Notes for my talk in the Bochum study group on [ABLRS], Section 4.1, 4.4-4.8. We start with a toy example that will inform the construction of the analytic de Rham stack later on. Namely, for algebraic de Rham stacks in positive characteristic the fpqc-sheafification turns out to agree with the effect of restricting a scheme to perfect test rings and then applying the right adjoint of the restriction of fpqc-sheaves on rings in positive characteristic to perfect rings. When working over $\mathbb{Q}_p$ and in non-archimedean analytic geometry, we replace perfect rings by perfectoid Tate rings. On perfectoid Tate rings there is a topology given by declaring covers to be those maps that are surjective on Berkovich spectra. It is called the *arc-topology*. Much of the technical work in the sections I had to cover was concerned with the interplay of the !-topology, used in the definition of Gelfand stacks, and with the arc-topology.

The *big analytic de Rham* stack will be defined as the right adjoint of restricting a Gelfand stack to perfectoid Tate rings. This construction does not have all properties we want and we have to impose a certain finiteness condition called *quasi-finite dimensional* on the test objects. This will then give the desired analytic de Rham stack. The main results concerning this construction are that it sends epimorphisms of arc-stacks to epimorphisms of Gelfand stacks and that for †-formally smooth Gelfand stacks X, the canonical map $X \to X^{\mathrm{dR}}$ is an epi.

An important technical notion is that of a *nilperfectoid* Gelfand ring.

1. Toy model

Let k be a field of characteristic $p > 0$. Let X be a k-scheme. Consider the functor $X^{\mathrm{dR},pre}$ given by $R \mapsto X(R_{\mathrm{red}})$ on k-algebras. This is an étale sheaf but not an fppf-sheaf (consider the covering $k(\!(t)\!) \longrightarrow k(\!(t^{1/p})\!)$). Let us denote by $X^{\mathrm{dR},f}$ the fpqc-sheafification of $X^{\mathrm{dR},pre}$. Observe that any k-algebra R has an fpqc-cover $R \to R'$ that is semiperfect (i.e. absolute Frobenius is surjective; take a surjection from a huge polynomial k-algebra onto R, then take colimit perfection of the polynomial k-algebra and then tensor product). For semiperfect rings, reduction is the same as colimit perfection. If $R \to R'$ is fpqc, then also $R_{\mathrm{perf}} \to R'_{\mathrm{perf}}$ is fpqc. It follows that

$$X^{\mathrm{dR},f}(R) = X(R_{\mathrm{perf}}).$$

Therefore, $X^{\mathrm{dR},f}$ can be interpreted as the result of applying to the restriction of X to perfect k-algebras the *right adjoint* of restricting an fpqc-sheaf on all k-algebras to all perfect k-algebras. This construction has the following properties

- If X' is a *locally finite type* k-scheme, then it sends fpqc-epimorphisms $X \to X'$ to fpqc-sheaf epimorphisms $X^{\mathrm{dR},f} \to X'^{,\mathrm{dR},f}$.
- If X is smooth over k, then $X^{\mathrm{dR},f}$ is the quotient of X by the equivalence relation given by the completion of the diagonal $\Delta\colon X \to X \times_k X$,
- If X is proper over k algebraically closed, one can check that

$$H^i(X^{\mathrm{dR},f}, \mathcal{O}) = H^i_{\text{ét}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k.$$

Now let us explain the proofs of these statements.

Proof. For the first, note that $X' \to \mathrm{Spec}(k)$ being locally of finite type implies that its functor of points commutes with filtered colimits of rings. This implies that $X'^{,\mathrm{dR},f} \simeq \mathrm{colim}_n X'^{,(p^n)}$, where the transition maps are the relative Frobenius of X' over k. Now let $x \in X'^{,\mathrm{dR},f}(R) = X'(R_{\mathrm{perf}})$. Then x is represented by a point $x_n \in X'^{,(p^n)}(R)$ for some n. If $X \to X'$ is an epi, also $X^{(p^n)} \to X'^{,(p^n)}$ is an epi, and we may therefore lift x_n, after passing to an fpqc cover $R \to R'$, to a point in $X^{(p^n)}(R')$. Its image under the natural map $X^{(p^n)}(R') \to X(R'_{\mathrm{perf}})$ is a point lifting x, as desired.

The second bullet means more precisely that $X^{\mathrm{dR},f}$ is the sheafification of the presheaf sending R to the quotient of $X(R)$ by the equivalence relation given by $x \equiv y$ iff $x = y \in X(R_{\mathrm{red}})$. In fact, this quotient is $X(R_{\mathrm{red}}) = X^{\mathrm{dR},pre}(R)$ even before sheafification: since $X \to \mathrm{Spec}(k)$ is locally of finite presentation and formally smooth, the map $X(R) \to X(R_{\mathrm{red}})$ is surjective.

For the last point, we write again $X^{\mathrm{dR},f} = \mathrm{colim}_n X^{(p^n)}$. Then

$$R\Gamma(X^{\mathrm{dR},f}, \mathcal{O}) = R\lim_n R\Gamma(X^{(p^n)}, \mathcal{O}).$$

Since the cohomology groups of the structure sheaf of all $X^{(p^n)}$ are finite dimensional k-vector spaces, the $R^1 \lim_n$ terms vanish and we have the formula

$$H^i(X^{\mathrm{dR},f}, \mathcal{O}) = \lim_n H^i(X^{(p^n)}, \mathcal{O}) = \lim_F H^i(X, \mathcal{O}).$$

Now $\lim_F H^i(X,\mathcal{O})$ agrees with $H^i(X,\mathcal{O})^{F-\mathrm{iso}}$, the largest subspace of this finite dimensional k-vector space on which the absolute Frobenius F is bijective. Since k is algebraically closed, such a vector space always has an F-invariant basis, so that $H^i(X,\mathcal{O})^{F-iso} = H^i(X,\mathcal{O})^{F=1}\otimes_{\mathbb{F}_p} k$ in a Frobenius-equivariant way. Finally, it follows from the Artin–Schreier exact sequence and the fact that $F-1$ is surjective on all $H^j(X,\mathcal{O})$, since k is algebraically closed, that $H^i(X,\mathcal{O})^{F=1} \simeq H^i_{\text{ét}}(X,\mathbb{F}_p)$, as desired. □

The last bullet point tells us that this construction does not see the de Rham cohomology of X but rather its étale $\mathbb{F}_p$-cohomology. However, this construction will guide the construction in the analytic context.

Let me point out that the actual construction closer to the spirit of the paper would be to take the sheafification of $X^{\mathrm{dR},pre}$ for the *descendable* topology. I wanted to stick to the better known topology for convenience. For finite type test objects, where the descendable topology agrees with the h-topology, the statement about epimorphisms still holds; in fact, becomes easier to prove.

2. Arc-stacks

It is a well-established heuristic that one should replace perfect rings in positive characteristic by perfectoid rings in mixed characteristic. Therefore, we would want to seek to construct the *analytic de Rham* stack by a putative *right adjoint* of restricting a sheaf on a category of analytic rings to perfectoid rings.

Recall that a *Tate* ring is a topological ring R that contains an open and bounded subring $R_0 \subset R$ and a topologically nilpotent unit $\varpi \in R^\times$. Then the induced topology on R_0 is the ϖ-adic one for any pseudo-uniformizer contained in R_0 (this statement has a converse). We call R *uniform* if R° is bounded. We call R *complete* if it is complete in its topology. The category of complete Tate rings R equipped with a continuous map $\mathbb{Q}_p \to R$, with continuous maps, is equivalent to the category of Banach $\mathbb{Q}_p$-algebras (with norms up to equivalence) with bounded morphisms. We call a Tate ring R *perfectoid* if R is complete, uniform and there exists a pseudo-uniformizer $\varpi \in R$ such that $p \in \varpi^p R^\circ$ and the absolute Frobenius induces an isomorphism $R^\circ/\varpi \simeq R^\circ/\varpi^p$. We call R *separable* if R°/ϖ is countable. It is easy to see that this condition is independent of the choice of the pseudo-uniformizer. We call a Gelfand ring A *separable* if $A^{u,\circ}/p$ is countable. We denote by $\mathrm{AffdPerfd}_{\omega_1} \subseteq \mathrm{AffdPerfd}$ the full subcategory of separable perfectoid Tate rings. We denote by $\mathrm{GelfRing}_{\omega_1} \subseteq \mathrm{GelfRing}$ the full subcategory of separable Gelfand rings.

Here let us recall that a *Gelfand ring* was a solid $\mathbb{Q}_{p,\square}$-algebra A that was *bounded* i.e. $A = \bigcup_r A^{\leq r}$, so that the *dagger nil-radical* $\mathrm{Nil}^\dagger(A) = A^{\leq 0}$ is actually an ideal and such that the *uniform completion* $A^u = \widehat{(A^{\leq 1})}_p[1/p]$ is a classical ring that is a Banach $\mathbb{Q}_p$-algebra.

A good example on underlying rings to keep in mind is the following: Let $\mathbb{Q}_p\langle T\rangle^{\leq 0} = \mathrm{colim}_{\rho>0}\,\mathcal{O}(\mathbb{D}_{[0,\rho],\mathbb{Q}_p})$ be the colimit of functions on the closed discs of radius ρ; these are power series $f = \sum_{n\geq 0} a_n T^n$ with $a_n \in \mathbb{Q}_p$ such that there exists a $\rho > 0$ with $\lim_{n\to\infty} |a_n|\rho^n = 0$. The dagger nil-radical is here the ideal generated by T and therefore the dagger-reduction is just $\mathbb{Q}_p$. Moreover, the unit ball is $\mathbb{Z}_p + T\cdot\mathbb{Q}_p\langle T\rangle^{\leq 0}$; since the second summand is p-divisible, the p-adic completion of the unit ball is $\mathbb{Z}_p$, so that the uniform completion is also $\mathbb{Q}_p$.

We call a Gelfand ring A *totally disconnected* if $\mathcal{M}(A) := \mathcal{M}(A^u)$ is profinite. We call A *strictly totally disconnected* if $\mathcal{M}(A)$ is profinite and all residue fields of A^u are algebraically closed.

A map $R \to R'$ in AffdPerfd is an arc cover if $\mathcal{M}(R') \to \mathcal{M}(R)$ is surjective.

Remark 2.1 (Berkovich spectrum and the maximal Hausdorff quotient)**.** Recall that for a Banach $\mathbb{Q}_p$-algebra A with Banach norm $||.||$, we have defined

$$\mathcal{M}(A) = \{||_x \colon ||_x \colon A \to \mathbb{R}_{\geq 0}, \text{bounded, multiplicative semi-norm}\}.$$

Note that then automatically $|p|_x = \frac{1}{p}$. Here, for a general perfectoid Tate ring R, we define $\mathcal{M}(R)$ to be the maximal Hausdorff quotient of $\mathrm{Spa}(R,R^\circ)$. For perfectoid Tate $\mathbb{Q}_p$-algebras, these notions agree.

Lemma 2.2. *Let R be a separable perfectoid Tate ring and let A be a separable Gelfand ring.*

(a): *$\mathcal{M}(R)$ is metrizable,*
(b): *there exists an arc cover $R \to R'$ such that R' is separable and strictly totally disconnected.*
(c): *More generally: There exists a separable perfectoid ring R'' that is strictly totally disconnected and a map $A \to R''$ surjective on Berkovich spectra.*

Proof. By assumption, the ring R°/ϖ is countable. Let $f_j \in R^\circ$ be a countable set of representatives and let g_j, $j \in J$, be the finite sums $\sum_{k=0}^n \varpi^k f_{j_k}$. This is a countable dense subset of R°. This gives a map $\mathcal{M}(R) \to \prod_J [0,1]$, $x \mapsto |g_j(x)|$, where J is countable. Since these functions lie dense, the induced map is injective and it is continuous by the definition of the topology on $\mathcal{M}(R)$. Since $\mathcal{M}(R)$ is compact and the target is Hausdorff, this map is a closed immersion. Since J is countable, $\prod_J [0,1]$ is metrizable and therefore $\mathcal{M}(R)$ as well. This proves (a).

For (b), we first produce an arc-cover $R \to S =: R^w$, where $S \in \mathrm{AffdPerfd}_{\omega_1}$ is totally disconnected. Since $\mathcal{M}(R)$ is metrizable, hence second countable, we can choose a sequence of finite covers of $\mathcal{M}(R)$ by rational subsets, each refining the previous one, which is cofinal among all such covers. For the i-th cover, let R_i be the product of the rings of functions of the rational localizations appearing in that cover. Then consider $R \to S$, where S is the completed colimit of the R_i. Note that $S \in \mathrm{AffdPerfd}_{\omega_1}$. Note that $\mathcal{M}(S) = \lim_i \mathcal{M}(R_i) \to \mathcal{M}(R)$ is surjective and that $\mathcal{M}(S)$ is profinite as a compact totally disconnected Hausdorff topological space.

Next, we explain how to make the residue fields algebraically closed. Note that for $x \in \mathcal{M}(S)$ and any finite étale cover of $\kappa(x)$, we may spread it out to a finite étale cover of a rational neighborhood of x (this uses that the crucial fact in non-archimedean analytic geometry that the local ring $\mathcal{O}_{S,x}$ is henselian with residue field dense in $\kappa(x)$; this does not work in algebraic geometry). Since $\mathcal{M}(S)$ is profinite, x has a clopen neighborhood inside this rational neighborhood, so that we may extend this cover to a finite étale S-algebra. By separability, countably many finite étale S-algebras suffice, and one takes R' to be the completed colimit of a countable filtered system of them (rigidified by a map to a fixed strictly totally disconnected S-algebra given by the usual 'product of points' construction).

Finally, for (c) note that this is just a question about the Banach $\mathbb{Q}_p$-algebra A^u. We will see the argument later when we talk about nilperfectoid rings: there is a map $A \to A'$, surjective on Berkovich spectra, such that $A'^{,u}$ is a separable perfectoid ring, and then we apply (b) to $A'^{,u}$. □

Remark 2.3. Note that in (c) we can only ensure the existence of an arc cover by a strictly totally disconnected perfectoid. On the side of Gelfand rings we rather work with !-covers or descendable covers (same notion for Gelfand rings) and much of the technical hassle in the sections I had to cover comes from the fact that without finiteness hypothesis there is no such !-cover.

Definition 2.4. The category of *arc-stacks* is the category of arc-hypersheaves on $\mathrm{AffdPerfd}_{\omega_1}^{\mathrm{op}}$. We will denote it by ArcStk. For a base such as $\mathbb{Z}_p$, $\mathbb{Q}_p$ or $\overline{\mathbb{F}}_p$, we denote by $\mathrm{ArcStk}_{\mathbb{Z}_p}$, $\mathrm{ArcStk}_{\mathbb{Q}_p}$, $\mathrm{ArcStk}_{\overline{\mathbb{F}}_p}$ the analogous category of arc-hypersheaves on separable perfectoid Tate rings over this base.

Example 2.5. We can consider $\mathrm{Bun}_G, \mathrm{Spd}(\mathbb{Q}_p), \mathrm{Div}^1 ... \in \mathrm{ArcStk}_{\overline{\mathbb{F}}_p}$.

Another example is the following. Let S be a light profinite set. Then we can consider the arc-stack $\underline{S} \in \mathrm{ArcStk}_{\mathbb{Q}_p}$ given by $\underline{S}(R) = \mathrm{Hom}_{cts}(\mathcal{M}(R), S)$. We can take the unique colimit-preserving extension to define a functor from light condensed anima to $\mathrm{ArcStk}_{\mathbb{Q}_p}$. We then have for any light condensed anima T that

$$\underline{T}(R) = \mathrm{Hom}_{CondAni}(\mathcal{M}(R), T).$$

This makes sense because $\mathcal{M}(R)$ is metrizable.

We want to discuss the relation between the arc-topology and the descendable resp. !-topology. Let us recall these notions.

Definition 2.6. Let $f\colon A \to B$ be a map of Gelfand rings. We call this map

(a): !-*cover* if $D(A) \simeq \lim_{[n]\in\Delta} D^!(B^{\otimes_A (n+1)})$,

(b): *descendable* if $\{\mathrm{Tot}_{\leq k}(B^{\otimes_A(\bullet+1)})\}_k$ is pro-isomorphic to A (i.e. we do not require that simply $A \simeq \mathrm{Tot}(B^{\otimes_A(\bullet+1)})$ but for any k, there exists $k' \geq k$ such that the map $I_{k'} \to I_k$ is (homotopic to) zero, where $I_k = \mathrm{cofib}(A \to \mathrm{Tot}_{\leq k}(B^{\otimes_A(\bullet+1)}))$. This is a condition that can be checked in the homotopy category of $D(A)$, i.e. in the derived category of solid A-modules).

These notions agree.

Lemma 2.7. *Let $f\colon A \to B$ be a map of Gelfand rings. Then f is a !-cover if and only if it is descendable.*

Proof. Recall that $D(B)$ is the category of B-modules in $D(A)$, that $f_! = f_*\colon D(B) \to D(A)$ is the forgetful functor, f^* corresponds to the base change $B \otimes_A .$ and $f^! = \underline{\mathrm{Hom}}_A(B, -)$ is the right adjoint of $f_!$. We consider the Check nerve $A \to B^\bullet$ with $B^\bullet$ the co-simplical object given by $B^{\otimes/_A n+1}$ and let $f_n\colon A \to B^{\otimes/_A n+1}$.

Let us first remark that an equivalent definition of descendability for f is that A lies in the thick tensor ideal $\langle B\rangle \subseteq D(A)$ generated by B (this means that A can be obtained from B inside $D(A)$ in finitely many steps by tensoring with arbitrary objects of $D(A)$, taking shifts and cones, and passing to direct summands). Then f^* and $f^!$ are conservative: if $M \otimes_A B = 0$ (resp. $\underline{\mathrm{Hom}}_A(B, M) = 0$), then the class of all N with $M \otimes_A N = 0$ (resp. $\underline{\mathrm{Hom}}_A(N, M) = 0$) is a thick tensor ideal containing B, hence it contains A, so that $M = 0$. A similar argument shows that the relevant (co)simplicial objects have pro-constant (resp. ind-constant) partial totalizations (resp. realizations), so that the Barr–Beck–Lurie theorem applies and gives descent for D^* and for $D^!$, see [ABLRS, Proposition 3.2.10] and the references given there.

Conversely, assume that f is a !-cover, i.e. $D(A) \simeq \lim_{[n]\in\Delta} D^!(B^{\otimes_A(n+1)})$. Passing to left adjoints, this means that $D(A)$ is the colimit of the $D(B^{\otimes/_A n+1})$ along the forgetful functors. For $M, N \in D(A)$ we therefore have

$$\mathrm{Hom}(M, N) = \lim_{[n]\in\Delta} \mathrm{Hom}(f_n^! M, f_n^! N) = \lim_{[n]\in\Delta} \mathrm{Hom}(f_{n,!} f_n^! M, N),$$

so that $M \simeq \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} f_{n,!} f_n^! M$. For $M = A$ this says that

$$A \simeq \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} \underline{\mathrm{Hom}}_A(B^{\otimes/_A n+1}, A).$$

Each term is a B-module and therefore lies in $\langle B\rangle$ (a B-module N is a retract of $B \otimes_A N$). The geometric realization is the sequential colimit of its partial realizations, which are finite colimits of the terms and hence also lie in $\langle B\rangle$. Since the unit A is a compact object of $D(A)$, the identity of A factors through one of the partial realizations. Thus A is a retract of an object of $\langle B\rangle$, i.e. A lies in the thick tensor ideal generated by B in $D(A)$ and therefore f is descendable. □

As remarked above already, to ensure the existence of a !-cover of a Gelfand ring by a (strictly) totally disconnected space, we have to impose the following finiteness hypothesis. For this, recall that a map of arc-stacks $f\colon X \to Y$ is called *quasi-pro-étale* if for all strictly totally disconnected R and all maps $\mathcal{M}_{\mathrm{arc}}(R) \to Y$, the fiber product $X \times_Y \mathcal{M}_{\mathrm{arc}}(R)$ is representable by a perfectoid space that is pro-étale over $\mathcal{M}_{\mathrm{arc}}(R)$.

Definition 2.8. (a): We call a map $f\colon X \to Y$ in $\mathrm{ArcStk}_{\mathbb{Z}_p}$ *quasi-finite dimensional* (qfd for short) if it factors over a quasi-pro-étale map $X \to \mathbb{A}^n_Y = \mathbb{A}^{n,\diamond}_{\mathbb{Z}_p} \times_{\mathcal{M}_{\mathrm{arc}}(\mathbb{Z}_p)} Y$.

(b): A map of Gelfand rings $f\colon A \to B$ is qfd if $\mathcal{M}_{\mathrm{arc}}(B) \to \mathcal{M}_{\mathrm{arc}}(A)$ is in the sense of (a).

We will explain and justify later what $\mathcal{M}_{\mathrm{arc}}(A)$ is for a general Gelfand ring.

Example 2.9. Let us consider the Gelfand ring $A = \mathrm{LC}(S, \mathbb{Q}_p)$ of locally constant functions from a light profinite set S to $\mathbb{Q}_p$ (Gelfand as a filtered colimit of Banach). Now $\mathcal{M}_{\mathrm{arc}}(A)$ will essentially be the functor co-represented by A on strictly totally disconnected perfectoid Tate $\mathbb{Q}_p$-algebras. These are Banach $\mathbb{Q}_p$-algebras that are uniform by definition and therefore $\mathcal{M}_{\mathrm{arc}}(A) = \mathcal{M}_{\mathrm{arc}}(A^u)$ depends only on the uniform completion of A^u. We have $A^u = \mathcal{C}(S, \mathbb{Q}_p)$ (unit ball in A is $\mathrm{LC}(S, \mathbb{Z}_p)$ and the p-adic completion is $\mathcal{C}(S, \mathbb{Z}_p)$); the ring of all continous maps from S to $\mathbb{Q}_p$. We claim that $\mathcal{M}_{\mathrm{arc}}(A^u) \to \mathcal{M}_{\mathrm{arc}}(\mathbb{Q}_p)$ is quasi-finite dimensional with $n = 0$- in fact, we claim that this map is already quasi-pro-étale. For this, let $T = \mathcal{M}_{\mathrm{arc}}(R) \to \mathcal{M}_{\mathrm{arc}}(\mathbb{Q}_p)$ be a strictly totally disconnected perfectoid. The pullback $T \times_{\mathcal{M}_{\mathrm{arc}}(\mathbb{Q}_p)} \mathcal{M}_{\mathrm{arc}}(A^u)$ is given by $\mathcal{M}_{\mathrm{arc}}(\mathcal{C}(S, R))$. This is a perfectoid space and it is pro-étale over T since $\mathcal{C}(S, R)$ is the completed colimit of the R^{S_n}.

The arc-stack $\mathcal{M}_{\mathrm{arc}}(A)$ agrees with the arc-stack $\underline{S}$ from Example 2.5 above.

For us, the main reason we impose this condition is the following.

Remark 2.10. Let A be a qfd Gelfand ring over $\mathbb{Q}_p$. Then the topological space $\mathcal{M}(A)$ has *finite cohomological dimension*. Since $\mathcal{M}_{\mathrm{arc}}(A)$ will turn out to be qcqs, the quasi-pro-étale map $\mathcal{M}_{\mathrm{arc}}(A) \to \mathbb{A}^{n,\diamond}_{\mathbb{Q}_p}$ will factorize over $\mathbb{D}^{n,r,\diamond}_{\mathbb{Q}_p}$ which is the diamond of the Tate algebra of functions of the closed poly-disc of dimension n over $\mathbb{Q}_p$ of some radius r. Therefore, we have a quasi-pro-étale map $f\colon \mathcal{M}_{\mathrm{arc}}(A) \to \mathbb{D}^{n,r,\diamond}_{\mathbb{Q}_p}$. On underlying topological spaces, this is a map of compact Hausdorff spaces (so it is proper) and for all geometric points of $\mathbb{D}^{n,r,\diamond}_{\mathbb{Q}_p}$ the fiber is pro-finite. Then it follows from the Leray-spectral sequence that the cohomological dimension of $\mathcal{M}(A)$ is at most the cohomological dimension of $|\mathbb{D}^{n,r}_{\mathbb{Q}_p}|$.

Definition 2.11. We define $\mathrm{ArcStk}^{\mathrm{qfd}}_{\mathbb{Z}_p} = \widehat{\mathrm{Shv}}(\mathrm{AffdPerfd}^{\mathrm{qfd}}_{\mathbb{Z}_p,\omega_1})$ and similiarly for other basis like $\mathbb{F}_p$ or $\mathbb{Q}_p$.

The following is the key technical statement for the interplay of the arc- and !-topology.

Proposition 2.12. (a): *Let $R \to R^\bullet$ be an arc-hypercover of a separable* totally disconnected *perfectoid ring R by separable perfectoid rings. Then it is a* !*-equivalence.*

(b): *Let $R \in \mathrm{AffdPerfd}_{\omega_1}$. If the topological space $\mathcal{M}(R)$ is finite cohomological dimension, then there exists a* !*-cover (=descendable cover) such that $\mathcal{M}(R')$ is strictly totally disconnected.*

Unfortunately, we are not able to cover the proof of these important results. Let us just point out that (a) is [ABLRS, Lemma 4.4.3] and it reduces to a vanishing result for Ext-groups [ABLRS, Lemma 4.4.4]. It is in that lemma where separability is used critically. The idea for (b) is the following: The sheaf of (almost) idempotent algebras $C \mapsto \mathbb{A}_{\mathrm{inf}}(C)$ on rational subsets $C \subseteq \mathcal{M}(R)$ gives a symmetric monoidal pullback functor ψ^* from sheaves on the topological space $\mathcal{M}(R)$ to almost $\mathbb{A}_{\mathrm{inf}}(R)$-modules. Let $f\colon S \to \mathcal{M}(R)$ be a light profinite cover built from rational covers. Since $\mathcal{M}(R)$ has finite cohomological dimension, f_*1 is descendable with index bounded in terms of the dimension, and $\psi^* f_* 1$ is $\mathbb{A}_{\mathrm{inf}}$ of a totally disconnected perfectoid cover of R. A further pro-finite-étale cover, which is descendable by almost purity, makes the residue fields algebraically closed.

Example 2.13. Let us explain why (b) fails without a finiteness hypothesis. We explain why $A = \mathbb{Q}_p\langle T_n\colon n \in \mathbb{N}\rangle$ does not have a descendable cover $A \to B$ with B separable such that $\mathcal{M}(B)$ is a light profinite set.

Let us assume that the cover is descendable of index $d < +\infty$. This means that $A \to \mathrm{Tot}_{\leq d-1}(B^{\otimes_A(\bullet+1)})$ is split in the derived category $D(A)$ of solid A-modules. For example if we consider a cover that is descendable of index one, we just require to have a splitting in Banach modules. Let us assume the existence of such a cover.

For all $n \geq 0$, consider the closed subspace $Z_n = \mathcal{M}(\mathbb{Q}_p\langle T_1, ...T_n\rangle) \hookrightarrow \mathcal{M}(A)$. Let $V_n \hookrightarrow Z_n$ be the open subspace which is the complement of the origin. Let us explain why $H^{n-1}(V_n, \mathcal{O}) \neq 0$ for $n \geq 2$. Cover V_n by the open subsets $V_{n,i} = \{T_i \neq 0\}$, $i = 1, ..., n$, and for $\emptyset \neq I \subseteq \{1, ..., n\}$ let $V_{n,I} = \bigcap_{i\in I} V_{n,i}$. Each $V_{n,I}$ is the increasing union of the affinoid subdomains $W_{m,I} = \{|T_i| \geq p^{-m}\colon i \in I\}$, $m \geq 0$, and the restriction maps $\mathcal{O}(W_{m+1,I}) \to \mathcal{O}(W_{m,I})$ have dense image. Therefore $V_{n,I}$ is quasi-Stein, so that it has no higher cohomology, and $R\Gamma(V_n, \mathcal{O})$ is computed by the Cech-complex of this cover. This complex ends with

$$\bigoplus_{i=1}^{n} \mathcal{O}(V_{n,\{1,...,n\}\setminus\{i\}}) \xrightarrow{d} \mathcal{O}(V_{n,\{1,...,n\}}), \quad (f_i)_i \mapsto \sum_{i=1}^{n} (-1)^i f_i,$$

in degrees $n-2$ and $n-1$, so that $H^{n-1}(V_n, \mathcal{O}) = \mathrm{coker}(d)$. Now recall that

$$\mathcal{O}(W_{m,I}) = \mathbb{Q}_p\langle T_1, ..., T_n, p^m T_i^{-1}\colon i \in I\rangle$$

consists of Laurent series $\sum_{\nu\in\mathbb{Z}^n} c_\nu T^\nu$ with uniquely determined coefficients $c_\nu \in \mathbb{Q}_p$ such that $c_\nu = 0$ unless $\nu_j \geq 0$ for all $j \notin I$ (and satisfying a convergence condition). The same description then holds for $\mathcal{O}(V_{n,I}) = \lim_m \mathcal{O}(W_{m,I})$, and the restriction maps are the identity on coefficients. In particular, a function f_i on $V_{n,\{1,...,n\}\setminus\{i\}}$ only involves non-negative powers of T_i. It follows that for any element in the image of d, the coefficient of T^ν vanishes whenever $\nu_i < 0$ for all $i = 1, ..., n$. But $T_1^{-1} \cdots T_n^{-1}$ is a function on $V_{n,\{1,...,n\}}$. Hence it is not in the image of d and defines a non-zero class in $H^{n-1}(V_n, \mathcal{O})$.

Let us restrict the descendable cover $\mathcal{M}(B) \to \mathcal{M}(A)$ to Z_n; this is still a descendable cover of $\mathcal{M}(B') \to Z_n$ by a light pro-finite set (closed in a light profinite is still light profinite) of index d. Now note that for *any* open $U \subset \mathcal{M}(B')$ and any B'-module N, the sections of N over U have no cohomology in degrees $i \geq 1$.

Namely, if we have a countable open cover of U by affinoids, it can be refined to a disjoint cover by clopens since $\mathcal{M}(B')$ is light profinite. Therefore, taking sections over U is exact on B'-modules. (Namely, if S is some light profinite set and $\mathcal{F}$ any abelian sheaf on S and U any open in S, then $H^i(U, \mathcal{F}) = 0$ for $i \geq 1$. This follows because clopen sets form the basis of the topology in any profinite set (pullback from the inverse limit presentation). If S is light profinite, it is second countable and any U open in S is a countable union of $U = \bigcup_n C_n$ of clopen sets. Now consider $E_n = C_n - (\bigcup_{i=1}^{n-1} C_i)$. Then U is a countable disjoint union of compact clopen sets E_n. Any open cover of the E_n can be refined to a countable partition; namely let $\{U_{\alpha(n)}\}$ be an open cover of E_n. If $x \in E_n$ we find a clopen neighborhood D_x of x such that $D_j \subset U_{\alpha(x)}$. Since E_n is compact, finitely many of the D_x cover it. Let these $D_{x_1}, ..., D_{x_k}$. Put $D_j = D_{x_j} - (\bigcup_{i=1}^{j-1} D_{x_i})$. These constitute a disjoint open covering by clopens refining the given open covering of E_n. This implies that global sections over U sends epimorphisms of sheaves to surjections and therefore vanishing of cohomology.)

Turning back to $V_n \hookrightarrow Z_n \hookrightarrow X$, this would imply that $H^i(V_n, \mathcal{O}_{V_n}) = 0$ for $i \geq d$, which is a contradiction for $n > d$. The argument for this is the following: Let us pull back V_n to $U_n \subset \mathcal{M}(B')$. Let us first explain the baby case when the cover is descendable of index 1. Let $\mathcal{C}^\bullet(\mathcal{V}_n, \mathcal{O})$ be again the Chech-complex calculating the cohomology of V_n for $\mathcal{O}$ as above. As explained there, we may calculate $H^\bullet(V_n, \mathcal{O})$ by the complex

$$\bigoplus_{i=1}^{n} \mathcal{O}(V_{n,i}) \to \bigoplus_{i,j} \mathcal{O}(V_{n,i,j}) ... \to \mathcal{O}(V_{n,1,...,n}).$$

(Here one can see visibly that for example for $n = 2$ the monomial $T_1^{-1}T_2^{-1}$ is not a diference of functions defined on $T_1 \neq 0$ and $T_2 \neq 0$). We can pullback the Chech-covering used here also to get a covering $\mathcal{U}_n$ of U_n. We have maps $\mathcal{C}^k(\mathcal{V}_{n,i_1,...i_k}, \mathcal{O}) \to \mathcal{C}^k(\mathcal{U}_{n,i_1,...i_k}, \mathcal{O})$ and by assumption, these split as topological modules so that $\mathcal{C}^k(\mathcal{U}_n, \mathcal{O}) \simeq \mathcal{C}^k(\mathcal{V}_n, \mathcal{O}) \oplus B_n$ leading in total to a splitting as complexes. The complex $\mathcal{C}^\bullet(\mathcal{U}_n, \mathcal{O})$ is exact is positive degrees and therefore the complex $\mathcal{C}^\bullet(\mathcal{V}_n, \mathcal{O})$ as well.

In general, one may argue as follows: Let us assume it is descendable of index d. Write $A_n = \mathbb{Q}_p\langle T_1, ..., T_n\rangle$ and $B' = B \otimes_A A_n$, so that $A_n \to B'$ is descendable of index $\leq d$. For $\emptyset \neq I \subseteq \{1, ..., n\}$ and $m \geq 0$ let $W_{m,I} = \{|T_i| \geq p^{-m} \colon i \in I\} \subset Z_n$, so that $V_{n,I} = \bigcup_m W_{m,I}$. For $M \in D(A_n)$ we define

$$\Phi(M) = \lim_{\emptyset \neq I \subseteq \{1,...,n\}} \mathrm{Rlim}_m \; M \otimes_{A_n} \mathcal{O}(W_{m,I}),$$

where the outer limit is the (finite) limit over the Cech-diagram of the cover $\mathcal{V}_n$. This is an exact functor on $D(A_n)$ and by the above $\Phi(A_n) = R\Gamma(V_n, \mathcal{O})$.

We now show that if N is a B'-module in cohomological degrees ≤ 0, then $\Phi(N)$ is in cohomological degrees ≤ 0.

Indeed, $N \otimes_{A_n} \mathcal{O}(W_{m,I}) = N \otimes_{B'} B'_{K_{m,I}}$, where $B'_{K_{m,I}}$ is the localization of B' at the preimage $K_{m,I} \subset \mathcal{M}(B')$ of $W_{m,I}$. Since $\mathcal{M}(B')$ is profinite, we find clopen subsets $E_{m,I}$ with $K_{m,I} \subseteq E_{m,I} \subseteq K_{m+1,I}$. If $e_{m,I} \in B'$ denotes the corresponding idempotent (and $e_{-1,I} = 0$), the towers $\{B'_{K_{m,I}}\}_m$ and $\{e_{m,I}B'\}_m$ are interleaved, so that

$$\mathrm{Rlim}_m \; N \otimes_{B'} B'_{K_{m,I}} = \mathrm{Rlim}_m \; e_{m,I}N = \prod_m (e_{m,I} - e_{m-1,I})N.$$

This is exact in N. Now choose a countable partition of U_n into clopens P_k such that $P_k \subseteq U_{n,i(k)}$ for some $i(k)$; this is possible because $\mathcal{M}(B')$ is light profinite. Then the Cech-diagram for N is the product over k of the Cech-diagrams of the covers $\{U_{n,i} \cap P_k\}_i$ of P_k. These covers contain P_k itself as a member, so that their Cech-complexes are contractible. Therefore $\Phi(N) = \prod_k e_{P_k} N$, which proves the claim.

Now by descendability of index $\leq d$, A_n is a retract of $\mathrm{Tot}_{\leq d-1}(B'^{,\otimes_{A_n}(\bullet+1)})$ in $D(A_n)$. As Φ is exact, $R\Gamma(V_n, \mathcal{O}) = \Phi(A_n)$ is a retract of

$$\mathrm{Tot}_{\leq d-1}(\Phi(B'^{,\otimes_{A_n}(\bullet+1)})).$$

Each $B'^{,\otimes_{A_n}(k+1)}$ is a B'-module in cohomological degrees ≤ 0, so by the claim each $\Phi(B'^{,\otimes_{A_n}(k+1)})$ is in cohomological degrees ≤ 0. A totalization $\mathrm{Tot}_{\leq d-1}$ of such objects is in cohomological degrees $\leq d-1$. Hence $H^i(V_n, \mathcal{O}) = 0$ for $i \geq d$, which contradicts the non-vanishing of $H^{n-1}(V_n, \mathcal{O})$ as soon as $n > d$.

3. Big analytic de Rham stack

Let $A \in \mathrm{GelfRing}_{\omega_1}$. Let $\mathcal{M}_{\mathrm{arc}}(A)$ be given on *totally disconnected* $R \in \mathrm{AffdPerfd}_{\mathbb{Q}_p,\omega_1}$ by $\mathrm{Hom}_{\mathrm{GelfRing}}(A, R)$. Since $\mathrm{GSpec}(A)$ is an analytic stack, so satisfies the condition for !-equivalence required in that definition, it follows from Proposition 2.12 that $\mathcal{M}_{\mathrm{arc}}(A)$ defines an arc-hypersheaf on totally disconnected $\mathrm{AffdPerfd}_{\mathbb{Q}_p}$. By Lemma 2.2, we can then extend this definition to obtain $\mathcal{M}_{\mathrm{arc}}(A) \in \mathrm{ArcStk}_{\mathbb{Q}_p}$.

Lemma 3.1. *This construction extends to a colimit-preserving left-exact functor*

$$()^\diamond \colon \mathrm{GelfStk} \longrightarrow \mathrm{ArcStk}_{\mathbb{Q}_p}$$

which is the left-adjoint of a morphism of ∞-topoi. It agrees with restricting $X \in \mathrm{GelfStk}$ to the totally disconnected rings in $\mathrm{AffdPerfd}^{\mathrm{op}}_{\mathbb{Q}_p,\omega_1}$ and we have $\mathcal{M}_{\mathrm{arc}}(A) = \mathcal{M}_{\mathrm{arc}}(A^u)$.

Proof. Recall that GelfStk are pre-sheaves of anima on $\mathrm{GelfRing}^{\mathrm{op}}_{\omega_1}$ localized at !-equivalences. We therefore need to see that the functor in the Lemma sends !-equivalences to ∞-connective maps in $\mathrm{ArcStk}_{\mathbb{Q}_p}$. The key mathematical content is the following: If $A \to B$ is a morphism of Gelfand rings that is a !-cover, then $\mathcal{M}(B) \to \mathcal{M}(A)$ is *surjective*. This follows because !-descent also implies $*$-descent. Then the functor $B \otimes_A .$ is conservative on A-modules. Namely, if $M \otimes_A B = 0$, then since $M \simeq \lim_{[n]\in\Delta} M \otimes_A B^{\otimes_A(n+1)}$,

we would have $M \simeq \lim_{[n]\in\Delta} M \otimes_A B^{\otimes_A(n+1)} = \lim_{[n]\in\Delta}(M \otimes_A B) \otimes_B B^{\otimes_A(n+1)} = 0$. Therefore, for a point $x \in \mathcal{M}(A)$, we have that $B \otimes_A \kappa(x) \neq 0$ and by good properties of Gelfand rings this implies that $f^{-1}(x) \neq \emptyset$. Since $\mathcal{M}_{\mathrm{arc}}(.)$ commutes with finite limits and finite coskeleta are finite limits, we see that $\mathcal{M}_{\mathrm{arc}}(.)$ sends !-hypercovers $A \to A_\bullet$ of Gelfand rings to arc-hypercovers of arc-sheaves. But these are ∞-connective in $\mathrm{ArcStk}_{\mathbb{Q}_p}$ by a result of Lurie. In general, a result of Lurie saying that any ∞-connective morphism in an ∞-topos (like !-equivalences in the topos of sheaves for the !-topology) is the geometric realization of a hypercover shows that this verification is enough.

By construction, our functor commutes with all colimits and is left-exact, so by the adjoint functor theorem we obtain a right adjoint. This gives the desired morphism of ∞-topoi.

Finally, only the first of the further assertions is non-trivial. Therefore, let us prove that this colimit preserving extension does indeed coincide with the restriction to totally disconnected rings. Let i^* denote the functor restricting a presheaf on $\mathrm{GelfRing}_{\omega_1}^{\mathrm{op}}$ to the totally disconnected rings in $\mathrm{AffdPerfd}_{\mathbb{Q}_p,\omega_1}^{\mathrm{op}}$. On presheaves, i^* commutes with all colimits and sends $\mathrm{GSpec}(A)$ to the restriction of $\mathcal{M}_{\mathrm{arc}}(A)$. Next, if $X \in \mathrm{GelfStk}$, then i^*X is an arc-hypersheaf: if $R \to R^\bullet$ is an arc-hypercover of totally disconnected rings, then it is a !-equivalence by Proposition 2.12, so that $X(R) \simeq \lim_{[n]\in\Delta} X(R^n)$ because X is an analytic stack. Since totally disconnected rings form a basis for the arc-topology by Lemma 2.2, we obtain a functor $i^*\colon \mathrm{GelfStk} \to \mathrm{ArcStk}_{\mathbb{Q}_p}$. We claim that it commutes with colimits. Indeed, a colimit in GelfStk is the stackification $L(P)$ of the colimit P of presheaves. We have seen above that arc-hypersheafification composed with i^* sends !-equivalences to isomorphisms; therefore it sends $P \to L(P)$ to an isomorphism. As $i^*L(P)$ is already an arc-hypersheaf, this shows that $i^*L(P)$ is the arc-hypersheafification of i^*P, i.e. the colimit in $\mathrm{ArcStk}_{\mathbb{Q}_p}$. Hence i^* and $()^\diamond$ are two colimit preserving functors $\mathrm{GelfStk} \to \mathrm{ArcStk}_{\mathbb{Q}_p}$ which agree on all $\mathrm{GSpec}(A)$, so that they agree. □

Definition 3.2 (Big analytic de Rham stack)**.** We denote by $()^{\mathrm{DR}}\colon \mathrm{ArcStk}_{\mathbb{Q}_p} \to \mathrm{GelfStk}$ the right adjoint of the functor in Lemma 3.1. For $X \in \mathrm{GelfStk}$, we let $X^{\mathrm{DR}} = (X^\diamond)^{\mathrm{DR}}$.

Remark 3.3. Let $\mathrm{GSpec}(A) \in \mathrm{GelfStk}$. Then $\mathrm{GSpec}(A)^\diamond = \mathrm{GSpec}(A^u)^\diamond$. This implies that the analytic de Rham stack only sees the uniform completion and therefore has *overconvergence* implemented into its construction. For example, the analytic de Rham stack of the closed unit disc and of its overconvergent neighborhood do agree. This is good because overconvergent de Rham cohomology works better for rigid spaces.

Note that we always have a map $X \to X^{\mathrm{DR}}$ just by the definition of a right adjoint. One can show that the big analytic de Rham stack functor is always fully faithful [ABLRS, Proposition 4.5.7]. In general, it seems hard to control the functor of points of X^{DR}. In the following special case, this is however possible: If A is a separable Gelfand ring such that A^u is perfectoid and such that $\mathcal{M}(A)$ has finite cohomological dimension, then $X^{\mathrm{DR}}(A) = X(A^u)$ [ABLRS, Corollary 4.5.10].

There are however key issues with this construction.

(a): The big analytic de Rham stack does not send arc-topology epimorphisms to !-topology epimorphisms.

(b): For a compact metrizable Hausdorff space X, $\underline{X}^{\mathrm{DR}}$ is not X_{Betti} in general.

In fact, if the big analytic de Rham stack sent arc-topology epimorphisms to !-topology epimorphisms, then $\underline{X}^{\mathrm{DR}} = X_{\mathrm{Betti}}$ would hold for all such X, but this is wrong.

Note the similiarity to our toy-model: there after we have restricted to maps of k-schemes with locally finite type targets the functor $(.)^{\mathrm{dR},f}$ did in fact respect fpqc-epimorphisms.

Example 3.4. Let us consider again $A = \mathbb{Q}_p\langle T_n\colon n \in \mathbb{N}\rangle$. Let $X = \mathcal{M}(A)$. This is a compact and metrizable Hausdorff space. In general, there is always a monomorphism $X_{\mathrm{Betti}} \to \underline{X}^{\mathrm{DR}}$. If the big analytic de Rham stack functor preserved epimorphisms, it would be an isomorphism. But this is wrong. Namely, the difference between $\underline{X}^{\mathrm{DR}}$ and X_{Betti} is the following. If $A' \in \mathrm{GelfRing}_{\omega_1}$, then $\underline{X}^{\mathrm{DR}}(A') = \mathrm{Hom}_{cts}(\mathcal{M}(A'), X)$ while $X_{\mathrm{Betti}}(\mathrm{GSpec}(A'))$ are those maps $\mathcal{M}(A') \to X$ such that there exists a !-cover $\mathcal{M}(A'') \to \mathcal{M}(A')$ such that we get a lifting $\mathcal{M}(A'') \to S$ to some light profinite set S covering X.

Let us assume that $X_{\mathrm{Betti}} \simeq \underline{X}^{\mathrm{DR}}$ and let $S \to X$ be some light profinite set covering X. The fiber product $\mathrm{GSpec}(A) \times_{X^{\mathrm{DR}}} S_{\mathrm{Betti}}$ is representable by an affinoid Gelfand stack $\mathrm{GSpec}(C)$ whose Berkovich spectrum is a light profinite set. Since $S_{\mathrm{Betti}} \to X_{\mathrm{Betti}} \simeq \underline{X}^{\mathrm{DR}}$ is an epimorphism, so is its base change $\mathrm{GSpec}(C) \to \mathrm{GSpec}(A)$. Then $A \to C$ would be a !-cover and this would contradict Example 2.13 above.

Definition 3.5 (analytic de Rham stack)**.** Let $\mathrm{GelfStk}^{\mathrm{qfd}}$ be the category of analytic stacks on separable qfd Gelfand rings. The *perfectoidization* functor

$$(.)^{\diamond}\colon \mathrm{GelfStk}^{\mathrm{qfd}} \longrightarrow \mathrm{ArcStk}^{\mathrm{qfd}}_{\mathbb{Q}_p}$$

is the restriction functor which is well-defined by Proposition 2.12. It is left-exact and colimit preserving so that it admits a right adjoint

$$(.)^{\mathrm{dR}}\colon \mathrm{ArcStk}^{\mathrm{qfd}}_{\mathbb{Q}_p} \longrightarrow \mathrm{GelfStk}^{\mathrm{qfd}}$$

called the *analytic de Rham stack* functor.

4. Nilperfectoid rings and main results

Recall that in our toy example we could control $X^{\mathrm{dR},f}$ by working with a basis for the fpqc topology where the reduction functor is just the colimit perfection functor; this basis was given by semiperfect rings. In mixed characteristic and integral p-adic Hodge theory a similar phenomenon is that quasi-syntomic rings can be covered by quasi-regular semiperfectoid rings and then it is often easier to control the latter. In our story, the role of these rings is given by *nilperfectoid rings*.

Definition 4.1. A Gelfand ring A is called *nilperfectoid* if the dagger reduction $A^{\dagger-\mathrm{red}}$ is perfectoid.

Observe that this is equivalent to the condition that $A \to A^u$ is surjective and that A^u is perfectoid.

Example 4.2. Consider $\mathbb{C}_p\langle T\rangle^{\leq 0}$ which is given by $\mathrm{colim}_{\rho>0}\,\mathcal{O}(\mathbb{D}_{[0,\rho],\mathbb{C}_p})$. This ring is obviously not perfectoid but it is Gelfand and the dagger-nil-radical is just the ideal generated by T. The ring $\mathbb{C}_p$ is perfectoid, so that $\mathbb{C}_p\langle T\rangle^{\leq 0}$ is nilperfectoid.

Proposition 4.3. (a): *If A is a separable Gelfand ring then there exists a descendable map $A \to A'$ such that A' is nilperfectoid.*

(b): *if A is separable and in addition qfd, then we can assume that A' is totally disconnected nilperfectoid.*

Proof. For (a): Since A^u is a separable Banach algebra, the ring $A^{u,\circ}/p = A^{\leq 1}/p$ is countable. Choose representatives $f_i \in A^{\leq 1}$, $i \in \mathbb{N}$, of all its elements. Then the map $\mathbb{Z}_p\langle T_n\colon n\in\mathbb{N}\rangle \to A^{u,\circ}$, $T_i \mapsto f_i$, is surjective modulo p and therefore surjective by p-adic completeness, so that $\mathbb{Q}_p\langle T_n\colon n\in\mathbb{N}\rangle \to A^u$ is surjective. We claim that there exists also a map $R = \mathrm{colim}_n\,\mathbb{Q}_p\langle T_1,...,T_n\rangle \to A$ which induces this surjection on uniform compeltions.

The maps $\mathbb{Q}_p\langle T_1,...,T_n\rangle \to R^u$ is descendable of index ≤ 1 as it has a section and by a result by Mann on colimits of descandable maps $R \to R^u$ is descendable (of index ≤ 2), so that after base change along it we may assume that $A \to A^u$ is surjective. Next, the map $\mathbb{Q}_p\langle T_n\colon n\in\mathbb{N}\rangle \to \mathbb{Q}_p^{cycl}\langle T_n^{1/p^\infty}\colon n\in\mathbb{N}\rangle$ is descendable because it admits a section as modules. Let A' be the base change of A along this map. Then $A'^{,u}$ is the uniform completion of a quotient of a perfectoid ring and therefore perfectoid, and $A' \to A'^{,u}$ is surjective [ABLRS, Proposition 3.1.19]. This step is quite interesting in it's own right and it uses the *prismatic input* via the perfectoidization. It is explained further down below in Lemma 4.4.

For (b): Since A is qfd, $\mathcal{M}(A)$ has finite cohomological dimension. Then there exists a descendable map $A \to A^w$, which is a colimit of finite products of rational localizations of A, such that $\mathcal{M}(A^w)$ is light profinite [ABLRS, Lemma 3.2.9, Proposition 3.2.10]. Now apply (a) to A^w; the maps used in (a) are pro-étale on arc-stacks, so that the Berkovich spectrum stays profinite. □

We used the following statement.

Lemma 4.4. *Let A be a perfectoid Banach $\mathbb{Q}_p$-algebra and let $A \to B$ be a quotient (so B is a Gelfand ring). Then $B^u \simeq B_{\mathrm{perfd}}$, where B_{perfd} is the initial perfectoid Banach B-algebra and $B \to B^u$ is a surjection.*

Proof. Let $I = \ker(A \to B)$, let $I^{\leq 1} = I \cap A^{\leq 1}$, $B_0 = A^{\leq 1}/\overline{I}^{\leq 1}$, where $\overline{I}^{\leq 1}$ is the p-adic closure of $I^{\leq 1}$ in $A^{\leq 1}$. Then recall that $A^{\leq 1}$ is an integral perfectoid ring and by definition B_0 is a semiperfectoid ring as a quotient of $A^{\leq 1}$. Now let B_0^{perfd} be the integral perfectoidization in the sense of prismatic cohomology and define $B_{\mathrm{perfd}} = B_0^{\mathrm{perfd}}[1/p]$. Note that B_0^{perfd} might have p-torsion but B_{perfd} is still a perfectoid $\mathbb{Q}_p$-Banach algebra. This object is clearly the initial perfectoid Banach B-algebra and $B \to B_{\mathrm{perfd}}$ is a surjection since $B_0 \to B_0^{\mathrm{perfd}}$ is a surjection. It remains to see that $B^u \simeq B_{\mathrm{perfd}}$. By Bhatt–Scholze we know that $\mathrm{Spa}(B_{\mathrm{perfd}}) \hookrightarrow \mathrm{Spa}(A)$ is a Zariski-closed immersion cut out by the vanishing locus of $\overline{I}^{\leq 1}$. It follows from [ABLRS, Lemma 3.1.18] that the vanishing locus of $\overline{I}^{\leq 1}$ agrees with the vanishing

locus of I. These statements also hold for the Berkovich spaces $\mathcal{M}(B_{\mathrm{perfd}}) \hookrightarrow \mathcal{M}(A)$. Furthermore, $\mathcal{M}(B) \hookrightarrow \mathcal{M}(A)$ is a Zariski-closed immersion cut out by I. Therefore, the map $B \to B_{\mathrm{perfd}}$ induces an isomorphism $\mathcal{M}(B_{\mathrm{perfd}}) \to \mathcal{M}(B)$ and we have established that it induces a surjection on uniform completions. Using [ABLRS, Lemma 3.1.18] again, we see that $B^u \simeq B_{\mathrm{perfd}}$, as desired. □

This allows us to prove the following important control statement which reconciles the construction of the analytic de Rham stack as a right adjoint to perfectoidization with the definition as the sheafification of the (dagger) reduction functor; just as in our toy model.

Proposition 4.5 (Analytic de Rham stack via dagger reduction)**.** *Let $X \in \mathrm{GelfStk}^{\mathrm{qfd}}$, then X^{dR} is given by the !-sheafification of the pre-sheaf sending $A \in \mathrm{GelfRing}^{\mathrm{qfd}}_{\omega_1}$ to $X(A^{\dagger-\mathrm{red}})$.*

Proof. We need to show that both functor agree !-locally. We may by Proposition 4.3 assume that we work with A totally disconnected nilperfectoid qfd. But then $X(A^{\dagger-\mathrm{red}}) = X(A^u) = X^\diamond(\mathcal{M}_{\mathrm{arc}}(A^u)) = X^{\mathrm{dR}}(A)$, as desired. Is there a problem with separatedness? □

This allows us to discuss the example of the affine line.

Example 4.6. Let $X = \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_p}$ be the analytic affine line over $\mathbb{Q}_p$. Then we have that $X^{\mathrm{dR}}(A) = A^u = A^{\dagger-\mathrm{red}}$ for any A nilperfectoid. This implies that $X \to X^{\mathrm{dR}}$ is an epimorphism of Gelfand-stacks. The kernel is by this formula just given by $\mathbb{G}_a^\dagger = (\mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_p})^{\leq 0} = \mathrm{GSpec}(\mathbb{Q}_p\langle T\rangle^{\leq 0})$. Therefore,

$$X^{\mathrm{dR}} = \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_p}/\mathbb{G}_a^\dagger.$$

The affine line is an example of a †-formally smooth space.

Definition 4.7 (†-formally smooth resp. étale)**.** A morphism $f\colon X \to Y$ in GelfStk is called †-formally smooth (resp. †-formally étale) if for any separable nilperfectoid ring A, the natural map

$$X(A) \to Y(A) \times_{Y^\diamond(\mathcal{M}_{\mathrm{arc}}(A^u))} X^\diamond(\mathcal{M}_{\mathrm{arc}}(A^u))$$

of anima is surjective (resp. an equivalence).

We summarize the key properties of the analytic de Rham stacks in the following result.

Theorem 4.8 (Properties of the analytic de Rham stack)**.** (a): *If $X \to X'$ is an epimorphism of qfd arc-stacks, then $X^{\mathrm{dR}} \to X'^{,\mathrm{dR}}$ is an epimorphism of qfd Gelfand-stacks.*
(b): *If X is †-formally smooth, then the natural map $X \to X^{\mathrm{dR}}$ is an epimorphism.*
(c): *If X is a compact metrizable Hausdorff space, then $\underline{X}^{\mathrm{dR}} = X_{\mathrm{Betti}}$.*

Proof. For (a), let $x \in X'^{,\mathrm{dR}}(A)$ with $A \in \mathrm{GelfRing}^{\mathrm{qfd}}_{\omega_1}$. By Proposition 4.3 we may assume that A is strictly totally disconnected nilperfectoid, so that $x \in X'(\mathcal{M}_{\mathrm{arc}}(A^u))$. Since $X \to X'$ is an epimorphism, there is an arc-cover $A^u \to C$ of qfd perfectoid rings such that x lifts to $X(\mathcal{M}_{\mathrm{arc}}(C))$. The difficult point is to show that there is a descendable map $A \to B$ with $B^u = C$ [ABLRS, Proposition 4.8.5]; then the lift defines a point of $X^{\mathrm{dR}}(B)$ as desired. The idea is that, as A^u is strictly totally disconnected, C can be approximated by †-formally smooth A-algebras B_n such that $A \to B_n$ admits a section; then $A \to B = \mathrm{colim}_n B_n$ is descendable of index ≤ 2.

Item (b) is easy given our previous work: let $x \in X^{\mathrm{dR}}(A)$ with $A \in \mathrm{GelfRing}^{\mathrm{qfd}}_{\omega_1}$. By Proposition 4.3 we find a !-cover $A \to B$, where B is totally disconnected nilperfectoid. For those we know that $X^{\mathrm{dR}}(B) = X^\diamond(\mathcal{M}_{\mathrm{arc}}(B^u))$. By definition of †-formal smoothness, we can lift the image of x to a point in $X(B)$, as desired.

For (c), choose a surjection $S \to X$ from a light profinite set and let $S_n = S \times_X \ldots \times_X S$ be the terms of its Cech nerve; these are light profinite sets. Then $\underline{S} \to \underline{X}$ is an epimorphism of qfd arc-stacks, so that $\underline{S}^{\mathrm{dR}} \to \underline{X}^{\mathrm{dR}}$ is an epimorphism by (a). Since $(.)^{\mathrm{dR}}$ commutes with fiber products (it is a right adjoint so it commutes with all limits), its Cech nerve is $\underline{S_\bullet}^{\mathrm{dR}}$, and therefore $\underline{X}^{\mathrm{dR}} = \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} \underline{S_n}^{\mathrm{dR}}$. For a light profinite set T we have $\underline{T}^{\mathrm{dR}} = T_{\mathrm{Betti}} = \mathrm{GSpec}(\mathrm{LC}(T, \mathbb{Q}_p))$, because both send A to $\mathrm{Hom}_{cts}(\mathcal{M}(A), T)$. Hence $\underline{X}^{\mathrm{dR}} = \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}}(S_n)_{\mathrm{Betti}}$, which is X_{Betti} by definition. □

Note again how (a) and (b) do in fact mirror the properties of our toy-model de Rham stack. However, we will see that for a smooth partially proper rigid analytic space $X \to \mathrm{Spa}(K)$ with relative analytic de Rham stack $X^{\mathrm{dR}/K}$ we have that $R\Gamma(X^{\mathrm{dR}/K}, \mathcal{O})$ does agree with *overconvergent* de Rham cohomology of X over K in the sense of Grosse-Klönne.

References

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