\bar{\mathbb{L}}_V = \bar{\mathbb{L}} \cap \mathbb{V}$ and $\bar{\mathbb{L}}_U = \bar{\mathbb{L}} \cap \bar{\mathbb{U}}$.

For $\alpha \in \Phi$, define $i(\alpha)$ to be the integer $0 \leq i \leq d$, such that $\alpha \in \Phi(G^i, T) \smallsetminus \Phi(G^{i-1}, T)$, and define $r(\alpha) = r_{i(\alpha)-1}$. Put

$$\bar{\mathbb{H}}^\alpha = (\mathbb{G}^\alpha)_r^{r(\alpha)/2} / (\mathbb{G}^\alpha)_r^{r(\alpha)+}.$$

Then we have

$$\bar{\mathbb{H}} = \bar{\mathbb{H}}_V \bar{\mathbb{T}}^{0+} \bar{\mathbb{H}}_{\overline{V}} = \bar{\mathbb{T}}^{0+} \oplus_\alpha \bar{\mathbb{H}}^\alpha,$$

where $\bar{\mathbb{H}}_V = \bar{\mathbb{H}} \cap \bar{V}$, $\bar{\mathbb{H}}_{\overline{V}} = \bar{\mathbb{H}} \cap \bar{\overline{V}}$. For $\alpha, \beta \in \Phi$ we have $[\bar{\mathbb{H}}^\alpha, \bar{\mathbb{H}}^\beta] = \{0\}$ if $\alpha \neq -\beta$ and $[\bar{\mathbb{H}}^\alpha, \bar{\mathbb{H}}^\beta] = (\bar{\mathbb{T}}^\alpha)^{r(\alpha)} \cong (\mathbb{T}^\alpha)_r^{r(\alpha)} / (\mathbb{T}^\alpha)_r^{r(\alpha)+}$ if $\alpha = -\beta$ and $\bar{\mathbb{H}}^\alpha \neq \{0\}$.

Thus we have

$$\bar{\mathbb{K}} = \bar{\mathbb{H}} \bar{\mathbb{L}} = \bigsqcup_{w \in W_{\mathbb{L}_r}(\mathbb{T}_r)} \bar{\mathbb{H}} \bar{\mathbb{L}}_V \dot{w} \bar{\mathbb{T}} \bar{\mathbb{L}}_U = \bigsqcup_{w \in W_{\mathbb{L}_r}(\mathbb{T}_r)} \bar{\mathbb{K}}_V \bar{\mathbb{H}}_{\overline{V},w} \dot{w} \bar{\mathbb{T}} \bar{\mathbb{H}}_U,$$

where $\bar{\mathbb{K}}_V = \bar{\mathbb{L}}_V \bar{\mathbb{H}}_V = \bar{\mathbb{K}} \cap \bar{\mathbb{U}}$ and $\bar{\mathbb{H}}_{\overline{V},w} = \bar{\mathbb{H}}_{\overline{V}} \cap {}^{\dot{w}}\bar{\overline{\mathbb{U}}}$.

Write $H_c^*(Z_{\phi,V,r}^{\mathbb{K}}, \overline{\mathbb{Q}}_\ell) = \sum_{i \in \mathbb{Z}} H_c^i(Z_{\phi,V,r}^{\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi]$.

**Proposition 6.4.** *We have*

$$\langle H_c^*(Z_{\phi,V,r}^{\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi], H_c^*(Z_{\phi,U,r}^{\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi] \rangle_{\mathbb{K}_{\phi,r}^F} = \sharp \mathrm{stab}_{W_{\mathbb{L}_r}(\mathbb{T}_r)^F}(\phi|_{\mathbb{T}_r^F}).$$

*Proof.* For $w \in W_{\mathbb{L}_r}(\mathbb{T}_r)$ we set

$$\Sigma_w = \{(x, x', v, \bar{v}, \tau, u) \in F\bar{\mathbb{K}}_V \times F\bar{\mathbb{K}}_U \times \bar{\mathbb{K}}_{V,w} \times \bar{\mathbb{H}}_{\overline{V},w} \times \bar{\mathbb{T}} \times \mathbb{K}_U; xF(\bar{v}\dot{w}\tau) = v\bar{v}\dot{w}\tau u x'\}.$$

Write $\Sigma_w = \Sigma_w' \sqcup \Sigma_w''$, where $\Sigma_w''$ is defined by condition that $\bar{v} = 0$.

Let $D = \{(t, s) \in \mathbb{T}_r \times \mathbb{T}_r; t^{-1}F(t) = s^{-1}F(s)\}$. Then $D$ acts on $\Sigma_w''$ by

$$(t, s) : (x, x', v, \bar{v}, \tau, u) \longmapsto (txt^{-1}, sx's^{-1}, svs, s\bar{v}s^{-1}, \dot{w}^{-1}(t)\tau s^{-1}, sus^{-1}).$$

It follows that

$$(\Sigma_w'')^{D_{\mathrm{red}}^\circ} \cong (\dot{w}\bar{\mathbb{T}})^F.$$

Hence $H_c^*(\Sigma_w'', \overline{\mathbb{Q}}_\ell)_{\phi^{-1},\phi} = \overline{\mathbb{Q}}_\ell$ if $w = F(w)$ and is trivial otherwise.

It remains to show $H_c^*(\Sigma_w', \overline{\mathbb{Q}}_\ell)_{\phi,\phi^{-1}} = 0$. Note that

$$\bar{\mathbb{H}}_{\overline{V},w} = \oplus_{\alpha \in \Phi_{\overline{V}} \cap {}^w\Phi_{\overline{U}}} \bar{\mathbb{H}}^\alpha,$$

where $\bar{\mathbb{H}}^\alpha = \bar{\mathbb{H}} \cap \bar{\mathbb{G}}^\alpha$. For $\bar{v} \in \bar{\mathbb{H}}_{V,w}$ and $\alpha \in \Phi_{\overline{V}} \cap {}^w\Phi_{\overline{U}}$ let $\bar{v}_\alpha \in \bar{\mathbb{H}}^\alpha$ such that $\bar{v} = \sum_\alpha \bar{v}_\alpha$. We fix a total order $\leqslant$ on $\Phi_{\overline{V}} \cap {}^w\Phi_U$. Let $\bar{\mathbb{H}}^\alpha_{\overline{V},w}$ be subset of elements $\bar{v}$ such that $\bar{v}_\alpha \neq 0$ and $\bar{v}_\beta = 0$ for all $\beta < \alpha$. Then we have

$$\bar{\mathbb{H}}_{\overline{V},w} - \{0\} = \coprod_\alpha \bar{\mathbb{H}}^\alpha_{\overline{V},w}.$$

The above decomposition induces a decomposition

$$\Sigma'_w = \coprod_\alpha \Sigma'^{,\geqslant\alpha}_w.$$

It remains to show $H^*_c(\Sigma'^{,\alpha}_w, \overline{\mathbb{Q}}_\ell)_{\phi,\phi^{-1}} = 0$ for all $\alpha$.

Let $\alpha \in \Phi_{\overline{V}} \cap {}^w\Phi_{\overline{U}}$ such that $\Sigma'^{,\alpha}_w \neq \varnothing$. Consider the restricted action of $(\mathbb{T}_r)^F \cong (\mathbb{T}_r)^F \times \{1\} \subseteq (\mathbb{T}_r)^F \times (\mathbb{T}_r)^F$ on $\Sigma'^{,\alpha}_w$ given by

$$t : (x, x', v, \bar{v}, \tau, u) \longmapsto (txt^{-1}, x', tvt^{-1}, t\bar{v}t^{-1}, w^{-1}(t)\tau, u).$$

It suffices to show the $\phi$-isotropic subspace $H^*_c(\Sigma'^{,\alpha}_w, \overline{\mathbb{Q}}_\ell)_\phi$ is trivial.

For $\bar{v} \in \bar{\mathbb{H}}^{\geqslant a}_{\overline{V},w}$ we fix an isomorphism

$$\lambda_{\bar{v}} : \bar{\mathbb{H}}^{-\alpha} \xrightarrow{\sim} (\bar{\mathbb{T}}^\alpha)^{r(\alpha)}, \quad \zeta \longrightarrow [\bar{v}, \zeta].$$

Let

$$\mathcal{H} = \{t \in \bar{\mathbb{T}}^{r(\alpha)}; t^{-1}F^{-1}(t) \in (\bar{\mathbb{T}}^\alpha)^{r(\alpha)}\}.$$

For $t \in \mathcal{H}$ we define an isomorphism $f_t : \Sigma'^{,\alpha}_w \to \Sigma'^{,\alpha}_w$ by

$$f_t : (x, x', v, \bar{v}, \tau, u) \longmapsto (x_t, x' F({}^{(\dot{w}\tau)^{-1}}\zeta), tvt^{-1}, t\bar{v}t^{-1}, w^{-1}(t)\tau, u)$$

with $\zeta = \lambda_{\bar{v}}^{-1}(tF^{-1}(t)^{-1})$ such that

$$x_t F(\bar{v}\dot{w}\tau) = tv\bar{v}\dot{w}\tau u x' F({}^{(\dot{w}\tau)^{-1}}\zeta).$$

The induced map of $f_t$ on each subspace $H^i_c(\Sigma'^{,\alpha}_w, \overline{\mathbb{Q}}_\ell)$ is trivial for $t \in N_F^{F^n}((\bar{\mathbb{T}}^\alpha)^{F^n}) \subseteq \mathcal{H}^\circ \cap ((\bar{\mathbb{T}}^\alpha)^{r(\alpha)})^F$. Here $n \in \mathbb{Z}_{\geqslant 1}$ such that $F^n(\mathbb{T}^\alpha) = \mathbb{T}^\alpha$, and $N_F^{F^n} : \bar{\mathbb{T}} \to \bar{\mathbb{T}}$ is the map given by $t \mapsto tF(t) \cdots F^{n-1}(t)$. On the other hand, we have

$$\phi|_{N_F^{F^n}((\bar{\mathbb{T}}^\alpha)^{F^n}} = \phi_{i(\alpha)-1}|_{N_F^{F^n}((\bar{\mathbb{T}}^\alpha)^{F^n}},$$

which is nontrivial since $\phi_{i(\alpha)-1}$ is $(\mathbb{G}^{i(\alpha)-1}, \mathbb{G}^{i(\alpha)})$-generic. Thus $H^*_c(\Sigma'^{,\alpha}_w, \overline{\mathbb{Q}}_\ell)_\phi$ is trivial as desired. □

6.3. **Concentration in one degree.** Let notation be as in §6.2. Let $Z^{\mathbb{H}}_{\phi,U} = Z_{\phi,U} \cap \mathbb{H}_\phi$ and $Z^{\mathbb{L}}_{\phi,U} = Z_{\phi,U} \cap \mathbb{L}$ with $\mathbb{L} = \mathbb{G}^0$. We set $\bar{Z}^{\mathbb{H}}_{\phi,U} = Z^{\mathbb{H}}_{\phi,U}/\mathbb{E}_\phi$ and $Z^{\mathbb{L}}_{\phi,U} = Z^{\mathbb{L}}_{\phi,U}/(\mathbb{E}_\phi \cap \mathbb{L})$.

By [Nie24, §7.1], each cohomology group $H^i_c(Z^{\mathbb{H}}_{\phi,U}, \overline{\mathbb{Q}}_\ell)[\phi|_{(\mathbb{T}^{0+})^F}]$ is equipped a natural $\mathbb{K}^F_\phi$-module structure, and we define

$$\kappa_\phi = \sum_{i \in \mathbb{Z}} (-1)^i H^i_c(Z^{\mathbb{H}}_{\phi,U}, \overline{\mathbb{Q}}_\ell)[\phi|_{(\mathbb{T}^{0+})^F}]$$

as a virtual $\mathbb{K}_\phi^F$-module. The following result is proved in [Nie24, Theorem 6.2].

**Theorem 6.5.** *There exists a unique integer $n_\phi \geqslant 0$ such that*

$$H_c^i(\bar{Z}_{\phi,U}^{\mathbb{H}}, \overline{\mathbb{Q}}_\ell)[\phi|_{(\mathbb{T}^{0+})^F}] \neq 0$$

*if and only if $i = n_\phi$.*

**Theorem 6.6.** *Let $T \subseteq B$ be as in Proposition 3.5 with $M = L = G^0$. Then for $i \in \mathbb{Z}$ we have an isomorphism of $\mathbb{H}_\phi^F$-modules*

$$H_c^i(\bar{Z}_{\phi,U}^{\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi] \cong (-1)^{n_\phi}\kappa_\phi \otimes H_c^{i-n_\phi}(\bar{Z}_{\phi,U}^{\mathbb{L}}, \overline{\mathbb{Q}}_\ell)[\phi_{-1}],$$

*where $n_\phi$ is as in Theorem 6.5.*

*Proof.* By the assumption on $B$ we have that $\bar{\mathbb{H}}_\phi \cap F\mathbb{I}_{\phi,U}$ is normalized by $\bar{\mathbb{L}}$. By [Nie24, Proposition 7.4] the map $(h,l) \mapsto hl$ induces a $(\mathbb{T}^{0+})^F/(\mathbb{E}_\phi \cap \mathbb{T})^F$-torsor

$$f : \bar{Z}_{\phi,U}^{\mathbb{H}} \times \bar{Z}_{\phi,U}^{\mathbb{L}} \longrightarrow \bar{Z}_{\phi,U}^{\mathbb{K}}.$$

Let $\phi^\flat$ be the pull-back of the natural multiplication map $(\mathbb{T}^{0+})^F \times \mathbb{T}^F \to \mathbb{T}^F$. Combining Theorem 6.5 and the arguments in the proof of loc. cit. we deduce that

$$\begin{aligned} & H_c^i(\bar{Z}_{\phi,U}^{\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi] \\ \cong\ & H_c^i(\bar{Z}_{\phi,U}^{\mathbb{H}} \times H_c^i(\bar{Z}_{\phi,U}^{\mathbb{L}}, \overline{\mathbb{Q}}_\ell)[\phi^\flat] \\ \cong\ & \bigoplus_{i'+i''=i} H_c^{i'}(\bar{Z}_{\phi,U}^{\mathbb{H}}, \overline{\mathbb{Q}}_\ell)[\phi|_{(\mathbb{T}^{0+})^F}] \otimes H_c^{i''}(\bar{Z}_{\phi,U}^{\mathbb{L}}, \overline{\mathbb{Q}}_\ell)[\phi] \\ \cong\ & (-1)^{n_\phi}\kappa_\phi \otimes H_c^{i-n_\phi}(\bar{Z}_{\phi,U}^{\mathbb{L}}, \overline{\mathbb{Q}}_\ell)[\phi_{-1}]. \end{aligned}$$

The proof is finished. □

**Corollary 6.7.** *Let $T \subseteq B$ be as in Proposition 3.5 with $M = L = G^0$. Suppose that $\phi_{-1}$ (viewed as a character of $\bar{\mathbb{T}}^F$) is non-singular for $\bar{\mathbb{L}}$ in the sense of* [DL76, Definition 5.15]*, then there exists a unique integer $N_\phi$ such that $H_c^i(X_r, \overline{\mathbb{Q}}_\ell)[\phi] \neq 0$ if and only if $i = N_\phi$.*

*Proof.* By Proposition 6.1 and that the quotient map $Z_{\phi,U}^{\mathbb{K}} \to \bar{Z}_{\phi,U}^{\mathbb{K}}$ is a $\mathbb{K}^F \times \mathbb{T}^F$-equivariant affine space bundle, it suffices to consider the cohomology groups $H_c^i(\bar{Z}_{\phi,U}^{\mathbb{K}}, \overline{\mathbb{Q}}_\ell)[\phi]$.

Let $W_L$ be the Weyl group of $L$. By assumption, the relative position in $W\sigma$ of $B$ and $FB$ is of minimal length in its $W_L$-conjugacy class. It follows by [He08] and [OR08] that the classical Deligne-Lusztig variety for $\bar{Z}_{\phi,U}^{\mathbb{L}}/\bar{\mathbb{T}}^F$ is an affine variety. By [DL76, Corollary 9.9], $H_c^i(\bar{Z}_{\phi,U}^{\mathbb{L}}, \overline{\mathbb{Q}}_\ell)[\phi_{-1}] \neq 0$ if and only if $i = \dim \bar{Z}_{\phi,U}^{\mathbb{L}}$. Thus the statement follows form Theorem 6.6. □

## 7. Pro-unipotent DL-variety for an elliptic torus

Let $\mathbb{G}^+ = L^+\mathcal{G}_{\mathbf{x}}^{0+}$ be the pro-unipotent radical of $\mathbb{G} = L^+\mathcal{G}_{\mathbf{x}}$ (this corresponds to $r = \infty$ in the notation of §2.2). Consider the perfect scheme

$$X^+ = \{g \in \mathbb{G}^+ \colon g^{-1}F(g) \in \overline{\mathbb{U}}^+ \cap F\mathbb{U}^+\}$$

which is the the inverse limit of its truncations $X_r^+ \subseteq \mathbb{G}_r^+ := \mathbb{G}_r^{0+}$. Then $X^+$ is acted on by $(\mathbb{G}^+)^F \times (\mathbb{T}^+)^F$ by left and right multiplication. In this section we are going to prove Theorem 1.5.

7.1. **Preparations.** Fix a total order on $\widetilde{\Phi}^+/\langle F\rangle$ such that $O < O'$, if either $O(\mathbf{x}) < O'(\mathbf{x})$ or $(O(\mathbf{x}) = O'(\mathbf{x})$ and $O \in \mathbb{Z}_{\geq 1}$, $O' \notin \mathbb{Z}_{\geq 1})$. As $T$ is elliptic, any orbit $O \in \widetilde{\Phi}^+/\langle F\rangle$ intersects $\widetilde{\Delta}^+$, where $\Delta = \Phi^+ \cap F\Phi^-$. For each orbit $O$, pick some $f \in O \cap \widetilde{\Delta}^+$ and extend the order to a total order on $\widetilde{\Phi}^+$ in the unique way such that $f < F(f) < \cdots < F^{\#O-1}(f)$. For $f \in \widetilde{\Phi}^+$, denote by $O_f$ its $F$-orbit, and denote by $f+$ (resp. $f-$) any member of the orbit, which is the ascendant (resp. descendant) of $O_f$ with respect to the order on $\widetilde{\Phi}^+/\langle F\rangle$.

We use the setup from [IN24, §5.1-2], which slightly differs from that of §5.1. In this section for $f \in \widetilde{\Phi}^+$ we put

$$\widetilde{\Phi}^f = \{f' \in \widetilde{\Phi}^+ \colon O_{f'} \geq O_f\}.$$

Note that if $f = r \in \mathbb{Z}_{\geq 1}$, then $\widetilde{\Phi}^f = \{f' \in \widetilde{\Phi}^+ \colon 0 < f'(\mathbf{x}) < r\}$, so our notation is not ambiguous. We let $\widetilde{\Phi}_f^+ = \widetilde{\Phi}^+ \smallsetminus \widetilde{\Phi}^f$. We let $\mathbb{G}^f \subseteq \mathbb{G}^+$ be the subgroup generated by the affine roots subgroups in $\widetilde{\Phi}^f$. It is easy to see that $\mathbb{G}^f \subseteq \mathbb{G}^+$ is normal and we put

$$\mathbb{G}_f^+ = \mathbb{G}^+/\mathbb{G}^f.$$

Note that $\widetilde{\Phi}^f, \widetilde{\Phi}_f^+$ are $F$-stable, so that $\mathbb{G}^f, \mathbb{G}_f^+$ are defined over $\mathbb{F}_q$.

Fix some $r \geq 1$. Let $\mathbb{A}[r] = \prod_{f \in \widetilde{\Phi}_r^+} \mathbb{A}_f$ (with $\mathbb{A}_f$ as in §5.1). As in (5.1) we have the isomorphism of varieties

$$u \colon \mathbb{A}[r] \xrightarrow{\sim} \mathbb{G}_r^+, \quad (x_f)_f \longmapsto \prod_f u_f(x_f),$$

where the product is taken with respect to the fixed order on $\widetilde{\Phi}^+$. For a subset $E \subseteq \widetilde{\Phi}_r^+$, set $\mathbb{A}_E = \prod_{f \in E} \mathbb{A}_f$, let $p_E \colon \mathbb{A}[r] \to \mathbb{A}_E$ denote the natural projection, and let $\mathrm{pr}_E \colon \mathbb{G}_r^+ \to u(\mathbb{A}_E)$ denote the map obtained from $p_E$ by transport of structure via $u$. For $f \in \widetilde{\Phi}_r^+$, write $p_f = p_{\{f\}}$ and $\mathrm{pr}_f = \mathrm{pr}_{\{f\}}$. When the context is clear, we sometimes will abuse the notation and identify $\mathrm{pr}_f \colon \mathbb{G}_r^+ \to u(\mathbb{A}_f)$ with $u^{-1} \circ \mathrm{pr}_f \colon \mathbb{G}_r^+ \to \mathbb{A}_f$.

Let $f \in -\widetilde{\Delta}^+$. Then there exists a unique sequence

$$0 = a_0 < a_1 < \cdots < a_{2b(f)} = \#O_f \tag{7.1}$$

of integers, such that $F^{a_{2i}}(f) \in \widetilde{\Pi}^+$, $F^{a_{2i-1}}(f) \in \widetilde{\Delta}^+$ for all $0 \leq i \leq b(f)$, and $F^k(f) \notin \widetilde{\Delta}^+ \cup -\widetilde{\Delta}^+$ if $k \not\equiv a_j \mod \#O_f$ for any $j$.

7.2. **A cartesian diagram.** Fix some $r \in \mathbb{Z}_{\geq 1}$. Let $\widetilde{\Phi}^r \subseteq B \subseteq A \subseteq \widetilde{\Phi}^+$ be two closed subsets with $A + B \subseteq B$, $\mathbb{Z}_{\geq 0} + A \subseteq A$, $\mathbb{Z}_{\geq 0} + B \subseteq B$, so that $\mathbb{G}_r^B \subseteq \mathbb{G}_r^A$ are subgroups, and the smaller one is normal in the bigger one. Put

$$X_B^A = \{g \in \mathbb{G}_r^A \colon g^{-1}F(g) \in (\overline{\mathbb{U}}_r^+ \cap F\mathbb{U}_r^+) \cdot \mathbb{G}_r^B\}/\mathbb{G}_r^B.$$

If $A = \widetilde{\Phi}^+$, $B = \widetilde{\Phi}^f$ for some $f \in \widetilde{\Phi}^+$, then we write $X_f^+ = X_f^A$. For any character $\chi\colon (\mathbb{T}_r^+ \cap \mathbb{G}_r^A)^F \to \overline{\mathbb{Q}}_\ell^\times$, we have the $\chi$-weight space $H_c^i(X_B^A, \overline{\mathbb{Q}}_\ell)[\chi]$. Just as in [IN24, §5.2] we have the map

$$\pi_f^{A:B} = u^{-1} \circ \mathrm{pr}_f \circ L \circ s_{A:B} \colon \mathbb{G}_r^A/\mathbb{G}_r^B \longrightarrow \mathbb{A}_{\mathcal{O}_f}.$$

Our first observation is that [IN24, Proposition 5.3] admits the following generalization.

**Proposition 7.1.** *Let $\widetilde{\Phi}^r \subseteq C \subseteq B \subseteq A \subseteq \widetilde{\Phi}^+$ be closed subsets with $A + B \subseteq B$, $A + C \subseteq C$. Let $f \in B$ and suppose that $C = B \smallsetminus \mathcal{O}_f$ and $A + \mathcal{O}_f \subseteq C$. Let $q_f \colon X_C^A \to X_B^A$ denote the natural projection. Then the following hold.*

*(1) Suppose that $f \in \Delta_{\mathrm{aff}}^+$. Then the map*

$$\psi = (q_f, \mathrm{pr}_f, \mathrm{pr}_{F^{a_2}(f)}, \dots, \mathrm{pr}_{F^{a_{2b(f)-2}}(f)}) : X_C^A \cong X_B^A \times \prod_{i=0}^{b(f)-1} \mathbb{A}_f$$

*is an isomorphism.*

*(2) If $f \in \mathbb{Z}_{\geqslant 1}$ (in which case $\mathbb{A}_{\mathcal{O}_f} = \mathbb{A}_f = V$), then there is a Cartesian diagram*

$$\begin{array}{ccc} X_C^A & \xrightarrow{\mathrm{pr}_f} & \mathbb{A}_f \\ {\scriptstyle q_f}\downarrow & & \downarrow{\scriptstyle -L} \\ X_B^A & \xrightarrow{\pi_f^{A:B}} & \mathbb{A}_f. \end{array}$$

*Proof.* The proof is the same as in [IN24, Proposition 5.3], with the only difference that in (1) the map inverse to $\psi$ is given by

$$\begin{aligned}\phi(g, y_f, &y_{F^{a_2}(f)}(y_f), \dots, y_{F^{a_{2b(f)-2}(f)}(f)}) \\ &= s_{A:B}(g) \prod_{j=0}^{a_2-1} F^j(u(y_f)) \cdot \prod_{j=a_2}^{a_4-1} F^j(u(y_{F^{a_2}(f)})) \cdot \dots \cdot \prod_{j=a_{2b(f)-2}}^{a_{2b(f)}-1} F^j(u(y_{F^{a_{2b(f)-2}}(f)})),\end{aligned}$$

and instead of [IN24, Lemma 5.4] we use Lemma 7.2. □

**Lemma 7.2.** *Let $f \in -\Delta_{\mathrm{aff}}^+$ and let $x = (x_i)_{0 \leq i < \#\mathcal{O}_f} \in \mathbb{A}_{\mathcal{O}_f}$ with $x_i \in \mathbb{A}_{F^i(f)}$. Suppose that $L(x) \in \prod_{i=0}^{b(f)-1} \mathbb{A}_{F^{a_{2i}}(f)}$. Then for each $0 \leq j < \#\mathcal{O}_f$, $x_{F^j(f)} = F^{j-a_{2i}}(x_{F^{a_{2i}}(f)})$ for $a_{2i} \leq j < a_{2i+2}$. In particular,* *(1)* $L(x)_f = x_{a_{2b(f)-2}}^{q^{a_{2b(f)}-a_{2b(f)-2}}} - x_0$, $L(x)_{a_{2i}} = x_{a_{2i-2}}^{q^{a_{2i}-a_{2i-2}}} - x_{2i}$ *for* $0 < i < b(f)$*, and* $L(x)_j = 0$ *if* $j \neq a_{2i}$ *for any* $i$*;*

*(2)* $x = 0$ *if and only if* $x_{a_{2i}} = 0$ *for all* $0 \leq i < b(f) - 1$*.*

*Proof.* The proof is a direct computation. □

For a character $\chi$ of $(\mathbb{T}_{f+}^{+})^F$ we denote by $\chi_{f+}^{f}$ the restriction of $\chi$ to $(\mathbb{T}_{f+}^{f})^F$. As in [IN24, Corollary 5.9], the previous proposition implies the following.

**Corollary 7.3.** *Let* $f \in \widetilde{\Phi}^+$ *and let* $\chi$ *be a character of* $(\mathbb{T}_{f+}^{+})^F$*.*

*(1) If* $f \in \Phi_{\mathrm{aff}}^{+}$*, then* $H_c^i(X_{f+}^{+}, \overline{\mathbb{Q}}_\ell)[\chi] \cong H_c^{i-2}(X_f^+, \overline{\mathbb{Q}}_\ell)[\chi]$*.*

*(2) If* $f \in \mathbb{Z}_{\geqslant 1}$*, then* $H_c^i(X_{f+}^{+}, \overline{\mathbb{Q}}_\ell)[\chi_{f+}^{f}] \cong H_c^i(X_f^+, \pi^*(\mathcal{L}_{\chi_{f+}^{f}}))$*, and hence*

$$H_c^i(X_{f+}^{+}, \overline{\mathbb{Q}}_\ell)[\chi] \cong H_c^i(X_f^+, \pi^*(\mathcal{L}_{\chi_{f+}^{f}}))[\chi].$$

*Here* $\pi = \pi_f^{\widetilde{\Phi}^+:\widetilde{\Phi}^f}$ *and* $H_c^i(X_{f+}^{+}, \overline{\mathbb{Q}}_\ell)[\chi_{f+}^{f}]$ *is the* $\chi_{f+}^{f}$*-weight space of* $(\mathbb{T}_{f+}^{f})^F$*.*

Write $H_i(X^+, \overline{\mathbb{Q}}_\ell) = H^{-i} f_\natural \overline{\mathbb{Q}}_\ell$, where $f\colon X^+ \to \operatorname{Spec} \overline{\mathbb{F}}_q$ is the structure map. As in [IN24, Corollary 5.10, §2.7], Corollary 7.3 implies that $H_i(X^+, \overline{\mathbb{Q}}_\ell)[\chi] = H_c^{2d_r - i}(X_r^+, \overline{\mathbb{Q}}_\ell)[\chi]$ for all $r \geq$ the depth of $\chi$, where $d_r$ is the dimension of $X_r^+$. In this way, Theorem 1.5(1),(2) reduce to the following.

**Theorem 7.4.** *Assume* $p$ *is not a torsion prime for* $G$*. Let* $f \in \widetilde{\Phi}^+$ *and let* $\chi\colon (\mathbb{T}_f^+)^F \to \overline{\mathbb{Q}}_\ell^\times$ *be a character. Then there exists some* $s = s_{f,\chi} \in \mathbb{Z}_{\geq 0}$ *such that*

$$H_c^i(X_f^+, \overline{\mathbb{Q}}_\ell)[\chi] \neq 0 \quad \Leftrightarrow \quad i = s.$$

*Moreover,* $F^N$ *acts on* $H_c^s(X_f^+, \overline{\mathbb{Q}}_\ell)[\chi]$ *by the scalar* $(-1)^s q^{sN/2}$*.*

7.3. **Reduction to semisimple simply connected case.** Let $\widetilde{G} \to G$ be the simply connected cover of $G$. Identify the reduced buildings of $G$ and $\widetilde{G}$ and write $\widetilde{G}, \widetilde{T}, \widetilde{U}, \widetilde{\mathbb{T}}_r, \widetilde{\mathbb{U}}_r, \widetilde{X}_f^+, \ldots$ for the objects corresponding to $\widetilde{G}$.

Following [DL76, 1.24], if $\alpha\colon A \to B$ is a homomorphism of finite groups and $Y$ is a space (scheme or fpqc-sheaf) on which $A$ acts, we let the *induced space* $\operatorname{Ind}_A^B Y$ be the (unique up to a unique isomorphism) $B$-space $I$ equipped with an $A$-equivariant map $Y \to I$ such that $\operatorname{Hom}_B(I, V) = \operatorname{Hom}_A(Y, V)$ for any $B$-space $V$. (Minor variation of) the following statement already appears in [DI24, proof of Lemma 4.3.3] without proof. We give the proof here.

**Lemma 7.5.** *We have* $X_f^+ = \operatorname{Ind}_{(\widetilde{\mathbb{T}}_f^+)^F}^{(\mathbb{T}_f^+)^F} \widetilde{X}_f^+$*.*

*Proof.* The kernel of $\widetilde{G} \to G$ is contained in the center of $\widetilde{G}$. Thus the maps $\widetilde{\mathbb{T}}_f^+ \to \mathbb{T}_f^+$ and $\widetilde{\mathbb{G}}_f^+ \to \mathbb{G}_f^+$ are injective, and we identify $\widetilde{\mathbb{T}}_f^+$, $\widetilde{\mathbb{G}}_f^+$ with their images. Also, $\mathbb{G}_f^+/\widetilde{\mathbb{G}}_f^+ \cong \mathbb{T}_f^+/\widetilde{\mathbb{T}}_f^+$; we denote this group by $C$.

Note that any $g \in X_f^+$ can be written as $g = \tau_1 g_1$ with $g_1 \in \widetilde{\mathbb{G}}_f^+$ and $\tau_1 \in \mathbb{T}_f^+$. Then $g_1^{-1}\tau_1^{-1}F(\tau_1)F(g_1) = g^{-1}F(g) \in \overline{\mathbb{U}}_f^+ \cap F\mathbb{U}_f^+ \subseteq \widetilde{\mathbb{G}}^+$. As $\widetilde{\mathbb{G}}_f^+$ is normal in $\mathbb{G}_f^+$, it follows that $\tau_1 = F(\tau_1) \in C$, i.e., $\tau_1 \in C^F$. But as $\widetilde{\mathbb{T}}_f^+$ is connected, we have $(\mathbb{T}^+)^F/(\widetilde{\mathbb{T}}^+)^F = C^F$. Thus, changing $g_1$ and $\tau_1$ if necessary, we may achieve that $\tau_1 \in (\mathbb{T}_f^+)^F$. But then it is clear that $g_1^{-1}F(g_1) = g^{-1}F(g) \in \overline{\mathbb{U}}_f^+ \cap F\mathbb{U}_f^+$, which implies that $g_1 \in \widetilde{X}_f^+$. Thus $X_f^+ \cong \coprod_{\tau \in (\mathbb{T}_f^+)^F/(\widetilde{\mathbb{T}}_f^+)^F} \tau \widetilde{X}_f^+$, which is precisely the induced space. □

**Remark 7.6.** The analogous statement holds for $\mathbb{G}, \mathbb{T}, X = \{g \in \mathbb{G} \colon g^{-1}F(g) \in \overline{\mathbb{U}} \cap F\mathbb{U}\}$ instead of $\mathbb{G}^+, \mathbb{T}^+, X^+$. There, $\widetilde{\mathbb{G}} \to \mathbb{G}$ can be non-injective, and its kernel equals the *perfection* of $\ker(\widetilde{G} \to G)$. The situation with cokernels is similar as in the above proof.

**Example 7.7.** If $k = \mathbb{F}_2((\varpi))$, $G = \mathrm{PGL}_2$, $\mathbf{x}$ hyperspecial, then $\widetilde{G} = \mathrm{SL}_2$, the maps $\widetilde{\mathbb{G}} \to \mathbb{G}$, $\widetilde{\mathbb{T}} \to \mathbb{T}$ are injective with cokernels isomorphic to $C = H^1((\mathrm{Spec}\,\overline{\mathbb{F}}_2[\![\varpi]\!])_{\mathrm{fppf}}, \mu_2) = \mathrm{coker}(\overline{\mathbb{F}}_2[\![\varpi]\!]^\times \stackrel{(\cdot)^2}{\to} \overline{\mathbb{F}}_2[\![\varpi]\!]^\times)$, and $C^F$ is an infinite-dimensional $\mathbb{F}_2$-vector space.

It follows that if $\chi \colon (\mathbb{T}_f^+)^F \to \overline{\mathbb{Q}}_\ell^\times$ is a character and $\widetilde{\chi}$ is it's pullback to a character of $(\widetilde{\mathbb{T}}_f^+)^F$, then $H_c^i(X_f^+, \overline{\mathbb{Q}}_\ell)[\chi] = \bigoplus_{(\mathbb{T}_f^+)^F/(\widetilde{\mathbb{T}}_f^+)^F} H_c^i(\widetilde{X}_f^+, \overline{\mathbb{Q}}_\ell)[\widetilde{\chi}]$ as vector spaces with Frobenius action. In particular, if Theorem 7.4 holds for $\widetilde{X}_f^+$, then it holds for $X_f^+$.

7.4. **Handling jumps.** As in [IN24, §5.6], fix a positive integer $h \leqslant r$ and a character $\chi$ of $(\mathbb{T}_{h+}^+)^F$. Recall that $\mathbb{T}_{h+}^h \cong \mathbb{A}_h = V = X_*(T) \otimes \overline{\mathbb{F}}_q$, so that for any $M \geq 1$ we have the map

$$\mathrm{Nm}_M : V \longrightarrow V,\ v \longmapsto v + F(v) + \cdots + F^{M-1}(v).$$

Note that for $\alpha \in \Phi$, the subgroup $\mathrm{Nm}_M(\alpha^\vee \otimes \mathbb{F}_{q^M})$ is independent of the choice of the integer $M \in \mathbb{Z}_{\geq 1}$ satisfying $F^M(\alpha) = \alpha$. Using the character $\chi$ we define the $F$-stable subset

$$\Phi_\chi = \{\alpha \in \Phi; \chi \circ \mathrm{Nm}_N(\alpha^\vee \otimes \mathbb{F}_{q^N}) = \{1\}\}.$$

of $\Phi$. Clearly, $-\Phi_\chi = \Phi_\chi$. However, $\Phi_\chi$ does not need to be closed under addition. This fact as well as (essentially) the proof of the following lemma was explained to us by David Schwein.

**Lemma 7.8.** *If $p > 3$ or if $p$ is not a torsion prime for $G$, then $\Phi_\chi$ is closed under addition.*

*Proof.* For $m \in \mathbb{Z}$ and $\alpha \in \Phi$, put

$$\chi_{m\alpha} = \chi \circ \mathrm{Nm}_N \circ (\alpha^\vee \otimes \mathbb{F}_{q^N}) \circ (m\cdot) \colon \mathbb{F}_{q^N} \xrightarrow{m} \mathbb{F}_{q^N} \longrightarrow V^{F^N} \longrightarrow V^F \longrightarrow \overline{\mathbb{Q}}_\ell^\times,$$

where the first map is multiplication by $m$. For $\alpha \in \Phi$, we have $\alpha^\vee = \frac{2}{(\alpha,\alpha)}$, where $(\alpha, \alpha) = |\alpha|^2$ is the square of the length of $\alpha$. For $\alpha, \beta \in \Phi$, write

$n_{\alpha,\beta} = \frac{(\alpha,\alpha)}{(\beta,\beta)}$. By [Bou68, Chap.VI, §4, Proposition 12(i)], the numbers $n_{\alpha,\beta}$ can only take the values $1, 2^{\pm 1}, 3^{\pm 1}$ and by inspection one checks that the values $2^{\pm 1}$ resp. $3^{\pm 1}$ can only appear if $p$ is a torsion prime for $G$. Suppose now that $\alpha, \beta \in \Phi_\chi$, that is $\chi_\alpha, \chi_\alpha$ are trivial. Suppose that $\gamma = \alpha + \beta \in \Phi$. Thus $\gamma^\vee = \frac{2}{(\gamma,\gamma)}\gamma = n_{\alpha,\gamma}\alpha^\vee + n_{\beta,\gamma}\beta^\vee$. Let $m \in \mathbb{Z}_{\geq 1}$ be the smallest positive integer such that $mn_{\alpha,\gamma}$ and $mn_{\beta,\gamma}$ lie in $\mathbb{Z}$. Then we have $m\gamma^\vee = mn_{\alpha,\gamma}\alpha^\vee + mn_{\beta,\gamma}\beta^\vee$. Note that we have $\chi_{m\gamma} = \chi_{mn_{\alpha,\gamma}\alpha} \cdot \chi_{mn_{\beta,\gamma}\beta}$ as characters of $\mathbb{F}_{q^N}$. As $\chi_\alpha, \chi_\beta$ are trivial, also $\chi_{mn_{\alpha,\gamma}\alpha}$ and $\chi_{mn_{\beta,\gamma}\beta}$ are trivial. Thus also $\chi_{m\gamma}$ is trivial and as $m$ invertible in $\mathbb{F}_q$ by assumption, it follows that $\chi_\gamma = 1$, that is $\gamma \in \Phi_\chi$. □

Let $M = M_\chi$ be the subgroup generated by $T$ and $U_\alpha$ for $\alpha \in \Phi_\chi$. By Lemma 7.8, $M$ is reductive with root system $\Phi_M = \Phi_\chi$. Let $\widetilde{\Phi}_M \subseteq \widetilde{\Phi}$ be the set of affine roots of $M$. Consider

$$D = (\Phi^+_{\text{aff}} \cap \Phi^+_h) \setminus \widetilde{\Phi}_M = \{f \in \Phi^+_{\text{aff}} \setminus \widetilde{\Phi}_M; f < h\}.$$

As $\widetilde{\Phi}_M$ is $F$-stable, $D$ is a union of $F$-orbits in $\widetilde{\Phi}$. Similar as in [IN24, §5.6], we can number the $F$-orbits of $D$ as

$$\mathcal{O}_1, \dots, \mathcal{O}_{m-1}, \mathcal{O}_m = \mathcal{O}^\flat_m, \dots, \mathcal{O}_n = \mathcal{O}^\flat_n, \mathcal{O}^\flat_{m-1}, \dots, \mathcal{O}^\flat_1$$

where $\mathcal{O}^\flat = \{h - f \colon f \in \mathcal{O}\}$, and such that

$$\mathcal{O}_1(\mathbf{x}) \leqslant \cdots \leqslant \mathcal{O}_{m-1}(\mathbf{x}) \leqslant \frac{h}{2} = \mathcal{O}_m(\mathbf{x}) = \cdots = \mathcal{O}_n(\mathbf{x}) = \frac{h}{2} \leqslant \mathcal{O}^\flat_{m-1}(\mathbf{x}) \leqslant \cdots \leqslant \mathcal{O}^\flat_1(\mathbf{x}),$$

$\mathcal{O}_i < \mathcal{O}^\flat_i$ for $1 \leqslant i \leqslant m-1$ and $\mathcal{O}^\flat_{m-1} < \cdots < \mathcal{O}^\flat_1$. Define $N_i = \#\mathcal{O}_i$.

Set $D^\flat_i = \bigcup_{j=1}^i \mathcal{O}^\flat_j$ for $1 \leqslant i \leqslant m-1$, and $D^\flat_m = \bigcup_{j=1}^n \mathcal{O}^\flat_j$. Define

$$A_i = \widetilde{\Phi}^+ \setminus \bigcup_{j=1}^{i-1} \mathcal{O}_j, \quad B_i = \widetilde{\Phi}^h \cup D^\flat_i, \quad C_{i-1} = B_{i-1} \setminus \{h\}.$$

Moreover, we set $A_0 = A_1 = \widetilde{\Phi}^+$, $B_0 = \widetilde{\Phi}^h$ and $C_0 = B_0 \setminus \{h\}$. Note that $A_m = B_m \cup \widetilde{\Phi}^+_M$, where $\widetilde{\Phi}^+_M = \widetilde{\Phi}_M \cap \widetilde{\Phi}^+$.

Let $g \in \mathbb{G}^+_r$, $x \in \mathbb{A}[r]$ and $E \subseteq \widetilde{\Phi}^+_r$. As in [IN24, §5.6] we set $g_E = \mathrm{pr}_E(g) \in u(\mathbb{A}_E)$, $x_E = p_E(x) \in \mathbb{A}_E$ and $\hat{x} = u(x) \in \mathbb{G}^+_r$. For $f \in \widetilde{\Phi}^+_r$ we will set $x_f = x_{\{f\}}$ and $x_{\geqslant f} = x_{\widetilde{\Phi}^f}$. We can define $g_f$ and $g_{\geqslant f} \in \mathbb{G}^+_r$ in a similar way. We identify $g_f \in u(\mathbb{A}_f)$ with $u^{-1}(g_f) \in \mathbb{A}_f$ according to the context.

Note that [IN24, Lemmas 5.12, 5.13 and 5.14] hold in our more general setup without any change and with literally the same proofs. (Note that the proof of [IN24, Lemmas 5.12] uses the following property of $\Phi_M \subseteq \Phi$: if $\alpha \in \Phi_M$, $\beta \in \Phi \smallsetminus \Phi_M$, then $\alpha + \beta \notin \Phi_M$. This holds when $\Phi_M \subseteq \Phi_G$ is a Levi subsystem, which follows from $p$ not being a torsion prime for $G$ by [Kal19, Lemma 3.6.1]. This is guaranteed by the assumptions in Theorem 7.4.)

We now generalize [IN24, Proposition 5.15]. Set $\pi = \pi_h^{\widetilde{\Phi}^+ : \widetilde{\Phi}^h} : \mathbb{G}^+_h = \mathbb{G}^+_r / \mathbb{G}^h_r \to \mathbb{A}_h \cong V$.

**Proposition 7.9.** *Let $1 \leqslant i \leqslant m$. Then there is an isomorphism*
$$\psi_i : X_h^{A_i} \cong X_{B_i}^{A_i} \times \mathbb{A}_{D_i^\flat \cap -\widetilde{\Delta}^+}.$$
*Moreover, for $(\hat{x}, y) \in X_{B_i}^{A_i} \times \mathbb{A}_{D_i^\flat \cap -\widetilde{\Delta}^+}$ with $x \in \mathbb{A}_{A_i \setminus B_i}$ we have*

*(1) Assume that $1 \leqslant i \leqslant m-1$, fix some $f \in \mathcal{O}_i \cap -\widetilde{\Delta}^+$ and let $f^\flat = h - f \in \mathcal{O}_i^\flat$. With notation of (7.1), we have*
$$\pi(\psi_i^{-1}(\hat{x}, y)) = \sum_{j=0}^{b(f)-1} \alpha_{F^{a_{2j}}(f)}^\vee \otimes (x_{F^{a_{2j+1}}(f)}^{q^{a_{2j+1}-a_{2j-1}}} - x_{F^{a_{2j-1}}(f)}) y_{F^{2j}(f^\flat)}^{q^{a_{2j}-a_{2j-1}}} + \pi(\psi_i^{-1}(\hat{x}, 0)) \in V.$$

*(2) Assume that $i = m$. For each $m \le k \le n$ fix some $f_k \in \mathcal{O}_k \cap -\widetilde{\Delta}^+$. Then $\pi(\psi_m^{-1}(\hat{x}, y))$ equals $\pi(\psi_m^{-1}(\hat{x}, 0))$ plus the sum over $m \le k \le n$ of the following term corresponding to $f = f_k$ (where $a_j = a_j(f_k)$ and $b = b(f_k)$ are as in (7.1)):*
$$-\alpha_f^\vee \otimes y_f \cdot y_{F^{a_{b-1}}(f)}^{q^{a_b - a_{b-1}}} + \sum_{j=1}^{(b-1)/2} \alpha_{F^{a_{2j}}(f)}^\vee \otimes (y_{F^{a_{2j}}(f)} - y_{F^{a_{2j-2}}(f)}^{q^{a_{2j}-a_{2j-2}}}) y_{F^{a_{2j-1+b}}(f)}^{q^{a_{2j+b}-a_{2j-1+b}}}.$$

Note that the formulas in Proposition 7.9 are similar to those in [Nie24, proof of Lemma 6.10]. We deduce the generalization of [IN24, Proposition 5.17] from this.

**Proposition 7.10.** *Write $X_h^M = X_h \cap \mathbb{M}_h^+$, and let $\pi_M$ be the restriction of $\pi$ to $\mathbb{M}_h^+$. The following statements hold:*

*(1) $H_c^j(X_h^{A_i}, \pi^*\mathcal{L}_{\chi_{h+}^h}) \cong H_c^j(X_h^{A_{i+1}}, \pi^*\mathcal{L}_{\chi_{h+}^h})^{\oplus q^{M_i}}$ for $1 \leqslant i \leqslant m-1$ and some $M_i \in \mathbb{Z}_{\geq 1}$;*

*(2) $H_c^j(X_h^{A_m}, \pi^*\mathcal{L}_{\chi_{h+}^h}) \cong H_c^{j-n-m+1}(X_h^M, \pi_M^*\mathcal{L}_{\chi_{h+}^h})^{\oplus q^{C_1}}((-1)^{C_2} q^{C_3})$, where $C_1 = \sum_{i=m}^n q^{\#O_i/2}$ and $C_2 = \sum_{i=m}^n N/\#O_i$ and $C_3 = q^{N(n+m-1)/2}$.*

Note that in general $C_2 \neq C_3$, in contrast to the special case of *loc. cit.*

*Proof.* We can proceed exactly as in [IN24, proof of Proposition 5.17], by noting that by Proposition 7.9 the local system $\pi^*\mathcal{L}_{\chi_{h+}^h}$ is trivial on a fiber over $\hat{x} \in X_{B_i}^{A_i}$ if and only if $x_{F^{a_{2j+1}}(f)}^{q^{a_{2j+1}-a_{2j-1}}} - x_{F^{a_{2j-1}}(f)} = 0$ for each $j$, which shows that for $0 \le i \le m-1$,
$$H_c^a(X_h^{A_i}, \pi^*\mathcal{L}) \cong H_c^a(Y_h^{A_{i+1}}, \pi^*\mathcal{L})^{\oplus q^{M_i}}$$
for some $M_i \in \mathbb{Z}_{\geq 1}$, and similarly for part (2). In course of proving (2), when repeating the computation from *loc. cit.*, we obtain

(7.2)
$$H_c^j(X_h^{A_m}, \pi^*\mathcal{L}) \cong \otimes_{i=m}^n H_c^1(\mathbb{A}_{f_i}, \tau_i^*\mathcal{L}) \otimes \otimes_{i=1}^{m-1} H_c^2(\mathbb{A}_{f_i^\flat}, \overline{\mathbb{Q}}_\ell) \otimes H_c^{j-n-m+1}(X_h^M, \pi^*\mathcal{L}).$$

Whereas for $i \le m-1$, $F^N$ acts on each $H_c^2(\mathbb{A}_{f_i^\flat}, \overline{\mathbb{Q}}_\ell)$ by $q^N$ as in *loc. cit.*, we have that $F^{\#O_i}$ acts on $H_c^1(\mathbb{A}_{f_i}, \tau_i^*\mathcal{L})$ by $-q^{\#O_i/2}$ (by [IN24, Proposition

5.16(2)]). Thus $F^N$ acts on this space by $(-q^{\#O_i/2})^{N/\#O_i} = (-1)^{N/\#O_i}q^{N/2}$. Altogether, we see that (7.2) equals

$$H_c^{j-n-m+1}(X_h^M, \pi^*\mathcal{L})^{\oplus \sum_{i=m}^n q^{\#O_i/2}}((-1)^{\sum_{i=m}^n N/\#O_i} q^{N(n+m-1)/2}),$$

where the number of summands again follows from [IN24, Proposition 5.16(2)]. □

Now, to prove Theorem 7.4 we may assume by §7.3 that $G$ is semisimple and simply connected. Then [IN24, Lemma 2.2] guarantees that $M_\chi \neq G$. Then exactly the same induction procedure as in [IN24, §5.7] finishes the proof.

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