LECTURE NOTES ON ADIC SPACES

ALEXANDER B. IVANOV

Contents

This is a (highly unfinished) version of my manuscript on adic spaces. I started to write it as as a preparation for a lecture course, which I planned to give at the University of Bonn. Due to moving to another university, I had not yet given this lecture, but continued to write this manuscript.

The manuscript does not contain any new/original results. I tried to find my own path through the theory of adic spaces. However, in big parts of the manuscript I heavily relied on other sources, specifically on the lecture notes of T. Wedhorn [Wed19], the lecture notes of S. Morel [Mor19] and some parts of the book of Scholze–Weinstein [SW20] and of the lecture notes of J. Anschütz [Ans]. Most of the time I tried to give precise references to these (or other) sources, however sometimes explicit references are missing.

1. Valuations and valuation rings

1.1. **Valuations.** The simpliest example of a valuation ring is a *discrete valuation ring (DVR)*. For example, if k is a field, then the ring of power series $k[\![t]\!]$ in the variable t over k is a DVR. The corresponding valuation is the map $v\colon k[\![t]\!] \to \mathbb{Z}_{\geq 0}$, $v(\sum_{i\geq 0} a_i t^i) = \min\{i \geq 0 \colon a_i \neq 0\}$. Note that this map extends to $v\colon k(\!(t)\!) = \operatorname{Frac} k[\![t]\!] \to \mathbb{Z}$, defined in the same way. One can extend this by drastically enlarging the group of possible values.

Definition 1.1. (1) A *totally ordered (abelian) group* is an abelian group[1] $(\Gamma, \cdot)$ equipped with a total order $\leq$ on the underlying set Γ, such that for all $x, y, z \in \Gamma$ one has $x \leq y \Rightarrow xz \leq yz$.

Given Γ as above, we often also consider the monoid $\Gamma \cup \{0\}$, with multiplication given by $\gamma \cdot 0 = 0$ for all $\gamma \in \Gamma$ and with $0 < \gamma$ for all $\gamma \in \Gamma$.

(2) A homomorphism $\Gamma \to \Gamma'$ of totally ordered groups is a homomorphism of groups, which preserves the order relation. (And similar for monoids $\Gamma \cup \{0\}$.)

Remark 1.2. A group homomorphism $f\colon \Gamma \to \Gamma'$ is a homomorphism of totally ordered groups if $x \leq 1 \Rightarrow f(x) \leq 1$ for all $x \in \Gamma$. (This is immediate.)

Example 1.3. (1) $(\mathbb{R}_{>0}, \times)$ with the usual order. Note that that the additive group $(\mathbb{R}, +)$ with the usual order is isomorphic to this group via exp and log maps.

[1] we write the group Γ always multiplicatively.

(2) Any subgroup of $(\mathbb{R}_{>0}, \times)$ with the usual order. For example, for any $q \in \mathbb{R}_{>0}$, the group $(q^{\mathbb{Z}}, \cdot)$. Or, the subgroup generated by 2 and π.
(3) For $n \geq 1$, the group $(\mathbb{R}^n_{>0}, \times)$ with lexicographic order, i.e., $(x_1, \dots, x_n) > (y_1, \dots, y_n)$ if and only if there is some $1 \leq k \leq n$ with $x_i = y_i$ for $i < k$, and $x_k > y_k$.
(4) More generally, if I is any totally ordered set, $\prod_{i \in I} \mathbb{R}_{>0}$ with lexicographic order is a totally ordered group. Even more generally, if for each $i \in I$, Γ_i is a totally ordered group, then $\prod_{i \in I} \Gamma_i$ is totally ordered (with lexicographic order).

Definition 1.4. Let R be a ring.

(1) A *valuation* on R is a map
$$|\cdot|\colon R \to \Gamma \cup \{0\},$$
into a totally ordered group Γ, which satisfies the following conditions:
  (a) $|ab| = |a| \cdot |b|$
  (b) $|a + b| \leq \max(|a|, |b|)$
  (c) $|1| = 1$ and $|0| = 0$.
(2) The *value group* of $|\cdot|$ is the (totally ordered) subgroup $\Gamma_{|\cdot|}$ of Γ generated by $\Gamma \cap |R|$.
(3) The *support* (or *kernel*) of $|\cdot|$ is the set $\operatorname{supp}|\cdot| = \{x \in R\colon |x| = 0\}$.
(4) Two valuations $|\cdot|_1$, $|\cdot|_2$ with values in Γ_1, Γ_2 are *equivalent*, if $|a|_1 \leq |b|_1 \Leftrightarrow |a|_2 \leq |b|_2$ holds for all $a, b \in R$.

Example 1.5. (1) Let K be a field. The *trivial valuation* on K is given by $|x| = 1$ for all $x \in K^\times$ and $|0| = 0$. (So, $\Gamma = \{1\}$ suffices. But actually we can take any Γ here, to obtain an equivalent valuation.)
(2) Let p be a prime. We have the p-adic valuation on the rings $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p$, all having the value group $p^{\mathbb{Z}} \subseteq \mathbb{R}_{>0}$.

We will see much more examples of valuations soon. Let us summarize some basic facts about valuations in the following lemma.

Lemma 1.6. *Let R be a ring. Let $|\cdot|, |\cdot|'$ be two valuations on R with values in totally ordered groups Γ, Γ' respectively.*

(1) $\operatorname{supp}|\cdot|$ *is a prime ideal of R.*
(2) Let $K_{|\cdot|} = \operatorname{Frac}(R/\operatorname{supp}|\cdot|)$. Then $|\cdot|$ factors uniquely as $R \to K_{|\cdot|} \xrightarrow{\widetilde{|\cdot|}} \Gamma \cup \{0\}$, where $\widetilde{|\cdot|}$ is a valuation of K. Moreover, $V_{|\cdot|} = \{x \in K_{|\cdot|}\colon \widetilde{|x|} \leq 1\}$ is a subring of K, called the valuation ring of $\widetilde{|\cdot|}$.
(3) The following are equivalent:
  (a) $|\cdot|, |\cdot|'$ are equivalent.
  (b) There is an isomorphism of ordered groups $f\colon \Gamma_{|\cdot|} \to \Gamma_{|\cdot|'}$, such that $|\cdot|' = f \circ |\cdot|$.
  (c) $\operatorname{supp}|\cdot| = \operatorname{supp}|\cdot|'$ and $V_{|\cdot|} = V_{|\cdot|'}$.

Proof. (1) is easy and (2) is easy too, once we note that a valuation $|\cdot|$ on a domain A admits a unique extension $|\cdot|$ to the quotient field given by $|a/b| = |a| \cdot |b|^{-1}$ for $a, b \in A$, $b \neq 0$. To prove (3), note first that all conditions imply that $\operatorname{supp}(|\cdot|) = \operatorname{supp}(|\cdot|')$ (clear for (b),(c); for (a): insert $b = 0$ in Definition 1.4(3)). Moreover, using part (2) of the lemma, we may replace R by the quotient by this prime ideal, and hence assume that R is a domain and the support of $|\cdot|, |\cdot|'$ is trivial. Moreover, we may extend $|\cdot|, |\cdot|'$ to the quotient field $K = \operatorname{Frac} R$ and it suffices to prove the claimed equivalence for valuations on a field K. It is clear that (b) $\Rightarrow$ (a) $\Rightarrow$ (c). Suppose (c) holds. Then, note that $|\cdot|\colon K^\times \to \Gamma_{|\cdot|}$ is a group homomorphism. As K is a field, $K \smallsetminus \{0\} = K^\times$, and it follows that $K^\times \to \Gamma_{|\cdot|}$ is surjective (and similar for $|\cdot|'$). Moreover, by (c) we have $\{x \in K^\times : |x| \leq 1\} = \{x \in K^\times : |x|' \leq 1\}$, and hence (applying the isomorphism $x \mapsto x^{-1}$ of $K^\times$) we get the equality of the similar sets with "$\leq$" replaced by "$\geq$". Intersecting both, we deduce $\{x \in K^\times : |x| = 1\} = \{x \in K^\times : |x|' = 1\}$. Altogether, $|\cdot| \colon K^\times \to \Gamma_{|\cdot|}$ and $|\cdot|' \colon K^\times \to \Gamma_{|\cdot|'}$ are surjective homomorphisms with equal kernels, i.e., there is an isomorphism $\Gamma_{|\cdot|} \xrightarrow{\sim} \Gamma_{|\cdot|'}$ as in (b). □

1.2. **Valuation rings.** We will deal a lot with local rings here. If A is a local ring, we usually will write $\mathfrak{m}_A$ for its maximal ideal and $\kappa_A = A/\mathfrak{m}_A$ for its residue field.

Lemma 1.6 (and its proof) suggests to study valuations on fields first. Therefore, let us make the following important definition.

Definition 1.7. Let K be a field. A subring V of K is called a *valuation ring of* K, if there exists a valuation $|\cdot|$ on V such that $V = \{a \in K : |a| \leq 1\}$ (with other words, V is the valuation ring of $|\cdot|$ in the sense of Lemma 1.6(2)).

We also say that an integral domain V is a *valuation ring*, if V is a valuation ring of its fields of fractions.

Note that a valuation ring V is clearly a local ring with maximal ideal $\mathfrak{m}_V = \{x \in V : |x| < 1\}$.

Example 1.8. The valuation ring of the trivial valuation on a field K is K itself.

Valuation rings are rings of very special kind; the next theorem shows that they can be characterized by a number of quite different looking, but equivalent, properties. If $A \subseteq B$ are local rings, then we say that B *dominates* A if $\mathfrak{m}_B \cap A = \mathfrak{m}_A$.[2] Note that if K is a field, then "B dominates A" defines an partial order on the set of all local subrings of K. In the rest of this section we write $A \leq B$ if B dominates A.

Theorem 1.9. *Let A be an integral domain contained in a field K. The following conditions are equivalent:*

(1) *A is a valuation ring of K.*
(2) *For any $a \in K^\times$ one has $a \in A$ or $a^{-1} \in A$.*
(3) $\operatorname{Frac} A = K$ *and the set of ideals of A is totally ordered by inclusion.*
(3)' $\operatorname{Frac} A = K$ *and the set of principal ideals is totally ordered by inclusion.*
(4) *A is a local and maximal with respect to respect to the dominance order.*
(5) *There exists an algebraically closed field L and a ring homomorphism $f \colon A \to L$, which is maximal in the set of homomoprphisms from subrings of K to L, i.e., if $A \subseteq A' \subseteq K$ is another subring of K and $f' \colon A' \to L$ restricts to f, then $A' = A$.*

There are canonical mutually inverse bijections

$$\{\textit{valuation subrings of } K\} \xleftrightarrow{\sim} \{\textit{eqivalence classes of valuations on } K\}$$
$$V \mapsto (|\cdot|_V \colon K \to K^\times/V^\times \cup \{0\}), \qquad \textit{cf. the proof of (1)} \Leftrightarrow \textit{(2) below}$$
$$\{x \in K : |x| \leq 1\} \leftarrow\!\shortmid |\cdot|.$$

Proof. (1) $\Leftrightarrow$ (2): "$\Rightarrow$" is obvious. For "$\Leftarrow$", note that $K^\times/A^\times$ is a totally ordered group with the order defined by $|a| \leq |b| :\Leftrightarrow ba^{-1} \in A$ (note that condition (2) is needed to justify the word "totally"), and that the map $|\cdot|_A \colon K \to (K^\times/A^\times) \cup \{0\}$ sending $x \neq 0$ to $xA^\times \in K^\times/A^\times$ and 0 to 0 is a valuation.

(2) $\Rightarrow$ (3): $\operatorname{Frac} A = K$ is clear; if $\mathfrak{a}, \mathfrak{b} \subseteq A$ are ideals with $\mathfrak{a} \not\subseteq \mathfrak{b}$, then there is some $a \in \mathfrak{a} \smallsetminus \mathfrak{b}$, and so for all $b \in \mathfrak{b}$, $b^{-1}a \notin A$. Hence, by (2), $ba^{-1} \in A$ for all $b \in \mathfrak{b}$, i.e., $Ab \subseteq Aa$ for all $b \in \mathfrak{b}$, i.e., $\mathfrak{b} \subseteq Aa \subseteq \mathfrak{a}$.

[2]The inclusion $A \to B$ induces a map $f \colon \operatorname{Spec} B \to \operatorname{Spec} A$ of local schemes and the condition $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ simply means that f maps the closed point to the closed point.

For other equivalences, all of which are standard facts in commutative algebra, cf. for example [Wed19, Prop. 2.2] or [Mat89, Thm. 10.2]. The last claim is immediate. □

Thus, if V is a valuation ring, then $\operatorname{Spec} V$ is a "linear" chain of points with specialization relations. Clearly, it has a generic and a closed point. Also apart from this condition, it cannot be completely arbitrary:

Corollary 1.10. *Let V be a valuation ring. Then the following hold:*

1. *If $I \subseteq V$ is any ideal, then $\sqrt{I}$ is a prime ideal. In particular, any (proper) reduced ideal of V is prime*
2. *Let $f \in V$ be non-zero and non-unit. Then $\mathfrak{q} = \sqrt{fV}$ is the smallest prime ideal containing f, $\mathfrak{p} = \bigcap_n f^n V$ is the biggest prime ideal contained in fV and $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ is an immediate specialization relation in $\operatorname{Spec} V$, i.e., there is no prime ideal $\mathfrak{p}_1$ in V with $\mathfrak{p} \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{q}$.*
3. *Any quotient of V by a prime ideal is a valuation ring.*
4. *Subrings of $K := \operatorname{Frac} V$ containing V are precisely the localizations of V at various prime ideals. All of them are valuation rings.*

Proof. (1)-(3): Exercise. (4): From characterization in Theorem 1.9(1) it is clear that any ring R with $V \subseteq R$ is a valuation subring of K. It thus is enough to show that any such R is the localization of V at a prime ideal. Let $x \in \mathfrak{m}_R$. Then $x^{-1} \notin R$, hence $x^{-1} \notin V$, hence (as $x \in V$) also $x \in \mathfrak{m}_V$. I.e., $\mathfrak{m}_R \subseteq \mathfrak{m}_V \subseteq V$. Now, $V \hookrightarrow R$ induces a map of spectra, under which $\mathfrak{m}_R \mapsto \mathfrak{m}_R \cap V$, i.e., $\mathfrak{m}_R \cap V = \mathfrak{m}_R$ is a prime ideal of V. Now, we have the localization $V_{\mathfrak{m}_R}$, and its universal property implies that $V_{\mathfrak{m}_R} \subseteq R$. Conversely, $R \subseteq V_{\mathfrak{m}_R}$. Indeed: as $V \subseteq V_{\mathfrak{m}_R}$, it suffices to check that any $x \in R \smallsetminus V$ lies in $V_{\mathfrak{m}_R}$. But for such x, $x^{-1} \in V$ and $x \notin \mathfrak{m}_R$, i.e., $x \in V \smallsetminus \mathfrak{m}_R$, so x^{-1} becomes invertible in $V_{\mathfrak{m}_R}$, i.e, $x \in V_{\mathfrak{m}_R}$. □

Exercise 1.11. Let V be a valuation ring. Describe all closed, all pro-open, all closed constructible and all pro-open constructible subsets of $\operatorname{Spec} V$.

Corollary 1.10(4) motivates the following definition.

Definition 1.12. Let $|\cdot|$ be a valuation of any ring. Let $V = A(|\cdot|)$ be its valuation ring (as in Definition 1.4 and Lemma 1.6(2)). The *rank* (sometimes also *height*) of $|\cdot|$ is the cardinality of $|\operatorname{Spec} V|$ minus 1, or, equivalently, the (Krull) dimension of $\operatorname{Spec} R$.

For example, the rank of a discrete valuation always equals 1. Valuations on fields allow quite some flexbility. An important example is given by the following construction.

Construction 1.13 (Concatenation of valuation rings)**.** Let V be a valuation ring of a field K, and let $\kappa_V = V/\mathfrak{m}_V$ be the residue field. Then

$$W \mapsto W' := \{x \in V : x \mod \mathfrak{m}_V \in W\}$$

induces a bijection between valuation rings W of κ_V and valuation rings of K, which are contained in V. We have $\operatorname{ht}(W') = \operatorname{ht}(V) + \operatorname{ht}(W)$.

Exercise 1.14. Prove this. What does this mean geometrically?

Thus if K is field and V a valuation ring of K, then Corollary 1.10(4) resp. Construction 1.13 (which are in a sense dual to each other) describe the less resp. the more fine valuations on K in terms of points of $\operatorname{Spec} V$ resp. of valuations of κ_V.

Example 1.15 (A valuation of rank 2)**.** All valuations we know from discrete valuation rings, and also all valuations coming from norms on rings are of rank 1 (essentially by definition). Using Construction 1.13 we can give an example of a rank 2 valuation. Therefore, let k be any field, let $K = K(\!(x)\!)(\!(u)\!)$ (it's a field) and let $V = k(\!(x)\!)[\![u]\!]$. So, V is the valuation ring of the

u-adic valuation on K. Now consider the x-adic valuation on $\kappa_V = k(\!(x)\!)$, with valuation ring $k[\![x]\!]$ and let

$$W = \{f = \sum_{n\geq 0} f_n u^n \in V = k(\!(x)\!)[\![u]\!] \colon f_n \in k[\![x]\!] \text{ for all } n\}.$$

By Constuction 1.13, W is a valuation ring of rank 2. Similarly, one can construct valuation of arbitrary big rank (including ∞).

Let us list some further consequences of the above characterization of valuation rings.

Corollary 1.16. *Let K be a field, R a subring of K and $\mathfrak{p}$ a prime ideal of R. Then there is some valuation ring V of K, such that $\mathfrak{m}_V \cap R = \mathfrak{p}$.*

Proof. We have $R \subseteq R_{\mathfrak{p}}$ ($R \subseteq K$, so R is a domain) and $\mathfrak{p}R_{\mathfrak{p}} \cap R = \mathfrak{p}$. Replacing R by $R_{\mathfrak{p}}$ and $\mathfrak{p}$ by $\mathfrak{p}R_{\mathfrak{p}}$, we thus may assume that R is local and $\mathfrak{p}$ is the maximal ideal of R. But in this case, the result follows from Theorem 1.9 (1) $\Leftrightarrow$ (4). □

Corollary 1.17. *Any valuation ring is integrally closed. If R is a subring of a field K, then the integral closure of R in K equals the intersection of all valuation rings of K containing R.*

Proof. Let V be a valuation ring and $K = \operatorname{Frac} V$. If $x \in K \smallsetminus V$ with $x^n + a_1x^{n-1} + \dots + a_n = 0$ with $a_i \in V$, then by Theorem 1.9(2), $x^{-1} \in V$. We even have $x^{-1} \in \mathfrak{m}_V$, as $x \notin V$ and so $x^{-1} \notin V^\times$. But then $-1 = a_1x^{-1} + a_2x^{-2} + \dots + a_nx^{-n} \in \mathfrak{m}_V$, which is a contradiction, showing the first claim. For the second claim, note that the first claim implies that the integral closure of R is contained in the intersection R' of all valuation rings of K containing R. For the converse, it suffices to show that if $x \in K$ is not integral over R, then $x \notin R'$, i.e., that there is a valuation ring V of K with $R \subseteq V$ and $x \notin V$. Consider the ring $R[x^{-1}] \subseteq K$. The then $x^{-1}R[x^{-1}]$ is a proper ideal: indeed, if $1 \in x^{-1}R[x^{-1}]$, then $1 = a_1x^{-1} + \dots a_mx^{-m}$ for some $m \geq 1$ and $a_i \in R$, which contradicts the non-integrality of x over R. Let $\mathfrak{p}$ be any maximal ideal of $R[x^{-1}]$ containing $x^{-1}R[x^{-1}]$. By Corollary 1.16 there is some valuation ring V of K containing $R[x^{-1}]$ and with $\mathfrak{m}_V \cap R[x^{-1}] = \mathfrak{p}$. But in particular, $R \subseteq V$ and $x^{-1} \in x^{-1}R[x^{-1}] \subseteq \mathfrak{m}_V$, i.e., $x \notin V$. □

1.2.1. *Appendix: Field extensions.* Valuations (on fields) are quite flexible with respect to extensions:

Proposition 1.18. *Let K'/K be a an extension of fields and $|\cdot|$ a valuation of K. Then there exists a valuation $|\cdot|'$ of K', such that its restriction $|\cdot|'_{|K}$ to K is equivalent to $|\cdot|$.*

Moreover, if $x_1, \dots, x_n \in K'$ are algebraically independent over K and $\gamma_1, \dots, \gamma_n \in \Gamma$, then there exists such an extension with $|x_i|' = \gamma_i$ for all i.

Proof. See [Bou, VI,§2.4,Prop.4]. □

Corollary 1.19. *Let $K_1 \hookrightarrow K_2$ be a map of fields. Let V_2 and $V_1 := V_2 \cap K_1$ be valuation rings of K_2 resp. of K_1. Then there are surjections:*

$$\{\textit{Val. rings of } K_2 \textit{ containing } V_2\} \twoheadrightarrow \{\textit{Val. rings of } K_2 \textit{ containing } V_1\}$$

$$\{\textit{Val. rings of } K_2 \textit{ contained in } V_2\} \twoheadrightarrow \{\textit{Val. rings of } K_2 \textit{ contained in } V_1\}$$

induced by $R_2 \mapsto R_2 \cap K_1$. Moreover, if K_2/K_1 is algebraic, the first map is bijective.

Proof. Cf. [Wed19, 2.25] for the precise argument. Essentially, this follows from the ability to extend valuations by Proposition 1.18 and the descriptions of valuations which are more fine resp. more coarse than the one given by V in Corollary 1.10(4) and Construction 1.13. The bijectivity part is equivalent by Construction 1.13 and passage to the limit over finite subextensions to the statement that if V, W are valuation rings then any generically finite and faithfully flat map $\operatorname{Spec} W \to \operatorname{Spec} V$ is bijective. □

1.3. **Convex subgroups.** It is common to reformulate the rank in terms of the (totally ordered) value group of the valuation. It turns out (not very surprisingly) that the structure of totally ordered groups is not very complicated.

Definition 1.20. Let Γ be a totally ordered group. A subgroup Δ of Γ is *convex* (or *isolated*) any element of Γ, which lies between two elements of Δ is itself in Δ, i.e., if for all $a, b, c \in \Gamma$ holds:

$$a \le b \le c \text{ and } a, c \in \Delta \quad \Rightarrow \quad b \in \Delta$$

The *height* of Γ, $\mathrm{ht}(\Gamma) \in \mathbb{Z} \cup \{\infty\}$, is the number of convex subgroups of Γ minus 1.

All one should know about convex subgroups is contained in the following example and the next result:

Example 1.21. For $n \ge 0$, let $\Gamma = (\mathbb{R}^n_{>0}, \le)$ be the lexicographically ordered group from Example 1.3(3). Then for any $0 \le k \le n$, $\{1\}^k \times \mathbb{R}^{n-k}_{>0}$ is a convex subgroup, and every convex subgroup is one of these. So, $\mathrm{ht}(\Gamma) = n$.

Proposition 1.22. *Let Γ be a totally ordered group.*

(1) The convex subgroups of Γ are precisely the kernels of homomorphisms of ordered groups with source Γ.

(2) Let $\Delta \subseteq \Gamma$ be a subgroup, and define for $\bar{a}, \bar{b} \in \Gamma/\Delta$,

$$\bar{a} \le \bar{b} \;\;:\Leftrightarrow\;\; \exists a \in \bar{a}, b \in \bar{b} \text{ with } a \le b.$$[3]

Then $(\Gamma/\Delta, \le)$ is a totally ordered group if and only if Δ is convex. Assume this holds. Then $\mathrm{ht}(\Gamma) = \mathrm{ht}(\Delta) + \mathrm{ht}(\Gamma/\Delta)$. Moreover, if $\bar{a}, \bar{b} \in \Gamma/\Delta$, then $\bar{a} < \bar{b} \Leftrightarrow$ for all lifts $a \in \bar{a}, b \in \bar{b}$ one has $a < b$.

(3) We have $\mathrm{ht}(\Gamma) = 0 \Leftrightarrow \Gamma = 1$.

(4) Suppose $\Gamma \neq 1$. Then the following are equivalent:
   - *(a) $\mathrm{ht}(\Gamma) = 1$.*
   - *(b) there exists an injective homomorphism of ordered groups $\Gamma \to \mathbb{R}_{>0}$.*
   - *(c) For all $a, b \in \Gamma$ with $a < 1$ and $b \le 1$, there exists some $n \in \mathbb{N}$ with $a^n < b$.*

Proof. (1) and (2): Exercise or [Bou, §4 № 2, Prop.3 (p.108) and §4 №4, Expl. (p.111)]. Last claim of (2): exercise! (3) is clear as 1 and Γ are convex subgroups of Γ. (4): (a) $\Rightarrow$ (c): Let H be the smallest convex subgroup of Γ containing a. By inspection, H is the set of all $\beta \in \Gamma$ which sit between a^m and 1 for some $m \in \mathbb{Z}$. On the other hand, as $a \neq 1$, condition (a) implies that $H = \Gamma$. Thus b is sandwiched between 1 and some a^n; as $b < 1$, we also must have $a^n < 1$, i.e., $n > 0$. (c) $\Rightarrow$ (a): the characterization above shows that (c) implies that the smallest convex subgroup containing an element $\neq 1$ must be Γ. (b) $\Rightarrow$ (c): clear. (c) $\Rightarrow$ (b): a somewhat technical but easy construction; see [Mat89, Thm. 10.6]. □

Remark 1.23. Note that if $\Delta \subseteq \Gamma$ is not convex, then Γ/Δ with the order defined in Proposition 1.22(2) is *not* a totally ordered group. Indeed, as Δ is not convex, there exist (after scaling) $\delta < a < 1$ in Γ with $\delta \in \Delta$ and $a \notin \Delta$. With respect to the given order in Γ/Δ, one thus has $a \le \bar{1}$ (as $a < 1$ in Γ) and $\bar{1} \le \bar{a}$ (as $1 \cdot \delta \le a$ in Γ). But as $a \notin \Delta$, we have $\bar{1} \neq \bar{a}$, and hence $\bar{1} < \bar{a} < \bar{1}$, contradiction. (Exercise: look at the example $\Gamma = \mathbb{R}^2_{>0} \supseteq \Delta = \mathbb{R}_{>0} \times \{1\}$ with lexicographic order.)

[3] This might not hold for all lifts a, b, as the example $\Gamma = \mathbb{R}^2_{>0} \supseteq \Delta = \{1\} \times \mathbb{R}_{>0}$ with lex. order shows.

Corollary 1.24. *Let V be a valuation ring of a field K, and let $\Gamma_V := K^\times/V^\times$ be the corresponding value group. We have a natural bijection*

$$\operatorname{Spec} V \overset{\text{Cor. 1.10(4)}}{\cong} \{V \subseteq R \subseteq K \colon R \text{ (valuation) ring}\} \xrightarrow{\sim} \{\text{convex subgroups of } \Gamma_V\}$$
$$R \mapsto \ker(\Gamma_V \twoheadrightarrow \Gamma_R)$$
$$V_\Delta = \{a \in K \colon \; \exists \delta \in \Delta \text{ with } |a|_\Gamma \leq \delta\} = V \cdot \widetilde{\Delta} \leftarrow\!\shortmid \Delta,$$

where $\widetilde{\Delta}$ is the preimage of Δ under $K^\times \twoheadrightarrow K^\times/V^\times = \Gamma$. Moreover, the maximal ideal of V_Δ is

$$\mathfrak{p}_\Delta = \{a \in K \colon |a|_V < \delta \; \forall \; \delta \in \Delta\}.$$

(Note that this agrees with $\mathfrak{p}_{V_\Delta} \subseteq \mathfrak{p}_V \subseteq V \subseteq V_\Delta \subseteq K$.)

Proof. By Proposition 1.22(1), $\ker(\Gamma_V \twoheadrightarrow \Gamma_R)$ is a convex subgroup of Γ_V. Conversely, the convex subgroup $\Delta \subseteq \Gamma_V$ goes to the subring $V_{\mathfrak{p}_\Delta}$ with $\mathfrak{p}_\Delta$. (Exercise: check the details. Use the description of the order on Γ/Δ in Proposition 1.22(2).) $\square$

Corollary 1.25. *Let K be a field. Let $V \subseteq K$ be a valuation subring. Then the V has rank* 1 *if and only if it is maximal among all proper subrings of K.*

Remark 1.26. A valuation is discrete (i.e., the value group is isomorphic to $(q^{\mathbb{Z}}, \geq)$ for some $q > 0$) if and only if the corresponding valuation ring is noetherian if and only if it is a principal ideal domain. Cf. [Bou, §3 № 6, Prop. 9 (p. 105)].

Note that a valuation of rank 1 needs not to be discrete. E.g., the p-adic valuation of $\overline{\mathbb{Q}}_p$ (normalized such that $|p| = p^{-1}$) is of rank 1, but has $p^{\mathbb{Q}}$ as its value group and hence is not discrete.

2. The valuation spectrum

2.1. **The valuation spectrum of a ring.** We first study the adic spectrum of discrete rings (that is with rings equipped with discrete topology).

Definition 2.1 (valuation spectrum and standard opens)**.** Let R be a ring and $S \subseteq R$ a subset.

(1) We denote by
$$\operatorname{Spv}(R, S) = \{|\cdot| \text{ valuation on } R \colon |S| \leq 1\}/\sim$$
the set of all equivalence classes of valuations on R which are ≤ 1 on S.

(2) We equip $\operatorname{Spv}(R, S)$ with the topology generated by the *standard open subsets*
$$\operatorname{Spv}(R, S)\left(\frac{f_1, \dots, f_n}{g}\right) := \{|\cdot| \in \operatorname{Spv}(R, S) \colon |f_1|, \dots, |f_n| \leq |g| \neq 0\}$$
for varying $n \geq 0$, $f_1, \dots, f_n, g \in R$.

(3) Let R', S' be another pair as above and let $\varphi \colon R \to R'$ be a map of rings with $\varphi(S) \subseteq S'$. The induced map $\operatorname{Spv} \varphi \colon \operatorname{Spv}(R', S') \to \operatorname{Spv}(R, S)$ sends $|\cdot|$ to $|\cdot| \circ \varphi$.

(4) The *valuation spectrum* of R is the space $\operatorname{Spv}(R, \mathbb{Z})$.

Notation 2.2. We sometimes denote points of $\operatorname{Spv} R$ by letters $x, y, \dots$, and write $|\cdot|_x$ or $|\cdot(x)|$ for the corresponding valuation on R (so, x and $|\cdot|_x$, $|\cdot(x)|$ are identical objects). As in the case of schemes, we wish to regard elements of R as functions on the space of all valuations, the function given by $f \in R$ being $x \mapsto |f(x)|$ – this explains the latter notation.

We collect some simple facts about the valuation spectrum:

Remark 2.3. (1) All appearing valuations are non-archimedean, so we always have $|\mathbb{Z}| \leq 1$ and so $\operatorname{Spv} R := \operatorname{Spv}(R, \mathbb{Z}) = \operatorname{Spv}(R, \varnothing)$.

(2) (Exercise!) Prove that the collection of all standard opens of $\operatorname{Spv}(R,S)$ is stable under finite intersections. In particular, they form a base for topology on $\operatorname{Spa}(R,S)$ (and not just a subbase), and every open is a union of standard opens. Moreover, one has

$$\operatorname{Spv}(R,S)\left(\frac{f}{g}\right) \cap \operatorname{Spv}(R,S)\left(\frac{f'}{g'}\right) = \operatorname{Spv}(R,S)\left(\frac{fg', f'g}{gg'}\right), \tag{2.1}$$

and so in particular, $\operatorname{Spv}(R,S)\left(\frac{f_1}{g}\right) \cap \operatorname{Spv}(R,S)\left(\frac{f_2}{g}\right) = \operatorname{Spv}(R,S)\left(\frac{f_1,f_2}{g}\right)$[4]. In particular, $\left\{\operatorname{Spv}(R,S)(\frac{f}{g})\colon f,g \in R\right\}$ form a subbase for the topology on $\operatorname{Spv}(R,S)$.

(3) For any ring map φ, the map $\operatorname{Spv}\varphi$ is continuous as $(\operatorname{Spv}\varphi)^{-1}(U\left(\frac{f}{g}\right)) = U\left(\frac{\varphi(f)}{\varphi(g)}\right)$.

(4) Note that $U\left(\frac{f}{0}\right) = \varnothing$ and that all $U\left(\frac{0}{f}\right) = \{x\colon |f(x)| \neq 0\}$ and $U\left(\frac{f}{1}\right) = \{x\colon |f(x)| \leq 1\}$ are open subsets of $\operatorname{Spv}(R,S)$. Thus the topology on $\operatorname{Spv}(R,S)$ unifies the flavor of the Zariski topology on schemes (where opens are given by non-vanishing sets) as well as the real topology (where opens are given by inequalities). However, here strict inequalities define *open* subsets.

(5) (Exercise!) Show that $\operatorname{Spv}(R,S) = \operatorname{Spv}(R,S')$, where S' is the smallest integrally closed subring of R containing S. Thus there is no loss in assuming that S is itself an integrally closed subring of R.

Example 2.4. (1) We have $\operatorname{Spv}\mathbb{Q} = \{|\cdot|_{\mathrm{triv}}\} \cup \{|\cdot|_p\colon p \text{ prime}\}$. Here $|\cdot|_{\mathrm{triv}}$ resp. $|\cdot|_p$ is the trivial resp. p-adic valuation on $\mathbb{Q}$. (Check that there are no further points!) Moreover, for $f,g \in \mathbb{Q}$ with $g \neq 0$, $U(f/g) = \{|\cdot|_{\mathrm{triv}}\} \cup \{|\cdot|_p\colon v_p(f/g) \geq 0\}$, where v_p is the (additive) p-adic order of $f/g \in \mathbb{Q}$; so $\operatorname{Spv}\mathbb{Q} \smallsetminus U(f/g)$ is finite. We deduce $\operatorname{Spv}\mathbb{Q} \cong \operatorname{Spec}\mathbb{Z}$.

(2) We have $\operatorname{Spv}\mathbb{Z} = \operatorname{Spv}\mathbb{Q} \cup \{|\cdot|_{p,\mathrm{triv}}\colon p \text{ prime}\}$, where $|\cdot|_{p,\mathrm{triv}}$ is the valuation given by $\mathbb{Z} \twoheadrightarrow \mathbb{F}_p \overset{\mathrm{triv}}{\to} \{0,1\}$. As $U(p/p) = \operatorname{Spv}\mathbb{Z} \smallsetminus \{|\cdot|_{p,\mathrm{triv}}\}$ is open, $|\cdot|_{p,\mathrm{triv}}$ is closed in $\operatorname{Spv}\mathbb{Z}$. Moreover, $\overline{\{|\cdot|_p\}} = \{|\cdot|_p, |\cdot|_{p,\mathrm{triv}}\}$ (indeed, this means that any open which contains $|\cdot|_{p,\mathrm{triv}}$ also contains $|\cdot|_p$. It suffices to show the last claim for the opens in a subbase; but if $|\cdot| := |\cdot|_{p,\mathrm{triv}} \in U(f/g)$ for some $f,g \in \mathbb{Z}$, then $|g| \neq 0$ (i.e., $(p,g) \neq 0$), and so $|g|_p = 1$ and then also $|f|_p \leq 1 = |g|_p$ for all $f \in \mathbb{Z}$.)

(3) Let k be a (say, algebraically closed) field and let K/k be a finitely generated field extension of transcendence degree 1 (e.g., $K = k(t)$, or $K = k(t)[\sqrt{t^2-1}]$). We then have the uniquely determined smooth projective curve C over k with function field K, cf. [Har77, I §6, p.42]. It is a classical observation that $\operatorname{Spv}(K,k) = C$ as topological spaces[5] (and in fact, also as ringed spaces, once we define the structure sheaf on $\operatorname{Spv}(K,k)$).

Note also that $\operatorname{Spv} K$ is in general far too big: we have the restruction map $\operatorname{Spv} K \to \operatorname{Spv} k$, the fiber over the trivial valuation on k is precisely $\operatorname{Spv}(K,k)$, but all the other fibers, as well as the base $\operatorname{Spv} k$, will be huge in general (the latter will hold whenever k

[4] Note that in general, we have $\operatorname{Spv}(R,S)(\frac{fh}{gh}) \subseteq U(\frac{f}{g})$, but the inclusion might be strict; however, if $h = g$, it is always an equality.

[5] To be precise, from [Har77, I §6, p.42] it only follows that the subset consisting of K itself and all *discrete* valuation rings $k \subseteq V \subseteq K$ is in bijection with C; moreover, Constuction 1.13 shows that each of the discrete ones does not contain further ones. Unfortunately, this does not exclude the possibility that there are some *non-microbial* (cf. §**??** below) valuation rings in K containing k ... However, it is not very hard to show that *all* valuation rings V of K with $k \subseteq V \subsetneq K$ are disrete, cf. for example [Sch, Lm. 23.1].

is not algebraic over $\mathbb{F}_p$). This observation indicates that $\mathrm{Spv}(R,S)$, and not $\mathrm{Spv}\,R$, is the right object to look at.

(4) Let K be a field and K^+ a valuation ring of K, then we have a homeomorphism

$$\mathrm{Spv}(K,K^+) \cong \mathrm{Spec}(K^+), \quad |\cdot| \mapsto \{a \in K^+ : |a| < 1\}. \tag{2.2}$$

Indeed, $\mathrm{Spv}(K,K^+)$ in bijection with valuation subrings of K containing K^+ (indeed, the support of any valuation of K is the zero ideal, and so the equivalence classes of valuations which are ≤ 1 on K^+ are in bijection with valuation subrings containing K^+) and hence with $\mathrm{Spec}(K^+)$ by Corollary 1.24. Exercise: check that this is a homeomorphism.

Let R be a ring and $S \subseteq R$ a subset. There is a natural map

$$\mathrm{supp}\colon \mathrm{Spv}(R,S) \to \mathrm{Spec}\,R, \quad x \mapsto \mathrm{supp}(x)$$

This map is continuous and in fact induces a homeomorphism $\mathrm{supp}'\colon \mathrm{Spv}(R,S)_{\mathrm{triv}} \to \mathrm{Spec}\,R$, where $\mathrm{Spv}(R,S)_{\mathrm{triv}}$ is the subset of trivial valuations at all prime ideals of R. (*Proof*: For $f \in R$, $\mathrm{supp}^{-1}(D(f)) = \{x \in \mathrm{Spv}(R,S) : f \notin \mathrm{supp}(x)\} = \{x \in \mathrm{Spv}(R,S) : |f|_x \neq 0\} = \mathrm{Spv}(R,S)\left(\frac{0}{f}\right)$. This proves continuity. Further, note that supp' is clearly bijective, and it suffices to show that it is open; but if $\mathrm{Spv}(R,S)\left(\frac{f_1,\dots,f_n}{g}\right) \subseteq \mathrm{Spv}(R,S)$ is a standard open, then supp' maps $\mathrm{Spv}(R,S)((f_i)_{i=1}^n/g) \cap (\mathrm{Spv}\,R)_{\mathrm{triv}} = \{|\cdot|_{\mathfrak{p},\mathrm{triv}} : g \notin \mathfrak{p}\}$ to $D(g)$).

2.2. **Digression: Riemann–Zariski space of a field.** In this section we consider the valuation spectrum of a field. By Theorem 1.9 this is the same as just looking at valuation rings of K. Furthermore, note that as K has no non-trivial ideals, the support of any valuation is 0 and so $|g| \neq 0$ for all $g \in K^\times$ and all $|\cdot| \in \mathrm{Spv}\,K$. Thus, $U(f/g) = U(fg^{-1}/1)$, i.e., a (sub)base for topology on $\mathrm{Spv}\,K$ is given by $\{|f| \leq 1\}$ for $f \in K^\times$ varying.

As Example 2.4(3) suggests, it makes sense to introduce some constrains (pass to a subset of $\mathrm{Spv}(R)$) to obtain a more adequate notion in relative situations.

Definition 2.5. Let K be a field and $A \subseteq K$ any subring.

(1) The *Riemann–Zariski space* $\mathrm{RZ}(K,A)$ of K over A is the set of all valuation subrings $A \subseteq V \subseteq K$.

(2) The *Zariski topology* on $\mathrm{RZ}(K,A)$ is defined by declaring

$$U(f_1,\dots,f_n) := \mathrm{RZ}(K, A[f_1,\dots,f_n]) = \{V \in \mathrm{RZ}(K,A) : f_1,\dots,f_n \in V\}$$

to be a base of open subsets (note that $U((f_i)_{i=1}^n) \cap U((g_j)_{j=1}^m) = U((f_i)_{i=1}^n, (g_j)_{j=1}^m)$, so it's indeed a base).

Furthermore, $\mathrm{RZ}(K) := \mathrm{RZ}(K,\mathbb{Z})$ is called the *Riemann–Zariski space* of K.

Example 2.6. (1) $\mathrm{RZ}(\mathbb{Q}) \cong \mathrm{Spv}(\mathbb{Q})$ (see Proposition 2.8 below for the isomorphism) we have already seen in Example 2.4. (Determine also $\mathrm{RZ}(\mathbb{Q}_p)$ and $\mathrm{RZ}(\mathbb{Q},\mathbb{Q})$.)

(2) If k is a field and K/k is a finitely generated extension of transcendence degree 1, then $\mathrm{RZ}(K,k)$ is homeomorphic to the unique smooth projective k-curve with function field K. (Cf. Example 2.4(3))

(3) If K/k is as in (1) but of transcendence degree 2, the situation is more complicated. It is still true, that all points of $\mathrm{RZ}(K,k)$ will be valuation rings of height ≤ 2. However, a unique smooth projective model X/k such that $X \cong \mathrm{RZ}(K,k)$ does not exist. The informal reason is that there are many different candidates for such a model, and none of them should be preferred. E.g., if $K = k(x,y)$, then both, $X_1 = \mathbb{P}^2$ and $X_2 = \mathbb{P}^1 \times \mathbb{P}^1$ are smooth projective integral k-schemes with function field K. Now the valuative criterion for properness (with arbitrary, not necessarily discrete, valuation rings!) shows that

there are maps $\mathrm{RZ}(K,k) \to X_i$ for $i = 1,2$, which are in fact continuous for the Zariski topology on both sides. Note that there exists a third smooth projective surface X_3 with the same fraction field and birational maps $X_1 \leftarrow X_3 \to X_2$ (in fact, blow-ups), and by the same reason as before there is a map $\mathrm{RZ}(K,k) \to X_3$.

The same argument as in Example 2.6(3) shows in fact the following:

Proposition 2.7. *Let K/k be a finitely generated field extension. Then $\mathrm{RZ}(K,k)$ is homeomorphic to the inverse limit over all projective integral k-schemes with function field K.*

Continuing Example 2.6(3), all possibilities for points of $\mathrm{RZ}(K,k)$ with K/k finitely generated of transcendence degree 2, were determined by Zariski in 1939.[6].

Note that we can reformulate the definition in terms of valuations: then $\mathrm{RZ}(K,A)$ becomes the set of equivalence classes of valuations $|\cdot|$ on K, for which $|A| \leq 1$, and similarly for the open subsets. From this remark and the observations in the beginning of §2.2 it is clear that the bijection from Theorem 1.9 induces a natural homeomorphism

$$\mathrm{Spv}(K,A) \xrightarrow{\sim} \mathrm{RZ}(K,A).$$

In fact even a more general statement is true (by essentially the same arguments):

Proposition 2.8. *Let R be any ring. The fiber of* $\mathrm{supp}\colon \mathrm{Spv}\, R \to \mathrm{Spec}\, R$ *over any* $\mathfrak{p} \in \mathrm{Spec}\, R$ *is homeomorphic to* $\mathrm{RZ}(\mathrm{Frac}(R/\mathfrak{p}))$*.*

Proof. Exercise. □

In particular, the Riemann–Zariski space of a field is a special case of a valuation spectrum. As already the space $\mathrm{RZ}(K,k)$ with $\mathrm{tr.deg}_k(K) = 2$ is quite complicated, we should not expect $\mathrm{Spv}\, R$ of a general ring R to be a very accessible object. However, we will show that $\mathrm{Spv}\, R$ is always a spectral space (i.e., homeomorphic to $\mathrm{Spec}\,\mathscr{R}$ for some –possibly horrible– ring $\mathscr{R}$). In particular, $\mathrm{RZ}(K,A)$ should be quasi-compact, and the proof of this fact is not very hard.

Proposition 2.9 ([Mat89], Theorem 10.5)**.** *Let K be a field and $A \subseteq K$ be any subring. The Riemann–Zariski space* $\mathrm{RZ}(K,A)$ *is quasi-compact.*

Proof. We omit the proof, because this result will be a special case of the quasi-compactness of valuation spectrum $\mathrm{Spv}(R,S)$ in §3. □

It is remarkable that using the classification of valuation rings in dimensions 2,3 (see above for a reference) and the quasi-compactness of $\mathrm{RZ}(K,k)$, Zariski gave a valuation theoretic proof of resolution of singularities in characteristic 0 for varieties of dimension ≤ 3.

3. $\mathrm{Spv}(R,S)$ is a spectral space

To do geometry on $\mathrm{Spv}(R,S)$, we need it to be a reasonably well-behaved space. Towards this, we show that it is spectral. Recall what spectrality means:

Theorem 3.1 (Hochster)**.** *For a topological space X the following conditions are equivalent:*

(a) X has the following properties:
- *X is quasi-compact (i.e., any open cover has a finite subcover),*
- *X is sober (i.e., any irreducible closed subset of X has a unique generic point),*
- *the intersection of two qc opens of X is qc open, and*
- *the collection of qc opens of X form a basis for the topology on X.*

[6]See [Har77, II §4 Ex. 4.12(b)]; for a brief overview (and the reference to Zariski's original work), see https://en.wikipedia.org/wiki/Zariski%E2%80%93Riemann_space.

(b) X can be written as an inverse limit of finite T_0-spaces.
(c) $X \cong \operatorname{Spec} A$ for some ring A.

Recall that a space X is T_0 (sometimes also called *Kolmogorov*), if for any two points in X there is an open subset of X, which contains exactly one of these points. For any topological space X, we have the map $X \to \{\text{closed irred. subsets of } X\}$, $x \mapsto \overline{\{x\}}$; then X is T_0 (resp. sober) if and only if this map is injective (resp. bijective). In particular, a finite space is T_0 if and only if it is sober.

Definition 3.2. A topological space X is called *spectral* if it satisfies the equivalent conditions of Theorem 3.1.

Our aim is to prove the following theorem.

Theorem 3.3 ($\operatorname{Spv}(R,S)$ is spectral)**.** *For any ring R, the space $\operatorname{Spv}(R,S)$ is spectral. Moreover,*

$$\Sigma = \left\{ \operatorname{Spv}(R,S)\left(\frac{f_1,\dots,f_n}{g}\right) : f_1,\dots,f_n,g \in R \right\}$$

is a basis for topology of $\operatorname{Spv} R$ which consists of quasi-compact opens and is stable under finite intersections.

The key steps in proving (c) $\Rightarrow$ (a) in Theorem 3.1 are: (i) show that $\operatorname{Spec} R$ is quasi-compact for any ring R; (ii) show that the collection of principal opens $\{D(f)\colon f \in R\}$ is a basis for topology consisting of qc opens and stable under finite intersection: indeed, $D(f)\cap D(g) = D(fg)$ implies that last part, $D(f)$ is homeomorphic to $\operatorname{Spec} R_f$, hence (i) implies quasi-compactness, and it is easy to show that they form a basis; finally, (iii) check that a closed subset $V(I) \subseteq \operatorname{Spec} R$ is irreducible if and only if I is a prime ideal, by noting that $V(I) \cong \operatorname{Spec} R/I \cong \operatorname{Spec} R/\sqrt{I}$.

Whereas step (i) can be done for $\operatorname{Spv} R$ (just as in Proposition 2.9), the problem with steps (ii) and (iii) is that $U\left(\frac{f}{g}\right)$ are their closed complements are usually not of the form $\operatorname{Spv} R'$ for a ring R' (e.g., in contrast to the case of spectra of rings, neither $U\left(\frac{f}{g}\right)$ nor $\operatorname{Spv} R \smallsetminus U\left(\frac{f}{g}\right)$ need be homeomorphic to valuation spectra themselves).[7]

We will need some preliminaries. First, note that the collection of *all open* subsets of a spectral space has a much nicer companion, namely the collection of all *qc open* subsets. For a general topological space, this latter might not be big enough to "see" the topology, but for a spectral space it is, essentially by definition. Maps preserving this collection are particularly nice:

Definition 3.4. (i) A continuous map of topological spaces $f\colon Y \to X$ is *quasi-compact*, if for any qc open $U \subseteq X$, $f^{-1}(U)$ is qc open.
(ii) A topological space is *quasi-separated*, if the intersection of any two qc opens is again qc open. Equivalently, X is quasi-separated if the diagonal embedding $X \to X \times X$ is quasi-compact.

Example 3.5. Let k be a field and let $\mathbb{A}^\infty = \operatorname{Spec} k[x_1,x_2,\dots]$. Then $\mathbb{A}^\infty$ is qc (as follows from Hochster's theorem), but e.g., the subset $D(x_1,x_2,\dots) = \mathbb{A}^\infty \smallsetminus \{(0,0,\dots)\}$ is a an open subset which is not quasi-compact. Moreover, if we glue two copies of $\mathbb{A}^\infty$ along the identity on U, we get a quasi-compact scheme, which is not quasi-separated.

[7]In fact, this problem does not appear for the standard opens of the spaces $\operatorname{Spv}(R,S)$, cf. the discrete case of Proposition 7.7. This is because the choice of S provides additional flexibility.

We abbreviate "quasi-compact and quasi-separated" by *qcqs*. Note that the definition of a spectral space (Theorem 3.1(a)) can be rephrased as: X is qcqs + sober + the qc opens form a basis of topology. In a qcqs space we have the following well-behaved notions:

Definition 3.6. Let X be a qcqs topological space.

(1) A subset of X is *locally closed constructible*, if it is of the form $U \cap (X \setminus V)$ with $U, V \subseteq X$ qc opens.
(2) A subset of X is *constructible*, if it is a finite union of subsets of locally closed constructible subsets.
(3) The *constructible topology* on X is topology generated by constructible subsets (they form a base for topology). We denote X^{cons} the set X equipped with the constructible topology.
(4) A subset of X is *pro-constructible* (resp. *ind-constructible*), if it an intersection (resp. a union) of constructible subsets of X.

Before diving into the study of the constructible topology, let us recall the classical reason to study it:

Theorem 3.7 (Chevalley)**.** *Let X be a qcqs scheme.*

(1) The pro-constructible sets in X are precisely the images of all morphisms $Y \to X$ with Y an affine (or, equivalently, qcqs) scheme.
(2) The constructible sets in X are precisely the images of all morphisms of finite presentation $Y \to X$ with Y affine (or, equivalently, qcqs) scheme.

Note that part (1) follows from part (2), as we can write any morphism $Y \to X$ between affine schemes as an inverse limit $Y = \varprojlim_i Y_i \to X$ with all Y_i affine and all $Y_i \to X$ of finite presentation, and as in such a situation $\mathrm{im}(Y \to X) = \bigcap_i \mathrm{im}(Y_i \to X)$.

Example 3.8. Let X be a smooth curve over a (say, algebraically closed) field k. Then X^{cons} is the one point compactification of the (discrete) set $X(k)$. So, each closed point $x \in X$ is clopen in X^{cons}, and if $\eta \in X$ is the generic point, then any open neighboorhood of η contains all but finitely many points in $X(k)$. Exercise: verify this and show that X^{cons} is a profinite set. (Note: this profinite set is similar to the one point compactification $\mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$ with discrete topology.)

Example 3.8 suggests the right intuition for the general case:

Proposition 3.9. *Let X be a spectral space. Then X^{cons} is profinite, i.e., quasi-compact, Hausdorff and totally disconnected space.*

Proof. Let $x, y \in X$ with $x \neq y$. As X is sober, it is T_0, so there is an open subset U containing x but not y or vice versa. Wlog, assume $x \in U, y \notin U$. As X has a basis of topology by qc opens, we may shrink U and assume it is qc open. But then $U, X \setminus U$ are both clopen in X^{cons} and $x \in U, y \in X \setminus U$. This shows that X^{cons} is Hausdorff and totally disconnected. A somewhat technical argument (similar to that in the proof of Proposition 2.9) shows that X^{cons} is quasi-compact, cf. [Sta14, 08YF]. □

Note that the collection of constructible subsets is stable under finite unions, finite intersections and complements. Using the latter property and the quasi-compactness part of Proposition 3.9, we deduce:

Lemma 3.10. *Let X be qcqs topological space. A subset of X is closed (resp. open; resp. clopen) in the constructible topology if and only if it is pro-constructible (resp. ind-constructible; resp. constructible).*

Proof. As the collection of constructible subsets is stable under complements and form a base for the constructible topology, it is clear that any constructible subset is clopen in X^{cons}. It follows that any pro- (resp. ind-) constructible subset is closed (resp. open) in X^{cons}. Conversely, it is clear that any open in X^{cons} is a union of constructibles, hence ind-constructible. Finally, if $Z \subseteq X$ is pro-constructible, then $Z = \bigcap_i C_i$ with $C_i \subseteq X$ constructibles. But then $X \smallsetminus Z = \bigcup_i (X \smallsetminus C_i)$. As all $X \smallsetminus C_i$ are constructible, $X \smallsetminus Z$ is ind-constructible, hence open in X^{cons} by the above, and hence $X \smallsetminus Z$ is closed in X^{cons}.

Finally, we have to show that any $C \subseteq X$, which is clopen in X^{cons}, is constructible. First, as C is open in X^{cons}, we can write $C = \bigcup_i Y_i$ with all Y_i constructible (hence open in X^{cons}). Second, as C is closed in X^{cons}, it is quasi-compact by Proposition 3.9, and hence the open covering $C = \bigcup_i Y_i$ (in X^{cons}) has a finite refinement. Thus C is the union of finitely many constructible subsets, hence itself constructible. ☐

Pro- and ind-constructible sets are also very useful because of the following properties:

Lemma 3.11. *Let X be a spectral space.*

1. *A subset $S \subseteq X$ is closed (resp. open) if and only if it is pro-constructible and stable under specialization (resp. ind-constructible and stable under generization).*
2. *Let $S \subseteq X$ be pro-constructible subset. Then $\overline{S} = \bigcup_{s \in S} \overline{\{s\}}$.*

Proof. Cf. [Sta14, 0903]. ☐

Before we come to the application of the constructible topology to the proof of spectrality of $\mathrm{Spv}\, R$, let us outline another important aspect of it:

Remark 3.12. By Hochster's Theorem 3.1, any spectral space is homeomorphic to $\mathrm{Spec}\, R$ for some ring R. One might ask, whether the same holds for continuous maps between spectral spaces. Hochster proved that it indeed does, but only for *quasi-compact* continuous maps between spectral spaces. This additional condition has a very natural interpretation in the constructible topology:

Proposition 3.13. *Let $f\colon Y \to X$ be a continuous map of spectral spaces. The following are equivalent:*

1. *f is quasi-compact.*
2. *$f^{\mathrm{cons}}\colon Y^{\mathrm{cons}} \to X^{\mathrm{cons}}$ is continuous.*
3. *Preimage under f of any constructible set is constructible.*

Proof. (1) $\Rightarrow$ (3) is clear; (3) $\Rightarrow$ (2): if (3) holds, then (as preimage commutes with taking intersections) the same condition as in (3) holds for pro-constructible sets, which are (by Lemma 3.10) the same as the closed ones in the constructible topology. Thus f^{-1} takes closed subsets to closed ones in the constructible topology. With other words, f^{cons} is continuous. (2) $\Rightarrow$ (1): Let $U \subseteq X$ be qc open. Then U is clopen in X^{cons}, and hence, by assumption, $f^{-1}(U)$ is clopen in Y^{cons}. As Y^{cons} is quasi-compact by Proposition 3.9, the closed subset $f^{-1}(U)$ is too. Now, the identity map $f^{-1}(U)^{\mathrm{cons}} \to f^{-1}(U)$ is[8] surjective and continuous (with respect to the usual on the right side). As the image of a quasi-compact subset under a continuous map is always quasi-compact, it follows that $f^{-1}(U)$ is also quasi-compact with the usual topology. ☐

Remark 3.12 makes the following definition natural.

[8]To be precise, $f^{-1}(U)^{\mathrm{cons}}$ means here "$f^{-1}(U)$ equipped with the subspace topology of X^{cons}", and not "the constructible topology on $f^{-1}(U)$", although (as one can prove) both notions coincide *a posteriori* (note that we actually aim to prove that $f^{-1}(U)$ is quasi-compact, hence itself a spectral space, so we cannot yet form its constructible topology).

Definition 3.14. A *spectral* map between spectral spaces is a continuous map, which satisfies the equivalent conditions of Proposition 3.13.

Now we prove Theorem 3.3.

Proof of Theorem 3.3. By Remark 2.3(2), the collection of standard opens is stable under finite intersections. We have to show that $X = \operatorname{Spv}(R, S)$ itself and all of them are quasi-compact. This follows from the combination of Lemma 3.15 and Proposition 3.16 (which is the cornerstone of the proof). Finally, by Lemma 3.17, X is sober. □

Lemma 3.15. *Let A be a ring and A^+ an integrally closed subring. For any $n > 0$ and $f_1, \dots, f_n, g \in A$, put $B := A\left[\frac{1}{g}\right]$ and $B^+ := A^+\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right]$. Then restriction of valuations induces a homeomorphism*

$$\operatorname{Spv}(B, B^+) \xrightarrow{\sim} \operatorname{Spv}(A, A^+)\left(\frac{f_1, \dots, f_n}{g}\right). \tag{3.1}$$

Proof. Any valuation $|\cdot|\colon A \to \Gamma \cup \{0\}$ with $|g| \neq 0$ extends uniquely to a valuation of $A\left[\frac{1}{g}\right]$ (by the universal property of localization) and the condition $|f_i| \leq |g|$ translates into the condition $\left|\frac{f_i}{g}\right| \leq 1$. With other words, this defines an inverse to the restriction map in (3.1), proving that it is a bijection.

Moreover (to simplify notation, assume $n := 1$ and $f := f_1$, general case is similar), if $f', g' \in A$, then

$$\operatorname{Spv}\left(A, A^+\right)\left(\frac{f}{g}\right) \cap \operatorname{Spv}\left(A, A^+\right)\left(\frac{f'}{g'}\right) = \operatorname{Spv}\left(A, A^+\right)\left(\frac{fg', f'g}{gg'}\right) = \operatorname{Spv}\left(A\left[\frac{1}{g}\right], A^+\left[\frac{f}{g}\right]\right)\left(\frac{f'}{g'}\right).$$

Thus standard opens on the left side of (3.1) map to opens on the right side, and conversely, opens on the right are determined by some pairs (in general $n+1$-tuples) of the form $f'g^i, g'g^j$, with $i, j \in \mathbb{Z}$. and such a pair determines the same open as $f'g^{i+N}, g'g^{j+N}$ for arbitrary $N \in \mathbb{Z}$ as g is invertible in $A[\frac{1}{g}]$, so we may assume that all these elements lie in A. Hence any open on the right side of (3.1) comes from the left. With other words, (3.1) is a homeomorphism. □

Proposition 3.16. *The space $\operatorname{Spv}(A, A^+)$ is quasi-compact.*

Proof. Write $X = \operatorname{Spv}(A, A^+)$. Consider the coarsest topology on X, in which all standard open subsets are open and closed.[9] We denote X equipped with this topology by X'.

Note that by Lemma 1.6(3) each valuation $x \in X$ is uniquely determined by the set $|_x := \{(a, b) \in A \times A\colon |b|_x \leq |a|_x\} \subseteq A \times A$. This defines an injection

$$i\colon X' \hookrightarrow 2^{A \times A} := \prod_{a,b \in A \times A} \{0, 1\}, \quad x \mapsto (\delta_x(a, b))_{a,b}, \quad \text{where } \delta_x(a, b) = 1 \Leftrightarrow (a, b) \in |_x.$$

The product of the finite discrete sets on the right side carries the product topology, with respect to which it is compact (=Hausdorff + quasi-compact) by Tychonoff's theorem. Now we claim that i is continuous and that the topology on X' coincides with the subspace topology of $2^{A \times A}$. For continuity, it suffices to show that the composition of i with each projection $p_{a,b}\colon 2^{A \times A} \to \{0, 1\}$ $(a, b \in A)$ is continuous. But $p_{a,b}^{-1}(1) \cap i(X')$ is the image under i of the union of subsets $\operatorname{Spv}(A, A^+)(\frac{a}{b})$, $\big(\operatorname{Spv}(A, A^+) \smallsetminus \operatorname{Spv}(A, A^+)(\frac{0}{b})\big) \cap \big(\operatorname{Spv}(A, A^+) \smallsetminus \operatorname{Spv}(A, A^+)(\frac{0}{a})\big)$ of X', both of which are open in X'. Thus i is continuous, i.e., the subspace topology on $i(X')$ is coarser or equal

[9] Once Theorem 3.3 is proven and we know that all $\operatorname{Spv}(A, A^+)$ are spectral, this topology is just the constructible topology on X. At the moment we do not know this.

to the defining topology of X'. Conversely, if $a, b \in A$, then $i(\mathrm{Spv}(A, A^+)(\frac{a}{b})) = p_{a,b}^{-1}(1) \smallsetminus p_{0,b}^{-1}(1)$ is clopen in the subspace topology on $i(X')$. This shows our claim.

Suppose for a second that we know that $i(X')$ is a *closed* subset of $2^{A\times A}$. As $2^{A\times A}$ is compact, also $i(X')$ is compact. By the above claim, also X' is compact, and in particular, quasi-compact. By definition of topology on X', the identity map $X' \to X$ is evidently continuous. As the image of a quasi-compact set under a continuous map is quasi-compact, the proposition follows.

It remains to show that $i(X)$ is closed in $2^{A\times A}$. Towards this, we claim that $i(X)$ is characterised as exactly the set of those $| \in 2^{A\times A}$, which satisfy the following conditions (we write $a|b$ for "$(a, b) \in |$", i.e., for $|a|_x \geq |b|_x$ if x is a valuation corresponding to $|$) for all $a, b, c \in A$:

- $a|b$ or $b|a$ holds, and $0 \not| 1$ holds,
- $a|b$ and $b|c \Rightarrow a|c$,
- $a|b \Rightarrow \forall d \in A\colon ad|bd$,
- $a|b$, $a|c \Rightarrow a|b + c$,
- $ac|bc$ and $c \neq 0 \Rightarrow a|b$,
- $\forall s \in A^+\colon 1|s$.

It is clear that any valuation on A satisfies all these conditions except possibly the last one, and all the valuations in X' also satisfy the last one. Thus, $i(X')$ is contained in the set of all $|$ satisfying this list of conditions. Conversely, suppose that $|$ satiesfies all conditions. Let $M := A/\sim$ where $\sim$ is the equivalence relation is defined by $a \sim b \Leftrightarrow a|b$ and $b|a$. Then one verifies quickly by means of (1)-(5) that M is a totally ordered commutative monoid with neutral element, with respect to a multiplication, which descends from A; that $M^\times := M \smallsetminus \{0\}$ is a totally ordered submonoid (with neutral element) and with the cancellation property (i.e., if $mn = mk$ for $m, n, k \in M^\times$, then $n = k$). For any commutative monoid with neutral element, there is a universal map into an abelian group; moreover, this map is injective if the monoid is has the cancellation property; moreover, if the monoid was totally ordered, it defines a unique compatible order on the "quotient group". Applying these considerations to $M^\times$ we deduce a totally ordered group Γ and a composed map $A \twoheadrightarrow M \to \Gamma \cup \{0\}$, which is in fact a valuation. This shows the claim.

After we have characterised $i(X)$ as the set those $|$ satisfying the above list of conditions, it remains to show that any condition cuts out a closed subset of $2^{A\times A}$, which is easy. Let us check this (for example) for the second condition above. For $a, b \in A$ consider the function $p_{a,b}\colon 2^{A\times A} \to \{0, 0\}$ be the projection to the (a, b)-component. Then $p^{-1}(1) = \{| \in 2^{A\times A}\colon a|b\}$ is closed. Then the set of all $|$ which satisfy the second condition is equal to $p_{a,c}^{-1}(1) \cup (p_{a,c}^{-1}(0) \cap (p_{a,b}^{-1}(0) \cup p_{b,c}^{-1}(0)))$. □

Lemma 3.17. *The space* $\mathrm{Spv}(A, A^+)$ *is sober. That is, any irreducible closed subset of* $\mathrm{Spv}(A, A^+)$ *has a unique generic point.*

Proof. Write $X = \mathrm{Spv}(A, A^+)$. If $x \neq y \in X$, then (up to exchanging x, y), there is some $f, g \in A$ with $|f(x)| \leq |g(x)|$ and $|f(y)| > |g(y)|$ (otherwise x, y are equivalent). If $|g(x)| \neq 0$, then $x \in X\left(\frac{f}{g}\right)$, $y \notin X\left(\frac{f}{g}\right)$. If $|g(x)| = 0$, then $x \notin X\left(\frac{0}{g}\right)$ but $y \in X\left(\frac{0}{g}\right)$. This shows that X is T_0.

It remains to show that any closed irreducible set $Z \subseteq X$ has a unique generic point. We define a valuation x_Z by the binary relation $|_Z$ which it determines (as in the proof of Proposition 3.16). Namely, for an element $a \in A$, let $V(a) = X \smallsetminus X\left(\frac{0}{a}\right)$ be the vanishing locus of a. For $a, b \in A$ then put

$$a|_Z b \;:\Leftrightarrow\; Z \subseteq V(a) \cap V(b) \text{ or } X\left(\frac{b}{a}\right) \cap Z \neq \varnothing,$$

that is either a, b are 0 on Z or there is some point $z \in Z$ with $|b(z)| \leq |a(z)| \neq 0$. Then, using irreducibility of Z, one checks that $|_Z$ is a valuation in X (we omit the details) and we denote the corresponding point in X by η. We have to show that $\overline{\{\eta\}} = Z$. Let $x \in Z$. If $x \in X\left(\frac{b}{a}\right)$, then $X\left(\frac{b}{a}\right) \cap Z \neq \varnothing$, so by definition of η, $\eta_Z(b) \leq \eta(a)$. Moreover, note that $\eta(a) \neq 0$, as $\eta(a) = 0$ (by definition of η) implies that either $Z \subseteq V(a) \cap V(b)$, or that $X\left(\frac{0}{a}\right) \cap Z \neq \varnothing$; but both cannot hold as $a \neq 0$ on the whole set Z). Thus $\eta \in X\left(\frac{a}{b}\right)$. This shows $Z \subseteq \overline{\{\eta\}}$. Conversely, suppose $x \notin Z$. As Z is closed, there must be some open $X\left(\frac{a}{b}\right)$ containing x and satisfying $X\left(\frac{a}{b}\right) \cap Z = \varnothing$. We claim that $\eta \notin X\left(\frac{a}{b}\right)$. Indeed, $\eta \in X\left(\frac{a}{b}\right)$ would mean that either $X\left(\frac{a}{b}\right) \cap Z = \varnothing$ (which is not true by assumption), or $Z \subseteq V(a) \cap V(b)$, i.e., $b = 0$ on Z – but then $\eta(b) \leq \eta(0)$, and so $\eta(b) = 0$, and so $\eta \notin X\left(\frac{a}{b}\right)$, contradiction. □

4. Specializations in $\mathrm{Spv}(R, R^+)$

Our goal in the following next lectures is to add topology on the ring R itself, introduce the adic spectrum $\mathrm{Spa}(R, R^+) \subseteq \mathrm{Spv}(R, R^+)$ (only looking at continuous valuations on R), prove that $\mathrm{Spa}(R, R^+)$ is a spectral space too (however, in general not closed and not even pro-constructible in $\mathrm{Spv}(R, R^+)$), study its properties and consider examples. Before doing so, we need some understanding of specialization relations in the the spectral space $\mathrm{Spv}(R, R^+)$.

Recall that in any T_0-space (in particular, in any spectral space), a point x *specializes* to a point y, or equivalently y *generalizes* to x (notation: $x \rightsquigarrow y$) if $y \in \overline{\{x\}}$. By the T_0 property, we cannot have $x \rightsquigarrow y$ and $y \rightsquigarrow x$ simultaneously, unless $x = y$. It follows that specialization defines a partial order, the *specialization order*, on the space. E.g., in an affine scheme $\mathfrak{p} \rightsquigarrow \mathfrak{q} \Leftrightarrow \mathfrak{q} \in \overline{\{\mathfrak{p}\}} \Leftrightarrow \mathfrak{p} \subseteq \mathfrak{q}$. Specialization in $\mathrm{Spv}\, R$ is more subtle. To understand it, we break it up in two special cases, the *horizontal* and the *vertical* specialization.

4.1. Vertical specializations.

Vertical specialization is easy and happens in the fibers of the support map (whence the name): Let $X = \mathrm{Spv}(R, R^+)$ and recall the map $\mathrm{supp}\colon X \subseteq \mathrm{Spv}\, R \to \mathrm{Spec}\, R$ from Proposition 2.8. Let $x \in X$ be a point with $\mathfrak{p} = \mathrm{supp}(x)$, so that $x \in \mathrm{supp}^{-1}(\mathfrak{p}) = \mathrm{Spv}(\mathrm{Frac}(R/\mathfrak{p}), \bar{R}^+)$, where $\bar{R}^+$ is the image of $R^+ \subseteq R \to R/\mathfrak{p} \to \mathrm{Frac}(R/\mathfrak{p})$. Then by Corollary 1.10(4) we understand all the generizations $y \in \mathrm{Spv}(R, R^+)$ of x which are contained in $\mathrm{supp}^{-1}(\mathfrak{p})$.

Definition 4.1. Let R be a ring. A specialization $x \rightsquigarrow y$ in $\mathrm{Spv}\, R$ is *vertical* if $\mathrm{supp}\, x = \mathrm{supp}\, y$.

Thus, vertical specialization with support $\mathfrak{p} \in \mathrm{Spec}\, R$ happens in the Riemann–Zariski space $\mathrm{RZ}(\mathrm{Frac}\, R/\mathfrak{p})$, where points are given by valuation rings, and we can describe them more explicitly.

Lemma 4.2. *Let R be a ring, $x \in \mathrm{Spv}\, R$ with corresponding valuation ring $k(x)^+ \subseteq k(x) = \mathrm{Frac}(R/\mathrm{supp}(x))$. Then there is a natural bijection*

$$\{\textit{vertical generizations of } x \textit{ in } X\} \cong \mathrm{Spec}\, k(x)^+ \cong \mathrm{Spv}(k(x), k(x)^+) \cong \{\textit{convex subgroups of } \Gamma\}$$

Under the first isomorphism, x itself corresponds to the closed point of $\mathrm{Spec}\, k(x)^+$ and the trivial valuation $|\cdot|_{\mathrm{triv},\mathrm{supp}(x)}$ with support $\mathrm{supp}(x)$ corresponds to the generic point.

Proof. All vertical generizations y of x lie in $\mathrm{RZ}(k(x))$, so correspond to valuation rings of $k(x)$. The condition that y specializes to x means that any open of $\mathrm{Spv}\, R$ –or, equivalently, of $\mathrm{RZ}(K)$– containing x also contains y. This is equivalent to the fact that the valuation ring $V_y \subseteq K$ of y contains $k(x)^+$. But those are in bijection with $\mathrm{Spec}\, k(x)^+$ by Corollary 1.10(4). The two last bijection follow from Corollary 1.24. □

We spell out which valuation corresponds to a convex subgroup $\Delta \subseteq \Gamma_x$ under the bijection in Lemma 4.2: it is

$$x_{/\Delta}\colon R \xrightarrow{|\cdot|_x} \Gamma \cup \{0\} \twoheadrightarrow \Gamma/\Delta \cup \{0\}, \tag{4.1}$$

where Γ/Δ is totally ordered as in Proposition 1.22(2).

Example 4.3. If $R = K$ is a field and S any subset, then $X = \operatorname{Spv}(K, S) = \operatorname{RZ}(K, S)$ admits only vertical specializations. In particlar, the set of all generizations of a point $x \in X$ corresponding to a valuation subring $V \subseteq K$ is homeomorphic to $\operatorname{Spec} V$, and thus looks quite simple. But given $y \in X$, the set $\overline{\{y\}}$ might be much more complicated. In particular, $|\cdot|_{\mathrm{triv}} \in X$ is the generic point of X.

4.2. **Horizontal specializations.** In *analytic* adic spaces, considered below, horizontal specializations do not appear: all specializations are vertical. However in discrete adic spaces, horizontal specializations do appear and are as important as vertical ones.

To define horizontal specializations we need the characteristic subgroup of a valuation. Let $x \in X = \operatorname{Spv} R$, given by $|\cdot|_x\colon R \to \Gamma_x \cup \{0\}$, where Γ_x is the value group of x (by definition generated by the submonoid $|R|_x \smallsetminus \{0\}$). Note that in general we can have $|R|_x \smallsetminus \{0\} \neq \Gamma_x$. The difference is measured by the characteristic subgroup:

Definition 4.4. Let R be a ring and $x \in \operatorname{Spv} R$. The convex subgroup $c\Gamma_x$ of Γ_x, generated by $\operatorname{im}(x) \cap \Gamma_{x,\geq 1}$ is called the *characteristic subgroup* of x.

Example 4.5.
(1) If $R = K$ is a field, then $|R|_x \smallsetminus \{0\}$ is a group, and hence $c\Gamma_x = \Gamma_x$ for all x. (This will translate to the fact that in Spv of a field there are no horizontal specializations.)
(2) If $R = V$ is a valuation ring and x is its defining valuation, then $|R|_x \smallsetminus \{0\} \subseteq \Gamma_{x,\leq 1}$, and so $c\Gamma_x = 1$.

To understand the characteristic subgroup better, recall the factorization of $|\cdot|_x$ as

$$R \to K_x \to \Gamma_x \cup \{0\}$$

and the valuation subring $V_x = \{a \in K_x\colon |a|_x \leq 1\}$. We then have $\Gamma_x \cong K_x^\times/V_x^\times$ (cf. the last statement of Theorem 1.9). Now, in the extreme case that $\operatorname{im}(R \to K_x) \subseteq V_x$, we have $c\Gamma_x = 1$; in the other extreme case when $R \to K_x$ is surjective, we have $c\Gamma_x = \Gamma_x$. In general, recall (Proposition 1.10 and Corollary 1.24) the bijections

$$\begin{array}{ccccc} \{V_x \subseteq W \subseteq K_x\colon W \text{ (valuation) ring}\} & \overset{\sim}{\leftarrow} & \operatorname{Spec} V_x & \overset{\sim}{\longleftarrow} & \{\text{convex subgroups of } \Gamma_x\} \\ V_{x,\Delta} := V_{x,\mathfrak{p}_{x,\Delta}} & \leftarrow\!\shortmid & \mathfrak{p} := \mathfrak{p}_{x,\Delta} & \leftarrow\!\shortmid & \Delta \end{array}$$

where

$$V_{x,\Delta} = \{a \in K_x\colon \ \exists\, \delta \in \Delta \text{ with } |a|_x \leq \delta\} \quad \text{and} \quad \mathfrak{p}_{x,\Delta} = \{a \in V_x\colon |a|_x < \delta\ \forall \delta \in \Delta\}$$

Lemma 4.7 below will show that $c\Gamma_x$ is the smallest convex subgroup $\Delta \subseteq \Gamma_x$ such that $\operatorname{im}(R \to K_x) \subseteq V_{x,\mathfrak{p}_{x,\Delta}}$.

Construction 4.6. Let R be a ring, $x \in \operatorname{Spv} R$, $\Delta \subseteq \Gamma_x$ a (convex) subgroup. Define the function $|\cdot|_{x|_\Delta}\colon R \to \Delta \cup \{0\}$ by

$$|a|_{x|_\Delta} = \begin{cases} |a|_x & \text{if } |a|_x \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

This turns out to define a valuation on R if and only if $\Delta \supseteq c\Gamma_x$. Let us study this.

Proposition 4.7. *Let $x \in \operatorname{Spv} R$ and $\Delta \subseteq \Gamma_x$ be a convex subgroup. Write $\mathfrak{p} = \mathfrak{p}_{x,\Delta}$. The following are equivalent:*

(1) *$|\cdot|_{x|_\Delta}$ is a valuation on R.*
(2) *$\Delta \supseteq c\Gamma_x$.*[10]
(3) *The image of $R \to K_x$ is contained in the localization $V_{x,\mathfrak{p}}$.*

If these hold, then we have

(a) *$x|_\Delta$ is a specialization of x,*
(b) *$\operatorname{supp}(x|_\Delta) = \pi^{-1}(\mathfrak{p}V_{x,\mathfrak{p}})(\supseteq \operatorname{supp}(x))$, where $\pi\colon R \to V_{x,\mathfrak{p}}$ is the map given by condition (3); equality holds if and only if $\Delta = \Gamma_x$,*
(c) *$V_x/\mathfrak{p} \subseteq V_{x,\mathfrak{p}}/\mathfrak{p}V_{x,\mathfrak{p}}$ is a valuation ring and $x|_\Delta$ is the composition of $R \to V_{x,\mathfrak{p}} \to V_{x,\mathfrak{p}}/\mathfrak{p}V_{x,\mathfrak{p}}$ with the corresponding valuation.*

Point (3) in Proposition 4.7 tells us that, rouhgly, the more "field-like" (as opposed to "valuation ring-like") the ring $R/\operatorname{supp}(x)$ looks, the less horizontal specializations will $x \in \operatorname{Spv}(R, R^+)$ have.

Proof. (1) $\Rightarrow$ (2): It suffices to show that $\Delta \supseteq \Gamma_{x,\geq 1} \cap |R|_x$, as the latter generates $c\Gamma_x$ as a convex subgroup and as Δ is convex. Let $a \in R$ with $|a|_x \geq 1$. If $|a|_x = 1$, then clearly $|a|_x \in \Delta$. Suppose $|a|_x > 1$ and $|a|_x \notin \Delta$. Then $|a+1|_x = |a|_x \notin \Delta$, and so $0 = |a+1|_{x|_\Delta} = \max(|a|_{x|_\Delta}, 1) = \max(0,1) = 1$, contradiction (second equality, as $|\cdot|_{x|_\Delta}$ is a valuation).

(2) $\Rightarrow$ (3): Recall that $V_\mathfrak{p} = \{y \in K_x \colon \exists\, \delta \in \Delta \text{ with } |y|_x \leq \delta\}$. Now, suppose $\Delta \supseteq c\Gamma_x$, and let $a \in R$. We have to show that there is some $\delta \in \Delta$ with $|a|_x \leq \delta$. If $|a|_x \leq 1$, we may take $\delta = 1$. If $|a|_x > 1$, then $|a|_x \in \Gamma_{x,\geq 1} \cap |R|_x = c\Gamma_x \subseteq \Delta$, and so we may take $\delta = |a|_x$.

(3) $\Rightarrow$ (1): We need a lemma.

Lemma 4.8. *Suppose (3) holds. If $a \in R$ with $|a|_x \notin \Delta$, then $|a|_x < 1$.*

Proof. Suppose $|a|_x \geq 1$. By (3) we must have $|a|_x \in V_{x,\mathfrak{p}} = V_{x,\Delta}$, that is there is some $\delta \in \Delta$ with $|a|_x \leq \delta$. Thus $1 \leq |a|_x \leq \delta$. As $1, \delta \in \Delta$ and $\Delta \subseteq \Gamma_x$ convex, we deduce $|a|_x \in \Delta$, contradicting the assumption. □

Now we show, using this observation, that $|\cdot|_{x|_\Delta}$ is a valuation. Clearly, $|0|_{x|_\Delta} = 0$ and $|1|_{x|_\Delta} = 1$. Let $a, b \in R$. If $|a|_x \in \Delta \cup \{0\}$ or $|b|_x \in \Delta \cup \{0\}$, then $|ab|_x \in \Delta \cup \{0\}$ (as Δ is a subgroup) and so $|a|_{x|_\Delta}|b|_{x|_\Delta} = |ab|_{x|_\Delta}$. Assume $|a|_x, |b|_x \notin \Delta \cup \{0\}$. By Lemma 4.8, $|a|_x, |b|_x < 1$. Thus $|ab|_x \leq |a|_x \leq 1$, and as Δ is convex, we would get a contradiction if $|ab|_x \in \Delta$. Thus $|ab|_x \notin \Delta$, and then $|a|_{x|_\Delta}|b|_{x|_\Delta} = 0 = |ab|_{x|_\Delta}$ by definition. Finally, we must check that $|a+b|_{x|_\Delta} = \max(|a|_{x|_\Delta}, |b|_{x|_\Delta})$. If $|a+b|_x \notin \Delta$, then this is clear. If not, let wlog $|a|_x \leq |b|_x$. Using the just proven multiplicativity, we may, multiplying with $|b|_x^{-1}$, assume $|b|_x \leq 1$. Then $|a+b|_x \leq |b|_x \leq 1$, and convexity of Δ implies that $|b|_x \in \Delta$. But then the result is clear. This finishes the proof of the equivalences.

(a): Let $f, g \in R$, such that $|f|_{x|_\Delta} \leq |g|_{x|_\Delta} \neq 0$ (i.e., $x|_\Delta \in U\left(\frac{f}{g}\right)$). We must check that $|f|_x \leq |g|_x \neq 0$. First, directly from definition, $|g|_{x|_\Delta} \neq 0$ implies $|g|_x \neq 0$. In particular, $|g|_x \in \Delta$. Assume that $|f|_x > |g|_x$. Then (from definition of $x|_\Delta$ and as $|f|_x \leq |g|_x \neq 0$) we must have $|f|_x \notin \Delta$. Then Lemma 4.8 gives $|f|_x < 1$. But then we have $|g|_x < |f|_x < 1$ with $|g|_x, 1 \in \Delta$ and $|f|_x \notin \Delta$, contradicting the convexity of Δ.

[10] Note that $R/\operatorname{supp}(x) \subseteq V_{x,\Delta}$ need not to hold in general: e.g. when $R = K$ is a field, so that necessarily $K_x = K$, we always have $R/\operatorname{supp}(x) = K_x$ but V_x and $V_{x,\Delta}$ might be strictly smaller than K_x. This just means that in the case of Spv of a field there are no horizontal specializations. Note that this agrees with the fact that for a field, the support map has only one fiber, so that all specializations are vertical.

(b): It is clear that $\mathrm{supp}(x) \subseteq \mathrm{supp}(x|_\Delta)$. Now, $V_{x,\mathfrak{p}} \subseteq K_x$ is the valuation ring corresponding to the valuation $K_x \xrightarrow{|\cdot|_x} \Gamma_x \cup \{0\} \twoheadrightarrow \Gamma_x/\Delta \cup \{0\}$. Thus its maximal ideal is

$$\mathfrak{p}V_{x,\mathfrak{p}} = \{\bar{a} \in K_x \colon |\bar{a}|_x < \gamma \ \forall \gamma \in \Delta\} \tag{4.2}$$

An element $a \in R$ lies in $\mathrm{supp}(x|_\Delta)$ if and only if $|a|_x \notin \Delta$. We would be done by (4.2) once we show that this is equivalent to $|a|_x < \gamma$ for all $\gamma \in \Delta$. Clearly, the implication $\Leftarrow$ holds. Conversely, suppse $|a|_x \notin \Delta$. Then by Lemma 4.8, $|a|_x < 1$. □

Let us consider a simple special case:

Example 4.9. Let V be a valuation ring with $K = \mathrm{Frac}(V)$ and with defining valuation $x \colon V \to \Gamma \cup \{0\}$, so that $x \in \mathrm{Spv}(V,V)$. Let $\Delta \subseteq \Gamma$ be a convex subgroup, let $\mathfrak{p}_\Delta \in \mathrm{Spec}\, V$ denote the corresponding prime ideal and let $x_{/\Delta} \in \mathrm{Spv}(V,V)$ be the corresponding valuation (note that the $V_{x_{/\Delta}} = V_{\mathfrak{p}_\Delta} \subseteq K$ contains $V = V_x$, so indeed $x_{/\Delta} \in \mathrm{Spv}(V,V)$). Then $x_{/\Delta}$ is a vertical generization of x.

Note that as $|V|_x \subseteq \Gamma_{\leq 1}$, $c\Gamma = 1$. Thus $x|_\Delta$ is a valuation by Proposition 4.7. Moreover, $x|_\Delta \in \mathrm{Spv}(V,V)$, and we explicate it now. Therefore note that we have

$$\{a \in V \colon |a|_x \notin \Delta\} = \mathfrak{p}_\Delta$$

by Proposition 4.7(b). So the valuation $x|_\Delta$ is given by $V \twoheadrightarrow V/\mathfrak{p}_\Delta \xrightarrow{|\cdot|_x} \Delta \cup \{0\}$ and in particular the support of $x|_\Delta$ is $\mathfrak{p}_\Delta$. Whereas $x, x_{/\Delta}$ lie over the generic point of $\mathrm{Spec}\, V$, the point x_Δ lies over the point $\mathfrak{p}_\Delta$.

Example 4.10. Suppose $c\Gamma_x = 1$. Then the minimal horizontal specialization of $x \in \mathrm{Spv}\, R$ is $x|_1$. By definition it is the valuation $R \to R/\mathrm{supp}(x) \to V_x \twoheadrightarrow V_x/\mathfrak{m}_x \xrightarrow{\mathrm{triv}} \{0,1\}$. With other words, $x|_1$ is the trivial valuation on R with support equal to the (preimage in R) of the maximal ideal of x.

Given $x \in \mathrm{Spv}\, R$, which prime ideals $\mathfrak{p} \in \mathrm{Spec}\, R$ can appear as supports of horizontal specializations of x?

Definition 4.11. Let $x \in \mathrm{Spv}\, R$. A prime $\mathfrak{p} \in \mathrm{Spec}\, R$ is called *x-convex* if for all $a \in R$ we have: $|a|_x \leq |t|_x$ for some $t \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$.[11]

Note that $\mathrm{supp}(x)$ is x-convex and is contained in any x-convex prime.

Proposition 4.12 ([Mor19], I.3.3.9-10)**.** *Let $x \in \mathrm{Spv}\, R$. Let*

$$S = \{\textit{horizontal specializations of } x\} \subseteq \overline{\{x\}} \subseteq \mathrm{Spv}\, R$$

$$C = \{x\textit{-convex primes in } \mathrm{Spec}\, R\} \subseteq \overline{\{\mathrm{supp}(x)\}} \subseteq \mathrm{Spec}\, R.$$

Then $\mathrm{supp} \colon \mathrm{Spv}\, R \to \mathrm{Spec}\, R$ restricts to a homeomorphism $S \xrightarrow{\sim} C$ and both sets are totally ordered.

Thus, for a point $x \in \mathrm{Spv}\, R$ the set of vertical generizations of x and the set of horizontal specializations of x are both totally ordered. Also note that both, vertical generization and horizontal specialization *decrease the rank* of the valuation.

Proof. Let $V_x \subseteq \mathrm{Frac}(R/\mathrm{supp}(x))$ is the valuation ring of x and Γ_x the value group. If $y = x|_\Delta$ is a horizontal specialization of x corresoponding to some convex subgroup $\Delta \subseteq \Gamma$, then the support of y is x-convex: indeed, it is the preimage under x of the prime ideal $\mathfrak{p}_\Delta \subseteq V_x$; it

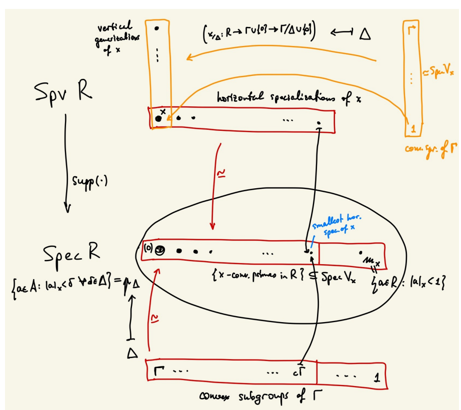
[11] We can rephrase this condition as follows: $|0|_x \leq |a|_x \leq |t|_x$ for $0 \in \mathfrak{p}$ and some $t \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$. This motivates the terminology.

suffices to check x-convexity of $\mathfrak{p}_\Delta = \{a \in V_x : |a|_x < \delta \ \forall\, \delta \in \Delta\} \subseteq V_x$, which is clear by definition of convexity of an ideal. Thus we get a map $S \to C$.

Next, recall from Proposition 4.7 that S is in bijection with convex subgroups $c\Gamma \subseteq \Delta \subseteq \Gamma$, or equivalently, with an open subset of the totally ordered $\operatorname{Spec} V_x$. Moreover, y specializes to y' in S if $\mathfrak{p}_{y'} \supseteq \mathfrak{p}_y$ in $\operatorname{Spec} V_x$, so that this bijection is a homeomorphism.

Moreover, $\operatorname{Spec} V_x$ is a (pro-open) subset of $\operatorname{Spec} R/\operatorname{supp}(x)$, and it remains to show that its open subset (Spec of the localization at $\mathfrak{p}_{c\Gamma}$) coincides with the set of x-convex primes in $\operatorname{Spec} R/\operatorname{supp}(x)$. We may assume here that $\operatorname{supp}(x) = 0$. By the first paragraph, any $\mathfrak{p} \in \operatorname{Spec} V_x$ with $\mathfrak{p} \subseteq \mathfrak{p}_{c\Gamma}$ is x-convex. Conversely, assume $\mathfrak{q} \in \operatorname{Spec} R$ is a x-convex prime. By definition of x-convexity, we have $|a|_x < |b|_x$ for all $a \in \mathfrak{q}$, $b \in R \smallsetminus \mathfrak{q}$. In particular, $|\mathfrak{q}|_x \subseteq \Gamma_{x,<1}$, and hence $|R \smallsetminus \mathfrak{q}|_x \supseteq |R|_x \cap \Gamma_{x,\geq 1}$. Passing to the convex subgroups of Γ_x generated by these sets, we deduce $\Delta := \langle |R \smallsetminus \mathfrak{q}|_x \rangle_{\text{convex}} \supseteq c\Gamma_x$. In particular, by Proposition 4.7 we deduce that $x|_\Delta$ is a valuation and that its support is the ideal $\operatorname{supp}(x|_\Delta) = \{a \in R : |a|_x < \delta \ \forall\, \delta \in \Delta\}$. As $\Delta \subseteq \Gamma$ is the convex subgroup generated by $|R \smallsetminus \mathfrak{q}|_x$ this is equal to $\{a \in R : |a|_x < |b|_x \ \forall\, b \in R \smallsetminus \mathfrak{q}\} = \mathfrak{q}$. □

Let us summarize what we know about specalizations/generizations so far in one picture. Let $x \in \operatorname{Spv} R$. When interested in horizontal specializations and vertical specalizations/generizations only (but not horizontal generizations, which might make the support smaller), we may and do pass to $R/\operatorname{supp}(x)$ and hence assume that $\operatorname{supp}(x) = 0$. Thus we have $R \subseteq V_x \subseteq K = \operatorname{Frac}(R)$ with V_x being the valuation ring of x with maximal ideal $\mathfrak{m}_x$.

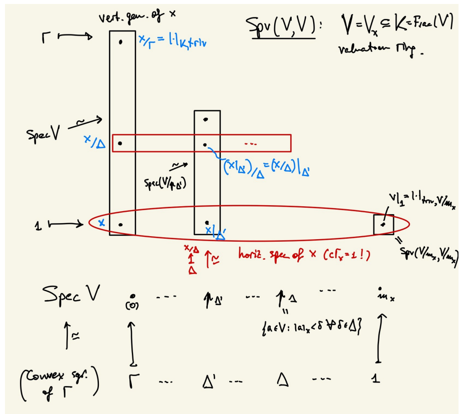


Thus both, the set of vertical generizations of x and the set of horizontal specializations of x are homeomorphic to $\operatorname{Spec} V_x$, but x corresponds to the zero ideal under the first and to the maximal ideal under the second isomorphism.

Example 4.13. Let K be a field, x a valuation on K with valuation ring $V \subseteq K$ and value group Γ. First, note that we have a diagram with cartesian squares:

$$\begin{array}{ccc} \operatorname{Spv}(K,V) & \hookrightarrow & \operatorname{Spv}(V,V) \\ \downarrow & & \downarrow \\ \operatorname{Spv} K & \hookrightarrow & \operatorname{Spv} V \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \hookrightarrow & \operatorname{Spec} V \end{array}$$

(easy exercise: check injectivity of the arrows and the cartesian property). Moreover, we already know that $\operatorname{Spv}(K,V) \cong \operatorname{Spec} V \cong \{\text{convex subgroups of } \Gamma\}$ (a "vertical isomorphism"). Let us now use our knowledge on vertical/horizontal specializations to understand $\mathrm{Spv}(V,V)$, proceeding fiber by fiber over $\mathrm{Spec}\, V$. (See figure 4.13). Note that the fiber over a point $\mathfrak{p}_\Delta \in \mathrm{Spec}\, V$ is $\mathrm{Spv}(\mathrm{Frac}(V/\mathfrak{p}_\Delta), V/\mathfrak{p}_\Delta)$. Thus we arrive at the following picture of $\mathrm{Spv}(V,V)$:



The top points in all columns (equally, the most right points in all rows) are precisely the trivial valuations.

4.3. Factoring arbitrary specializaions.

Remark 4.14. (transitivity for vertical generizations) Let y be a vertical generization of x and let z be a vertical generization of y. Then y corresponds to an ideal $\mathfrak{p}_y$ of the valuation ring V_x, and $V_y = (V_x)_{\mathfrak{p}_y}$; z correpsonds to a prime ideal of V_y and hence to a prime ideal of V_x. Thus z is a vertical generization of x.

The same holds for horizontal specializations, but is less obvious:

Lemma 4.15 ([Mor19], I.3.3.11)**.** *If $x \rightsquigarrow y \rightsquigarrow z$ are two horizontal specializaions, then the specialization $x \rightsquigarrow z$ is also horizontal.*

Proof. Omitted. □

Let us collect further properties of vertical/horizontal specializaions without proofs. Note that they (all) can be visualized in the example $\mathrm{Spv}(V,V)$.

Theorem 4.16. *Let $x, y, z, \ldots$ denote points of $\mathrm{Spv}(R)$.*

(i) ([Mor19, Prop.I.3.4.1(i)] or [Wed19, Lemma 4.19(2)]) If $x \rightsquigarrow y$ is a horizontal specialization and $y \rightsquigarrow z$ is a vertical specalization, then there exist a vertical specialization $x \rightsquigarrow y'$ that admits z as a horizontal specialization:

$$\begin{array}{ccc} x & \rightsquigarrow & y \\ \exists \, \downarrow & & \wr\downarrow \\ y' & \dashrightarrow & z \end{array}$$

(ii) ([Mor19, Prop.I.3.4.1(ii)] or [Wed19, Lemma 4.19(1)]) If $x \rightsquigarrow y$ is a horizontal specialization and $x \rightsquigarrow y'$ is a vertical specialization, then there eixsts a unique horizontal specialization $y' \rightsquigarrow z$, such that z is a vertical specalization of y:

$$\begin{array}{ccc} x & \rightsquigarrow & y \\ \wr\downarrow & & \downarrow \, \exists ! \\ y' & \dashrightarrow & z \end{array}$$

(iii) *[Mor19, Cor I.3.4.2]. Let* $y \mapsto \mathfrak{p}_y = \operatorname{supp}(y)$ *under* $\operatorname{Spv} R \to \operatorname{Spec} R$*. Any generization* $\mathfrak{p} \rightsquigarrow \mathfrak{p}_y$ *in* $\operatorname{Spec} R$ *may be lifted to some horizontal generization* $x \rightsquigarrow y$ *with* $\mathfrak{p} = \operatorname{supp}(x)$*.*

(iv) *[Mor19, I.3.4.3(i)] or [Wed19, 4.21](1). Any specialization* $x \rightsquigarrow y$ *in* $\operatorname{Spv} R$ *can be factored as*

$$\begin{array}{ccc} x & & \\ \wr\downarrow & \searrow & \\ x' & \rightsquigarrow & y \end{array}$$

with $x \rightsquigarrow x'$ *vertical and* $x' \rightsquigarrow y$ *horizontal.*

(v) *[Mor19, I.3.4.3(ii)] or [Wed19, 4.21](2). For any specialization* $x \rightsquigarrow y$ *in* $\operatorname{Spv} R$*, there exist a vertical generization* $y' \rightsquigarrow y$ *such that either*

$$\begin{array}{ccc} x & \rightsquigarrow & y' \\ & \searrow & \wr\downarrow \\ & & y \end{array}$$

or $c\Gamma_x = 1$ $(\Leftrightarrow |R_x| \subseteq \Gamma_{x,\leq 1})$ *and* y' *is the trivial valuation with* $\operatorname{supp}(y') \supseteq \operatorname{supp}(x|_1)$*.*[12]

(v') *[Mor19, I.3.4.4] Assume* $|R|_x \not\subseteq \Gamma_{x,\leq 1}$ *(that is* $c\Gamma_x \neq 1$*). Then the minimal horizontal specialization* $x|_{c\Gamma_x}$ *of* x *admits only vertical specializations.*

(vi) *[Mor19, I.3.4.5] Assume* $|R|_x \not\subseteq \Gamma_{x,\leq 1}$ *(that is* $c\Gamma_x \neq 1$*). Then the image of* $\overline{\{x\}} \hookrightarrow \operatorname{Spv} R \to \operatorname{Spec} R$ *is the set* $\operatorname{conv}(x)$ *of* x*-convex primes in* R*, the fibers of*

$$\overline{\{x\}} \twoheadrightarrow \operatorname{conv}(x)$$

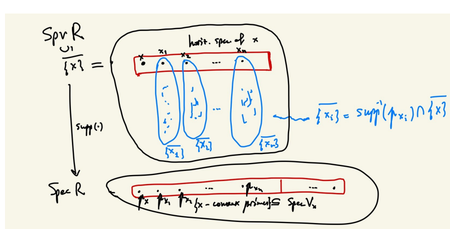
are irreducible, and the (unique) generic points of the fibers of this map are precisely the horizontal specializations of x*.*

The condition $c\Gamma_x \neq 1$ $(\Leftrightarrow |R_x| \subsetneq \Gamma_{x,\leq 1})$ simply means that the minimal horizontal specialization of x in $\operatorname{Spv}(R)$ is not a trivial valuation. (This tends to hold when $R/\operatorname{supp}(x)$ tends to be more field-like than valuation ring-like.)

Remark 4.17. Note that if R contains a subfield on which the valuation $|\cdot|_x$ is non-trivial, then clearly $|R|_x \not\subseteq \Gamma_{x,\leq 1}$, and so $c\Gamma_x \neq 1$. Then the minimal horizontal specialization of x is a non-trivial valuation (cf. Example 4.10).

Exercise 4.18. Visualize all of properties of Theorem 4.16 in the above picture of $\operatorname{Spv}(V, V)$!

Theorem 4.16(vi) refines Figure 4.2: In particular, this says that any point in $\overline{\{x_n\}}$ has the



same support as x_n, that is is a vertical specialization of x_n.

[12]In the latter case, y' can be thought of as being "even further away" in the horizontal direction from x than the most special horizontal specialization $x|_1$ of x. Note that this already happens in the picture of $\operatorname{Spv}(V, V)$!

5. Tate and Huber rings

Most rings naturally appearing in our setup carry a natural topology (think of affinoid Tate algebras, formal schemes, Witt vectors, ...). But $\operatorname{Spv} R$ does not catch the information about the topology on R. We wish to consider valuations on R, which are continuous, i.e., compatible with the topology on R, ultimately ending up with a certain subset $\operatorname{Spa}(R, R^+) \subseteq \operatorname{Spv}(R, R^+)$. To carry out this program, we need a suffiently well-behaved notion of topological rings, which however covers all cases appearing in practice. This leads us to the notion of Huber and Tate (and analytic Huber) rings.

Recall that a topological ring is a ring, equipped with a topology, such that addition and multiplication maps are continuous. As (additive) translation by an element is a homeomorphism, the topology on a topological ring is determined by a *fundamental system of open neighboorhoods*[13] of 0. We will have to make several somewhat technical definitions regarding special topological rings.

Definition 5.1. Let A be a topological ring.

(1) A is called *non-archimedean* if it has a fundamental system of open neighboorhoods consisting of subgroups.
(2) A is called *adic*, if there is an ideal I (called *ideal of definition*), such that $\{I^n\}_{n\geq 0}$ is a fundamental system of open neighboorhoods of 0.
(3) A is called *Huber*[14], if there exists an open subring $A_0 \subseteq A$ (called *ring of definition*), which is adic with a finitely generated ideal of definition. I.e., there is a finitely generated ideal $I \subseteq A_0$, such that $\{I^n : n \geq 0\}$ is a basis of open neighboorhoods of 0 in A_0 (and hence also in A). The pair (A_0, I) is sometimes called a *couple of definition*.
(4) A Huber ring is called *Tate*, if it contains a topologically nilpotent[15] unit. Any such element is called a *pseudo-uniformizer*.

Note that I in (1) and A_0, I in (2) are far from being unique (example later). They are also not part of the datum defining an adic resp. Huber ring (only their existence is). In the case of fields, let us also make the following definition.

Definition 5.2. (a) A *topological field* is a topological ring K, which is a field, and for which the inversion map $x \mapsto x^{-1} : K^\times \to K^\times$ is continuous ($K^\times$ equipped with subspace topology).[16]
(b) A *non-archimedean field* is a topological field K, whose topology is induced by a non-trivial valuation of rank 1.

Example 5.3. (1) Let A be a discrete ring. Then A is Huber, but not Tate. Indeed, we can take $A_0 = A$, $I = 0$ as ring/ideal of definition. Clearly, no non-zero element is topologically nilpotent.
(2) Let A be a discrete ring. For any $n \geq 1$, $A[\![T_1, \dots, T_n]\!]$, equipped with $(T_1, \dots, T_n)$-adic topology, is a Huber ring, which is not Tate. Indeed, $A[\![T_1, \dots, T_n]\!]$, $(T_1, \dots, T_n)$ are ring/ideal of definition. Clearly, any unit $u \in A[\![T_1, \dots, T_n]\!]^\times$ must have a unit of A

[13]Recall what this means: if R is the topological ring, then a system $\{U_\alpha\}_{\alpha\in I}$ of open neighboorhoods of 0 is fundamental if any neighboorhood V of 0 contains U_α for some α.
[14]Huber's original terminology: *f-adic*
[15]An element $x \in A$ is topologically nilpotent, if $\lim x^n = 0$ (meaning that 0 is *a* limit of this sequence; no uniqueness claim, in particular, the topology on A is not required to be Hausdorff).
[16]Note that the condition in the definition is not automatic: for example, there is a topology on $\mathbb{Q}$, making it a topological ring, such that inversion is not continuous, cf. https://math.stackexchange.com/questions/1393303/group-of-units-in-a-topological-ring.

as the constant coefficient, thus each u^n ($n \geq 1$) is not in $(T_1, \dots, T_n)$, thus u is not topologically nilpotent.

(3) Let $(K, |\cdot|)$ be a non-archimedean field. As the norm $|\cdot|$ is non-trivial, there is some element $\varpi \in K$ with $0 < |\varpi| < 1$. Note that K is a Huber ring with ring of definition $K^\circ := \{x \in K : |x| \leq 1\}$ and ideal of definition (ϖ).[17] Moreover, ϖ is a pseudo-uniformizer, hence K is Tate. Examples of non-archimedean fields are: any algebraic extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Completion $\mathbb{C}_p$ of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

(4) Let $(K, |\cdot|)$ be a non-archimedean field with pseudo-uniformizer ϖ. For $n \geq 1$, we have the *Tate algebra*

$$K\langle T\rangle := K\langle T_1, \dots, T_n\rangle := \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu T^\nu \in K[\![T_1, \dots, T_n]\!] : |a_\nu| \to 0 \text{ as } |\nu| := \sum_{i=1}^n \nu_i \to \infty \right\}.$$

For $\varepsilon > 0$, let $U_\varepsilon \subseteq K\langle T\rangle$ be the set of all elements $\sum_\nu a_\nu T^\nu$ with $|a_\nu| \leq \varepsilon$ for all ν. Declaring U_ε as a fundametal set of open neighboorhoods of 0 makes $K\langle T\rangle$ a topological K-algebra. It is a Tate Huber ring, with ring of definition $K^\circ\langle T\rangle := U_1$, ideal of definition $(\varpi) = U_{|\varpi|}$ and pseudo-uniformizer ϖ.

(5) More generally than in (4), if A_0 is any ring, $g \in A$ a non-zero divisor, and $A = A_0[g^{-1}]$ is equipped with the topology for which $\{g^n A_0\}_{n \geq 0}$ is a fundamental system of open neighboorhoods of 0, then A is Huber with ring of definition A_0 and ideal of definition gA_0. Moreover, A is Tate with pseudo-uniformizer g.

In particular, if K is a non-archimedean field and A is a *Banach K-algebra*(= complete normed K-algebra), then taking $A_0 = \{x \in A : |x| \leq 1\}$ and $g \in A_0$ any element with $|g| < 1$, makes A a Huber ring of this type. In particular, all rings which give rise to affinoid rigid-analytic spaces are also Huber.

(6) Let K, ϖ, n be as in (4). Then $A = K^\circ[\![T_1, \dots, T_n]\!]$, equipped with $(\varpi, T_1, \dots, T_n)$-adic topology, is a Huber ring, having itself as a ring of definition and $(\varpi, T_1, \dots, T_n)$ as an ideal of definition. Note however that A is not Tate.

(7) If A is adic with a finitely generated ideal of definition, then A is Huber. Conversely, let A be a Huber ring. Then A is adic $\Leftrightarrow$ A is bounded in itself. [Proof: $\Rightarrow$: if A is bounded, then A is a ring of definition (in itself) by Proposition 5.6(5) below. $\Rightarrow$: We show, more generally, that if A is any topological ring and $B \subseteq A$ is an open adic subring, then B is bounded. Indeed, let $J \subseteq B$ be an ideal such that the topology on B (the one induced from A) is the J-adic topology. As $B \subseteq A$ is open, J^n also form a fundamental system of neighboorhoods of 0 in A. Let $U \subseteq A$ be open, then $J^n \subseteq U$ for some $n \gg 0$. Then $J^n.B = J^n \subseteq U$, i.e., B is bounded.]

(8) Let us give also an example of perfectoids. For example, $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ (completion of algebraic closure of $\mathbb{Q}_p$) is a perfectoid field, and the p-adic completion of $\mathbb{C}_p\langle T^{1/p^\infty}\rangle := \bigcup_{n \geq 0} \mathbb{C}_p\langle T^{1/p^n}\rangle$ is a perfectoid $\mathbb{C}_p$-algebra. Any such is a Huber and Tate.

(9) Let K be a non-archimedean field. There is no topology on $K[\![T]\!]$ such that $K^\circ[\![T]\!]$ is a ring of definition and (ϖ, T) is a ideal of definition. Indeed, suppose there is such a topology. Then we have $T^n \to 0$, and hence also $\varpi^{-1}T^n \to 0$ (as multiplication by ϖ^{-1} is continuous). But no member of the sequence $\varpi^{-1}T^n$ lies in $K^\circ[\![T]\!]$, which contradicts the fact that $K^\circ[\![T]\!]$ (being a ring of definition) is open.

[17] For non-uniqueness of ring/ideal of definition, observe that $(\varpi) \neq (\varpi^2)$ and also that if $\iota : \mathbb{Z} \to K$ is the natural map, then $\iota(\mathbb{Z}) + \varpi.K^\circ$ is a ring of definition

Definition 5.4. (1) A subset S of a topological ring is *bounded* if for any open neighboorhoods U of 0, there is some open neighboorhood V of 0, such that $SV \subseteq U$.
(2) Let A be a Huber ring. An element $f \in A$ is called *power-bounded*, if $\{f^n \colon n \geq 0\}$ is a bounded subset of A. The set of power-bounded elements is denoted by A°.
(3) The set of topologically nilpotent elements of a Huber ring A is denoted by $A^{\circ\circ}$.

Clearly, in a Huber ring one has $A^{\circ\circ} \subseteq A^\circ \subseteq A$.

Exercise 5.5. Show that A° is a subring of A and $A^{\circ\circ}$ is an ideal of A°. Show that $A^{\circ\circ}$ is open in A and contains any ideal of definition of any ring of definition.

To check that a subset $S \subseteq A$ of a Huber ring is bounded, it suffices to consider only sets $U = I^n$ in Definition 5.4(1), as those form a fundamental system of open neighboorhoods of 0.

Proposition 5.6. *Let A be Huber ring, A° the subset of bounded elemenets.*
(1) Any ring of definition is contained in A°.
(2) Let $B \subseteq A$ be a subring. Then B is a ring of definition $\Leftrightarrow$ B is open and bounded.
(3) The collection of all subrings of definition of A is filtered; their union is A°.
(4) A° is a open subring of A.
(5) A° is a ring of definition $\Leftrightarrow$ A° is bounded.
(6) If I is any ideal of definition of A, then $I \subseteq A^{\circ\circ}$.

Proof. (1): Let A_0, I be a ring and an ideal of definition in A. Let $x \in A_0$. Then $S = \{x^n\}_{n\geq 0}$ is bounded, as $S.I^m = I^m$ for all $m \geq 0$ (as I^m is an ideal).

(2): If B is a ring of definition (with ideal of definition I), then B is open (by definition), and bounded as $BI^n \subseteq I^n$ for each n (cf. the observation preceeding the proposition), which holds as each I^n is an ideal of B. For the converse, assume B is open and bounded subring of A. Let A_0, I_0 be any ring+ideal of definition in A. Let T be a finite set of generators of I_0 (recall that I_0 is finitely generated). As B is open, there is some $k > 0$ with $B \supseteq I_0^k \supseteq T(k) := \{t_1 \dots t_k \colon \text{all } t_i \in T\}$. Put $I := T(k) \cdot B$. As B is open, it suffices to show that $\{I^n\}_{n\geq 0}$ form a fundamental system of open neighboorhoods of 0 in A. First, we check that for any $n \geq 0$, I^n is open: as B is open, there is some $\ell > 0$ with $I_0^\ell \subseteq B$. Then $I^n = T(kn).B \supseteq T(kn)I_0^\ell = I_0^{\ell+kn}$, i.e., I^n contains an open neighboorhood of 0, and hence is itself open (as it is stable under addition). Next, we check that any open neighboorhood $V \subseteq A$ of 0 contains I^n for $n \gg 0$. As B is bounded, there is some $m \geq 0$ with $I_0^m.B \subseteq V$. Clearly, I_0^m contains $T(k')$ for all $k' \gg 0$. Hence $I_0^m.B \supseteq T(kn).B = I^n$ for all $n \gg 0$. Thus $V \supseteq I^n$ for $n \gg 0$. This proves (2).

(3): We first claim that the set of all rings of definition of A is filtered, i.e., any two such, A_0, A_0', are contained in a third. Let A_0'' be the subring of A generated by A_0, A_0'. As $A_0'' \supseteq A_0$, A_0'' is open. By (2) it suffices to check that A_0'' is bounded. Let U be an open neighboorhood of 0, which is stable under addition. As A_0 and A_0' are bounded, there exists some open nieghboorhoods U_1, V of 0 with $U_1 A_0 \subseteq U$ and $VA_0' \subseteq U_1$. Let $\sum_{i=1}^r x_i y_i$ be any element of A_0'' with $x_i \in A_0$, $y_i \in A_0'$. Then

$$\left(\sum_{i=1}^r x_i y_i\right) V \subseteq \sum_{i=1}^r x_i y_i V \subseteq \sum_{i=1}^r x_i U_1 \subseteq \sum_{i=1}^r U = U,$$

i.e., $A_0''V \subseteq U$ and hence A_0'' is bounded, proving our claim. Now, a similar argument shows that if A_0 is any ring of definition, and $x \in A^\circ$, then $A_0[x]$ is again open and bounded, hence by (2) a ring of definition. Thus any element of A° lies in a ring of definition, proving (3). (4) follows from (3). (5) follows from (2). (6) is easy. □

Remark 5.7 (Completion)**.** A Huber ring is not assumed to be complete (:= Hausdorff and complete), but one always may pass to completion without big harm. More precisely, if A is Huber with pair of definition A_0, I, then we may form $\hat{A}$ (= Cauchy sequences in A modulo zero sequences) and $\hat{A}_0 := \lim_n A_0/I^n$ (I-adic completion of A_0). Then $\hat{A}_0$ is complete and the induced topology on it is $I \hookrightarrow A_0$-adic by the standard commutative algebra fact [Sta14, 05GG] (which uses that I is finitely generated!) Then $\hat{A}_0 \subseteq \hat{A}$ and $\hat{A}_0, I.\hat{A}_0$ is a pair of definition of $\hat{A}$, which makes it a complete Huber ring. Moreover, $\hat{A}_0 =$ closure of A_0 in $\hat{A}$ and $\hat{A} = \hat{A}_0 \otimes_{A_0} A$ (see [Hub93, Lm.1.6] for more details).

In the case of a Tate ring, the topology is particularly nice:

Lemma 5.8. *Let A be a Tate ring with pseudo-uniformizer g. Let A_0 be any ring of definition in A. Then $g^n \in A_0$ for some $n > 0$, and the topology on A_0 is g^n-adic. Moreover, a subset $S \subseteq A$ is bounded if and only if $S \subseteq g^{-m} A_0$ for some $m > 0$. Finally, $A = A_0[g^{-1}]$.*

Proof. Let $I \subseteq A_0$ be an ideal of definition. As g is topologically nilpotent, there exists some $k > 0$ with $g^k \in I$. Replacing g by g^k, we may assume that $g \in I$. Now, multiplication by g^{-1} on A is a homeomorphism and A_0 is open, hence gA_0 is open too. Hence there is some $\ell > 0$ with $I^\ell \subseteq gA_0$. On the other side, $g \in I$ and I is an ideal in A_0, so $g^\ell A_0 \subseteq I^\ell \subseteq A_0$. Thus $g^\ell A_0 \subseteq I^\ell \subseteq gA_0$. It follows that gA_0 and I define the same topology on A_0. For the last claim, if S is bounded, then there is some $N \gg 0$ with $g^N S \subseteq A_0$ (as the topology is g^n-adic by the previous claim), and hence $S \subseteq g^{-N} A_0$. Conversely, if A_0 is bounded, hence $g^{-N} A_0$ is for any N, and hence any of its subsets is. The last claim follows from the second, as for any $a \in A$, $\{a\}$ is finite and hence contained in $g^{-n} A_0$ for some $n > 0$. ∎

Remark 5.9 (Tate vs. analytic rings)**.** Tate rings are more convenient than general Huber rings. All classical rigid-analytic spaces give rise to Tate rings (whence the name), and also all affinoid perfectoids are Tate. However, there are important examples of adic spaces which are not Tate, like discrete adic spaces and formal schemes. Also, for an affinoid adic space $\mathrm{Spa}(A, A^+)$ the condition to be Tate is not local[18].

To repair this failure of locality, we can use the following notion (introduced by Kedlaya): a Huber ring A is *analytic* if the topologically nilpotent elements $A^{\circ\circ}$ generate the unit ideal in A. Now the expectation would be that an analytic ring is Zariski-locally Tate: indeed, $\mathrm{Spec}\, A = \bigcup_{t \in A^{\circ\circ}} D(t)$, and the part of $\mathrm{Spa}(A, A^+)$ over $D(t)$ should have $A[t^{-1}]$ as its Huber "coordinate ring", which should be Tate with pseudo-uniformizer t. This argumentation does not work on the nose, because the map $\mathrm{Spa}(A, A^+) \to \mathrm{Spec}\, A$ is bad-behaved in general (non-spectral). However, we will show in Proposition 8.5 that $\mathrm{Spa}(A, A^+)$ for an analytic Huber pair (A, A^+) is covered by finitely many opens $\mathrm{Spa}(B_i, B_i^+)$ with (B_i, B_i^+) Tate.

On the other side, we will define the notion of analytic points of an adic space $X = \mathrm{Spa}(A, A^+)$. Then A is analytic if and only if all points of X are analytic. The notion for a point to be analytic is local, and so the notion of a Huber ring to be analytic is local.

A major technical problem of adic spaces is that it is not true in general that the structure presheaf $\mathcal{O}_X$ on $X = \mathrm{Spa}(A, A^+)$ (to be defined later, similarly as for schemes) is not a sheaf. However, it is a sheaf under various finiteness conditions; moreover, it also is a sheaf under the uniformity condition on A, which includes the case of perfectoid Huber rings (which are far from being of finite type):

Definition 5.10. A Huber ring A is called *uniform*, if A° is bounded (hence, by Proposition 5.6(4), a ring of definition).

[18]There exists an affinoid adic space $\mathrm{Spa}(A, A^+)$ with A not Tate, which is covered by two open affinoid subsets $\mathrm{Spa}(B, B^+)$, $\mathrm{Spa}(C, C^+)$ which are Tate.

Remark 5.11. (1) All Huber rings in Example 5.3 are uniform. Here is an example of a non-uniform Huber ring. Let $A = \mathbb{Q}_p[T]/T^2$ with ring of definition $\mathbb{Z}_p[T]/T^2$ and ideal of definition (p). Then $A^\circ = \mathbb{Z}_p + \mathbb{Q}_p \cdot T$ is not bounded.

(2) If A is separated, Tate and uniform, then A is reduced. Proof: let $x \in A$ nilpotent and let $g \in A$ be a pseudo-uniformizer. Then for all $n \geq 0$, $g^{-n}x$ is nilpotent, hence power-bounded, hence in A°. So, we have $x \in g^n A^\circ$ for all $n \geq 0$. But A° is bounded, hence has the g-adic topology (by Lemma 5.8), and is separated (since A is), hence $\bigcap_{n\geq 0} g^n A^\circ = 0$. Thus $x = 0$.

Remark 5.11(2) is in particular responsible for the fact that all perfectoid spaces are reduced. Let us rightaway mention one criterion for uniformity. Recall that a ring R of characteristic p is *perfect*, if $x \mapsto x^p \colon R \to R$ is an isomorphism.

Lemma 5.12 (Perfect implies uniform)**.** *Let A be a complete Tate ring of characteristic p which is perfect. Then A is uniform.*

Proof. Let A_0 be a ring of definition and $\varpi \in A_0$ a pseudo-uniformizer. For any $n \geq 0$ we have the subring $A_n = A_0^{1/p^n}$. Then $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ and $A^{\mathrm{perf}} = \bigcup_n A_n$ is the perfection of A_0.

First we claim that A^{perf} is bounded. The Frobenius $x \mapsto x^p \colon A \to A$ is open by the (analogue for A) of the Banach open mapping theorem (use that F is surjective (even bijective) and A complete Tate), hence homeomorphism. Thus $A_1 = F^{-1}(A_0)$ is bounded, i.e., there is some $n \geq 0$ with $\varpi^n A_1 \subset A_0$ by Lemma 5.8. Applying F^{-r} for all $r \geq 1$, we get $\varpi^{\frac{n}{p^r}} A_{r+1} \subseteq A_r$. Applying this formula succesively r times, we get $\varpi^{\sum_{i=1}^r \frac{n}{p^i}} A_{r+1} \subseteq A_0$. It follows that for any $c > \frac{np}{p-1}$ we have $\varpi^c A^{\mathrm{perf}} \subseteq A_0$, showing the boundedness of A^{perf}.

Now let $a \in A^\circ$. Then $a^{\mathbb{N}} := \{a^k\}_{k\in\mathbb{N}}$ is bounded, so there is some $N > 0$ with $\varpi^n a^{\mathbb{N}} \subseteq A_0 \subseteq A^{\mathrm{perf}}$. But if $x \in A^{\mathrm{perf}}$, then also $x^{1/p^n} \in A^{\mathrm{perf}}$. Extracting the p^n-th root from $\varpi^n a^n$ we deduce that $\varpi a \in A^{\mathrm{perf}}$. Thus $\varpi A^\circ \subseteq A^{\mathrm{perf}}$, i.e., $A^\circ \subseteq \varpi^{-1} A^{\mathrm{perf}}$ is bounded too. □

5.1. **Morphisms of Huber rings.** Let $f \colon A \to B$ be a continuous homomorphism of Huber rings. If B_0, J is any pair of definition of B, then $f^{-1}(B_0) \supseteq f^{-1}(J)$ is an open subring and an open ideal of A, and so if $A_0' \subseteq A$ is any ring of definition of A, then $A_0 = A_0' \cap f^{-1}(B_0)$ is bounded and open, and $f(A_0) \subseteq B_0$. Clearly, we also may find an ideal of definition $I \subseteq A_0$ with $f(I) \subseteq J$. However, $f(I)B_0$ needs not be an ideal of definition of B. This, leads to the fact that not all continuous morphisms between Huber rings are appropriate, cf. Example 5.15(1),(2). The relevant definition is the following.

Definition 5.13. A morphism $f \colon A \to B$ of Huber rings is called *adic*, if there exists a pair of definition (A_0, I) of A and a ring of definition B_0 of B, such that $f(A_0) \subseteq B_0$ and $B_0, f(I)B_0$ is a pair of definition of B.

This is a reasonable definition:

Lemma 5.14. *Let $f \colon A \to B$ be an adic morphism of Huber rings. Then the following hold:*

(1) f is continuous,
(2) if A_0, B_0 are rings of definition of A, B such that $f(A_0) \subseteq B_0$, then for every ideal of definition I in A_0, $f(I)B_0$ is an ideal of definition of B_0.
(3) if $S \subseteq A$ is bounded, then $f(S)$ is bounded.

Proof. Fix A_0, B_0, I as in Definition 5.13. Then $J := f(I)B_0$ is an ideal of definition of B_0. Thus, $f^{-1}(J^n) \supseteq I^n$ and hence f is continuous, showing (1). For (3), we have to show that for each $n > 0$ there is some $m > 0$ with $f(S)J^m \subseteq I^n$. Let m be such that $EI^m \subseteq I^n$. Then $f(S)J^m = f(S)f(I)^m B_0 = f(SI^m)B_0 \subseteq f(I^n)B_0 = J^n$, proving (3). (2): Let A_0', B_0' be rings of definition of A, B with $f(A_0') \subseteq B_0'$, and let I' be an ideal of definition in A_0'. Then there are some $a, b > 0$ with $I \supseteq I'^a \supseteq I^b$; the same holds after applying f, i.e. $f(I) \supseteq f(I')^a \supseteq f(I)^b$. Then $f(I')^a B_0$ is a ideal of definition (note that it is finitely generated as I' is). If J' denotes any ideal of definition of B_0', then ...

$J^M \subseteq (f(I')^a B_0)^N = f(I'^{aN})B_0 \subseteq J'$ for some $N, M \gg 0$. (TODO: finish proof.) □

Example 5.15.
(1) If A is discrete Huber ring and B is any Huber ring, then any homomorphism $A \to B$ is continuous, but it is adic if and only if B is discrete.
(2) The inclusion $\mathbb{Z}_p \to \mathbb{Z}_p[\![T]\!]$ is not adic, when the left side carries p-adic topology and the right side the (p, T)-topology.
(3) Any continuous surjective and open morphism is adic.

Luckily, for Tate rings the situation is less subtle:

Lemma 5.16. *Let $f\colon A \to B$ be a homomorphism between Huber rings. If A is Tate, then f is adic if and only if it is continuous. Moreover, in this case, B is Tate and if B_0 is any ring of definition of B, then $f(A)B_0 = B$.*

Proof. Let B_0 be a ring of definition of B, and let A_0 be a ring of definition of A with $f(A_0) \subseteq B_0$. Let $g \in A_0$ be a pseudo-uniformizer in A, i.e., a topologically nilpotent unit. Now, $f(g)$ is again a topologically nilpotent unit, as f is continuous and a homeomorphism. It follows that B is a Tate ring. Now, by Lemma 5.8, it follows that gA_0 is an ideal of definition of A_0 and $f(g)B_0$ is an ideal of definition of B_0. Hence f is adic. Again, by Lemma 5.8 we have $B = B_0[g^{-1}] = f(A)B_0$ □

Note that the map $\mathbb{Z}_p \to \mathbb{Z}_p[\![T]\!]$ in Example 5.15(2) is not a counterexample to the above lemma, as $\mathbb{Z}_p$ is not Tate, and when we make it Tate by inverting p, then there is again no contradiction, as in the resulting map $\mathbb{Q}_p \to \mathbb{Q}_p[\![T]\!]$ there is no (p, T)-adic topology making the target a Huber ring, cf. Example 5.3(9).

5.2. **Huber pairs.** For a Huber ring A, the space $\operatorname{Spv} A$ contains too many points (think for example of all the trivial valuations). We will sort them out in two ways: by considering only valuations which are continuous with respect to the topology on A, and by considering valuations which are ≤ 1 on some subring A^+. Moreover, as we already have seen in the proof of Theorem 3.3, it is very convenient to make this ring A^+ part of our datum. Thus, ultimately, we will roughly define later

$$\operatorname{Spa}(A, A^+) := \operatorname{Spv}(A, A^+) \cap \operatorname{Cont}(A) \subseteq \operatorname{Spv} A,$$

where $\operatorname{Cont}(A)$ will be the set of continuous valuations, studied in §6 below.

Definition 5.17. Let A be a Huber ring.
(1) A subring $A^+ \subseteq A$ is called a *ring of integral elements* in A, if it is open, contained in A°, and integrally closed in A.
(2) A *Huber pair* (or *affinoid ring* in Huber's original terminology) is a pair (A, A^+) consisting of a Huber ring A together with a ring of integral elements A^+.
(3) A Huber pair (A, A^+) is called *adic*, resp. *complete*, resp. *Tate*, if A has the same property.
(4) A morphism of Huber pairs $(A, A^+) \to (B, B^+)$ is a continuous ring homomorphism $A \to B$, which maps A^+ into B^+. It is called *adic* if the underlying map $A \to B$ is adic.

Lemma 5.18. *Let A be a Huber ring. Then*
(1) A° is integrally closed and hence a ring of integral elements.
(2) Any ring of integral elements in A contains $A^{\circ\circ}$.

(3) $A^+ \mapsto A^+/A^{\circ\circ}$ *induces a bijection between all possible rings of integral elements and all integrally closed subrings of* $A^\circ/A^{\circ\circ}$.

Proof. (1): If $x \in A$ integral over A°, then $x^n = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ for some $n \geq 1$ and $a_i \in A^\circ$. Let A_0 be any ring of definition of A containing $a_0, \ldots, a_{n-1}$ (exists by Proposition 5.6(3)). Thus $x^{\mathbb{N}}$ is contained in the A_0-submodule of A generated by the set $\bigcup_{i=0}^{n-1} a_i^{\mathbb{N}}$, which is bounded by assumption. Thus x is power-bounded, i.e., $x \in A^\circ$. (2): If $x \in A^{\circ\circ}$, then $x^n \to 0$ for $n \to \infty$. Thus, as A^+ is open, $x^n \in A^+$ for $n \gg 0$. But then also $x \in A^+$, as A^+ is integrally closed. (3): Clearly, the image of any $A^\circ \supseteq A^+ \supseteq A^{\circ\circ}$ in $A^\circ/A^{\circ\circ}$ is integrally closed. Conversely, if $\bar{A} \subseteq A^\circ/A^{\circ\circ}$ is intergrally closed with preimage A^+, then A^+ is integrally closed. Indeed, if $f \in A$ is integral over A^+, then it is so over A°, hence lies in A° by part (1); from integrality of $\bar{A}$, it follows that $f \mod A^{\circ\circ}$ lies in A^+, i.e., $f + \alpha \in A^+$ for some $\alpha \in A^{\circ\circ}$. Thus $f \in A^+$ as $A^{\circ\circ} \subseteq A^+$. □

Any ring of integral elements is the filtered union of all rings of definition contained in it (by Proposition 5.6(3)) and the intersection of any two rings of integral elements is again one (by Lemma 5.18(3)).

Example 5.19. (1) To get the classical rigid-analytic geometry setup one takes $A^+ = A^\circ$ throughout.

(2) A prototypical example at the other extreme: let K be a topological field whose topology is defined by a valuation x. Then for any vertical generization y of x we have the valuation subring $K_y \subseteq K$. It is open since it contains the maximal ideal of K_x, and it is integrally closed, since it is a valuation ring. Hence it is a ring of integral elements. We will come back to this example later.

(3) Note that in any Huber ring there is always the biggest and the smallest ring of integral elements, namely A° and $\mathbb{Z} + A^{\circ\circ}$.

6. Continuous valuations

6.1. **Continuous valuations.** The desired adic spectrum of the Tate ring $\mathbb{Q}_p$ should consist of one point, whereas $\mathrm{Spv}(\mathbb{Q}_p, \mathbb{Z}_p) = \{|\cdot|_{\mathrm{triv}}, |\cdot|_p\}$ has two points. To rule out the slightly disturbing trivial valuation, which "does not belong to the p-adic world", we introduce the notion of continuity of valuations. For example $|\cdot|_{\mathrm{triv}}$ on $\mathbb{Q}_p$ is not continuous with respect to the p-adic topology.

Definition 6.1. Let A be any topological ring. A valuation $x \in \mathrm{Spv}\, A$ is *continuous* if for all $\gamma \in \Gamma_x$, the set $\{a \in A \colon |a| < \gamma\}$ is open in A. We denote by $\mathrm{Cont}(A) \subseteq \mathrm{Spv}\, A$ the subset of all continuous valuations, equipped with the subspace topology.

Some remarks are in order, hopefully helping to clarify this definition.

Remark 6.2. Let A be a topological ring, x a valuation on A with value group Γ.

(1) Evidently, continuity of x only depends on the equivalence class of x.

(2) If $|\cdot|\colon A \to \Gamma' \cup \{0\}$ is a valuation, then the condition "for all $\gamma \in \Gamma'$ the set $\{a \in A\colon |a| < \gamma'\}$ is open" is in general stronger than continuity[19]. It depends on the choice of a particular representative of an equivalence class of valuations.

(3) If A is discrete, then any valuation is continuous, and so $\mathrm{Cont}(A) = \mathrm{Spv}\, A$.

[19] For example, let $|\cdot|\colon K \to \Gamma_1 := \mathbb{R}^\times_{\geq 0}$ be a discrete valuation on a field (with topology defined by the valuation), $\Gamma_2 := \mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{>0}$ with lexicographic order, and $\iota\colon \Gamma_1 \hookrightarrow \Gamma_2$, $r \mapsto (1, r)$ the inclusion of a convex subgroup. Then the valuation $|\cdot|' := \iota \circ |\cdot|$ is continuous, just as $|\cdot|$ is. On the other hand, $|\cdot|$ satisfies the condition in question, whereas $|\cdot|'$ does not, as $\{a \in K\colon |a|' < (\frac{1}{2}, 1)\} = \{0\}$ is not open in K.

(4) The support of a continuous valuation is a closed prime ideal. Indeed, it is the intersection of the open subsets $\{a \in A \colon |a|_x < \gamma\}$ for all $\gamma \in \Gamma$. All of these opens are ideals, hence they are also closed.

Remark 6.3 (Valuation topology)**.** (1) Let Γ be a totally ordered group. Assume $\Gamma \neq 1$. Topologize $\Gamma \cup \{0\}$ by declaring all subsets of Γ to be open and a subset $0 \ni U \subseteq \Gamma \cup \{0\}$ to be open if and only if there is some $\gamma \in \Gamma$ with $\{\delta \in \Gamma \colon \delta < \gamma\} \subseteq U$.[20]

(2) Let A be any ring and let x be a valuation on A with value group Γ. Then there is an associated *valuation topology* on A, with basis of opens given by the open balls $B_\gamma(a) = \{b \in A \colon |b - a| < \gamma\}$ for all $\gamma \in \Gamma$. It is the coarsest topology such that the map $|\cdot| \colon A \to \Gamma \cup \{0\}$ is continuous, where $\Gamma \cup \{0\}$ is equipped with the topology as in (1). (Check this!)

(3) Let A, x, Γ be as in (2). Assume that A is a topological ring. Then x is continuous if and only if $|\cdot|_x \colon A \to \Gamma \cup \{0\}$ is continuous, where the topology on $\Gamma \cup \{0\}$ is as in (1). (This follows from (2).) Equivalently, x is continuous if and only if its valuation topology is coarser of equal to the given topology on A.

(4) Let A, x, Γ be as in (2). We also have the closed balls $\overline{B}_\gamma(a) = \{b \in A \colon |b - a| \leq \gamma\}$. As the valuations are non-archimedean, $\overline{B}_\gamma(a) \supseteq B_\gamma(x)$ for all $x \in \overline{B}_\gamma(a)$. If $|\cdot|$ is a non-trivial valuation, then the closed balls also form a basis for topology on A, as then $B_\gamma(a) = \bigcup_{\beta < \gamma} \overline{B}_\beta(a)$.

Remark 6.4 (Continuity vs. trivial valuations)**.** Let A be a topological ring.

(1) Let x be the trivial valuation with support $\mathfrak{p}$ on A, that is $|f|_x = 0$ if $f \in \mathfrak{p}$ and $|f|_x = 1$ otherwise. Then x is continuous if and only if $\mathfrak{p}$ is an open prime ideal. (If we would change "$<$" in the definition to "$\leq$", any trivial valuation would be continuous).

(2) Suppose A is Huber. An ideal $\mathfrak{a}$ is open if and only if $\sqrt{\mathfrak{a}} \supseteq A^{\circ\circ}$. Indeed, if $\mathfrak{a}$ open, and $f \in A^{\circ\circ}$, then $f^n \in \mathfrak{a}$, and so $f \in \sqrt{\mathfrak{a}}$. Conversely, if $\sqrt{\mathfrak{a}} \supseteq A^{\circ\circ}$, then let (A_0, I) be some couple of definition of A. As $I \subseteq A^{\circ\circ}$, also $I \subseteq \sqrt{\mathfrak{a}}$. As I is finitely generated, $I^m \subseteq \mathfrak{a}$ for some $m \gg 0$. But I^m is open and so $\mathfrak{a}$ is too.

(3) Parts (1),(2) above imply that if A is Tate, or more generally, analytic (cf. Remark 5.9), then no of the trivial valuations on A is continuous.

(4) Let x be a non-trivial valuation. Then changing "$<$" to "$\leq$" in Definition 6.1 would not affect the definition. Indeed, this follows from Remark 6.3(4).

6.2. **Cofinal elements and microbial valuations.** For Huber rings continuity can be reformulated in terms of cofinal elements.

Definition 6.5. Let Γ be a totally ordered group. Then $\gamma \in \Gamma \cup \{0\}$ is called *cofinal* if for any $\delta \in \Gamma$, there is some $n > 0$ with $\gamma^n < \delta$.

Remark 6.6. Note that $\gamma \in \Gamma$ is cofinal if and only if $\gamma^n \to 0$ in the topology of Remark 6.3(2). Further, notice that if V is a valuation ring of rank ≥ 1 and $K = \operatorname{Frac} V$, then $x \in K$ is topologically nilpotent (in the valuation topology of K) if and only if $|x| \in K^\times / V^\times \cup \{0\}$ is cofinal.

The following lemma is very useful to remember. It says that a valuation is continuous, if it preserves topological nilpotence of elements.

Lemma 6.7. *Let (A, A^+) be a Huber pair and let $|\cdot| \colon A \to \Gamma \cup \{0\}$ be a valuation with $|A^+| \leq 1$. Then $|\cdot|$ is continuous if and only if the image of any element $a \in A^{\circ\circ}$ in $k(x)$ is topologically nilpotent.*

[20] Note that for $\Gamma = 1$ this definition does not output a topological space as $\Gamma \cup \{0\} = \{0, 1\}$ would not be open.

It also suffices to check topological nilpotency for a set of generators of any ideal of definition I of any ring of definition $A_0 \subseteq A^+$.

Proof. The valuation $|\cdot| = |\cdot|_x$ factors as

$$A \to k(x) = \operatorname{Frac} A/\operatorname{supp}(x) \to \Gamma_x \cup \{0\}.$$

All maps are continuous with repsect to the valuation topologies on A and $k(x)$ and the discrete topology on $\Gamma_x \cup \{0\}$, by Remark 6.3(1). By Remark 6.3(2), continuity of $|\cdot|$ means that the valuation topology on A is coarser of equal to the defining topology on A. Thus $A \to k(x)$ is continuous when A has its given topology and $k(x)$ its valuation topology. Now if $a \in A^{\circ\circ}$, then $a^n \to 0$ and hence also $\bar{a}^n \to 0$, where $\bar{a} \in k(x)$ is the image of a.

Conversely, assume that A_0, I is a couple of definition (then necessarily $I \subseteq A^{\circ\circ}$), with $A_0 \subseteq A^+$, so that $|A_0| \subseteq |A^+| \leq 1$. As I is finitely generated, we may assume the set T of generators of I for which topological nilpotency is known to be finite. Let $\gamma \in \Gamma_x$. As the image of each $t \in T$ in $k(x)$ is topologically nilpotent and T is finite, it follows that there is some $n > 0$ with $|T^n| < \gamma$ (where $T^n = \{t_1 \dots t_n \colon t_i \in T\}$). As $|A_0| \leq 1$ it follows that $|I^n| = |T^n A_0| < \gamma$. □

Let A be a analytic Huber ring (cf. Remark 5.9). Then $\operatorname{Cont}(A)$ does not contain any trivial valuation (Remark 6.4(3)), which is good, since trivial valuations should not show up in adic spectra (except discrete cases, where this is OK "by design"). Now we are ready to show that the valuations in $\operatorname{Cont}(A)$ will all automatically have a further nice proprty:

Definition 6.8. A valuation ring V is called *microbial* if it has a prime ideal of height 1. A valuation of a ring is called *microbial*, if its valuation ring has this property.

Equivalently, V is microbial if it contains a non-zero topologically nilpotent element. The (unique) height 1 prime ideal in V is precisely the collection of all topologically nilpotent elenents. (Note that by Corollary 1.10, if V is a valuation ring and $f \in V \smallsetminus \{0\}$ topologically nilpotent, then $\sqrt{fV}$ is the prime ideal of height 1 and $\bigcap_{n\geq 0} f^n V = 0$.) Note also that f is a pseudo-uniformizer of V if and only if $\operatorname{Frac}(V) = V[f^{-1}]$.

Proposition 6.9. *Let A be an analytic Huber ring. Then all points in* $\operatorname{Cont}(A)$ *have rank* ≥ 1 *and are microbial.*

Proof. By Remark 6.4(3), the trivial valuations are not in $\operatorname{Cont}(A)$, so the rank of all points in $\operatorname{Cont}(A)$ is ≥ 1. Let $x \in \operatorname{Cont}(A)$. Note that $\operatorname{supp}(x) \not\supseteq A^{\circ\circ}$ as $A^{\circ\circ}$ generate the unit ideal in A. Let $a \in A^{\circ\circ} \smallsetminus \operatorname{supp}(x)$. Then the image $\bar{a} \in k(x)$ is non-zero and topologically nilpotent for the valuation topology on $k(x)$ (as by continuity of x the map $A \to k(x)$ is continuous, cf. Remark 6.3(3)). But the existence of a non-zero topologically nilpotent element (in the valuation topology) shows that the valuation ring of x in $k(x)$, and hence also x itself, are microbial. □

It's time for an example.

Example 6.10. Let V be a valuation ring with $K = \operatorname{Frac}(V)$, let x denote the valuation on V and let $\Gamma = K^\times/V^\times$ denote the value group of x. Equip V with the valuation topology given by x. So, a basis of open neighboorhoods of $0 \in V$ is given by $B_\gamma(0) = \{a \in V \colon |a|_x < \gamma\}$ with γ varying through Γ. In Example 4.13 we studied how $\operatorname{Spv}(V, V)$ looks like. Now may ask two questions:

A. When is V a Huber (and (V, V) a Huber pair)?
B. How $\operatorname{Cont}(V) \cap \operatorname{Spv}(V, V)$ looks like? (If (V, V) is a Huber pair, then $\operatorname{Spa}(V, V) = \operatorname{Cont}(V) \cap \operatorname{Spv}(V, V)$ by definition.)

(Note that V is never Tate as $|a|_x = 1$ for all $a \in V^\times$, so $V^{\circ\circ} \cap V^\times = \varnothing$.) There are three disjoint cases: $V = K$ is a field; V is microbial; V is non-microbial.

1) *$V = K$ is a field.* Then $x = |\cdot|_{\mathrm{triv}}$ is trivial and the topology on V is discrete. As any discrete ring, V is Huber, and as $|a|_x < 2$ for all $a \in V$, V is bounded, so that (V, V) is a Huber pair. This answers question A. For question B note that x is the only point of $\mathrm{Spv}(V, V)$ and that $0 \subseteq V$ is an open prime ideal, so x is continuous by Remark 6.4(1). Thus $\mathrm{Spa}(V, V) = \mathrm{Spv}(V, V) = \{x\}$.
2) *V is microbial.* Let $\varpi \in V$ be a topologically nilpotent element. Then the valuation topology on V agrees with the ϖ-adic topology (by cofinality). Thus V is an adic ring and hence a Huber ring and (V, V) is a Huber pair.

   Now consider the prime ideals of V. Clearly, $\{0\}$ is not open (as the valuation topology on V is not discrete), but all other $\mathfrak{p} \in \operatorname{Spec} V$ are open. (Indeed, let $\mathfrak{p} = \mathfrak{p}_\Delta = \{a \in V : |a|_x < \delta \ \forall\, \delta \in \Delta\}$ for some convex subgroup $\Delta \subsetneq \Gamma$. Then we may find some $n \gg 0$ with $|\varpi^n| < \delta$ for all $\delta \in \Delta$ (otherwise, by convexity of Δ top.nilpotency of ϖ, we wuold deduce the contradiction $\Delta = \Gamma$). This implies $\mathfrak{p} \supseteq B_{|\varpi^n|}(0)$, so that being a subgroup of V, $\mathfrak{p}$ is open.

   As $\{0\} \subseteq V$ is not open, the maximal vertical generization $x_{/\Gamma} = |\cdot|_{\mathrm{triv},K}$ of x is not continuous. On the other hand, all other points in $\mathrm{Spv}(V, V)$ are reached from x by vertical generizations resp. horizontal specializations. Thus all of them (including the trivial ones, for which we have checked above) are continuous by Proposition **??** below (Exercise: check directly.) Thus $\mathrm{Spa}(V, V) = \mathrm{Spv}(V, V) \setminus \{|\cdot|_{\mathrm{triv},K}\}$.
3) *V is non-microbial.* First note that V is not discrete as $\{0\} \subseteq \Gamma \cup \{0\}$ is not open (topology as in Remark 6.4) and $\{0\} \subseteq V$ is not open. Next note that $V^{\circ\circ} = \{0\}$.[21] Thus, if V would be a Huber ring, then any ideal of definition $I \subseteq V^{\circ\circ} = 0$. Thus the topology on V is discrete; contradiction. Thus V is not Huber.

7. The topological space $\mathrm{Spa}(A, A^+)$

Definition 7.1 (Adic spectrum)**.** Let (A, A^+) be a Huber pair. The *adic spectrum* of (A, A^+) is the subset

$$\mathrm{Spa}(A, A^+) = \{x \in \mathrm{Spv}(A, A^+) : x \text{ continuous}\}$$

of $\mathrm{Spv}(A, A^+)$, equipped with the subspace topology.

A morphism $\varphi\colon (A, A^+) \to (B, B^+)$ of Huber pairs induces a map

$$\mathrm{Spa}\,\varphi\colon\ \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+),$$

sending $|\cdot|$ to $|\cdot| \circ \varphi$. It is clearly continuous (as $\mathrm{Spv}\,\varphi$ is).

Remark 7.2. (1) If $S \subseteq A$ is any subset (contained in A°), then one may consider the subset $\mathrm{Spa}(A, S) \subseteq \mathrm{Spv}\, A$ consisting of all continuous valuations $|\cdot|$ satisfying $|f| \leq 1$ for all $f \in S$. Then $\mathrm{Spa}(A, S) = \mathrm{Spa}(A, A^+)$, where A^+ is the smallest ring of integral elements containing S (i.e., the intersection of all such).

(2) If A is a discrete Huber ring and (A, A^+) a Huber pair, then any valuation is continuous and so $\mathrm{Spa}(A, A^+) = \mathrm{Spv}(A, A^+)$.

---

[21] Indeed, suppose $a \in V \setminus \{0\}$ topologically nilpotent. Let $\mathfrak{p} = \sqrt{(a)}$, which is a non-zero prime ideal of V. We claim $\mathrm{ht}\mathfrak{p} = 1$. Thus, we have to show that any prime ideal $\{0\} \subsetneq \mathfrak{q} \subseteq \mathfrak{p}$ satisfies $\mathfrak{q} = \mathfrak{p}$. Pick some $0 \neq b \in \mathfrak{q}$, let $c \in \mathfrak{p} = \sqrt{(a)}$ arbitrary. Then $c^n \in (a)$ for $n \gg 0$; and $a^N \in (b)$ for $N \gg 0$ (as a topologically nilpotent). Thus $c^{nN} \in (b)$, i.e., $c \in \sqrt{(b)} \subseteq \mathfrak{q}$, proving the claim. As V is non-microbial, we get a contradiction.

7.1. **Rational open subsets.** If $X = \mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+). For $f_1, \dots, f_n, g \in A$, we denote the corresponding open of X by

$$X\left(\frac{f_1, \dots, f_n}{g}\right) = X \cap \mathrm{Spv}(A, A^+)\left(\frac{f_1, \dots, f_n}{g}\right).$$

There is one subtlety here, namely $X(\frac{f}{g})$ does not need to be quasi-compact (in contrast to the standard opens of $\mathrm{Spv}(A, A^+)$, cf. Theorem 3.3:

Example 7.3. Let K be a non-archimedean field with $K^\circ = \mathcal{O}_K$, then $(K\langle T\rangle, \mathcal{O}\langle T\rangle)$ is a Huber pair and $X = \mathbb{B} = \mathrm{Spa}(K\langle T\rangle, K^\circ\langle T\rangle) = \{|T| \leq 1\}$ is the *adic closed ball* over K (see below §9). Then $X(\frac{0}{T}) = \mathbb{B}^1_K \smallsetminus \{0\}$ is not quasi-compact, as it admits the open covering by subsets $X(\frac{\varpi^n}{T}) = \{|T| \geq |\varpi|^n\}$, which has no finite refinement.

Note that this example also shows that the map $\mathrm{Spa}(A, A^+) \to \mathrm{Spv}\, A$ is not necessarily quasi-compact (Question: Is $\mathrm{Spa}(A, A^+) \to \mathrm{Spv}(A, A^+)$ also not qc in general?), and hence not spectral (once we show that $\mathrm{Spa}(A, A^+)$ is spectral). Also the map $\mathrm{Spa}(A, A^+) \to \mathrm{Spec}\, A$ is in general not spectral (for the same reason: in the above example, the preimage of the qc open $D(T) \subseteq \mathrm{Spec}\, K\langle T\rangle$ in $\mathbb{B}$ is not quasi-compact).

However, adding one more condition related to the topology of A will rescue the desired quasi-compactness again:

Definition 7.4 (Rational subsets)**.** Let (A, A^+) be a Huber pair. A *rational open subset* of $X = \mathrm{Spa}(A, A^+)$ is a subset of the form $X(\frac{f_1, \dots, f_n}{g})$ with $f_1, \dots, f_n, g \in A$ such that the ideal of A generated by $f_1, \dots, f_n$ is open.

Remark 7.5.
(1) If A is an analytic Huber ring, then "$f_1, \dots, f_n \in A$ generate an open ideal" $\Leftrightarrow$ "$f_1, \dots, f_n$ generate the unit ideal"[22]. This implies that we can write the rational opens without the condition "$g(x) \neq 0$" (cf. Definition 2.1). That is, if (A, A^+) is an analytic Huber pair, $X = \mathrm{Spa}(A, A^+)$ and $(f_1, \dots, f_n)_A$ is open in A, then

$$X\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in X : |f_i(x)| \leq |g(x)| \neq 0\, \forall i\} = \{x \in X : |f_i(x)| \leq |g(x)|\, \forall i\}$$

(2) If (A, A^+) is a Tate–Huber pair with pseudo-uniformizer ϖ, then any rational open of $X = \mathrm{Spa}(A, A^+)$ is of the form $X\left(\frac{f_1, \dots, f_n}{g}\right)$ with $f_1, \dots, f_n, g \in A^+$ (moreover, one also may assume that $f_1 = \varpi^N$ for $N \gg 0$). Indeed, for an arbitrary rational open we have $X\left(\frac{f_1, \dots, f_n}{g}\right) = X\left(\frac{f_1\varpi^N, \dots, f_n\varpi^N}{g\varpi^N}\right)$ for any $N > 0$, just as ϖ is a unit of A.

Lemma 7.6 (Rational opens form basis stable under finite intersections)**.** *Let A, A^+ be a Huber pair. The rational open subsets of* $\mathrm{Spa}(A, A^+)$ *form a basis for topology of* $\mathrm{Spa}(A, A^+)$*, stable under intersections.*

Proof. Product of two open ideals is open (why?). If $f_1, \dots, f_n = g, f'_1, \dots, f'_m, g' \in A$ are such that the ideals $(f_i) \cdot A, (f'_i) \cdot A$ are open in A, then $(f_i)_i \cdot (f'_j)_j = (f_i f_j)_{ij}$ is also open and formula (2.1) shows that intersections of rational opens are again rational.

Fix a couple of definition A_0, I in A. Let T be a finite set of generators of I. Write $T^r = \{t_1 \cdot \dots \cdot t_r : t_i \in T\}$. Then we have, for any $f_1, \dots, f_n, g \in A$:

$$\mathrm{Spa}(A, A^+)\left(\frac{f_1, \dots, f_n}{g}\right) = \bigcup_{n \geq 1} \mathrm{Spa}(A, A^+)\left(\frac{f_1, \dots, f_n, T^r}{g}\right),$$

[22] Indeed, by Remark 6.4(2), an ideal is open if and only if its radical contains $A^{\circ\circ}$; but in an analytic Huber ring, $A^{\circ\circ}$ generate the unit ideal of A, so the radical of $(f_1, \dots, f_n)_A$ is the unit ideal, and so $(f_1, \dots, f_n)_A$ itself is too.

Indeed, $\supseteq$ is clear, and $\subseteq$ holds by (the easy direction of) Lemma 6.7. Now the ideal of A_0 generated by T^r contains I^r, hence is open, and so $(f_1, \ldots, f_n, T^r) \cdot A$ is also open. $\square$

A fundamental fact in the theory of schemes is that in Spec A we have $D(f) \cong \operatorname{Spec} A[f^{-1}]$ for any $f \in A$. The analogous statement for Spv is Lemma 3.15. For Spa we have the following, slightly more complicated analogue, which takes into acount the topology.

Proposition 7.7 (rational opens are adic spectra)**.** *Let (A, A^+) be a Huber pair. Let $f_1, \ldots, f_n, g \in A$ such that $f_1, \ldots, f_n$ generate an open ideal of A. Then the pair of rings $(B, B^+) = (A[\frac{1}{g}], A^+[\frac{f_1,\ldots,f_n}{g}])$ can be equipped with a topology making it a Huber pair, such that there is a natural homeomorphism*

$$\operatorname{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right) \cong \operatorname{Spa}(B, B^+).$$

Proof. Neglecting the topology on A, A^+, we have the homeomorphism

$$\operatorname{Spv}\left(A\left[\frac{1}{g}\right], A^+\left[\frac{f_1, \ldots, f_n}{g}\right]\right) \cong \operatorname{Spv}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right)$$

from Lemma 3.15, induced by restriction of valuations.

To prove the proposition, it remains to endow $B = A[\frac{1}{g}]$ with a topology, such that B, B^+ becomes a Huber pair, and such that a valuation on B which is ≤ 1 on B^+ is continuous if and only if its restriction to A is so. Therefore, fix a couple of definition A_0, I in A, set

$$B_0 := A_0\left[\frac{f_1}{g}, \ldots, \frac{f_n}{g}\right] \subseteq B = B_0\left[\frac{1}{g}\right]$$

and let $J = I \cdot B_0$. We wish that B has (B_0, J) as a couple of definition. Therefore, equip B –as a topological group under addition– with the (coarsest) topology such that $\{J^n\}_{n\geq 1}$ form a fundamental system of open neighboorhoods of 0. In particular, the subgroup B_0 carries the J-adic topology.

Lemma 7.8. *With the above topology, B is a topological ring.*

Proof. We need to check that the multiplication map $B \times B \to B$ is continuous. Therefore, it suffices to show that for all $b \in B$ and $n > 0$, there is an $m > 0$ with $bJ^m \subseteq J^n$ (check this!). For all $b \in A$ this is clear as the same claim holds for A and I. If it holds for two elements, then also for their product and sum. Thus, if it holds for some $b \in B$, then for all elements in the A-subalgebra $A[b] \subseteq B$. Thus, it suffices to show the claim for $b = g^{-1}$. Suppose we know that for any $n > 0$, there is some $m > 0$ with

$$f_1 I^n + \cdots + f_n I^n \supseteq I^m. \tag{7.1}$$

Then $g^{-1}I^m \subseteq \frac{f_1}{g}I^n + \cdots + \frac{f_n}{g}I^n \subseteq I^n A_0[\frac{f_1,\ldots,f_n}{g}] = I^n B_0 = J^n$, and hence also $g^{-1}J^m \subseteq J^n$ and we are done. So, it remains to show (7.1). Write $T := \{f_1, \ldots, f_n\}$, so we have to show that for given n, there is some m with $TI^n \supset I^m$. By assumption, $T \cdot A$ is open in A, so there is some $k \geq 0$ with $I^k \subseteq T \cdot A$. We may replace I by I^k. Now, I is finitely generated, so let $S \subseteq I$ be a finite set of generators. Let also $V \subseteq A$ be a finite set such that $V \cdot T \supseteq S$. A finite set is bounded, so there is some $m \geq 1$ with $V \cdot I^m \subseteq I^n$. But then

$$T \cdot I^n \supseteq T \cdot V \cdot I^m \supseteq S \cdot I^m = I^{m+1},$$

and we are done. $\square$

Now, B_0 with the induced topology is J-adic, hence in particular open and bounded, and J is finitely generated (as I is), so B_0, J is a couple of definition and B is therefore a Huber ring. Now let $|\cdot|$ be a valuation of B satisfying $|B^+| \leq 1$. Then also $|A^+| \leq 1$, and it follows from Lemma 6.7 applied twice to the (same) set of generators of $I \subseteq A_0$ resp. $J \subseteq B_0$, that $|\cdot|$ is continuous if and only if its restriction to A is. □

Remark 7.9. (1) The topology on B, B^+ does not depend on the choice of the couple A_0, I, as follows from the the fact that the set of all such couples is cofiltered (Proposition 5.6(3)).

(2) The ring B in the proposition is not unique, as it depends on $f_1, \dots, f_n, g$ and not just the rational open of $\mathrm{Spa}(A, A^+)$ which they cut out. However, after passing to completion, this ambiguity will vanish, cf. **??** below.

(3) It is the additional flexibility of choosing B^+ that ensures us that one can describe $\mathrm{Spa}(A, A^+)(\frac{f_1,\dots,f_n}{g})$ as Spa of some pair (B, B^+).

7.2. **Spectrality of** $\mathrm{Spa}(A, A^+)$**.**

Theorem 7.10 ($\mathrm{Spa}(A, A^+)$ is spectral)**.** *Let (A, A^+) be a Huber ring. Then $\mathrm{Spa}(A, A^+)$ is spectral and the set of rational open subsets forms a basis of qc opens, which is stable under intersections.*

Remark 7.11. Ideally, we would like to prove Theorem 7.10 by showing that $\mathrm{Spa}(A, A^+)$ is pro-constructible subset of $\mathrm{Spv}(A, A^+)$ (of which we know that it is spectral by Theorem 3.3), and using that spectrality passes to pro-constructible subsets. However, in general, the inclusion $\mathrm{Spa}(A, A^+) \hookrightarrow \mathrm{Spv}(A, A^+)$ is not a spectral map (this follows from Example 7.3), hence the image is not pro-constructible (as inclusions of pro-constructible subsets are spectral).

Proof of Theorem 7.10. First we note that $X = \mathrm{Spa}(A, A^+)$ is sober: indeed, therefore it suffices to look again at the proof of Lemma 3.17 and notice that the T_0-part of the proof goes through without change for X, and it only suffices to check that the valuation η constructed in the second half of the proof is continuous, if any element of Z is so[23].

To finish the proof we will construct a continuous retraction from a closed subset of $\mathrm{Spv}(A, A^+)$ to $\mathrm{Spa}(A, A^+)$; then from quasi-compactness of all $\mathrm{Spv}(A, A^+)$ –which we know by Theorem 3.3– (and hence their closed subsets) it follows that also $\mathrm{Spa}(A, A^+)$ is quasi-compact. This and Proposition 7.7 implies that all rational opens of $\mathrm{Spa}(A, A^+)$ are quasi-compact, and by Lemma 7.6 they form a basis of topology. □

Thus, to finish the proof of Theorem 7.10 it remains to construct a continuous retraction from a certain closed subset of $\mathrm{Spv}(A, A^+)$ to $\mathrm{Spa}(A, A^+)$. First note that by Lemma 6.7 for each $a \in A^{\circ\circ}$ we have $\mathrm{Spa}(A, A^+) \cap \mathrm{Spv}(A, A^+)\left(\frac{1}{a}\right) = \varnothing$. Thus $\mathrm{Spa}(A, A^+) \subseteq \mathrm{Spv}(A, A^+) \smallsetminus \bigcup_{a \in A^{\circ\circ}} \mathrm{Spv}(A, A^+)\left(\frac{1}{a}\right)$. We now will construct a continuous retraction

$$r\colon \mathrm{Spv}(A, A^+) \smallsetminus \bigcup_{a \in A^{\circ\circ}} \mathrm{Spv}(A, A^+)\left(\frac{1}{a}\right) \to \mathrm{Spa}(A, A^+). \tag{7.2}$$

Therefore, fix a valuation $x \in \mathrm{Spv}(A, A^+) \smallsetminus \bigcup_{a \in A^{\circ\circ}} \mathrm{Spv}(A, A^+)(\frac{1}{a})$. In particular, we have $|A^+|_x \leq 1$ and $|A^{\circ\circ}|_x < 1$. We have to attach to x a continuous valuation $r(x)$ on A satisfying $|A^+|_{r(x)} \leq 1$. Let $k(x)^+ \hookrightarrow k(x)$ be the valuation ring of x and its field of fractions. As $|A|_x \leq 1$,

[23] To do this, notice that by Lemma 6.7 the continuity of a valuation $x \in X$ is encoded in the following property of the corresponding relation $| = |_x \in 2^{A \times A}$ (as in the proof of Proposition 3.16): $\forall\, t \in A^{\circ\circ}$ and $\forall\, a \in A$ with $a \not| 0$, there exists some $n > 0$ with $t^n | a$.

the natural map $A^+ \hookrightarrow A \to k(x)$ factors through a map $\varphi_x\colon A^+ \to k(x)^+$. Fix a couple of definition A_0, I in A, such that $A_0 \subseteq A^+$. Then consider the composed map

$$\varphi_{r(x)}\colon A^+ \xrightarrow{\varphi_x} k(x)^+ \twoheadrightarrow V_{r(x)} := k(x)^+ / \bigcap_{n\geq 0} (\varphi_x(I)\cdot k(x)^+)^n. \tag{7.3}$$

The ideal I is finitely generated in A_0, thus $\varphi_x(I)\cdot k(x)^+$ is a finitely generated –hence principal– ideal of the valuation ring $k(x)^+$. Moreover, as $I \subseteq A^{\circ\circ}$, we have $|I|_x < 1$, so that $\varphi_x(I)\cdot k(x)^+ \subseteq k(x)^+$ is a proper ideal and the quotient ring is not the zero ring. Thus $\varphi_{r(x)}$ is indeed a valuation of A^+.[24] Note that $\varphi_{r(x)}$ does not depend on the choice of I. We even can be more precise: the principal ideal $\varphi_x(I)\cdot k(x)^+$ of $k(x)^+$ sits between two prime ideals of $k(x)^+$ (as in Corollary 1.10!): $\sqrt{\varphi_x(I)} = \varphi_x A^{\circ\circ} \supseteq \varphi_x(I)\cdot k(x)^+ \supseteq \bigcap_{n\geq 0}(\varphi_x(I)\cdot k(x)^+)^n$ and the specialization relation between these two ideals is *immediate*. Thus the image of $\varphi_x(A^{\circ\circ})$ in the quotient $k(x)^+/\bigcap_{n\geq 0}(\varphi_x(I)\cdot k(x)^+)^n$ is a prime ideal of height one.

Exercise 7.12. Show that $V_{r(x)}$ is a microbial valuation ring, and the image of any element I under $\varphi_{r(x)}$ is topologically nilpotent. (Use Corollary 1.10)

As $A^{\circ\circ} = \sqrt{I}$ (and so the same holds for their images in $k(x)^+$), we also have that the image under $\varphi_{r(x)}$ of any element in $A^{\circ\circ}$ is topologically nilpotent[25]. Thus, by Lemma 6.7 it becomes clear that $\varphi_{r(x)}$ defines a *continuous* valuation on A^+, which we denote by $x'\colon A^+ \to \Gamma_{x'} \cup \{0\}$. We now extend it to a continuous valuation $r(x)$ on A. If $|t(x)| = 0$ for all $t \in I$, then $I \subseteq \operatorname{supp}(x)$ and $V_{r(x)} = k(x)^+$. In this case, x' is the restriction of x to A^+ and we define $r(x) = x$. In the other case we have the following lemma:

Lemma 7.13. *Let $|\cdot|\colon A^+ \to \Gamma \cup \{0\}$ be a valuation. Assume that there exists some $t \in A^{\circ\circ}$ with $|t| \neq 0$. Then $|\cdot|$ extends uniquely to a valuation on A. Moreover, $|\cdot|$ is continuous if and only if its extension is.*

Proof. As t is topologically nilpotent and $A^+ \subseteq A$ open, for any $a \in A$ there is some $n \gg 0$ with $t^n a \in A^+$. Put $|a| := |t|^{-n}|t^n a|$. Clearly, this is an extension of $|\cdot|$ to A. Moreover, each extension has to satisfy this equation, hence is unique. The last assertion follows from the characterization of continuity in Lemma 6.7. □

Denote by $r(x)$ the unique continuous valuation on A attached to x' by Lemma 7.13. As $|A^+|_{r(x)} \leq 1$, this finishes the construction of a map (7.2).

Remark 7.14. Note the splitting in two cases: if $\varphi_x(I) = 0$, then $r(x) = x$, and so $r(x)$ might be microbial or non-microbial. If $\varphi_x(I) \neq 0$, then $r(x)$ is necessarily microbial. If A is not analytic, there might be non-microbial valuations in $\operatorname{Spa}(A, A^+)$, and all of them must land in the first case.

Lemma 7.15. *The map*

$$r\colon \operatorname{Spv}(A, A^+) \setminus \bigcup_{a\in A^{\circ\circ}} U\left(\frac{1}{a}\right) \to \operatorname{Spa}(A, A^+)$$

is a continuous retraction (i.e., $r(x) = x$ if $x \in \operatorname{Spa}(A, A^+)$). In particular, $\operatorname{Spa}(A, A^+)$ is quasi-compact.

[24] Note that $\varphi_{r(x)}$ is a horizontal specialization of (restriction to A^+ of) x.

[25] Actually, $\varphi_{r(x)}$ is the largest (say, by rank) horizontal specialization of x, for which this holds

Proof. Note that the quasi-compactness assertion follows from the preceeding ones and Theorem 3.3, as a retraction is in particular surjective and the image of a quasi-compact space under a continuous map is quasi-compact again.

First we show that r is a retraction. Indeed, using Lemma 6.7 we see that if x was continuous, $|I(x)| \subseteq \Gamma_x \cup \{0\}$ are cofinal, hence $\varphi_x(I)k^+(x)$ is the principal ideal generated by a topologically nilpotent element, hence $\bigcap_{n\geq 1}(\varphi_x(I)k^+(x))^n = 0$, and hence $V_{r(x)} = k(x)^+$ and $r(x) = x$.

By Lemma 7.6 it suffices to show that $r^{-1}(U)$ is open for any rational open $U \subseteq \mathrm{Spa}(A, A^+)$. We claim that we have

$$r^{-1}(\mathrm{Spa}(A, A^+)\left(\frac{f_1, \dots, f_n}{g}\right)) = \mathrm{Spv}(A, A^+)(\frac{f_1, \dots, f_n}{g}) \smallsetminus \bigcup_{a \in A^{\circ\circ}} \mathrm{Spv}(A, A^+)(\left(\frac{1}{a}\right).$$

whenever $f_1, \dots, f_n$ generate an open ideal in A. The inclusion $\supseteq$ is immediate from the fact that the map r commutes with the "localization" map $(A, A^+) \to (B, B^+)$ from in Proposition 7.7. For the converse, let $x \in r^{-1}(\mathrm{Spa}(A, A^+)\left(\frac{f_1,\dots,f_n}{g}\right))$, that is $r(x) \in \mathrm{Spa}(A, A^+)\left(\frac{f_1,\dots,f_n}{g}\right)$, so $|f_i(r(x))| \leq |g(r(x))| \neq 0$ for all $1 \leq i \leq n$. Then either $|t(x)| = 0$ for all $t \in I$ (I is an ideal of definition as above), in which case $r(x) = x$ and so $x \in \mathrm{Spv}(A, A^+)\left(\frac{f_1,\dots,f_n}{g}\right)$ is automatic, or there is some $t \in I$ with $1 > |t(r(x))| \neq 0$, in which case we may, replacing all f_i, g by $f_i t^N, g t^N$ with $N \gg 0$, assume that all $f_i, g \in A^+$. But then, by definition of $x' = r(x)|_{A^+}$, $|f_i(r(x))| \leq |g(r(x))|$ simply means that $\varphi_{r(x)}\left(\frac{f_i}{g}\right) \in V_{r(x)}$ (a priori it lies in only in $\mathrm{Frac}\, V_{r(x)}$). Then, looking at (7.3), it follows that $\varphi_x\left(\frac{f_i}{g}\right) \in k(x)^+$, i.e., $|f_i(x)| \leq |g(x)|$ for all i. To show that x lies in $\mathrm{Spv}(A, A^+)\left(\frac{f_1,\dots,f_n}{g}\right)$ it thus remains to show that $|g(x)| \neq 0$. Suppose that $|g(x)| = 0$. But then $|f_i(x)| = 0$ for all i, and as $f_1, \dots, f_n$ generate an open ideal of A, $|t(x)| = 0$ for all $t \in I^N$ with $N \gg 0$. But then also $|t(x)| = 0$ for all $t \in I$, contradicting our assumption on x. ☐

(Note that a posteriori $r^{-1}(\mathrm{Spa}(A, A^+)\left(\frac{f_1,\dots,f_n}{g}\right))$ is also open when the ideal generated in A by $f_1, \dots, f_n$ is not open, but it need not be a standard open.) Theorem 7.10 is finally proven.

Remark 7.16. In [Mor19] a more conceptual/more general treatment of the retraction r is given. Here we –relying on the presentation in [Ans]– have essentially considered the relevant special case of [Mor19].

8. First examples and properties of analytic $\mathrm{Spa}(A, A^+)$

We have seen examples of $\mathrm{Spa}(A, A^+) = \mathrm{Spv}(A, A^+)$ for a discrete Huber pair (A, A^+) before. Now let us make the first steps towards understanding the structure of $\mathrm{Spa}(A, A^+)$ for analytic (A, A^+). First we recall some old and fix some new notation.

8.1. **Residue fields and further notation.** Let A, A^+ be a (for now, arbitrary) Huber pair and let $X = \mathrm{Spa}(A, A^+)$. Given $x \in X$ (and choosing a representative of the equivalence class of valuations), we have the field $k(x) = \mathrm{Frac}\, A/\mathrm{supp}(x)$, which we call the (uncompleted) *residue field* of x. The valuation x factors as

$$A \xrightarrow{\varphi_x} k(x) \xrightarrow{|\cdot|_x} \Gamma_x \cup \{0\}.$$

We denote the valuation of $k(x)$ induced by x again by $a \mapsto |a(x)| \in \Gamma_x$ If $f \in A$, we write $f(x) := \varphi_x(f) \in k(x)$, and refer to it as the *value of f at x*.

Inside $k(x)$ we have the *valuation ring*

$$k(x)^+ = \{a \in k(x) \colon |a(x)| \leq 1\}$$

of x. Note that the value group of x can be written as $\Gamma_x \cong k(x)^\times/(k(x)^+)^\times$. As $x \in X$, $|f(x)| \leq 1$ for all $f \in A^+$. Thus φ_x induces a map

$$\varphi_x^+ \colon A^+ \longrightarrow k(x)^+ \quad \text{and} \quad \overline{\varphi}_x \colon A^+ \to k(x)^+ \twoheadrightarrow \kappa(x),$$

where $\kappa(x) = k(x)^+/\mathfrak{m}_{k(x)^+}$ is the residue field of the valuation ring $k(x)^+$ (not to be confused with $k(x)$!)

We equip the field $k(x)$ and its subring $k(x)^+$ with a topology:

- If $\mathrm{supp}(x)$ is not open in A[26] (i.e., x is analytic, cf. Definition 10.1), then we equip $k(x)$ with the topology induced by the valuation ring $k(x)^+$.
- If $\mathrm{supp}(x)$ is an open ideal in A (i.e., x is not analytic), then we equip $k(x)$, $k(x)^+$ with the discrete topology.

In particular, $k(x)$ is a topological field. The maps φ_x, φ_x^+ are continuous.

8.2. **Case of a non-archimedean field.** Let K be a non-archimedean field with defining valuation $|\cdot|$ of rank 1. Then

$$K^\circ = \{a \in K \colon |a| \leq 1\}$$

are the integers of K, and

$$K^{\circ\circ} = \{a \in K \colon |a| < 1\}$$

is its maximal ideal. We also denote by ϖ a fixed pseudo-uniformizer of K (so $\varpi \in K^{\circ\circ} \smallsetminus \{0\}$ is an arbitrary element) and by $\overline{K} = K^\circ/K^{\circ\circ}$ the residue field of K. (It would also be natural to denote $K^\circ, K^{\circ\circ}$ by $\mathcal{O}_K, \mathfrak{m}_K$. To prevent confusion, we try to avoid this notation.)

We want to understand $\mathrm{Spa}(K, K^+)$ for a *valuation ring* K^+ of K. This will turn out to be a quite fundamental example of an analytic adic space. We want K^+ to be a ring of integral elements, so we better assume it to be open and bounded. In particular $K^\circ \supseteq K^+ \supseteq K^{\circ\circ}$. All possible such K^+ are in bijection with all valuation rings of the residue field $\overline{K}$ by Lemma 5.18(3) and Theorem 1.9(2), so in general there are very many such.

Proposition 8.1. *With K, K^+ as above, there is a natural homeomorphism*

$$\mathrm{Spa}(K, K^+) \xrightarrow{\sim} \mathrm{Spec}(K^+/K^{\circ\circ}). \tag{8.1}$$

obtained from $\mathrm{Spv}(K, K^+) \xrightarrow{\sim} \mathrm{Spec}\, K^+$ (cf. (2.2)) by removing the trivial valuation on the left and the generic point on the right.

In particular, taking $K^+ = K^\circ$, we get the Huber pair (K, K°) and $\mathrm{Spa}(K, K^\circ) = \{*\}$, the only point being $|\cdot|$.

Proof. Note that K^+ is microbial with $K^{\circ\circ}$ being its prime ideal of height 1 (as $K^+ \subseteq K^\circ \subseteq K$, and so by Corollary 1.10(4), K° is a rank 1 localization of K^+). Now, let $x \in \mathrm{Spv}(K, K^+)$ corresponding to a valuation ring $K^+ \subseteq V \subseteq K$ under the bijection (2.2). If $V \subseteq K^\circ$, then $K^{\circ\circ} \subseteq \mathfrak{m}_V$ is the height 1 prime ideal and so x is continuous (by Lemma 6.7; as by the above $K^{\circ\circ}$ coincides with the set of topologically nilpotent elements in K with respect to the valuation topology defined by V). On the other side, if $V = K$, then x is the trivial valuation, which is not continuous, cf. Remark 6.4(3). This shows bijectivity. As topologies on both sides are induced by the topologies on from the corresponding Spv's, the result follows from (2.2). □

Thus all points of $\mathrm{Spa}(K, K^+)$ have the same residue field K, but the valuation rings vary: the most generic point has the valuation ring K° and the closed point has the valuation ring K^+.

[26]Note that in this case there is some $a \in A^{\circ\circ} \smallsetminus \mathrm{supp}(x)$, and then $|a|_x$ is a topologically nilpotent element of $k(x)^+$, which forces $k(x)^+$ to be microbial!

Remark 8.2. A point of a scheme is a map of schemes $\operatorname{Spec} k \to X$ where k is a field. By analogy a point in the analytic adic world should be something like a map $\operatorname{Spa}(K, ?) \to X$ with K a non-archimedean field. It turns out that it is not enough to take $? = K^\circ$, but it "is enough" (for the understanding of analytic adic spaces) to let $? = K^+$ to vary through open bounded valuation subrings $K^+ \subseteq K^\circ$. A map $\operatorname{Spa}(K, K^+) \to X$ with $\operatorname{Spa}(K, K^+)$ as in Proposition 8.1 turns out to be a good notion of a "point" of an analytic adic space X.

8.3. Generizations and specializations in analytic $\operatorname{Spa}(A, A^+)$.

Proposition 8.3 (Only vertical specializations). *Let $X = \operatorname{Spa}(A, A^+)$ for an analytic Huber pair (A, A^+). Then any specialization $x \rightsquigarrow y$ in X is vertical, that is, preserves the support. We then have $k(x) = k(y)$ as topological fields, and within this field, the valuation ring $k(x)^+$ of x is a localization of the valuation ring $k(y)^+$ of y.*

Proof. Let $x \rightsquigarrow y$ be a specialization in X. The map $\operatorname{supp}\colon \operatorname{Spa}(A, A^+) \to \operatorname{Spec} R$ is continuous hence preserves specialization, hence $\operatorname{supp}(x) \subseteq \operatorname{supp}(y)$. Let $f \in \operatorname{supp}(y)$. Then $|f(y)| = 0$. Note that there must be some $t \in A^{\circ\circ}$ with $|t(y)| \neq 0$ (otherwise $|\cdot(y)|$ is zero on $A^{\circ\circ}$ and hence 0 on A, as A is analytic; but $|1(y)| = 1$, contradiction). Fix such $t \in A^{\circ\circ}$. As $|f(y)| = 0$, we have $y \in U(\frac{f}{t^n})$ for all $n > 0$. As x is a generization of y, this implies $x \in U(\frac{f}{t^n})$, i.e., $|f(x)| \leq |t(x)|^n$, for all n. But as x is continuous and $t \in A^{\circ\circ}$, Lemma 6.7 shows that $t(x)$ is topologically nilpotent in $k(x)$, and so $f(x) = 0$, i.e., $f \in \operatorname{supp}(x)$, proving $\operatorname{supp}(x) = \operatorname{supp}(y)$ and hence also $k := k(x) = k(y)$ (as abstract fields).

Now $k(x)^+$, $k(y)^+$ are valuation subrings of k and we check that $k(y)^+ \subseteq k(x)^+$. Indeed, let $a \in k(y)^+$; we can represent $a = \varphi_x(\frac{f}{g})$ with $f \in A$, $g \in A \smallsetminus \operatorname{supp}(y)$. As $a \in k(y)^+$, we have $|f(y)| \leq |g(y)|$ and as $g \notin \operatorname{supp}(y)$, $|g(y)| \neq 0$. Thus $y \in X(\frac{f}{g})$. As $x \rightsquigarrow y$, we also have $x \in X(\frac{f}{g})$, i.e., $|f(x)| \leq |g(x)| \neq 0$, and so $|a(x)| = |\frac{f}{g}(x)| \leq 1$, i.e., $a \in k(x)^+$. This proves $k(y)^+ \subseteq k(x)^+$. Now, Corollary 1.10(4) shows that $k(x)^+$ is a localization of $k(y)^+$.

It remains to show that the topologies on k induced by x and by y agree. Recall the definition of the valuation topology (Remark 6.3(2)) defined by y on k: a basis of opens neighboorhoods of 0 is given by $\{a \in k\colon |a| < \gamma\}$ for $\gamma \in \Gamma_y$. Now, y microbial (cf. Proposition 6.9; x is the rank 1 generization) and if $\gamma_0 \in \Gamma_y$ is a cofinal element, then we replace the above basis of neighboorhoods by $\{a \in k\colon |a| < \gamma_0^n\}$ for all $n > 0$. But this is also a basis of neighboorhoods of 0 which define the valuation topology attached to x. This finishes the proof. ☐

(With other words, the valuation topology on $k(x)$ is equal to the valuation topology defined by the maximal generization of x, i.e., the one defined by the valuation subring $k(x)^\circ \subseteq k(x)$, and the same for y. – This is a feature of microbial valuation rings.)

Now we look at all generizations (resp. all specializations) of a point of $\operatorname{Spa}(A, A^+)$ for an analytic Huber pair (A, A^+).

Proposition 8.4 (Generalizations and specializations). *Let $X = \operatorname{Spa}(A, A^+)$ for an analytic Huber pair (A, A^+).*

(1) For a point $y \in X$,

$$\{\textit{generizations of } y \textit{ in } X\} \cong \operatorname{Spa}(k(y), k(y)^+),$$

where $k(y)$ carries the valuation topology. Moreover, y has a unique rank 1 generization, which corresponds to the unique hight 1 prime ideal of $k(y)^+$.

(2) Let $x \in X$ be of rank 1. Then $k(x)^+ = k(x)^\circ$, $\mathfrak{m}_{k(x)^+} = k(x)^{\circ\circ}$ and

$$\{\textit{specializations of } x \textit{ in } X\} \cong \operatorname{Spv}(\kappa(x), \overline{\varphi}_x(A^+)),$$

where $\kappa(x) = k(x)^\circ / k(x)^{\circ\circ}$ *is the residue field of the valuation ring* $k(x)^\circ$*, equipped with discrete topology, and* $\overline{\varphi}_x\colon A^+ \to k(x)^\circ \twoheadrightarrow \kappa(x)$ *is as in §8.1. The trivial valuation of* $\kappa(x)$ *corresponds to* x*.*

Proof. (1): It is clear from Proposition 8.3 that the LHS is the subset of $\operatorname{Spec} k(y)^+$, consisting of those valuation rings, which give rise to continuous valuations. Proposition 6.9 shows that $k(y)^+$ is microbial, so there is precisely one rank 1 generization. We now conclude by Proposition 8.1: the generic point of $\operatorname{Spec} k(y)^+$ corresponds to the trivial valuation with support $\operatorname{supp}(y)$, and this is the only non-continuous one.

(2): Write K for $k(x)$. As x is of rank 1, $K^+ = K^\circ$ and $\mathfrak{m}_{K^+} = K^{\circ\circ}$ are clear from Proposition 8.1. Specializations of x correspond by Proposition 8.3 to valuation rings V of K, which

(i) are contained in K°,
(ii) the corresponding valuation is continuous (w.r.t. the valuation topology on K), and
(iii) contain $\varphi_x^+(A^+)$.

By (i), K° is a localization of V, and so we have $K^\circ \supseteq V \supseteq \mathfrak{m}_V \supseteq K^{\circ\circ}$. The valuation topologies on K defined by K° and V coincide (cf. Proposition 8.3), so $K^{\circ\circ}$ is also the set of topologically nilpotent elements in K with respect to the topology induced by V. Thus, by Lemma 6.7, the valuation $|\cdot|_V$ corresponding to V is continuous, i.e., condition (ii) is now automatic. It is immediately checked that taking preimage under the map $K^\circ \twoheadrightarrow K^\circ/K^{\circ\circ} = \kappa(x)$ induces a bijection between valuation subrings of $\kappa(x)$ and those of K, which are contained in K°. So, the set of all V's satisfying (i) and (ii) equals $\operatorname{Spv}\kappa(x)$. Condition (iii) then cuts out the subset $\operatorname{Spv}(\kappa(x), \overline{\varphi}_x(A^+))$. The trivial valuation on $\kappa(x)$ corresponds to $V = K^\circ$, and so to x. □

By Proposition 8.4(2), the set of all specilizations of a fixed rank 1 point of an analytic affinoid adic space $\operatorname{Spa}(A, A^+)$ has itself the structure of a (discrete) adic space.

8.4. **Analytic vs. Tate.**

Proposition 8.5. *Let* (A, A^+) *be an analytic Huber pair. Then* $\operatorname{Spa}(A, A^+)$ *admits a finite covering by rational open subsets* $\operatorname{Spa}(B_i, B_i^+)$ *with* (B_i, B_i^+) *Tate.*

Proof. Write $X = \operatorname{Spa}(A, A^+)$. Let $t \in A^{\circ\circ}$. The standard open $X\left(\frac{t}{t}\right)$ is not necessarily quasi-compact (as the ideal $t \cdot A$ is not necessarily open), but if I is any ideal of definition in A, which contains t, and T a finite set of generators of I, then

$$X\left(\frac{t}{t}\right) = \bigcup_{N>0} X\left(\frac{T^N}{t}\right),$$

where each $X\left(\frac{T^N}{t}\right)$ is rational open, as T^N generates the (open) ideal I^N. The equality holds by Lemma 6.7 as any $x \in X$ is continuous. As (A, A^+) is analytic, for any $x \in X$ there is some $t \in A^{\circ\circ}$ with $|t(x)| \neq 0$. Thus, varying $t \in A^{\circ\circ}$, we obtain a cover of X by qc opens of the form $X\left(\frac{T^N}{t}\right)$ with $t \in A^{\circ\circ}$. By quasi-compactness of X, there is a finite subcover. Now, by Proposition 7.7, $X\left(\frac{T^N}{t}\right) \cong \operatorname{Spa}(B, B^+)$, with $(B = A\left[\frac{1}{t}\right], B^+)$ a Huber pair. By construction (cf. proof of Proposition 7.7), $t \in B^{\circ\circ}$. As t is also a unit of B, B is Tate. □

9. The (closed) adic unit ball

Let the notation be as in §8.2. Assume that the non-archimedean field K is algebraically closed. We have the Tate algebra in one variable over K,

$$K\langle T\rangle := \left\{ \sum_n a_n T^n \in K[\![T]\!] \colon |a_n| \to 0 \text{ as } n \to \infty \right\},$$

and the of bounded elements

$$K\langle T\rangle^\circ = K^\circ\langle T\rangle = \left\{ \sum_n a_n T^n \in K^\circ[\![T]\!] \colon |a_n| \to 0 \text{ as } n \to \infty \right\}$$

We wish to understand the space

$$\mathbb{B} = \mathbb{B}^1_K = \operatorname{Spa}(K\langle T\rangle, K^\circ\langle T\rangle).$$

First of all, note that we have the natural map $\mathbb{B} \to \operatorname{Spa}(K, K^\circ) = \{*\}$. The point $*$ corresponds to the equivalence class of the valuation $|\cdot|$ on K, and it is immediate to check that any point (=equivalence class of valuations) of $\mathbb{B}$ contains precisely one valuation, which extends $|\cdot|$. For simplicity, we identify any point with the corresponding valuation (so that we now have to look at actual valuations, not equivalence classes).

Next, note that

$$K^\circ\langle T\rangle = K^\circ[T]^\wedge_\varpi \quad \text{and} \quad K\langle T\rangle = K^\circ\langle T\rangle[\varpi^{-1}]$$

is the ϖ-adic completion of the polynomial algebra. Passing to completion of a Huber pair does not change the adic spectrum (Theorem 11.8 below), so the natural map

$$\mathbb{B} = \operatorname{Spa}(K\langle T\rangle, K^\circ\langle T\rangle) \cong \operatorname{Spa}(K[T], K^\circ[T])$$

is a homeomorphism, and we have to describe the RHS. It turns out that $\mathbb{B}$ contains points of rank 1 and 2. Let us start by describing the points of rank 1:

Lemma 9.1. *(1) For any $c \in K^\circ$ and $r \in [0,1]$, the map*

$$\nu_{c,r} \colon K[T] \to \mathbb{R}_{\geq 0}, \quad f = \sum_{i=0}^n f_i (T-c)^i \mapsto \max_i \{|f_i| r^i\}$$

lies in $\mathbb{B}$ and is of rank 1.

(2) We have

$$\nu_{c,r}(f) = \sup\{|f(x)| \colon x \in D(c,r)\},$$

where

$$D(c,r) := \{x \in K \colon |x-c| \leq r\}$$

is the "closed" unit disc of radius r with center c. In particular $\nu_{c,r}$ only depends on this disc and not on the choice of c.

(3) For any point $\nu \in \mathbb{B}$ of rank 1, there exists a sequence of nested discs $D_{1,r_1} \supseteq D_{2,r_2} \supseteq \dots$ of radii $r_i \in [0,1]$ contained in $\mathbb{B}$, such that

$$\nu(f) = \inf_{n\to\infty} \sup_{y \in D_{n,r_n}} |f(y)|$$

Remark 9.2. We wrote "closed" in Lemma 9.1(2), as the condition $|x - r| \leq c$ looks like a closed condition in the archimedean world. However, for any $r = |\alpha| \in |K^\times|$ the disc $D(0,r) = \{x \in \mathbb{B} \colon |T(x)| \leq r\} = \mathbb{B}(\frac{T}{\alpha}) \subseteq \mathbb{B}$ is in fact a rational *open* subset, and is not closed (as we will see).

Proof. (1): It is clear that $\nu_{c,r}(0) = 0$, $\nu_{c,r}(1) = 1$, $\nu_{c,r}(f + g) \leq \max\{\nu_{c,r}(f), \nu_{c,r}(g)\}$. Further, $\nu_{c,r}$ is continuous as it sends the topologically nilpotent elements $K[T]^{\circ\circ} = K^{\circ\circ}[T]$ to elements $[0, 1) \subseteq \mathbb{R}_{\geq 0}$, all of which are cofinal. It remains to show that $\nu_{c,r}(fg) = \nu_{c,r}(f)\nu_{c,r}(g)$ for all $f, g \in K[T]$. Replacing $f(\cdot)$ by $f(\cdot - c)$ (and same for g) we may assume $c = 0$. Using that K is algebraically closed, we may factor f into linear polynomials and it is clear that the claim holds for arbitrary f, g, if it holds for all $f = T - a$ ($a \in K$ arbitrary) and arbitrary g. Assume we are in this situation. Then there are two cases: either $|a| \neq r$ or $|a| = r$. In the first case we deduce

$$\nu_{0,r}(Tg) = r\nu_{0,r}(g) \neq |a|\nu_{0,r}(g) = \nu_{0,r}(ag).$$

Then we can use the sharp triangle inequality (which holds, as the usual ultrametric one holds):

$$\begin{aligned}\nu_{0,r}((T - a)g) &= \max\{\nu_{0,r}(Tg), \nu_{0,r}(ag)\} = \max\{r\nu_{0,r}(g), |a|\nu_{0,r}(g)\} = \max\{r, |a|\} \cdot \nu_{0,r}(g)\\ &= \nu_{0,r}((T - a) \cdot \nu_{0,r}(g),\end{aligned}$$

finishing the proof in this case. For the other case note that $r \mapsto \nu_{0,r}(f)\colon [0, 1] \to \mathbb{R}_{\geq 0}$ is continuous (indeed, each function $r \mapsto |f_i| \cdot r^i$ is, and taking the maximum does not destroy continuity). Now, regarding both sides of the last displayed equation as functions in r (with a, g fixed), we deduce from the proven case that these functions agree at all points except possibly $r = |a|$, and by continuity they must also agree for $r = |a|$, proving the other case.

(2): Exactly the same argument as in the proof of (1), one shows that $f \mapsto \sup\{|f(x)|\colon x \in D(c, r)\}$ is a continuous valuation. This valuation agrees with $\nu_{c,r}$ on $T - a$ for all $a \in K$, so they must agree on all $f \in K[T]$.

(3): It suffices to show the equivalent (by (2)) claim that there is a family of nested discs $D(c_i, r_i)$, $i \in I$ such that

$$\nu(f) = \inf_{i\in I}\{\nu_{c_i,r_i}(f)\} \tag{9.1}$$

Given $\nu\colon K[T] \to \mathbb{R}_{\geq 0}$, let $I = \mathcal{O}_K$, $c_i = i$ and $r_i = \nu(T - c_i)$. This family of discs is nested. Indeed, fix two pairs $(c_i, r_i), (c_j, r_j)$. Wlog suppose $r_i \leq r_j$. Then $|c_i - c_j| = \nu(c_i - c_j) \leq \max\{T - c_i, T - c_j\}$, and so for all $a \in D(c_i, r_i)$ we have

$$|a - c_j| \leq \max\{|a - c_i|, |c_i - c_j|\} \leq \max\{|a - c_i|, \nu(T - c_i), \nu(T - c_j)\} = \max\{r_i, r_j\} = r_j.$$

Thus $D(c_i, r_i) \subseteq D(c_j, r_j)$, proving the claim. On both sides of (9.1) we have valuations, so it suffices to check (9.1) on $f = T - y$ ($y \in K$). Fix an $i \in I$. There are two cases: either $\nu(T - y) \leq r_i$ or $\nu(T - y) > r_i$. Suppose the first option holds. Then, similar as above, we compute

$$|y - c_i| = \nu(y - c_i) \leq \max\{\nu(T - y), \nu(T - c_i)\} = \max\{\nu(T - y), r_i\} = r_i,$$

and so $D(y, \nu(T - y)) \subseteq D(y, r_i) = D(c_i, r_i)$. Thus, from (2) we deduce

$$\nu(T - y) = \sup_{z\in D(y,\nu(T-y))}\{|z - y|\} \leq \sup_{z\in D(c_i,r_i)}\{|z - y|\} \overset{(2)}{=} \nu_{c_i,r_i}(T - y).$$

In the case $\nu(T - y) > r_i$, we can use the strict triangle inequality:

$$|c_i - y| = \nu(c_i - y) = \max\{\nu(T - c_i), \nu(T - y)\} = \nu(T - y).$$

(here we use that $\nu(T - c_i) \leq r_i$ by the calculation at the beginning of proof of (3)). On the other side,

$$\nu_{c_i,r_i}(T - y) \overset{\text{Def.}}{=} \max\{r_i, |y - c_i|\} = \max\{r_i, \nu(T - y)\} = \nu(T - y),$$

where the second equality follows from the preceeding computation. This verifies (9.1) for $T - y$ for each $y \in K$ and we are done. ∎

The sequence of nested discs in Lemma 9.1(3) can either converge to a point in $\mathbb{B}$ or not. In fact, there are four options for points of rank 1 in $\mathbb{B}$:

(1) classical points:

$\lim_n r_n \to 0$ and $\bigcap_n D_{n,r_n} = \{c\}$ for some (necessarily unique) point $c \in K^\circ$. Then ν is simply the valuation $\nu(f) = |f(c)|$.

Note that the set $\operatorname{SpecMax} K\langle T\rangle$ of the maximal ideals of the Tate algebra $K\langle T\rangle$ is is in bijection with the set of classical points, as any maximal ideal is of the form $(T-c)K\langle T\rangle$ for some $c \in K^\circ$. (This is a fact from the classical rigid-analytic geometry, proven in [BGR84, 6.1.2 Cor. 3])

(2) Points on the limbs with radii $\in |K^\times|$:

$\lim_n r_n = r > 0$ and $r \in |K^\times|$. Then $\nu = \nu_{c,r}$ for some $c \in \mathbb{B}$. As proven in Lemma 9.1, ν only depends on the disc, not on the choice of the point c in it. In particular, there is exactly one disc with $r = 1$, namely $K^\circ = D_{c,1}$ itself ($c \in K^\circ$ can be chosen arbitrarily!) and the corresponding $\nu_{0,1}$ is called the *Gauß point*.

(3) Points on the limbs with radii $\notin |K^\times|$:

$\lim_n r_n = r > 0$ and $r \notin |K^\times|$. Then still, $\nu = \nu_{c,r}$ for some $c \in \mathbb{B}$. But these points behave differntly than those of type (2).

Such points can only exist if $|K^\times| \neq \mathbb{R}_{>0}$. We can get rid of them by enlarging K: if $K' \supseteq K$ is some alagebraically closed non-archimedean extension of K with $r \in |K'^\times|$, then we have a map $\mathbb{B}_{K'} \to \mathbb{B}_K$ and there will be a point of type (2) which maps to ν. Ultimately, one can arrange $|K'^\times| = \mathbb{R}_{>0}$, in which case there are no points of type (3).

(4) "dead ends": points which become "visible" after spherical completion

$\lim_n r_n = 0$ and $\bigcap_n D_{n,r_n} = \varnothing$. This points "look like" the classical ones, except that there is no corresponding point in K°. The appearence of such points corresponds to the fact that that an algebraically closed non-archimedean field K needs not be spherically complete.[27] An example is $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$. More precisely, one can construct an algebraically closed non-archimedean field extension K^{sc} of K, with the same residue field, such that K^{sc} is spherically complete. Then, for any point $\nu = \inf_n \nu_{c_n,r_n} \in \mathbb{B}$ with $r_n \to 0$, there is some $a \in (K^{\mathrm{sc}})^\circ$ such that $\nu(f) = |f(a)|^{\mathrm{sc}}$, where $|\cdot|^{\mathrm{sc}}$ is the (unique) extension of $|\cdot|$ to K^{sc}.

Remark 9.3. The fact that $r_n \to 0$ for a point ν of type (4) is a bit misleading for the intuition, as it suggests that after replacing K by K^{sc} (notation as above), $\nu = |\cdot(a)|^{\mathrm{sc}}$ becomes a point of type (1). However, this is not true, and ν becomes a point of type (2) or (3)! Indeed, note that as K is complete, it is closed in K^{sc}. Thus $r := \inf_{c\in K^\circ} |c-a| > 0$ is always positive. Then ν becomes the point $\nu_{a,r} \in \mathbb{B}_{K^{\mathrm{sc}}}$, which is of type (2) or (3). If we enlarge K further, we can ensure that $r \in |K^\times|$, so that ν becomes a point of type (2).

Thus, if one enlarges K, such that it becomes algebraically closed non-archimedean spherically complete field with $|K^\times| = \mathbb{R}_{>0}$, then there are only rank 1 points of type (1) and (2).

Sofar (ignoring points of higher rank) we can think of $\mathbb{B}$ as a tree having the Gauß point ($r = 1$) as its root, points of type (1) ($r = 0$) as its leaves and the tree-trunk and branches consisting of points of type (2),(3) ($r \in (0,1)$) and (4).

[27]A non-archimedean field is called *spherically complete* if the intersection of any family of nested balls in K° is non-empty.

Remark 9.4. The branches of this tree are in a (non-precise) sense "continuous", meaning that we may vary the real variable $r \in [0,1]$ moving along the branches from the root to the leaves (or back). Let us make this more precise. For fixed $c \in K^\circ$ define a function

$$\gamma_c \colon [0,1] \to \mathbb{B}, \quad r \mapsto \nu_{c,r}.$$

Thus when r moves from 0 to 1, $\gamma_c(r)$ moves from the Gauß point to c along the branches of the tree. Note that if $c, d \in K^\circ$, then γ_c, γ_d coincide for $r \in [0, |c-d|]$ and do not coincide thereafter (as $D(c,r) = D(d,r) \Leftrightarrow r \le |c-d|$). This corresponds to the branching of the tree.

The function γ_c is *not* continuous. Indeed, take wlog $c = 0$. Then for any $\alpha \in K^\circ \smallsetminus \{0\}$, $\gamma_0^{-1}(\mathbb{B}(\frac{T}{\alpha})) = [0,r]$ is a closed intervall, whereas $\mathbb{B}(\frac{T}{\alpha})$ is an open subset of $\mathbb{B}$. Thus, we should rather think of γ_c as being anti-continuous, i.e., the preimage of qc opens are closed. A precise analysis of continuity of γ_c shows that it is continuous at r if and only if $r \notin |K^\times|$, see [Con15, Lecture 11, Prop. 11.2.10] for a proof.

Now we determine the residue fields, the value groups and the residue fields of valuation rings of all points of types (1)-(4). Write $\overline{K} = K^\circ / K^{\circ\circ}$.

(1) For $c \in K^\circ$, $\operatorname{supp} \nu_{c,0} = (T - c)$, so it is clear that

$$k(\nu_{c,0}) \cong K \quad \text{and} \quad \kappa(\nu_{c,0}) = \overline{K} \quad \text{and} \quad \Gamma_{\nu_{c,0}} = |K^\times|.$$

(2) Wlog assume $c = 0$ and let $r \in (0,1] \cap |K^\times|$. Write $\nu = \nu_{0,r}$. We claim that

$$k(\nu) = K(T) \quad \text{and} \quad \kappa(\nu) = \overline{K}\left(\frac{T}{a}\right) \quad \text{and } \Gamma_\nu = |K^\times|$$

for an arbitrary $a \in K^\circ$ with $|a| = r$.

Any $f \in K[T]$ has only finitely many zeros, so $\operatorname{supp} \nu_{0,r} = 0$ by Lemma 9.1(2). Thus $k(\nu_{c,r}) = K(T)$.[28] We have $k(\nu)^+ = \{\frac{f}{g} \in K(T) \colon \nu(f) \le \nu(g)\}$ and its maximal ideal $\mathfrak{m}_{k(\nu)^+}$ is given by replacing '$\le$' by '$<$'. Note that $k(\nu)^+ \cap K[T] = K^\circ[\frac{T}{a}]$, and $\mathfrak{m}_{k(\nu)^+} \cap K[T] = K^{\circ\circ}[\frac{T}{a}]$, for an arbitrary $a \in K^\times$ with $|a| = r$. Putting $\overline{K} = K^\circ/K^{\circ\circ}$, this induces a map on quotients $\overline{K}\left[\frac{T}{a}\right] \to \kappa(\nu)$, which is injective, as clearly $K^\circ[\frac{T}{a}] \cap \mathfrak{m}_{k(\nu)^+} = K^{\circ\circ}[\frac{T}{a}]$. Now $\kappa(\nu)$ is a field, so this injection factrs through an injection

$$\overline{K}\left(\frac{T}{a}\right) \hookrightarrow \kappa(\nu).$$

of fields. This injection is in fact an isomorphism. Let us check this for $r = 1$ (for simplicity; to deduce the general case apply the transformation $T \mapsto \frac{T}{a}$), in which case $\nu(h)$ is the maximum of all valuations coefficients of $h \in K[T]$. Start with $f, g \in K[T]$ such that $\frac{f}{g} \in k(\nu)^+$, i.e., $\nu(f) \le \nu(g)$. If $g \in K^\circ[T]$ (and hence also $f \in K^\circ[T]$), nothing is to do. Assume $g \notin K^\circ[T]$ and let $b \in K^\circ$ be any element with $|b| = \nu(g)^{-1}$. Then $\frac{f}{g} = \frac{bf}{bg}$ and $bf, bg \in K^\circ[T]$, i.e., $\frac{f}{g} \in \operatorname{Frac} \overline{K}[T]$, proving the claim. Fnially, for Γ_ν, we again may rescale and assume $r = 1$, in which case $\Gamma_\nu = |K^\times|$ is clear.

(3) Wlog assume $c = 0$ and let $r \in (0,1] \smallsetminus |K^\times|$. Write $\nu = \nu_{0,r}$. We claim that

$$k(\nu) = K(T) \quad \text{and} \quad \kappa(\nu) = \overline{K} \quad \text{and} \quad \Gamma_\nu = |K^\times| \times r^{\mathbb{Z}}.$$

The claim about Γ_ν is immediate from definition. Also, as for type (2) points, it is clear that the support is trivial and so $k(\nu) = K(T)$. For $f = \sum_{i=1}^n f_i T^i$, $\nu(f) = \max_i\{|f_i| r^i\}$. We have $\nu(f) \le 1$ (resp. $\nu(f) < 1$) if and only if $f_0 \in K^\circ$ and $|f_i| < r^{-i}$ for all $i > 0$

[28] Note that we work with the Huber pair $(K[T], K^\circ[T])$, not $(K\langle T\rangle), K^\circ\langle T\rangle)$: otherwise, the residue field $k(\nu_{c,r})$ would be bigger. Only the *completed* residue field (cf. **??** below) is an invariant of the adic space!

(resp. $f_0 \in K^{\circ\circ}$ and $|f_i| < r^{-i}$ for all $i > 0$). Note that equality can never hold as $r \notin |K^\times|$, which is divisible as K is algebraically closed. We deduce

$$K[T] \cap k(\nu)^+ = \left\{ \sum_i f_i T^i \colon f_0 \in K^\circ; |f_i| < r^{-i}\, \forall i > 0 \right\},$$

and the same formula with K° replaced by $K^{\circ\circ}$ for $K[T] \cap \mathfrak{m}_{k(\nu)^+}$. Thus $K[T] \cap k(\nu)^+/K[T] \cap \mathfrak{m}_{k(\nu)^+} \xrightarrow{\sim} K$, $f \mapsto \overline{f(0)} = \overline{f_0}$. This quotient is (as in case (2) above) just a subring of $\kappa(\nu) = k(\nu)^+/\mathfrak{m}_{k(\nu)^+}$, so we are not yet done, but essentially the same argument applies to determine $\kappa(\nu)$ using the following observation: an $\frac{f}{g} \in K(T)$ with $f = \sum_i f_i T^i, g = \sum_i g_i T^i \in K[T]$, lies in $k(\nu)^+ \smallsetminus \mathfrak{m}_{k(\nu)^+}$ if and only if $\nu(f) = \nu(g)$, which can only be the case if the maximum computing $\nu(f)$, $\nu(g)$ is attained by the same value of i. It then follows that $\kappa(\nu) \cong \overline{K}$.

(4) If ν is of type (4), then one can verify that

$$k(\nu) = K(T) \quad \text{and} \quad \kappa(\nu) = \overline{K} \quad \text{and} \quad \Gamma_\nu = |K^\times| \times r^{\mathbb{Z}}.$$

We omit the details.

Now we determine the higher rank points of $\mathbb{B}$. By Proposition 8.4(2), each such point is the specialization of a unique point of rank 1. We have to consider the image of

$$\overline{\varphi}_x \colon A^+ = K^\circ[T] \to k(x)^+ \twoheadrightarrow \kappa(x).$$

If x is of type (1),(3) or (4), then $\kappa(x) = \overline{K}$ and $\overline{\varphi}_x(K^\circ[T]) = \overline{K}$, and so $\mathrm{Spv}(\overline{K}, \overline{K}) = \{*\}$. With other words, x has no proper specializations. When x was of type (2), we obtain some rank 2 points, which are called of type (5):

(5) Wlog assume $c = 0$ and let $r > 0$. Let $x = \nu_{0,r}$ be a point of type (2). Then $\kappa(X) = \overline{K}\left(\frac{T}{a}\right)$ for any $a \in K^\times$ with $|a| = r$. Concerning $\overline{\varphi}_x(K^\circ[T])$ there are two different cases:

- x is the Gauß point of $\mathbb{B}$ ($r = 1$): then $\overline{\varphi}_x(K^\circ[T]) = \overline{K}[T] \subseteq \overline{K}(T) = \kappa(x)$, and the specializations of x are in bijection with points of

$$\mathrm{Spv}(\overline{K}(T), \overline{K}[T]) \cong \mathrm{Spec}\, \overline{K}[T] \cong \mathbb{A}_{\overline{K}}$$

The generic point of $\mathbb{A}_{\overline{K}}$ corresponds to x, and all closed points correspond to points of rank 2 of $\mathbb{B}$, with rank 1 generization x.

- x is not the Gauß point of $\mathbb{B}$ ($r < 1$): then $\overline{\varphi}_x(K^\circ[T]) = \overline{K} \subseteq \overline{K}(T) = \kappa(x)$, and the specializations of x are in bijection with points of

$$\mathrm{Spv}(\overline{K}(T), \overline{K}) \cong \mathbb{P}_{\overline{K}}.$$

To make the rank 2 valuations explicit, let $c \in K^\circ$, $r \in (0,1] \cap |K^\times|$. Let $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$ with γ infinitisimally less than 1, i.e., $s < \gamma < 1$ for any $s \in (0,1) \subseteq \mathbb{R}_{>0}$ (so $\Gamma \cong \mathbb{R}_{>0} \times \mathbb{Z}$ with lexicographical order). Then we can form the valuations

$$\nu_{c,<r} \colon f = \sum_i f_i (T-c)^i \mapsto \max_i \{|f_i| \cdot (r\gamma)^i\}$$
$$\nu_{c,>r} \colon f = \sum_i f_i (T-c)^i \mapsto \max_i \{|f_i| \cdot (r\gamma)^{-i}\}$$

One shows (as above) that these are continuous valuations, and all of them, except $\nu_{c,>1}$, are ≤ 1 on $K^\circ[T]$ (note that $\nu_{c,>1}(T) = \gamma^{-1} > 1$). Moreover, $\nu_{c,<r}$ only depends on the *open* disc $D^\circ(c,r) = \{a \in K^\circ \colon |a-c| < r\}$.

It remains to state which points of type (2) specialize to which ones of type (5). First note that a point $\nu_{c,r}$ of rank 2 only depends on the disc $D = D(c,r)$, not on its center c (any point $c' \in D$ could be the center!), So, if D is any of the discs $D(c,r)$, then let us write

$$\nu_D$$

for the corresponding type 2 point. Then ν_D specializes to $\nu_{c,<r}$ if and only if $D = D(c,r)$.

We can relate this to open discs: note that each open disc $D^\circ = D^\circ(c,r)$ only depends on $c \mod aK^{\circ\circ}$, where $a \in K^\times$ satisfies $|a| = r$. For example, for $r = 1$, the open discs are in bijection with $K^\circ/K^{\circ\circ} \cong \overline{K}$; for $r < 1$ the picture is similar. Thus, if D is a closed disc, then the collection of open subdiscs of radius r contained in D is in bijection with $\overline{K}$: for $c \in D$, the disc $D^\circ(c,r)$ goes to $c \mod |a|K^{\circ\circ}$.

With other words, the specializations of type (5) of ν_D are in bijection with the open discs contianed in D (those sit "below" ν_D), resp., if $r < 1$, then there is one more point $\nu_{c,>r}$ (independent on the choice of $c \in D$), which sits "above" ν_D.

Remark 9.5 (Higher rank points replace admissible covers)**.** Let $|\varpi| = \varepsilon$, so that $\varepsilon \in (0,1)$. We have the opens

$$U := \mathbb{B}\left(\frac{1}{T}\right) = \{x \in \mathbb{B} \colon |T(x)| = 1\}, \qquad \text{and}$$
$$V := \bigcup_{n>0} \mathbb{B}\left(\frac{T}{\varpi^{1/n}}\right) = \bigcup_{n>0} \left\{x \in \mathbb{B} \colon |T(x)| \le \varepsilon^{1/n}\right\}$$

of $\mathbb{B}$. When we only look at points of type (1) (that is the classical points, so we pass to the rigid-analytic unit ball), then U, V cover the whole ball set-theoretically. This even holds when we look at all points of rank 1. In the rigid-analytic world this issue had to be resolved by introducing the G-topology of admissible coverings and showing that $U \cup V$ is not an covering of $\mathbb{B}$ in the G-topology.

In adic spaces, –and this is an imporant advantage– the rank 2 point $\nu_{0,<1}$ does not lie in $U \cup V$: $\nu_{0,<1} \notin U$ as $\nu_{0,<1}(T) = \gamma < 1$ and $\nu_{0,<1} \notin V$ as $\nu_{0,<1}(T) > \lambda$ for any $\lambda \in (0,1) \subseteq \mathbb{R}_{>0}$. Thus, in the adic world we have $U \cup V \neq \mathbb{B}$.

Remark 9.6 (The "open" and the "closed" closed unit balls)**.** We can make the point $\nu_{0,>1}$ part of our ball by shrinking $A^+ = K^\circ[T]$. Let $A^{++} = K^\circ + K^{\circ\circ}[T]$ in $K[T]$. Then

$$\mathbb{B}' = \operatorname{Spa}(K[T], A^{++}) = \mathbb{B} \cup \{\nu_{0,>1}\}$$

(set-theoretically). One might call $\mathbb{B}$ the "open closed unit ball" and $\mathbb{B}'$ the "closed closed unit ball", for the following reason. When we embed $\mathbb{B}$ into the adic unit ball $\mathbb{B}_2(0) = \{|T| \le 2\}$ of radius 2, then the image will be the rational *open* subset $\mathbb{B}_2(0)\left(\frac{T}{1}\right)$. Its closure will be exactly $\mathbb{B}'$.

Remark 9.7 (The "closed open unit ball")**.** The complement in $\mathbb{B}$ of the rational open $\mathbb{B}\left(\frac{1}{T}\right) = \{x \colon |T(x)| = 1\}$ is the *closed* (even closed constructible) subset

$$\mathbb{B}^\circ = \{x \colon |T(x)| < 1\}.$$

To it all points belong, whose correponding disc (or family of discs) is contained in the open unit disc $D^\circ(0,1) = \{c \in K^\circ \colon |c| < 1\}$. Note that by Proposition 8.4(1), $\mathbb{B}^\circ$ cannot be of the form $\operatorname{Spa}(A, A^+)$ for an (at least analytic) Huber pair (A, A^+), as its point $\nu_{0,<1}$ has rank two but has no rank 1 generization. Thus, a closed subset of an affinoid adic space need not to carry the structure of an affinoid (analytic) adic space (which is quite annoying if one compares to the situation for schemes).

Remark 9.8 (The adic affine line)**.** To construct the adic analogue of the affine line over K, we can take the union of concentric closed balls of growing radii: for $n \geq 0$, let

$$\mathbb{B}_n = \operatorname{Spa}\left(K\left\langle \varpi^n T\right\rangle, K^\circ\left\langle \varpi^n T\right\rangle\right),$$

where $K\left\langle \varpi^n T\right\rangle = \left\{\sum_{i=0}^{\infty} a_i \left(\varpi^n T\right)^i : |a_i| \to 0\right\}$. Then $\mathbb{B}^n$ is a closed unit ball with radius $\frac{1}{\varpi^n}$ (just as on $\mathbb{B}$ the inequality $|T| \leq 1$ is forced, on $\mathbb{B}^n$ we must have $|\varpi^n T| \leq 1$, i.e., $|T| \leq \frac{1}{|\varpi^n|}$). For $n < m$, the inclusion $K\left\langle \varpi^m T\right\rangle \hookrightarrow K\left\langle \varpi^n T\right\rangle$ induces the inclusion of balls $\mathbb{B}_n \hookrightarrow \mathbb{B}_m$. Then –at least as a topological space– we can construct the *adic affine line* as the union $\mathbb{A}^{1,\mathrm{ad}} = \bigcup_{n\geq 0} \mathbb{B}_n$. Note that $\mathbb{A}^{1,\mathrm{ad}}$ is not quasi-compact.

Remark 9.9 (Maximal Hausdorff quotient)**.** Due to the presence of non-closed points, $\mathbb{B}$ is not Hausdorff. The maximal Hausdorff quotient $\mathbb{B} \twoheadrightarrow \mathbb{B}^{\mathrm{Hd}}$ will just identify all points of type (5) with the corresponding point of type (2), and change nothing otherwise. $\mathbb{B}^{\mathrm{Hd}}$ can be identified with the Berkovich spectrum of $K\langle T\rangle$.

More generally, for a Tate–Huber pair (A, A^+) the maximal Hausdorff quotient of $X = \operatorname{Spa}(A, A^+)$ coincides with the quotient $\overline{X}$ of X by the equivalence relation killing all specializations (cf. [Bha17, Proposition 7.4.13])[29]. Moreover, if A is an affinoid K-algebra and $A^+ = A^\circ$, then $\overline{X}$ also coincides with the Berkovich space of A. In this sense, adic spaces generalize Berkovich spaces.

Remark 9.10 (Differences to spectra of rings)**.** Spaces $\operatorname{Spa}(A, A^+)$ for analytic Huber rings A, A^+ are spectral, as we have shown. However, they behave quite differently from schemes. In particular:

- There are actually not so many specialization relations in $\mathbb{B}$, e.g., it does not have a generic point (and is, in partuclar, not irreducible). Instead there is the Gauß point.
- More precisely, the space $\mathbb{B}$ is quite close to its Hausdorff quotient (cf. Remark 9.9), whereas the Hausdorff quotient of the topological space of a noetherian scheme X is just $\pi_0(X)$, the finite discrete set of connected components.
- The set of all generizations of a point $y \in \operatorname{Spa}(A, A^+)$ is a totally ordered chain of points (Proposition 8.3). This is very different in schemes, where the set of generizations of $x \in \operatorname{Spec} A$ is homeomorphic to the local scheme $\operatorname{Spec} \mathcal{O}_{X,x}$, which can be arbitrarily complicated (even if X is of finite type over a field).

10. Affinoid fields and adic points

10.1. **Analytic points.** We can single out the collection of points of $\operatorname{Spa}(A, A^+)$ which behave like points in some $\operatorname{Spa}(B, B^+)$ with B analytic:

Definition 10.1. Let (A, A^+) be a Huber pair. A point $x \in \operatorname{Spa}(A, A^+)$ is called *analytic* if $\operatorname{supp}(x)$ is not open in A. The subset of all analytic points with subspace topology is denoted $\operatorname{Spa}(A, A^+)_{\mathrm{an}}$.

Remark 10.2. If A is analytic, then any point of $\operatorname{Spa}(A, A^+)$ is analytic. Indeed, if $\operatorname{supp}(x)$ is open, then it contains $A^{\circ\circ}$ (by Remark 6.4(2)), and so the unit ideal, contradiction.

Lemma 10.3. *Let $x \in \operatorname{Spa}(A, A^+)$. Then the following are equivalent:*

(i) x is not analytic
(ii) $|t(x)| = 0$ for any $t \in A^{\circ\circ}$

[29]The situation is a bit subtle: by Proposition 8.4 we know that the subset $X^{\mathrm{rk}=1} \subseteq X$ of rank 1 points is in bijection with $\overline{X}$, but the subspace topology on it differs from the quotient topology on $\overline{X}$, so the natural map $X^{\mathrm{rk}=1} \to \overline{X}$ is not a homeomorphism in general.

(iii) If $A_0, I = (t_1, \dots, t_n)_{A_0}$ is some couple of definition of A, then $|t_i(x)| = 0$ for all i.
Moreover, A is analytic if and only if $\mathrm{Spa}(A, A^+) = \mathrm{Spa}(A, A^+)_{\mathrm{an}}$.

Proof. (i) $\Leftrightarrow$ (ii): By Remark 6.4(2), $\mathrm{supp}(x)$ is open if and only if it contains $A^{\circ\circ}$. (ii) $\Rightarrow$ (iii) is clear; (iii) $\Rightarrow$ (ii): any $t \in A^{\circ\circ}$ is nilpotent, so $t^N \in I$ for $N \gg 0$, and then (iii) implies that $|t^N(x)| = 0$, and hence also $|t(x)| = 0$. For the last statement, note that if A is analytic, then any point in $x \in \mathrm{Spa}(A, A^+)$ is analytic (by equivalence (i) $\Leftrightarrow$ (ii)). Suppose A is not analytic. Then we may find some prime ideal $A^{\circ\circ} \subseteq \mathfrak{p} \subseteq A$. By Remark 6.4(2), $\mathfrak{p}$ is open. Hence by Remark 6.4(1), the trivial valuation $x_{\mathfrak{p}}$ on A with support $\mathfrak{p}$ is continuous. As also $|f(x_{\mathfrak{p}})| \leq 1$ for all $f \in A^+$, we have $x_{\mathfrak{p}} \in \mathrm{Spa}(A, A^+)$. As the support $\mathrm{supp}(x_{\mathfrak{p}}) = \mathfrak{p}$ is open, $x_{\mathfrak{p}}$ is not analytic. □

Lemma 10.4. *Let (A, A^+) be a Huber pair with a couple of definition $A_0, I = (t_1, \dots, t_n)_{A_0}$. Then*

$$\mathrm{Spa}(A, A^+)_{\mathrm{an}} = \bigcup_{i=1}^{n} \mathrm{Spa}(A, A^+)\left[\frac{f_1, \dots, f_n}{f_i}\right].$$

In particular, $\mathrm{Spa}(A, A^+)_{\mathrm{an}}$ *is qc open subset of* $\mathrm{Spa}(A, A^+)$ *(and hence spectral).*

Proof. Let $x \in \mathrm{Spa}(A, A^+)$. Then x lies in the RHS if and only if $|t_i(x)| \neq 0$ for some i. By Lemma 10.3 this is equivalent to x being analytic. □

We can generalize Proposition 6.9:

Exercise 10.5. Let (A, A^+) be a Huber pair. Any point in $\mathrm{Spa}(A, A^+)_{\mathrm{an}}$ is microbial and has rank ≥ 1.

Proposition 10.6 (Description of analytic points)**.** *Let (A, A^+) be a Huber pair. There is a natural bijection*

$$\mathrm{Spa}(A, A^+)_{\mathrm{an}} \xrightarrow{\sim} \left\{\varphi\colon A^+ \to V \middle| 0 \neq \varphi(A^{\circ\circ}) \subseteq \mathfrak{p}_V\right\} / \sim$$

where V is a microbial valuation ring, $\mathfrak{p}_V$ its prime ideal of height one, φ a homomorphism, and the equivalence relation is generated by identifying $\varphi\colon A^+ \to V$ with $\alpha \circ \varphi\colon A^+ \to V \to W$, whenever α is a faithfully flat map of microbial valuation rings, preserving the height one prime ideals.

Proof. For $x \in \mathrm{Spa}(A, A^+)_{\mathrm{an}}$, we have the attached map $\varphi^+\colon A^+ \to k(x)^+$. Then $k(x)^+$ is microbial by Exercise 10.5 (cf. also Proposition 6.9), $\varphi_x^+(A^{\circ\circ}) \subseteq \mathfrak{p}_V$ by Lemma 6.7, and $\varphi_x^+(A^{\circ\circ}) \neq 0$ (by Lemma 10.3 as x is analytic). This induces the map from left to right side. Conversely, start with some $\varphi\colon A^+ \to V$ on the right side. By Lemma 6.7, this is a continuous valuation of A^+. By assumption, for some $t \in A^{\circ\circ}$ we have $\varphi(t) \neq 0$. Thus, by Lemma 7.13, φ extends to a map $\widetilde{\varphi}\colon A \to K = \mathrm{Frac}(V)$ (defined by $\widetilde{\varphi}(a) = \frac{\varphi(t^N a)}{\varphi(t^N)}$ for $N \gg 0$), corresponding to a continuous valuation x of A. Its restriction to A^+ is φ, so in particular $|f(x)| \leq 1$ for all $f \in A^+$, and $|f(x)|$ are cofinal for all $f \in A^{\circ\circ}$. With other words, $x \in \mathrm{Spa}(A, A^+)$. By Lemma 10.3, x is analytic as there is some $t \in A^{\circ\circ}$ with $|t(x)| \neq 0$. This defines an inverse to the above map. □

Exercise 10.7. Provide a similar description for non-analytic points. (Solution: $Spa(A, A^+)_{\mathrm{non-an}} = \mathrm{Spv}(A/A^{\circ\circ} \cdot A, A^+/A^{\circ\circ} \cdot A^+)$, see [Gle24, 1.19].

10.2. **Adic points and affinoid fields.** In fact, one can unify both cases (analytic and non-analytic), by introducing the following notion.

Definition 10.8. An *affinoid field* is a Huber pair (K, K^+) where K is either a (complete!) non-archimedean or a discrete field, and K^+ is an open and bounded valuation subring.

Note that in both cases $\mathrm{Spa}(K, K^+)$ is a chain of specializations:

- if K is non-archimedean, $\mathrm{Spa}(K,K^+) \cong \mathrm{Spec}\, K^\circ/K^{\circ\circ}$, cf. Proposition 8.1.
- if K is discrete, $K^{\circ\circ} = 0$ and $\mathrm{Spa}(K,K^+) \cong \mathrm{Spec}\, K^+$.

To any point of an adic spectrum we can attach an affinoid field:

Construction 10.9. Let $X = \mathrm{Spa}(A,A^+)$ for Huber pair (A,A^+). Let $x \in X$. Attached to x we have the pair $(k(x), k(x)^+)$ from §8.1. Recall that both are equipped with a topology: non-archimedean in the analytic case and discrete in the other case. Then we have the completion

$$(K(x), K(x)^+) = (\widehat{k(x)}, \widehat{k(x)^+})$$

(so, in the discrete case $K(x) = k(x)$), which is an affinoid field. Clearly, there is a natural map $(A,A^+) \to (K(x), K(x)^+)$, making $(K(x), K(x)^+)$ an (A,A^+)-algebra.

In fact, we will see soon (Theorem 11.8) that completion does not change the adic spectrum, so we have

$$\mathrm{Spa}(K(x), K(x)^+) \cong \mathrm{Spa}(k(x), k(x)^+).$$

Recall that for an (affine) scheme $X (= \mathrm{Spec}\, A)$, the topological space of X can be described as the set of equivalence classes of maps $A \to k$ to fields. For adic spaces we have something similar:

Proposition 10.10 (Points correspond to maps to affinoid fields)**.** *We have a natural bijection*

$$\mathrm{Spa}(A,A^+) \xrightarrow{\sim} \left\{ \text{Maps } \varphi\colon (A,A^+) \to (K,K^+) \text{ into affinoid fields s.t. } \mathrm{im}(\mathrm{Frac}\,\varphi(A) \to K) \text{ dense in } K \right\} / \sim$$
$$x \mapsto (K(x), K(x)^+)$$

Here, two maps $\varphi_i\colon (A,A^+) \to (K_i, K_i^+)$ $(i = 1,2)$ are equivalent $(\varphi_1 \sim \varphi_2)$ if there is an isomorphism $\psi\colon (K_2,K_2^+) \to (K_1,K_1^+)$ of affinoid fields, such that $\varphi_1 = \psi \circ \varphi_2$.

Under this bijection, analytic (resp. non-analytic) points correspond to non-archimedean (resp. discrete) affinoid fields.

Proof. We construct an inverse. Let $\varphi\colon (A,A^+) \to (K,K^+)$ be an element of the RHS. Composing it with the valuation of K, we get a valuation x of A which is continuous (as φ is continuous) and satisfies $|f(x)| \leq 1$ on A^+ (as $\varphi(A^+) \subseteq K^+$). Thus $x \in \mathrm{Spa}(A,A^+)$. By construction x is the image of the closed point of $\mathrm{Spa}(K,K^+)$ under the induced map on adic spectra. As $k(x) = \mathrm{Frac}\, A/\mathrm{supp}(x)$, we get a map $(k(x), k(x)^+) \to (K,K^+)$. This map is continuous in both cases (analytic and non-analytic), as can be checked directly (in the non-analytic case $A^{\circ\circ} \mapsto 0 \in k(x)$, so the topology on $k(x)$ is discrete). As (K,K^+) is complete, it then factors through completion to give a map $(K(x), K(x)^+) \to (K,K^+)$. As by assumption the image of $\mathrm{Frac}(\varphi(A)) = k(x) \to K$ was dense in K, we must have $K(x) = K$ and then also $K(x)^+ = K^+$. □

Note that if $x \in \mathrm{Spa}(A,A^+)$ is non-analytic, $|t(x)| = 0$ for all $x \in A^{\circ\circ}$, so $A^{\circ\circ} \subseteq \mathrm{supp}(x)$ and so the map $A \to k(x)$ is indeed continuous, when $k(x)$ has the discrete topology. This is the reason why we may put the discrete topology on $k(x)$.

11. Further features of $\mathrm{Spa}(A,A^+)$

11.1. Zariski closed subsets.

Proposition 11.1. *Let (A,A^+) be a Huber pair and $J \subseteq A$ an ideal.*

1. *The category of Huber pairs (B,B^+) over (A,A^+) with the property $J \cdot B = 0$ has an initial object $(A/J, A/J^+)$, where A/J^+ is the integral closure of the image of A^+ in A/J.*

(2) The natural map $Z = \operatorname{Spa}(A/J, A/J^+) \to \operatorname{Spa}(A, A^+) = X$ is a homeomorphism onto a closed subset, whose image consists of precisely those valuations whose support contains J.

(3) If in (2) A is analytic, Z is pro-(qc open) in X.

Proof. (1): Let (A_0, I) be a couple of definition of A with $A_0 \subseteq A^+$. Let $J_0 = J \cap A_0$. Then $A_0/J_0 \subseteq A/J$. Put $\overline{I} := \operatorname{im}(I \hookrightarrow A_0 \twoheadrightarrow A_0/J_0) = I + J_0/J_0$, an ideal of A_0/J_0. As in the proof of Proposition 7.7, we make A/J into a topological group under addition by declaring $\overline{I}^n$ to be a system of fundamental neighboorhoods of 0 (so in particular A_0/J_0 has the $\overline{I}$-adic topology). With this topology A/J is a topological ring. Indeed, as in the proof of *loc. cit.* it suffices to prove that for any $\bar{a} \in A/J$, $n > 0$, there is some $m > 0$ with $\bar{a}\overline{I}^m \subseteq \overline{I}^n$; but $\overline{I}^n = I^n + J_0/J_0$; so, fix some lift a of $\bar{a}$ and some $m > 0$ with $aI^m \subseteq I^n$. Then $a(I^m + J_0) = aI^m + aJ_0 \subseteq I^n + J$ (as $J \subseteq A$ is an ideal containing J_0), which maps to $\overline{I}^n \subseteq A/J$.

Thus, A/J is Huber ring with $(A_0/J_0, I + J_0/J_0)$ a couple of definition. Now A/J^+ contains the image of A^+, hence of some power of I, hence is open; moreover, by construction it is intergrally closed, so it is a ring of definition and $(A/J, A/J^+)$ is a pair of definition. Also, it is clear that the map $(A, A^+) \to (A/J, A/J^+)$ is continuous, as I^n maps into $\overline{I}^n$. The universal property is straightforward.

(2): Injectivity, continuity and description of the image of $Z \to X$ is clear from construction. In particular, Z is the preimage under $\operatorname{supp}\colon \operatorname{Spa}(A, A^+) \to \operatorname{Spec} A$ of the closed subset $V(J)$. Evidently, standard opens in Z come from standard opens of X, so the map is a homeomorphism.

(3): If X is analytic, it is covered by finitely many rational opens X_i, all of which are Tate (Proposition 8.5); $Z \cap X_i \hookrightarrow X_i$ is then is cut out by the image of J; as a finite union of pro-(qc open) subsets is pro-(qc open), we may thus assume that X is Tate. In this case, let $\varpi \in A$ be a pseudo-uniformizer. Then

$$i(Z) \subseteq \bigcap_{f \in J} \bigcap_{n \geq 0} X\left(\frac{f, \varpi^n}{\varpi^n}\right),$$

as $|f(x)| = 0$ for all $f \in J$. Conversely, if x is in the RHS, then $|f(x)| \leq |\varpi^n(x)|$ for all $f \in J$ and all $n > 0$, so $f(x) = 0$ for all $f \in J$, and so $x \in Z$. □

Note that pro-(qc opens) are stable under generization, so Zariski closed subsets of analytic adic spectra have this property. This is quite different in the world of schemes!

Example 11.2. Only very few closed subsets are Zariski closed. For example, the proper Zariski closed subsets of $X = \mathbb{B}$ are the finite unions of classical (type (1)) points.

11.2. **The specialization map.** Let (A, A^+) be an analytic Huber pair. We construct a natural spectral map[30]

$$\operatorname{sp}\colon \operatorname{Spa}(A, A^+) \longrightarrow \operatorname{Spec}(A^+/A^{\circ\circ}), \tag{11.1}$$

called the *specialization map*. For $x \in \operatorname{Spa}(A, A^+)$ we have the corresponding map $\varphi_x^+\colon A^+ \to k(x)^+$, which induces the map

$$\operatorname{Spec} \varphi_x^+\colon \operatorname{Spec} k(x)^+ \to \operatorname{Spec} A^+.$$

Now $k(x)^+$ is a local ring, and we define $\operatorname{sp}(x) \in \operatorname{Spec}(A^+)$ to be the image $(\varphi_x^+)^{-1}(\mathfrak{m}_{k(x)^+})$ of the closed point $\mathfrak{m}_{k(x)^+}$ in $\operatorname{Spec} A^+$. By continuity (Lemma 6.7), $\varphi_x^+(A^{\circ\circ}) \subseteq \mathfrak{p}$, where $\mathfrak{p}$ is the smallest non-zero prime ideal of $k(x)^+$ (recall that $k(x)^+$ is microbial). In particular, $(\operatorname{Spec} \varphi_x^+)(\mathfrak{m}_{k(x)^+}) = (\varphi_x^+)^{-1}(\mathfrak{m}_{k(x)^+}) \supseteq A^{\circ\circ}$. With other words, $\operatorname{sp}(x)$ lies in the closed subset $\operatorname{Spec}(A^+/A^{\circ\circ})$.

[30] Recall that $\operatorname{Spa}(A, A^+) \to \operatorname{Spec} A$ is in general not spectral.

Lemma 11.3. *For an analytic Huber pair* (A, A^+)*, the map* sp *from* (11.1) *is spectral.*

Proof. We have to show that sp is continuous and quasi-compact. Therefore, it suffices to show that for any $\bar{f} \in A^+/A^{\circ\circ}$, $\mathrm{sp}^{-1}(D(\bar{f})) \subseteq \mathrm{Spa}(A, A^+)$ is qc open. Let $f \in A^+$ be any lift of $\bar{f}$ (so $D(\bar{f}) = D(f) \cap \mathrm{Spec}(A^+/A^{\circ\circ}) \subseteq \mathrm{Spec}(A^+)$). Now, for $x \in \mathrm{Spa}(A, A^+)$, $\mathrm{sp}(x) = (\varphi_x^+)^{-1}(\mathfrak{m}_{k(x)^+}) \in D(\bar{f})$ if and only if $\varphi_x^+(f) \notin \mathfrak{m}_{k(x)^+}$, that is $|f(x)| = 1$. That is, $\mathrm{sp}^{-1}(D(\bar{f})) = \mathrm{Spa}(A, A^+)\left(\frac{1}{f}\right)$ is a qc open of $\mathrm{Spa}(A, A^+)$. □

Example 11.4. Consider $X = \mathrm{Spa}(K, K^+)$ for a non-archimedean field K and an open and bounded valuation subring K^+. Then sp is nothing else than the isomorphism of Proposition 8.1.

Example 11.5. Consider the closed unit ball $\mathbb{B} = \mathrm{Spa}(K\langle T\rangle, K^\circ\langle T\rangle)$ from §9. Then $A^+/A^{\circ\circ} = \overline{K}[T]$ and the specialization map

$$\mathrm{sp}\colon \mathbb{B} \to \mathbb{A}^1_{\overline{K}} = \mathrm{Spec}\, \overline{K}[T]$$

Then $\mathrm{sp}(\nu_{0,1})$ = the generic point of $\mathbb{A}^1_{\overline{K}}$, and for $r < 1$, $\mathrm{sp}(\nu_{c,r}) = c \mod K^{\circ\circ} \in \overline{K}$. Indeed, let $x \in \mathbb{B}$. Let $\bar{\alpha} \in \overline{K}$ with (arbitrary) lift $\alpha \in K^\circ$. As in the proof of Lemma 11.3, $\mathrm{sp}(x) = \bar{\alpha}$ (closed point of $\mathbb{A}^1_{\overline{K}}$) $\Leftrightarrow \mathrm{sp}(x) \notin D(T - \alpha) \Leftrightarrow |(T - \alpha)(x)| < 1$. If $x = \nu_{c,r}$, then

$$|(T - \alpha)(\nu_{c,r})| = \max\{|c - \alpha|, r\}.$$

So, if $r = 1$, then $\nu_{0,1} \in D(T - \alpha)$ for any α, that is $\mathrm{sp}(\nu_{0,1})$ is the generic point of $\mathbb{A}^1_{\overline{K}}$. If $r < 1$, $|(T - \alpha)(\nu_{c,r})| < 1$ if and only if $|\alpha - c| < 1$ and so $\mathrm{sp}(\nu_{c,r}) = \bar{\alpha} \Leftrightarrow \bar{c} = \bar{\alpha} \in \overline{K}$. Similarly, we can determine $\mathrm{sp}(x)$ for x of type (3), and by continuity extend from type (2) points to their type (5) specializations.

Example 11.6. $\mathbb{B}' = \mathrm{Spa}(A, A^{++})$ be the "closed closed unit ball", as in Remark 9.6. Then $A^{++} = K^\circ + K^{\circ\circ}[T]$ and so $A^{++}/A^{\circ\circ} \cong \overline{K}$. Thus the specialization map $\mathrm{sp}\colon \mathbb{B}' \to \{*\}$ is constant.

11.3. **Passage to completion.**

Definition 11.7. The *completion* of a Huber pair (A, A^+) is the Huber pair $(\widehat{A}, \widehat{A}^+)$, where $\widehat{A}$ is the completion of A and $\widehat{A}^+$ is the integral closure of the image of A^+ in $\widehat{A}$.

The adic spectrum does not change when we pass to completion:

Theorem 11.8 (Completion of a Huber pair)**.** *Let* A, A^+ *be a Huber pair. Then the natural map*

$$\mathrm{Spa}(\widehat{A}, \widehat{A^+}) \to \mathrm{Spa}(A, A^+)$$

identifies the rational open sets. In particular, it is a homeomorphism.

Proof. Each continuous valuation $|\cdot|$ on A extends uniquely to one on $\widehat{A}$ by setting $|a| := \lim_{n\to\infty} |a_n|$ for a Cauchy sequence $a = (a_n)_{n\geq 1}$. Clearly, if $|a| \leq 1$ for $a \in A^+$, then also $|a| \leq 1$ for $a \in \widehat{A^+}$. This gives the natural bijection. It is clear that any rational open of $\mathrm{Spa}(A, A^+)$ pulls back to a rational open of $\mathrm{Spa}(\widehat{A}, \widehat{A^+})$, defined by the same elements. A priori, $\mathrm{Spa}(\widehat{A}, \widehat{A^+})$ can have more rational opens, but Proposition 11.9 shows that it does not, finishing the proof. □

In fact, we show that if we slighty change the defining elements $f_1, \dots, f_n, g$, this does not affect the corresponding rational open:

Proposition 11.9 (Perturbation of rational opens)**.** *Let (A, A^+) be a complete Huber pair. Let $f_1, \dots, f_n, g \in A$ be such that $f_1, \dots, f_n$ generate an open ideal of A. Then there exists an open neighboorhood V of 0, such that for all $f_i' \in f_i + V$ $(1 \le i \le n)$, $g' \in g + V$, the ideal generated by $f_1', \dots, f_n'$ is open and*

$$\mathrm{Spa}(A, A^+)\left(\frac{f_1, \dots, f_n}{g}\right) = \mathrm{Spa}(A, A^+)\left(\frac{f_1', \dots, f_n'}{g'}\right).$$

Proof. First of all, adding the element g to $f_1, \dots, f_n$ (this does not change the rational open), and renumbering, we may assume that $f_1 = g$.

Choose a couple of definition (A_0, I) of A such that I is contained in the (open!) ideal of A generated by $f_1, \dots, f_n$ (first choose any I and then take a high enough power which lies in that open ideal). Let $t_1, \dots, t_r$ be a finite set of generators of I.

First, we find an open $W \subseteq A$ such that perturbation of $f_1, \dots, f_n$ by W does not change the openness of the ideal generated by the f_i. As $I \subseteq (f_1, \dots, f_n)_A$, we may write, for $1 \le j \le r$, $t_j = \sum_{i=1}^n a_{ij} f_i$ with some $a_{ij} \in A$. The set $S = \{a_{ij}\}_{ij}$ is finite, hence bounded, and hence we may find a small open $W \subseteq A$, such that $S \cdot W \subseteq I^2$. Then, if for each i, $f_i' \in f_i + W$, then $t_j' = \sum_{i=1}^m a_{ij} f_i' \in t_i + I^2$, so in particular $t_j' \in A_0$; Lemma 11.10 then implies that the ideal of A_0 generated by the t_j''s is equal to I. In particular $(f_1', \dots, f_n')_A$ contains I, and is therefore open.

Lemma 11.10. *Let $I = (t_1, \dots, t_r)$ be an ideal of definition of a complete adic ring A_0. If $t_1', \dots, t_r' \in A_0$ are elements such that $t_j - t_j' \in I^2$ for all $1 \le j \le r$, then $I = (t_1', \dots, t_r')$.*

Proof. Clearly, $t_j' \in I$ for all j. We have to show the surjectivity of the A-linear map $u' \colon A^r \to I$, $(a_j)_j \mapsto \sum_{j=1}^r a_j t_j'$. Now, A^r and I are complete A_0-modules, whose I-adic topology defines the filtrations $\{I^n A^r\}_{n \ge 0}$ and $\{I^{n+1}\}_{n \ge 0}$ on them. Clearly, u' preserves this filtrations: $u'(I^n A^r) \subseteq I^{n+1}$. By completness (this is the only place in the proof of Proposition 11.9 where we used the completeness assumption on A, A^+), it suffices to show that the induced maps on graded objects

$$\mathrm{gr}_n(u') \colon (I^n A^r)/(I^{n+1} A^r) \to I^{n+1}/I^{n+2}.$$

are surjective for all $n \ge 0$. But if $u \colon A^r \to I$ is the (by assumption, surjective) map $(a_j)_j \mapsto \sum_{j=1}^r a_j t_j$, then the assumption of the lemma implies that $\mathrm{gr}_n(u') = \mathrm{gr}_n(u)$. ☐

For $1 \le i \le n$, let $U_i = \mathrm{Spa}(A, A^+)\left(\frac{f_1, \dots, f_n}{f_i}\right)$ (so, our original rational open is U_1). Each U_i is quasi-compact and $|f_i(x)| \neq 0$ for all $x \in U_i$. We now need a lemma:

Lemma 11.11. *Let (A, A^+) be a Huber pair and $X \subseteq \mathrm{Spa}(A, A^+)$ a quasi-compact subset. Let $t \in A$ be such that $|t(x)| \neq 0$ for all $x \in X$. Then there exists an open neighboorhood N of 0 in A, such that $|a(x)| < |t(x)|$ for all $a \in N$ and all $x \in X$.*

Proof. Let S be a finite set of generators of some ideal of definition J of A, and let $S^r = \{s_1 \dots s_r \colon s_i \in S\}$. Consider

$$X_r = \{x \in \mathrm{Spa}(A, A^+) \colon \forall |s(x)| \le |t(x)| \neq 0 \, \forall \, s \in S^r\}$$

This is a rational open(i.e., S^r generates an open ideal of A. Indeed, this ideal contains J^r. Note, however, that we just use that X_r is open below). Any $x \in \mathrm{Spa}(A, A^+)$ is continuous, hence $|a(x)|$ is cofinal for any $a \in J$ (Lemma 6.7). Hence, as S^r is finite, any x for which $|t(x)| \neq 0$ lies in X_r for $r \gg 0$. Thus, $X \subseteq \bigcup_{r > 0} X_r$. But as X is quasi-compact, there is some r with $X \subseteq X_r$. Replacing the ideal of definition J by J^r, we may replace S^r by S and X_r by X_1. Take now $N = J \cdot A^{\circ\circ}$, which is an additive subgroup of A, and evidently open, as it contains

J^2. If $S = \{s_1, \dots, s_m\}$, any $a \in N$ can be written as $a = \sum_{i=1}^m \alpha_i s_i$ with some $\alpha_i \in A^{\circ\circ}$. Then for any $x \in X_1$,

$$|a(x)| \leq \max_i |\alpha_i(x)| \cdot |s_i(x)| < \max_i 1 \cdot |t(x)| = |t(x)|,$$

where $|\alpha_i(x)| < 1$ is by continuity of x. □

Continuing with the proof of Proposition 11.9, note that by Lemma 11.11, for every i there is some open neighboorhood $N_i \subseteq A$, such that for all $x \in U_i$ and all $f \in N_i$, $|f_i(x)| > |f(x)|$.

Now, let $V := A^{\circ\circ} \cap W \cap \bigcap_{i=1}^n N_i$, which is an open neighboorhood of 0 in A. Let now $f_i' \in f_i + V$ for all $1 \leq i \leq n$. By the first part of the proof, the f_i' generate an open ideal of A. Let $U' = \mathrm{Spa}(A, A^+)(\frac{f_1', \dots, f_n'}{g})$ be the corresponding rational open. We must show that $U' = U_1$. Let $x \in U_1$. For $1 \leq i \leq n$, $f_i' - f_i \in V \subseteq N_1$, so $|g(x)| = |f_1(x)| > |(f_i' - f_i)(x)|$. For $i = 1$, we then get (sharp triangle inequality) $|g(x)| = |g'(x)|$. For $2 \leq i \leq n$, $|f_i'(x)| \leq |g(x)| = |g'(x)|$ (as $x \in U_1$), and this together with $|g(x)| > |(f_i' - f_i)(x)|$ implies $|f_i'(x)| \leq \max_i\{|f_i(x)|, |(f_i' - f_i))(x)|\} \leq |g'(x)|$. Thus $x \in U'$.

Conversely, let $x \in \mathrm{Spa}(A, A^+) \smallsetminus U_1$. If $|f_i(x)| = 0$ for all $1 \leq i \leq n$, then $\mathrm{supp}(x) \supseteq (f_1, \dots, f_n)_A$ must be open, hence $\mathrm{supp}(x) \supseteq A^{\circ\circ}$. Then $g' - g \in \mathrm{supp}(x)$, i.e. $|g'(x)| = |g(x)| = 0$, and hence $x \notin U'$ and we are done. So, assume there is some $1 \leq i \leq n$ with $|f_i(x)| \neq 0$. Replacing i, we may additionally assume that $|f_i(x)| = \max_j\{|f_j'(x)|\}$, so that $x \in U_j$ holds by construction. As $x \notin U_1$, we have $|f_i(x)| > |g(x)|$. For all $1 \leq j \leq n$ we have $f_j' - f_j \in V \subseteq N_i$, and so $|f_i(x)| > |(f_j' - f_j)(x)|$. Taking $j = i$ and again using the sharp triangle inequality, we get $|f_j'(x)| = |f_j(x)|$. Using this we estimate: $|g'(x)| = \max\{|g(x)|, |(g' - g)(x)|\} < |f_i(x)| = |f'(x)|$. With other words, $x \notin U'$. The proposition is proved. □

11.4. $\mathrm{Spa}(A, A^+)$ **has "enough points".** We wish to show that $\mathrm{Spa}(A, A^+)$ for a complete analytic Huber pair (A, A^+) has enough points to detect vanishing of A, units in A and the ring A^+.[31]

Theorem 11.12 ($\mathrm{Spa}(A, A^+)$ has enough points)**.** *Let $X = \mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+). Then*

(1) If $\widehat{A} \neq 0$, then $\mathrm{Spa}(A, A^+)$ is non-empty.
(2) $A^+ = \{f \in A \colon |f(x)| \leq 1 \ \forall x \in X\}$
(3) If A is complete, $A^\times = \{f \in A \colon |f(x)| \neq 0 \ \forall x \in X\}$
(4) If A is Tate(Question: is analytic enough?), then $A^{\circ\circ} = \{f \in A \colon |f(x)|^n \to 0 \ \forall x \in X\}$

Proof. We only prove the theorem in the analytic case (for some comments on the non-analytic case, see Remark 11.17). So assume that A is analytic.

(1): By Theorem 11.8, we may replace A by its completion. By Lemma 11.14 below, $\mathrm{Jac}(A^+) \supseteq A^{\circ\circ}$. So, if $\mathfrak{m}$ is any maximal ideal of A^+, then $\mathfrak{m} \supseteq A^{\circ\circ}$. The map $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^+$ is dense, so there is some point $\mathfrak{p} \in \mathrm{Spec}\, A$, such that $\mathfrak{p} \cap A^+ \subseteq \mathfrak{m}$ in $\mathrm{Spec}\, A^+$ (use that the image of $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^+$ is pro-constructible and the closure of a pro-constructible set is the union of closures of its points). We then obtain inclusions

$$\overline{A} := A^+/\mathfrak{p} \cap A^+ \hookrightarrow A/\mathfrak{p} \hookrightarrow \mathrm{Frac}(A/\mathfrak{p}) = \kappa(\mathfrak{p}).$$

Now, the domain $\overline{A}$ contains the maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/\mathfrak{p} \cap A^+$. As valuation rings are maximal among all local subrings of a field with respect to dominance relation (Theorem 1.9(4) applied to the localization $\overline{A}_{\overline{\mathfrak{m}}}$), we may find a valuation ring $\overline{A} \subseteq V \subseteq \kappa(\mathfrak{p})$ with $\mathfrak{m}_V \cap \overline{A} = \overline{\mathfrak{m}}$. With other words, under the composition $\mathrm{Spec}\, V \to \mathrm{Spec}\, \overline{A} \hookrightarrow \mathrm{Spec}\, A^+$, the generic point goes to $\mathfrak{p} \cap A^+$

[31]Note the similarity to the case of schemes, where $\mathrm{Spec}\, A = \varnothing$ if and only if $A = 0$; $f \in A$ is a unit if and only if $f \neq 0 \in \kappa(x)$ for any $x \in \mathrm{Spec}\, A$; $f \in A$ lies in $\mathrm{nil}(A)$ if and only if $f = 0 \in \kappa(x)$ for any x.

and the special to $\mathfrak{m}$. In particular, the corresponding map $\nu\colon A^+ \to V$ satisfies $\ker(A^+ \to V) = \mathfrak{p} \cap A^+$. So, $A^{\circ\circ} \not\subseteq \ker(A^+ \to V)$ (otherwise we would have $A^{\circ\circ} \subseteq \mathfrak{p}$, contradicting the analyticity of A). Let $t_1, \dots, t_n \in A^{\circ\circ}$ be a set of generators of some ideal of definition of a ring of definition of A contained in A^+. Let J be the ideal of V generated by $\nu(t_1), \dots, \nu(t_n)$. As J is finitely generated, $J = \tau V$ is principal. By the above $J \neq 0$. Let now $W = V/\bigcap_r \tau^r V$ be the τ-adic completion of V, which is microbial (cf. 1.10). Moreover, at least one t_i maps to non-zero element in W (as $JW \neq 0$). Alltogether, $\bar{\nu}\colon A^+ \to V \to W$ is a map into a microbial valuation ring satisfying $0 \neq \bar{\nu}(A^{\circ\circ}) \subseteq \mathfrak{p}_V$. By Proposition 10.6 this gives a point of $\operatorname{Spa}(A, A^+)$.

(2): By definition $|f(x)| \leq 1$ for all $f \in A^+$ and all $x \in X$. Conversely, let $f \in A$ with $|f(x)| \leq 1$ for all $x \in X$. If $f^{-1} \in A^+[f^{-1}]$ is a unit, then $f \in A^+[f^{-1}]$, which imlies that f is integral over A^+, and hence $f \in A^+$. Thus, we may assume that f^{-1} is not a unit in $A^+[f^{-1}]$. We will derive a contradiction. Let $\mathfrak{m} \subseteq A^+[f^{-1}]$ be a maximal ideal containing f^{-1} and let $\mathfrak{p} \subseteq A^+[f^{-1}]$ be a minimal prime ideal contained in $\mathfrak{m}$. Let $V \subseteq \operatorname{Frac}(A^+[f^{-1}]/\mathfrak{p})$ be a valuation ring dominating $A^+[f^{-1}]_{\mathfrak{m}}$; that is we have a map $\alpha\colon A^+[f^{-1}] \to V$ such that $\operatorname{Spec}\alpha$ sends the closed point to $\mathfrak{m}$ and the generic to $\mathfrak{p}$. This gives a map

$$\nu\colon A^+ \to A^+[f^{-1}] \xrightarrow{\alpha} V$$

of A^+, to which there correpsponds a valuation x of A^+.

Claim. We have $0 \neq \nu(A^{\circ\circ}) \subseteq \mathfrak{m}_V$ in V.

Proof of claim. The map $A^+[f^{-1}] \to A[f^{-1}]$ is injective, so $\operatorname{Spec} A[f^{-1}] \to A^+[f^{-1}]$ is dominant. As its image is pro-constructible (Theorem 3.7), and as the closure of a pro-constructible set is the union of closures of its elements (Lemma 3.11), it follows that the image of $\operatorname{Spec} A[f^{-1}] \to A^+[f^{-1}]$ contains all generic points, and in particular, $\mathfrak{p}$. Thus there is some $\mathfrak{P} \in \operatorname{Spec} A[f^{-1}]$ with $\mathfrak{P} \cap A^+[f^{-1}] = \mathfrak{p}$. But as A analytic, $A^{\circ\circ} \not\subseteq \mathfrak{P}$, and hence also $A^{\circ\circ} \not\subseteq \mathfrak{p}$. As $\mathfrak{p} = \ker\nu$, $\nu(A^{\circ\circ}) \neq 0$.

Further, let $t \in A^{\circ\circ}$. Then $t^n f \in A^+$ for $n \gg 0$. As $\nu(A^+) \subseteq V$, we have $|t^n f(x)| \leq 1$. On the other side, x is the restriction of a valuation of $A[f^{-1}]$ and by construction $|f^{-1}(x)| < 1$. This implies $|t^n(x)| < 1$ and hence $|t(x)| < 1$, that is $\nu(t) \in \mathfrak{m}_V$. This proves the claim.

"Microbilizing" x as in (1) above, we obtain some continuous valuation $\bar{x}$ of A^+ which, by Proposition 10.6 gives a point of X. Clearly, we still have $|f^{-1}(\bar{x})| < 1$ (as "microbilization" does not affect the maximal ideal of V). But by the initial assumption of f we have $|f(x)| \leq 1$. As $|f \cdot f^{-1}(x)| = 1$, this gives a contradiction. This proves (2).

(3): This is the special case $T = \{f\}$ of Corollary 11.15 (which is a consequence of part (1) of the theorem, so there is no circular reasoning).

(4): It is clear that for $f \in A^{\circ\circ}$, $|f(x)|^n \to 0$ for any $x \in X$ by Lemma 6.7. Conversely, assume $f \in A$ and for all $x \in X$, $|f(x)|^n \to 0$. Let $\varpi \in A$ be a pseudo-uniformizer. Then

$$X = \bigcup_{n>0} X\left(\frac{f^n}{\varpi}\right).$$

Indeed, for any $x \in X$, there $|\varpi(x)| \neq 0$ as t is a unit in A, then by assumption there is some $n > 0$ with $|f^n(x)| < |\varpi(x)|$. Thus, by quasi-compactness of X (Theorem 7.10), there is some $n > 0$ with $X = X\left(\frac{f^n}{\varpi}\right)$. Thus $|f(x)|^n \leq |\varpi(x)|^n$, and so $\frac{f^n}{\varpi} \in A^+$, that is $f^n \in \varpi A^+$. But A^+ is the cofiltered limit of open bounded subrings contained in it, so $f^n \in \varpi A_0$ for some open bounded A_0. As A_0 is Tate, the topology on A_0 is ϖ-adic, and so f^n is topologically nilpotent in A_0. Hence $f \in A^{\circ\circ}$. $\square$

Example 11.13. In Theorem 11.12(3) the completeness assumption can be weakened, but not omited completely, as the following example shows. Let $(A, A^+) = (K[T], K^\circ[T])$. Then $1 + \varpi T$ is non-zero at all points of the closed unit ball $\mathbb{B}$, but is not invertible. (Note that it becomes invertible after completion.)

Lemma 11.14. *Let (A, A^+) be a complete Huber pair. Then $A^{\circ\circ} \subseteq \mathrm{Jac}(A^+)$.*

Proof. Recall the notion of a *Henselian pair*: a pair (R, I) consisting of a ring and an ideal $I \subseteq R$ is Henselian if, essentially, it satisfies Hensel's lemma; cf. [Sta14, 09XE]; in particular $I \subseteq \mathrm{Jac}(R)$. Thus it suffices to show that $(A^+, A^{\circ\circ})$ is Henselian. But filtered colimits of Henselian pairs are again Henselian [Sta14, 0FWT]. Also, $(A^+, A^{\circ\circ})$ is the filtered colimit of (A_i, I_i), where A_i goes through all rings of definition contained in A^+ and I_i through (appropriate) ideals of definition in A_0. So we are reduced to show that any couple of definition (A_0, I) of A is Henselian, which holds as A_0 is I-adically complete. □

From Theorem 11.12(1) it follows that in the complete case evaluation at points in $\mathrm{Spa}(A, A^+)$ may detect whether an ideal of A is a unit ideal:

Corollary 11.15. *Let (A, A^+) be a complete Huber pair. Let $T \subseteq A$. Then the following are equivalent:*

(i) The ideal generated by T is A.
(ii) For any $x \in \mathrm{Spa}(A, A^+)$ there exists some $t \in T$ with $|t(x)| \neq 0$.

Moreover, if these conditions hold and T is finite, then $\mathrm{Spa}(A, A^+) = \bigcup_{t \in T} \mathrm{Spa}(A, A^+)\left(\frac{T}{t}\right)$ is an open covering of $\mathrm{Spa}(A, A^+)$.

Proof. (i) implies (ii): If for some $x \in \mathrm{Spa}(A, A^+)$, $|t(x)| = 0$ for all $t \in T$, then also $|1(x)| = 0$, which is absurd. (ii) implies (i): If $T \cdot A \neq A$, then let $T \cdot A \subseteq \mathfrak{m} \subseteq A$ be a maximal ideal. Then $\mathfrak{m}$ is closed in A by Lemma 11.16, and so $A/\mathfrak{m} \neq 0$ is Hausdorff, i.e., the map $A/\mathfrak{m} \hookrightarrow \widehat{A/\mathfrak{m}}$ is injective. In particular $\widehat{A/\mathfrak{m}} \neq 0$ and so the Zariski closed subset $\mathrm{Spa}(A/\mathfrak{m}, A/\mathfrak{m}^+) \neq \varnothing$ is non-empty by Theorem 11.12 for $\widehat{A/\mathfrak{m}}$ (and Theorem 11.8 to pass to completion). By Proposition 11.1, $\mathrm{Spa}(A/\mathfrak{m}, A/\mathfrak{m}^+) = \mathrm{Spa}(A, A^+) \cap V(\mathfrak{m})$. Thus, there is a valuation x in $\mathrm{Spa}(A, A^+)$ whose support contains $\mathfrak{m}$. Then $|t(x)| = 0$ for all $t \in T$. □

Lemma 11.16. *Let A be a complete Huber ring. Then $A^\times$ is open and any maximal ideal is closed in A.*

Proof. For every $a \in A^{\circ\circ}$, $(1 - a)$ is invertible with inverse $\sum_{n \geq 0} a^n$ (as $a \in A_0$ for some ring of definition, which is I-adically complete for some ideal of definition I). Thus $1 + A^{\circ\circ} \subseteq A^\times$. Hence, as $A^{\circ\circ}$ is open in A, $A^\times$ is a neighboorhood of 1. Multiplication by a unit is a homeomorphism of A preserving $A^\times$, and so $A^\times$ is also a neighboorhood of any other $a \in A^\times$. Thus $A^\times \subseteq A$ is open. Let $\mathfrak{m} \subseteq A$ be a maximal ideal. Then $\mathfrak{m}$ is contained in the closed subset $A \smallsetminus A^\times$. So the closure of $\mathfrak{m}$, which is again an ideal (this is easy) also lies in $A \smallsetminus A^\times$, i.e., is a proper ideal. As $\mathfrak{m}$ is maximal, $\mathfrak{m}$ is equal to its closure. □

Remark 11.17. Essentially the same argument as in the analytic case shows the following more general claim: for a Huber pair (A, A^+), the set of analytic points $\mathrm{Spa}(A, A)_{\mathrm{an}}$ is empty if and only if $A/\overline{\{0\}}$ is discrete. Cf. [Mor19, Proposition III.4.4.1]. From this Theorem 11.12(1) is easy to deduce: if $\mathrm{Spa}(A, A^+) = \varnothing$, then in particular $\mathrm{Spa}(A, A^+)_{\mathrm{an}} = \varnothing$, so $A/\overline{\{0\}}$ is discrete. If $A/\overline{\{0\}} \neq 0$, then there is a prime ideal $\mathfrak{p}$ of A containing $\overline{\{0\}}$; $\mathfrak{p}$ is automatically open, so that the trivial valuation with support $\mathfrak{p}$ lies in $\mathrm{Spa}(A, A^+)$, contradiction. So $A/\overline{\{0\}} = 0$, or equivalently, $\widehat{A} = 0$.

From Remark 11.17 it follows:

Corollary 11.18. *If (A, A^+) is a complete Huber pair with $\mathrm{Spa}(A, A^+)_{\mathrm{an}} = \varnothing$, then A is discrete.*

By Theorem 11.12, regarding elements of A as functions on $\mathrm{Spa}(A, A^+)$, we can recover the subsets $A^{\circ\circ} \subseteq A^+ \subseteq A$ from $\mathrm{Spa}(A, A^+)$. Note that in the claim about $A^{\circ\circ}$ there is no contradiction to Theorem 11.8: if A is such that $\widehat{A} = 0$, e.g., A has the indiscrete topology (and so $\mathrm{Spa}(A, A^+) = \varnothing$), then any element in A is topologically nilpotent, i.e., $A^{\circ\circ} = A$.

12. The structure presheaf on $\mathrm{Spa}(A, A^+)$

12.1. **Structure presheaf.** Analogously to the case of schemes, to globalize the notion of adic spectra, we have to define some structure sheaf of rings on the topological space $X = \mathrm{Spa}(A, A^+)$. Actually, we want a sheaf $(\mathcal{O}_X, \mathcal{O}_X^+)$ taking values in Huber pairs, whose global sections are (A, A^+). As rational opens form a basis of topology stable under finite intersections (Lemma 7.6), the following lemma says that it suffices only to consider rational opens:

Lemma 12.1. *Let X be a topological space, and $\mathscr{B}$ a basis of topology, stable under finite intersections. Then the restriction functor from sheaves on X to sheaves on $\mathscr{B}$ is an equivalence of categories.*

With other words, a sheaf (resp. morphism of sheaves) on X is uniquely determined by its values on all elements of $\mathscr{B}$, and any sheaf on $\mathscr{B}$ extends uniquely to one on X. Explicitly, if $\mathscr{F}$ is a sheaf on X, then for all $U \subseteq X$ open,

$$\mathscr{F}(U) = \varprojlim_{\mathscr{B} \ni V \subseteq U} \mathscr{F}(V),$$

which defines a quasi-inverse to the functor in the lemma.

We already know (Proposition 7.7) that rational opens are adic spectra of some localizations of (A, A^+) again, so our first guess for $\mathcal{O}_X, \mathcal{O}_X^+$ on rational opens would be

$$\mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right) \mapsto A\left[\frac{1}{g}\right], A^+\left[\frac{f_1, \ldots, f_n}{g}\right]^{\mathrm{int}}, \tag{12.1}$$

where $(\cdot)^{\mathrm{int}}$ denotes the integral closure within $A[g^{-1}]$. Thus is indeed a Huber pair by Proposition 7.7. However, this will not satisfy the universal property similar to the one for principal opens of an affine scheme: indeed, there are many such pairs with the same completion, so with equal adic spectra (by Theorem 11.8). This suggests that one should pass to completion. And indeed, this works:

Proposition 12.2. *Let $X = \mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+). Let U be a rational open subset. Then there exists a (necessarily unique)* complete *Huber pair (A_U, A_U^+) over (A, A^+), such that $\mathrm{Spa}(A_U, A_U^+)$ has image U in X and which satisfies the following universal property. For any complete Huber pair $(A, A^+) \to (C, C^+)$, such that the image of $\mathrm{Spa}(C, C^+) \to X$ is contained in U, there exists a unique factorization*

$$\begin{array}{ccc} (A, A^+) & \longrightarrow & (A_U, A_U^+) \\ & \searrow & \downarrow \\ & & (C, C^+) \end{array}$$

Moreover, if $U = X\left(\frac{f_1, \ldots, f_n}{g}\right)$, then (A_U, A_U^+) is the completion of (12.1).

Proof. Everything except the universal property follows by combining Proposition 7.7 and Theorem 11.8. Let $U = X\left(\frac{f_1,\dots,f_n}{g}\right)$ with $f_1,\dots,f_n,g \in A$ with $(f_1,\dots,f_n)_A$ open in A. As the map $\widetilde{\varphi} = \mathrm{Spa}(\varphi)$ induced by $\varphi\colon (A,A^+) \to (C,C^+)$ factors through U, we have for all $x \in \mathrm{Spa}(C,C^+)$: $|\varphi(g)(x)| = |g(\widetilde{\varphi}(x))| \neq 0$. Thus, by completeness of (C,C^+) and by Theorem 11.12(3), $\varphi(g) \in C^\times$. Thus, $\varphi\colon A \to C$ factors through $\varphi'\colon A\left[\frac{1}{g}\right] \to C$.

Similarly, for all $1 \leq i \leq n$ and all $x \in C$, we have $|\varphi(f_i)(x)| = |f_i(\widetilde{\varphi}(x))| \leq |g(\widetilde{\varphi}(x))| = |\varphi(g)(x)|$, and so $\frac{\varphi(f_i)}{\varphi(g)} \in C^+$ by Theorem 11.12(2). Thus $\varphi'(A^+\left[\frac{f_1,\dots,f_n}{g}\right]) \subseteq C^+$. Altogether, we get a map

$$\left(A\left[\frac{1}{g}\right], A^+\left[\frac{f_1,\dots,f_n}{g}\right]\right) \to (C,C^+) \tag{12.2}$$

of pairs of rings. Moreover, $\frac{\varphi(f_i)}{\varphi(g)} \in C^+ \subseteq C^\circ$ are powerbounded for all i. From this (and the cofinality of rings and ideals of definition of C within $C^\circ, C^{\circ\circ}$), it is easy to see that the map $A[g^{-1}] \to C$ is continuous (recall from the proof of Proposition 7.7 that if (A_0, I) is some couple of definition of A, then $A[g^{-1}]$ has $B_0 := A_0\left[\frac{f_1,\dots,f_n}{g}\right], I \cdot B_0$ as a couple of definition), and we leave this as an exercise (cf. e.g. [Wed19, Proof of Prop.-Def. 5.51]).

Thus (12.2) is a map of Huber pairs, and in fact it is the unique one through which $(A,A^+) \to (C,C^+)$ factors, by the following lemma (which does not make use of completeness of C, because it rightaway assumes that the image of g is invertible!):

Lemma 12.3. *The map $(A,A^+) \to \left(A\left[\frac{1}{g}\right], A^+\left[\frac{f_1,\dots,f_n}{g}\right]\right)$ is universal among all maps of Huber pairs $\alpha\colon (A,A^+) \to (B,B^+)$ for which*
- *(i) $\alpha(g)$ is a unit in B, and*
- *(ii) $\frac{\alpha(f_i)}{\alpha(g)}$ is power-bounded in B for any i.*

Proof. This is not very hard. See [Wed19, Prop.-Def. 5.51] or [Mor19, Proposition II.3.4.1] for details. □

Now, combining the uniqueness of the factorization of φ through φ' with the universal property of the completion (exploiting again that (C,C^+) is complete), we deduce that there is a unique factorization of φ through a map $(A_U, A_U^+) \to (C,C^+)$, where (A_U, A_U^+) is the completion of $\left(A\left[\frac{1}{g}\right], A^+\left[\frac{f_1,\dots,f_n}{g}\right]\right)$. □

Clearly, the universal property of Proposition 12.2 implies that if $V \subseteq U \subseteq X$ are two rational opens, then there is a unique map $r_{U,V}\colon (A_U, A_U^+) \to (A_V, A_V^+)$ of complete Huber pairs over (A,A^+).

Definition 12.4 (Structure presheaf)**.** Let $X = \mathrm{Spa}(A,A^+)$ for a Huber pair (A,A^+). The *structure presheaf* $\mathcal{O}_X$ resp. its $+$-version $\mathcal{O}_X^+$ on X are the presheaves (with values in complete topological rings) on the basis of rational open subsets of X, given by

$$\mathcal{O}_X\colon U \mapsto A_U \quad \text{and} \quad \mathcal{O}_X^+\colon U \mapsto A_U^+$$

with restriction maps $r_{U,V}$. A Huber pair (A,A^+) is called *sheafy*, if $\mathcal{O}_X$ is a sheaf.

Remark 12.5. (1) It is clear what it means for $\mathcal{O}_X$ to be a sheaf of (abstract) rings. The condition to be a sheaf of topological rings is more restrictive. It means that for any open $U \subseteq X$, any open coveing $U = \bigcup_i U_i$, and any topological ring T, $\mathrm{Hom}(T, \mathcal{O}_X(U)) = \mathrm{Eq}(\prod_i \mathrm{Hom}(T, \mathcal{O}_X(U_i)) \to \prod_{ij} \mathrm{Hom}(T, \mathcal{O}_X(U_{ij})))$ where the Hom's are taken in the category of topological rings. With other words, the topology on $\mathcal{O}_X(U)$ must be the weakest one such that all maps $\mathcal{O}_X(U) \to \mathcal{O}_X(U_i)$ are continuous. Also, it suffices to check this property on a collection of opens forming a base of topology. Cf. [DG67, I, chap. 0,(3.1.4) and (3.2.2)] for a discussion.

(2) If $\mathcal{O}_X$ is a sheaf, then also $\mathcal{O}_X^+$ is a sheaf. Indeed, as $\mathcal{O}_X^+ \subseteq \mathcal{O}_X$, it is clear that $\mathcal{O}_X^+$ is a separated presheaf; moreover, by Theorem 11.12(2), for any rational open $U \subseteq X$,
$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \colon |f(x)| \leq 1 \ \forall x \in U\},$$
and the sheaf property for $\mathcal{O}_X^+$ follows from that for $\mathcal{O}_X$.

(3) For any Huber pair (A, A^+) we can extend $\mathcal{O}_X$, $\mathcal{O}_X^+$ to presheaves on all opens $V \subseteq X$, putting
$$(\mathcal{O}_X, \mathcal{O}_X^+)(V) = \varprojlim_{\substack{U \subseteq V \\ U \text{ rational open}}} (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$
Note that if V is rational open, then the limit is taken over a category with a final object, namely V itself, so there is no ambiguity in this extension. By Lemma 12.1 this extension is well-behaved under the sheafiness assumption.

(4) For any (A, A^+) we have $(\mathcal{O}_X(X), \mathcal{O}_X^+(X)) = (\widehat{A}, \widehat{A}^+)$.

(5) There exist examples of non-sheafy (Tate-)Huber pairs (A, A^+) (only A matters, A^+ can be arbitrary). The (probably) first one, –of a Tate ring, which is of finite type over $\mathbb{Z}$– was given by Rost, see [Hub94, end of §1]. There are many examples, even in uniform Tate algebras over a non-archimedean field, cf. [BV18, §4] and [Mih16]. (TODO: add this.)

(6) Sheafiness of (A, A^+) only depends on the Huber ring A, not on A^+. (TODO: add proof, see Kedlaya.)

Directly from Proposition 12.2 we deduce the following corollary:

Corollary 12.6. *Let $(A, A^+) \to (B, B^+)$ be a map of complete Huber pairs, and let $f \colon Y \to X$ be the corresponding map on adic spectra. Let $U \subseteq X$ be a rational subset and let $V = f^{-1}(U)$ be its preimage in Y. Then*
$$\begin{array}{ccc} (A, A^+) & \longrightarrow & (B, B^+) \\ \downarrow & & \downarrow \\ (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) & \longrightarrow & (\mathcal{O}_X(V), \mathcal{O}_X^+(V)) \end{array}$$
is a pushout in the category of complete Huber pairs.

12.2. **Stalks, and the three residue fields at a point.** Let $X = \mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+). For $x \in X$ we may form the stalk of the structure presheaf and its +-version:
$$\mathcal{O}_{X,x} = \varinjlim_{x \in U \subseteq X} \mathcal{O}_X(U) \quad \text{and} \quad \mathcal{O}_{X,x}^+ = \varinjlim_{x \in U \subseteq X} \mathcal{O}_X^+(U),$$
where the limit is taken

- in the category of abstract rings (that is, ignoring the topology),
- over all rational opens of X containing x (or equivalently over all opens of X containing x).

For any $x \in U \subseteq X$, the valuation x of A extends to a valuation of $\mathcal{O}_X(U)$ (by Proposition 12.2 and the definition of the adic spectrum), and so $\mathcal{O}_{X,x}$ comes equipped with a valuation, denoted $|\cdot(x)|$, for which $|f(x)| \leq 1$ for all $f \in \mathcal{O}_{X,x}^+$.

Proposition 12.7. *Let $X = \mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+). Let $x \in X$.*

(1) *$\mathcal{O}_{X,x}$ is a local ring with maximal ideal*
$$\mathfrak{m}_{X,x} = \{f \in \mathcal{O}_{X,x} \colon |f(x)| = 0\}.$$
The valuation x on $\mathcal{O}_{X,x}$ factors through a valuation x of the residue field
$$\Bbbk(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$$
and the natural map $A \to \mathcal{O}_{X,x}$ induces a morphism $k(x) \to \Bbbk(x)$ of topological fields, which is an isomorphism after completion.[32] *Moreover, $k(x)^+$ is dense in the valuation subring*
$$\Bbbk(x)^+ = \{f \in \Bbbk(x) \colon |f(x)| \leq 1\}.$$
of $\Bbbk(x)$.

(2) *$\mathcal{O}^+_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}^+_{X,x}$,*
$$\mathfrak{m}_{X,x} \subseteq \mathfrak{m}^+_{X,x} \subseteq \mathcal{O}^+_{X,x} \subseteq \mathcal{O}_{X,x},$$
where
$$\mathcal{O}^+_{X,x} = \{f \in \mathcal{O}_{X,x} \colon |f(x)| \leq 1\}$$
$$\mathfrak{m}^+_{X,x} = \{f \in \mathcal{O}_{X,x} \colon |f(x)| < 1\}$$
In particular, $\mathcal{O}^+_{X,x}$ is the preimage of $\Bbbk(x)^+$ under $\mathcal{O}_{X,x} \twoheadrightarrow \Bbbk(x)$. The (discrete) residue fields of $\mathcal{O}^+_{X,x}$ and of $\Bbbk(x)^+$ are canonically isomorphic.

(3) *The constructions in (1) and (2) are contravariantly functorial in X, x.*

Proof. (1): It is clear that $\mathfrak{m}_{X,x}$ is an ideal of $\mathcal{O}_{X,x}$. Thus, to show the first claim of (1), it suffices to prove that $\mathcal{O}_{X,x} \smallsetminus \mathfrak{m}_{X,x} \subseteq \mathcal{O}^\times_{X,x}$. Let $x \in U \subseteq X$ be a rational open and let $f \in \mathcal{O}_X(U)$ whose image in $\mathcal{O}_{X,x}$ does not lie in $\mathfrak{m}_{X,x}$, i.e., $|f(x)| \neq 0$. By Lemma 11.11 applied to the quasi-compact subset $\{x\} \subseteq U$, we may find an open subset $N \subseteq \mathcal{O}_X(U)$ such that $|t(x)| < |f(x)|$ for all $t \in N$. Replacing N by some finitely generated ideal of definition it contains, we get a finite set $T \subseteq \mathcal{O}_X(U)$ (of generators of this ideal), such that T generates an open ideal of $\mathcal{O}_X(U)$ and $|T(x)| \leq |f(x)| \neq 0$. With other words, $x \in V := U\left(\frac{T}{f}\right) \subseteq U$. But by definition of rational opens, $|f(v)| \neq 0$ for any $v \in V$. Thus, by Theorem 11.12(3), f is invertible in $\mathcal{O}_X(V)$, and hence also in $\mathcal{O}_{X,x}$.

It is clear that $A \to \mathcal{O}_{X,x}$ induces the map $k(x) \to \Bbbk(x)$. Towards the isomorphism of completions, we need a lemma.

Lemma 12.8 (Density of residue fields)**.** *Let $x \in U \subseteq X$ be a rational open neighboorhood of x, with corresponding Huber pair $(\mathcal{O}_X(U), \mathcal{O}^+_X(U)) = (A_U, A_U^+)$. Write $k_U(x) = \mathrm{Frac}(A_U/\operatorname{supp}(x))$ and $k_U(x)^+$ for the corresponding valuation subring. The map $(A, A^+) \to (A_U, A_U^+)$ induces a map $(k(x), k(x)^+) \to (k_U(x), k_U(x)^+)$. This map is an isomorphism after completion.*

Proof. The first claim is clear (as the valuation x on A is the restriction of the valuation, again called x, of $\mathcal{O}_X(U)$). For the second claim it suffices to show that $k(x)$ is dense in $k_U(x)$. Write $U = X\left(\frac{f_1,\dots,f_n}{g}\right)$. The map $A \to A_U$ factors as $A \to B = A[g^{-1}] \to A_U$, where A_U is the completion of the Huber ring B. Further, the induced maps on residue fields factors as
$$k(x) = \mathrm{Frac}(A/\operatorname{supp}_A(x)) \to \mathrm{Frac}(B/\operatorname{supp}_{A_U}(x) \cap B) \to \mathrm{Frac}(A_U/\operatorname{supp}_{A_U}(x)) = k_U(x).$$

[32] Moreover, the map $\mathcal{O}_{X,x} \twoheadrightarrow \Bbbk(x)$ is also an isomorphism after $|\cdot(x)|$-completion, as $|f(x)| = 0$ for any $f \in \mathfrak{m}_{X,x}$ and so $\mathfrak{m}_{X,x}$ gets killed by completion.

The first of this maps is an isomorphism as $B = A[g^{-1}]$ and $g \notin \operatorname{supp}_A(x)$ (as $|g(x)| \neq 0$), and it suffices that the image of the second map is dense.

By Lemma 12.9, the map $A \to \mathcal{O}_X(U)$ is adic. Thus, by Lemma 12.10, x is analytic as a point of X if and only it is analytic as a point of U.

First suppose that x is non-analytic. Then $\operatorname{supp}_{A_U}(x)$ is open in A_U and the image of $B \to \widehat{B} = A_U$ is dense, so $B + \operatorname{supp}_{A_U}(x) = A_U$ (as additive groups), and so the injection of discrete rings $B/\operatorname{supp}_{A_U}(x) \cap B \to A_U/\operatorname{supp}_{A_U}(x)$ is an isomorphism; it remains so after passing to the fraction fields.

Next, suppose that x is analytic. Let R resp. R_U denote the image of $B \to k(x)$ resp. $A_U \to k_U(x)$ (all equipped with valuation topology). Then $R \subseteq R_U$ and if we know that R is dense in R_U, then an easy approximation argument shows that also $\operatorname{Frac}(R) = k(x)$ is dense in $\operatorname{Frac}(R_U) = k_U(x)$. Now, the image of $B \to A_U$ is dense by construction (both sides have the natural topology); also, the map $A_U \to R_U \subseteq k_U(x)$ is continuous ($k_U(x)$ has the valuation topology), hence the image of $B \to A_U \twoheadrightarrow R_U$ is also dense; but this coincides with the image of $R \to R_U$, which is therefore also dense.

Finally, consider the rings $k(x)^+ \subseteq k_U(x)^+$. Both are the unit discs in $k(x)$, $k_U(x)$; using the non-archimedean triangle inequality the density of $k(x)$ in $k_U(x)$ implies the density of $k(x)^+$ in $k_U(x)^+$. □

Now the density of the image of $k(x) \to \Bbbk(x)$ is clear from $\Bbbk(x) = \varinjlim_{x \in U \subseteq X} k_U(x)$, with notation as in the lemma; similarly for $k(x)^+ \to \Bbbk(x)^+$. This proves (1).

(2): It is clear that $\mathcal{O}^+_{X,x} \subseteq \mathcal{O}_{X,x}$ and also that $\mathcal{O}^+_{X,x} \subseteq \{f \in \mathcal{O}_{X,x} \colon |f(x)| \leq 1\}$. Let, conversely, $f_x \in \mathcal{O}_{X,x}$ be a germ satisfying $|f_x(x)| \leq 1$. Then f_x is represented by a function $f \in \mathcal{O}_X(U)$ for some rational open $x \in U \subseteq X$. The locus $U(\frac{f}{1}) \subseteq U$ of all points $y \in U$ where $|f(y)| \leq 1$ is open (but not necessarily rational open) in $U = \operatorname{Spa}(\mathcal{O}_X(U), \mathcal{O}^+_X(U))$ by the definition of its topology. So, replacing U by some rational open neighboorhood of x contained in $U(\frac{f}{1})$, we see –by Theorem 11.12(2)– that f comes from $\mathcal{O}^+_X(U)$, i.e., $f_x \in \mathcal{O}^+_{X,x}$. This proves that $\mathcal{O}^+_{X,x}$ is the preimage of $\Bbbk(x)^+$ under $\mathcal{O}_{X,x} \twoheadrightarrow \Bbbk(x)$. Clearly, $\mathfrak{m}^+_{X,x}$ is the preimage of the maximal ideal of $\Bbbk(x)^+$. If $f \in \mathcal{O}^+_{X,x} \smallsetminus \mathfrak{m}^+_{X,x}$, then $|f(x)| = 1$, so by (1), $f \in \mathcal{O}^\times_{X,x}$ and we clearly have $|f^{-1}(x)| = 1$, so by the above, $f^{-1} \in \mathcal{O}^+_{X,x}$; with other words, f is a unit in $\mathcal{O}^+_{X,x}$, proving that $\mathcal{O}^+_{X,x}$ is local.

Part (3) is immediate. □

In course of the proof we used the following two lemmas.

Lemma 12.9. *Let $X = \operatorname{Spa}(A, A^+)$ for a Huber pair (A, A^+). Let U be a rational open subset. Then the natural map $A \to \mathcal{O}_X(U)$ is adic.*

Proof. Write $U = X\left(\frac{f_1,\dots,f_n}{g}\right)$ for some $f_1, \dots, f_n, g \in A$, such that $(f_1, \dots, f_n)_A$ is open in A. Let A_0, I be a couple of definition of A. By construction (cf. Propositions 12.2 and 7.7), the map $A \to \mathcal{O}_X(U)$ factors as $A \to B := A[g^{-1}] \to A_U$, where A_U is the completion of B, and $B = A[g^{-1}]$ has $B_0 := A_0\left[\frac{f_1,\dots,f_n}{g}\right]$ as a ring of definition and $I \cdot B_0$ as an ideal of definition. If suffices to show that both maps are adic. The first map is adic, because (as recalled above) $I \cdot B_0$ is an ideal of definition. The completion map $B \to \widehat{B} = A_U$ is adic, –basically by construction,– as $\widehat{B}$ has $I \cdot \widehat{B}_0$ as an ideal of definition. □

Lemma 12.10. *Let $\varphi \colon (A, A^+) \to (B, B^+)$ be a map of Huber pairs, and let $f \colon Y = \operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+) = X$ be the iduced map on adic spectra. Let $y \in Y$.*

(1) *If y is not analytic, then $f(y)$ is not analytic.*
(2) *Suppose φ is adic. If y is analytic, then $f(y)$ is analytic.*

Proof. (1): $\operatorname{supp}(f(y)) = \varphi^{-1}(\operatorname{supp}(y))$. So, if $\operatorname{supp}(y)$ is open, then $\operatorname{supp}(f(y))$ is open by continuity of φ.

(2): Let (A_0, I) be a couple of definition of A, and let B_0 be a ring of definition of B, such that $\varphi(I) \cdot B_0$ is an ideal of definition of B. Suppose that $f(y)$ is not analytic. Then $\operatorname{supp}(f(y)) = \varphi^{-1}(\operatorname{supp}(y))$ is an open prime ideal of A, so $I \subseteq \varphi^{-1}(\operatorname{supp}(y))$, and then $\varphi(I) \subseteq \operatorname{supp}(y)$. Then $\varphi(I) \cdot B_0 \subseteq \operatorname{supp}(y)$, i.e., $\operatorname{supp}(y)$ is open, contradiction. □

Note that even if A is complete, the topological field $k(x) = \operatorname{Frac} A/\operatorname{supp}(x)$ must not be complete with respect to the valuation topology.

Remark 12.11 (Residue fields; cf. [Bha17], Rem.7.5.7)**.** All in all we have now three residue fields (along with the valuation rings in them) attached to a point $x \in X = \operatorname{Spa}(A, A^+)$:

(1) The residue field
$$(k(x), k(x)^+)$$
of the pair (A, A^+) at x. That is $k(x) = \operatorname{Frac}(A/\operatorname{supp}(x))$.

(2) The residue field
$$(\Bbbk(x), \Bbbk(x)^+) = \varinjlim_{x \in U \subseteq X} (k_U(x), k_U(x)^+)$$
of the stalk of the structure presheaf, with $k_U(x)$ as in the proof of Proposition 12.7.

(3) the *completed residue field*
$$(K(x), K(x)^+) := (\widehat{k(x)}, \widehat{k(x)^+}) = (\widehat{\Bbbk(x)}, \widehat{\Bbbk(x)^+}),$$
where the completion is taken with respect to the valuation topology. The second equality follows from Proposition 12.7. Note also that if x is not analytic, then we have $k(x) = \Bbbk(x) = K(x)$ (as follows from the proof of Proposition 12.7).

In practice, the completed residue field is most convenient: Firstly, it is complete, in contrast to $k(x)$, $\Bbbk(x)$. Secondly, $k(x)$ has the disadvantage of being dependent on the neighboorhood of x; and $\Bbbk(x)$ is hard to compute (to compute it one needs to control a cofinal family of rational neighboorhoods of x), whereas $K(x)$ is the completion of $k(x)$.

For analytic points we have the following surprising (when compared to the world of schemes) property:

Corollary 12.12. *Suppose that A is Tate and let $\varpi \in A$ be a pseudo-uniformizer. We have $(\mathcal{O}^+_{X,x})^\wedge_\varpi \cong K(x)^+$.*

Proof. As $\Bbbk(x)^+$ is microbial with pseudo-uniformizer ϖ (by Proposition 12.7, and as $k(x)^+$ is), its valuation topology coincides with the ϖ-adic topology, and so we have
$$K(x)^+ \stackrel{\text{def.}}{=} (\Bbbk(x)^+)^\wedge_{x-\text{adic}} \cong (\Bbbk(x)^+)^\wedge_\varpi$$
So it suffices to show that the natural map $\mathcal{O}^+_{X,x} \to \Bbbk(x)^+$ induces an isomorphism on ϖ-adic completion. But this is true (Exercise: check why!) as $\ker(\mathcal{O}^+_{X,x} \to \Bbbk(x)^+) = \mathfrak{m}_{X,x}$ is ϖ-divisible. Indeed, divisibility follows by noting that $\mathfrak{m}_{X,x}$ is also an ideal of $\mathcal{O}_{X,x}$, and ϖ is a unit of this ring. □

Remark 12.13. Suppose that A is complete and Tate with uniformizer ϖ. The ϖ-adic completion is in general finer than the $|\cdot|_x$-adic completion. In particular, the preceeding corollary only works with the stalk $\mathcal{O}^+_{X,x}$, not in general with A^+:

$$\begin{array}{ccccccc}
A^+ = (A^+)^\wedge_\varpi & \longrightarrow & (A^+)^\wedge_{x-\mathrm{adic}} & & (\mathcal{O}^+_{X,x})^\wedge_\varpi & \xrightarrow{\cong} & (\mathcal{O}^+_{X,x})^\wedge_{x-\mathrm{adic}} \\
\downarrow{\scriptstyle \neq} & & \downarrow & & {\scriptstyle \text{Cor. 12.12}}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
(k(x)^+)^\wedge_\varpi & \xrightarrow{\cong} & (k(x)^+)^\wedge_{x-\mathrm{adic}} & = & (\Bbbk(x)^+)^\wedge_\varpi & \xrightarrow{\cong} & (\Bbbk(x)^+)^\wedge_{x-\mathrm{adic}}
\end{array}$$

where the lower right isomorphism is because $\Bbbk(x)$ is microbial and the isomorphism on the right is obvious as the x-adic completion always factors through the support of x.

13. Pre-adic and adic spaces. Sheafiness

13.1. **Definition of adic spaces.**

Definition 13.1 (Huber's categories $\mathcal{V}^{\mathrm{pre}}$ and $\mathcal{V}$)**.** The category $\mathcal{V}^{\mathrm{pre}}$ is defined as follows. Its objects are triples $(X, \mathcal{O}_X, (|\cdot(x)|)_{x\in X})$, where

- X is a topological space
- $\mathcal{O}_X$ is a presheaf of complete topological rings on X, subject to the condition that the stalk (computed in abstract rings) is a local ring
- for any $x \in X$, $|\cdot(x)|$ is a valuation on $\mathcal{O}_{X,x}$, whose support is equal to the maximal ideal of $\mathcal{O}_{X,x}$.

A morphism $(Y, \mathcal{O}_Y, (|\cdot(y)|)_{y\in Y}) \to (X, \mathcal{O}_X, (|\cdot(x)|)_{x\in X})$ in $\mathcal{V}^{\mathrm{pre}}$ consists of

- a continuous map $f\colon Y \to X$ and
- a map of presheaves of topological rings $f^\flat\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$, such that for any $y \in Y$, $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is compatible with the valuations $|\cdot(y)|$, $|\cdot(f(x))|$.

The category $\mathcal{V}$ is the full subcategory of $\mathcal{V}^{\mathrm{pre}}$ consisting of all triples $(X, \mathcal{O}_X, (|\cdot(x)|)_{x\in X})$ such that $\mathcal{O}_X$ is a sheaf.

Remark 13.2.
(1) From the condition on the support of $|\cdot(x)|$ it follows that for any morphism $(f, f^\flat)$ in $\mathcal{V}^{\mathrm{pre}}$, $f^\flat$ induces *local* maps on stalks.
(2) By what we have done in the previous lectures, any Huber pair (A, A^+) gives rise to an object of $\mathcal{V}^{\mathrm{pre}}$, denoted $\mathrm{Spa}(A, A^+)$. Any map of Huber pairs $\varphi\colon (A, A^+) \to (B, B^+)$ induces a map $\mathrm{Spa}(\varphi)$ in $\mathcal{V}^{\mathrm{pre}}$.
(3) Let (A, A^+) be a Huber pair and let $X = \mathrm{Spa}(A, A^+)$ be the corresponding object in $\mathcal{V}^{\mathrm{pre}}$. The presheaf $\mathcal{O}^+_X$ of Definition 12.4 is not part of the datum of $\mathcal{V}^{\mathrm{pre}}$. However, it can easily be recovered as the subsheaf of $\mathcal{O}_X$ of all sections $f \in \mathcal{O}_X$ which satisfy $|f(x)| \leq 1$ at all points. Indeed, this follows from Theorem 11.12(2).
(4) Let $\varphi\colon (A, A^+) \to (\widehat{A}, \widehat{A^+})$ be the completion of a Huber pair. Then $\mathrm{Spa}(\varphi)$ is an isomorphism in $\mathcal{V}^{\mathrm{pre}}$. With other words, the functor Spa from Huber pairs to $\mathcal{V}^{\mathrm{pre}}$ factors through the completion functor on Huber pairs. This follows from the construction of $\mathrm{Spa}(A, A^+)$, cf. Theorem 11.8 and Definition 12.4.

Proposition 13.3. *Let (A, A^+), (B, B^+) be Huber pairs with (B, B^+) complete. Then*

$$\mathrm{Hom}((A, A^+), (B, B^+)) \to \mathrm{Hom}_{\mathcal{V}^{\mathrm{pre}}}(\mathrm{Spa}(B, B^+), \mathrm{Spa}(A, A^+))$$

is a bijection. In particular, the functor Spa *from complete Huber pairs to $\mathcal{V}^{\mathrm{pre}}$ is fully faithful.*

Proof. If $\varphi \in \mathrm{Hom}((A, A^+), (B, B^+))$, then (by definition) taking the global sections of $\mathrm{Spa}\,\varphi$ gives back the map of completed Huber pairs $\widehat{A} \to \widehat{B} = B$, and precomposing with $A \to \widehat{A}$ gives the map $\varphi\colon A \to B$. Write $Y = \mathrm{Spa}(B, B^+), X = \mathrm{Spa}(A, A^+)$. We have to show that any $(f, f^\flat) \in \mathrm{Hom}_{\mathcal{V}^{\mathrm{pre}}}(Y, X)$ is equal to $\mathrm{Spa}\,\varphi$ for $\varphi := f^\flat_X \circ (A \to \widehat{A})$. (Note that it is only at this point, where we use that B is complete: a priori we only get a map $A \to \widehat{A} \to \widehat{B}$, and by completeness this gives a map $A \to B$.)

As rational opens in X form a basis of topology (Lemma 7.6), it suffices to show that $f^\flat_U$ and the localization φ_U of φ coincide for all rational opens $U \subseteq X$. Let $U = X\left(\frac{s_1,\dots,s_n}{g}\right) \subseteq X$ be a rational open. Its preimage is the open

$$f^{-1}(U) = \{y \in Y \colon |s_i(f(y))| \leq |g(f(y))| \neq 0 \;\forall\, i\} \;=\; \{y \in Y \colon |\varphi(s_i)(y)| \leq |\varphi(g)(y)| \neq 0 \;\forall\, i\}\,,$$

By Lemma 7.6, $f^{-1}(U)$ is the union of all rational opens V of Y contained in it, and $\mathcal{O}_Y(f^{-1}(U)) = \varprojlim_{V \subseteq f^{-1}(U)} \mathcal{O}_Y(V)$. For each such V, $f(V) \subseteq U$, so $f^\flat$ (resp. φ) induces the map $f^\flat_V \colon \mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ (resp. $(\mathrm{Spa}\,\varphi)^\flat_V$); moreover, the maps $f^\flat_U, (\mathrm{Spa}\,\varphi)^\flat_U \colon \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ are the inverse limits of these maps over all V. Thus it suffices to show that $f^\flat_V = (\mathrm{Spa}\,\varphi)^\flat_V \colon \mathcal{O}_X(U) \to \mathcal{O}_Y(V)$. But these maps are equal when pulled back along $A \to \mathcal{O}_X(U)$. As $\mathcal{O}_X(U)$ is a completion of a localization of A, and both maps are continuous, they must be equal. ☐

We (finally!) can define adic spaces:

Definition 13.4 (Adic spaces)**.**
1. An *affinoid adic space* is an object of $\mathcal{V}$, which is isomorphic to $\mathrm{Spa}(A, A^+)$ for some (sheafy) Huber pair (A, A^+).
2. An *adic space* is an object $(X, \mathcal{O}_X, (|\cdot(x)|)_{x\in X})$ of $\mathcal{V}$, which admits an open covering $X = \bigcup_{i \in I} U_i$, such that $(U_i, \mathcal{O}_X\big|_{U_i}, (|\cdot(x)|)_{x \in U_i})$ is an affinoid adic space.
3. A *morphism of adic spaces* is a morphism in $\mathcal{V}$. With other words, adic spaces form a full subcategory of $\mathcal{V}$.

We denote the affinoid adic space attached to a sheafy Huber pair (A, A^+) by $\mathrm{Spa}(A, A^+)$.

Remark 13.5. Not every Huber pair is sheafy, so not every $\mathrm{Spa}(A, A^+)$ will be an (affinoid) adic space. Actually, we do not yet have a single (non-empty) example of an adic space.

Remark 13.6. We have the commutative diagram

$$\begin{array}{ccc} \{\text{cpl. sheafy Huber pairs}\} & \longrightarrow & \mathcal{V} \\ \big\downarrow & & \big\downarrow \\ \{\text{cpl. Huber pairs}\} & \longrightarrow & \mathcal{V}^{\mathrm{pre}} \end{array}$$

with both vertical functors being inclusions, and both horizontal ones sending (A, A^+) to $\mathrm{Spa}(A, A^+)$. Note that all four functors are fully faithful.

One might try to avoid the problem of non-sheafy Huber pairs by sheafifying the structure sheaf of $\mathrm{Spa}(A, A^+)$. Apart from the problem that it is not clear (and probably not true) that such a sheafification gives an object of $\mathcal{V}$ (and not just an –in general ill-behaved– sheaf of topological rings) this sheafification is not a fully faithful functor, so that the composition of functors from complete Huber pairs to $\mathcal{V}$ would also not be fully faithful.

13.2. **Sheafiness.** Recall the Tate algebra over a non-archimedean topological ring A:

$$A\langle T\rangle := A\langle T_1, \dots, T_n\rangle := \left\{ \sum_\nu a_\nu T^\nu \colon |a_\nu| \to 0 \text{ for } \nu \to +\infty \right\}$$

We now make two observations about it, although we will use them only a bit later. See [Mor19, II.3.3] for a (more general and more detailed) discussion.

Remark 13.7. (1) Let A be a non-archimedean topological ring. Then $A\langle T\rangle$ admits a unique non-archimedean topology with $\{\sum_\nu a_\nu T^\nu \in A\langle T\rangle : a_\nu \in U\}$ being a fundamental system of neighboorhoods of 0, where $U \subseteq A$ varies though a fundamental system of neighboorhoods of 0 in A.

(2) Suppose A is complete. For any complete non-archimedean ring B, any continuous map $\varphi\colon A \to B$ and any tuple $(b_1, \dots, b_n) \in B^n$ of power bounded elements, there exist a unique continuous extension $\tilde{\varphi}\colon A\langle T\rangle \to B$ of φ mapping T_i to b_i. (Exercise: check this.)

Now, we use the above construction to define strongly Noetherian Tate rings.

Definition 13.8. A complete Tate ring is called *strongly Noetherian* if $A\langle T_1, \dots, T_n\rangle$ is Noetherian for any $n \geq 1$.

Proposition 13.9 ([BGR84],§5.2.6, Theorem 1)**.** *Any complete non-archimedean field k is strongly Noetherian.*

Recall that a Huber ring A is uniform if A° is bounded.

Definition 13.10. A Huber pair (A, A^+) is called *stably uniform*, if for all rational opens $U \subseteq X = \mathrm{Spa}(A, A^+)$, the Huber ring $\mathcal{O}_X(U)$ is uniform.

Theorem 13.11. *Let (A, A^+) be a complete Huber pair and let $X = \mathrm{Spa}(A, A^+)$. Suppose that at least one of the following conditions is satisfied:*

(1) A is discrete ($\rightsquigarrow$ schemes)
(2) A has a Noetherian ring of definition ($\rightsquigarrow$ Noetherian formal schemes)
(3) A is strongly Noetherian analytic ring ($\rightsquigarrow$ rigid spaces)
(4) (A, A^+) is Tate and stably uniform ($\rightsquigarrow$ perfectoid spaces)

Then (A, A^+) is sheafy and for rational opens $U \subseteq X$ and all $i \geq 1$, $H^i(U, \mathcal{O}_X) = 0$.[33]

In parentheses the class of examples is outlined to which the item naturally applies. Ultimately one shows that the respective category (of schemes, rigid spaces, ...) embeds fully faithfully into adic spaces. In particular, if k is a non-archimedean field, then (3) applies to all topologically finitely generated (k, k°)-algebras. Note that if (say) k is algebraically closed, then k° is not noetherian and moreover, k does not admit a Noetherian ring of definition, so part (2) does not apply to those rings.

The case (2) covers Noetherian formal schemes over $\mathbb{Z}_p$. However, if (say) $k = \mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$, then $\mathcal{O}_{\mathbb{C}_p}$ is non-Noetherian, and (2) does not cover formal schemes over it. Those are recently covered by [**?**].

Proof of Theorem 13.11 in the discrete case. The map $\mathrm{supp}\colon X = \mathrm{Spa}(A, A^+) \to \mathrm{Spec}\, A$ is surjective, as for any $\mathfrak{p} \in \mathrm{Spec}\, A$, the trivial valuation with support $\mathfrak{p}$ maps to $\mathfrak{p}$. If $U = X\left(\frac{f_1,\dots,f_n}{g}\right)$ is a rational open, then $\mathrm{supp}(U) = D(g)$, the principal open given by g (for the same reason as above), so that

$$(\mathrm{supp}^* \mathcal{O}_{\mathrm{Spec}\, A})(U) = \mathcal{O}_{\mathrm{Spec}\, A}(D(g)) = A[g^{-1}].$$

(a priori, this formula computes the presheaf pullback of $\mathcal{O}_{\mathrm{Spec}\, A}$. But it also shows that this presheaf pullback is already a sheaf (as $\mathcal{O}_{\mathrm{Spec}\, A}$ is), and so is equal to its sheafification $\mathrm{supp}^* \mathcal{O}_{\mathrm{Spec}\, A}$.) On the other hand, $\mathcal{O}_X(U) = A[g^{-1}]$ with the discrete topology. Thus $\mathcal{O}_X = \mathrm{supp}^* \mathcal{O}_{\mathrm{Spec}\, A}$ as presheaves on the collection of all rational opens. As $\mathrm{supp}^* \mathcal{O}_{\mathrm{Spec}\, A}$ is a

[33] Here, we consider the cohomology of a sheaf of abelian groups on a topological space.

sheaf of abelian groups, also $\mathcal{O}_X$ is. It is then clear that it is a sheaf of topological (discrete) abelian groups. Similarly, the vanishing of higher cohomology follows from the vanishing of cohomology of affine schemes, by using Čech cohomology (see the proof in other cases below). □

Below we prove Theorem 13.11 in cases (3) and (4). We will not discuss case (2). We do case (4) in this chapter; we then deduce case (3) from the Banach open mapping theorem in §14. We always work with a complete Huber pair (A, A^+), with pseudo-uniformizer $\varpi \in A$ and we put $X = \mathrm{Spa}(A, A^+)$.

13.3. **Reduction to simple Laurent covers.** It turns out that to verify the sheaf property of $\mathcal{O}_X$ it suffices to do so for the following very special and very explicit type of coverings. Let $X = \mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+), and let $t \in A$. Then a *simple Laurent cover* of X is the cover by two rational open subsets

$$X = X\left(\frac{1,t}{1}\right) \cup X\left(\frac{1,t}{t}\right), \tag{13.1}$$

where $X\left(\frac{1,t}{1}\right) = \{x \in X : |t(x)| \leq 1\}$ and $X\left(\frac{1,t}{t}\right) = \{x \in X : |t(x)| \geq 1\}$.

Proposition 13.12. *Assume that (A, A^+) is complete Tate. Suppose that for all rational opens $U \subseteq X$ and all simple Laurent coverings $U = U_1 \cup U_2$, the sequence of abelian groups*

$$0 \to \mathcal{O}_X(U) \to \mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1 \cap U_2) \to 0$$

is exact. Then $\mathcal{O}_X$ is a sheaf of abelian groups on the collection of rational opens of X. Moreover, for any rational open $U \subseteq X$ and $i \geq 1$, $H^i(U, \mathcal{O}_X) = 0$.

Proof. The collection of all covers of rational opens (of X) by rational opens (of X) forms a cofinal collection of open covers, and the collection of rational opens is stable under finite intersections (Lemma 7.6). Thus, by general formalism [Sta14, 01EW][34] one can reduce to Čech cohomology: i.e., it suffices to show that for any rational open $U \subseteq X$, $\mathcal{O}_X(U) \to \check{H}^0(U, \mathcal{O}_X)$ is an isomorphism (of abelian groups!) and that $\check{H}^i(U, \mathcal{O}_X) = 0$ for $i > 0$. For this it suffices to show that for any rational open cover $\mathscr{U}/U$ of a rational open $U \subseteq X$, one has

$$\mathcal{O}_X(U) \stackrel{\sim}{\to} \check{H}^0(\mathscr{U}/U, \mathcal{O}_X) \quad \text{and} \quad \check{H}^i(\mathcal{U}/U, \mathcal{O}_X) = 0 \text{ for all } i > 0 \tag{13.2}$$

This property (of covers of varying rational opens) is *local* and *transitive*. That is, if a cover $\mathscr{V}/U$ refines a cover $\mathscr{U}/U$, and (13.2) holds for $\mathscr{V}/U$, then it also holds for $\mathscr{U}/U$. Moreover, let $\mathscr{U}/U$ be a cover, and for each $V \in \mathscr{U}$, let $\mathscr{V}_V/V$ be a cover. If (13.2) holds for $\mathscr{U}/U$ and for $\mathscr{V}_V/V$ for each $V \in \mathscr{U}$, then it also holds for $\bigcup_{V \in \mathscr{U}} \mathscr{V}_V/U$.

In order to reduce (13.2) for general $\mathscr{U}/U$ to simple Laurent covers, we need to introduce some special types of covers and prove refinement relations between them.

(a) *Standard rational cover*: it is a cover $X = \bigcup_{i=1}^n X\left(\frac{t_1,\dots,t_n}{t_i}\right)$, where $t_1, \dots, t_n \in A$ generate the unit ideal of A.
(b) *Standard rational cover generated by units*: a cover as in (a), subject to the condition that $t_i \in A^\times$ for all i.
(c) *Laurent cover*: let $t_1, \dots, t_n \in A$ and for each $I \subseteq \{1, \dots, n\}$, let $U_I = \{x \in X : |t_i(x)| \leq 1 \text{ if } i \in I \text{ and } |t_i(x)| \geq 1 \text{ if } i \notin I\}$. Then U_I is an intersection of finitely many rational opens, so itself a rational open (check it by hand if $n = 2$!). Moreover, $X = \bigcup_I U_I$ is a finite cover by 2^n rational opens.

[34] and by Lemma 12.1 to ensure that $\mathcal{O}_X$ satisfies the sheaf property for all open covers of all open subsets

For example, if in (c) $n = 2$, then

$$U_\varnothing = X\left(\frac{1, t_1, t_2}{t_1 t_2}\right), \quad U_{\{1\}} = X\left(\frac{1, t_1 t_2}{t_2}\right), \quad U_{\{1,2\}} = X\left(\frac{1, t_1, t_2}{1}\right)$$

For a formula in the general case see [Mor19, Definition IV.2.3.1].

Lemma 13.13 (Refinements of special covers)**.** *Suppose (A, A^+) is complete Tate.*

(1) Any open cover admits a refinement by a standard rational cover.
(2) For any standard rational cover $X = \bigcup_i U_i$ there exists a Laurent cover $X = \bigcup_V V$, such that for each V, the cover $V = \bigcup_i (U_i \cap V)$ is a standard rational cover generated by units.
(3) Any standard rational cover generated by units admits a refinement by a Laurent cover.

Proof. (1): Let $X = \bigcup_i U_i$ be an open covering. By Lemma 7.6 we may assume that that all U_i are rational opens and as X is quasi-compact, that this covering is finite, i.e., i ranges from 1 to some $n \geq 1$. Write $U_i = X\left(\frac{T_i}{s_i}\right)$ for some $T_i \subseteq A$ generating an open ideal of A. This open ideal contains some power ϖ^N of a pseudo-uniformizer $\varpi \in A$, hence is the unit ideal, that is $T_i \cdot A = A$. Also, adding s_i to T_i, we may assume that $s_i \in T_i$. Let $S \subseteq A$ be the subset

$$S = \left\{ \prod_{i=1}^n f_i \colon t_i \in T_i, \text{ such that for at least one } i, \, t_i = s_i \right\}$$

As each $T_i \cdot A = A$, we have $S \cdot A = s_1 A + \cdots + s_n A$. On the other side, U_i cover X, so for any $x \in X$ there is some i with $|s_i(x)| \neq 0$. Then Corollary 11.15 implies $S \cdot A = A$. It remains to show that the standard rational covering defined by S refines $X = \bigcup_i U_i$. Let $s = \prod_i t_i \in S$ and let $X\left(\frac{S}{s}\right)$ be the correponding member of the standard covering. Let i_0 be an index such that $t_{i_0} = s_{i_0}$. Then $X\left(\frac{S}{s}\right) \subseteq U_{i_0} = X\left(\frac{T_{i_0}}{s_{i_0}}\right)$ (check this!).

(2): Let $t_1, \dots, t_n \in A$ generating the unit ideal, with $U_1, \dots, U_n$ the corresponding standard open cover. Write $1 = \sum_i a_i t_i$ with $a_i \in A$. Let $\varpi \in A$ be a pseudo-uniformizer. As A^+ is open, there is some $N \gg 0$, such that $\varpi^N a_i \in A^+$ for each i. Let $(V_I)_I$ be the Laurent cover of X given by the elements $\varpi^{-(N+1)} t_1, \dots, \varpi^{-(N+1)} t_n$. Exercise: verify the statement of (2) for this Laurent cover.

(3): If $t_1, \dots, t_n \in A^\times$ and $U_1, \dots, U_n$ is the corresponding stadard rational cover generted by units, then take $I = \{(i,j) \in \{1, \dots, n\} \colon i < j\}$, let $t_{ij} = t_i t_j^{-1}$, and then one checks that the Laurent cover refines $X = \bigcup_i U_i$. □

Now we can continue with the proof of Proposition 13.12. Note that its assumption is precisely (13.2) for simple Laurent covers (as any simple Laurent cover $\mathscr{U}/U$ has two mebers, one automatically has $\check{H}^i(\mathscr{U}/U, \mathcal{O}_X) = 0$ for all $i > 1$).

Let $\mathscr{U}/U$ be any rational open cover of a rational open $U \subseteq X$. By Lemma 13.13(1) we can refine it by a standard rational cover. By locality of (13.2), if it holds for this refinement, then also for the original cover. Thus we may assume that $\mathscr{U}/U$ is a standard rational cover. By Lemma 13.13(2), there is a Laurent cover $\mathscr{V}/U$, such that for each $V \in \mathscr{V}$, $\mathscr{U} \cap V/V$ is a rational cover generated by units. Thus, by transitivity and locality of (13.2), it suffices to show it for Laurent covers and for standard covers generated by units. Finally, any standard cover generated by units admits a refinement by a Laurent cover by Lemma 13.13(3), so using locality again, it suffices to check (13.2) only for Laurent covers.

By induction on the number n of elements $t_1, \dots, t_n$ generating the Laurent cover (and using the assumption of the proposition as induction start for $n = 1$), we may assume that we know the assertion for all Laurent covers generated by $< n$ elements. For $t_1, \dots, t_n \in \mathcal{O}_X(U)$, let $(U_I)_I$ be the corresponding Laurent cover. If $U = U_1 \cup U_2$ is the simple Laurent cover generated by t_1, then for $i = 1, 2$, $(U_I)_I \cap U_i/U_i$ is a Laurent cover generated by $n-1$ elements $t_2, \dots, t_n$, so by induction assumption and by transitivity of (13.2), we are done. $\square$

13.4. **Exactness for simple Laurent covers: stably uniform case.** Let $X = \mathrm{Spa}(A, A^+)$ for a complete Tate Huber pair (A, A^+). Let us take a closer look at the Čech complex of a simple Laurent covering. Therefore, let $t \in A$, and consider

$$\begin{aligned} U &= X\left(\frac{1,t}{1}\right) && = \{x \in X \colon |t(x)| \leq 1\} \\ V &= X\left(\frac{1,t}{t}\right) && = \{x \in X \colon |t(x)| \geq 1\} \\ U \cap V &= X\left(\frac{1,t,t^2}{t}\right) && = \{x \in X \colon |t(x)| = 1\} \end{aligned}$$

as in (13.1). Recall from Propositions 7.7 and 12.2 how $\mathcal{O}_X(U)$, $\mathcal{O}_X(V)$, $\mathcal{O}_X(\cap V)$ are constructed. Let A_0, I be a couple of definition of A. First, we localized to get the (non-complete) Huber pairs, with their respective rings of definitions

$$\begin{aligned} B_U &= A & B_U^+ &= A^+[t] & B_{U,0} &= A_0[t] \\ B_V &= A[t^{-1}] & B_V^+ &= A^+[t^{-1}] & B_{V,0} &= A_0[t^{-1}] \\ B_{U\cap V} &= A[t^{-1}] & B_{U\cap V}^+ &= A^+[t, t^{-1}] & B_{U\cap V,0} &= A_0[t, t^{-1}] \end{aligned}$$

Note that the resulting Čech complex

$$0 \to A \to B_U \oplus B_V \to B_{U\cap V} \to 0 \tag{13.3}$$

(where the right map is given by $x, y \mapsto \psi_U(x) - \psi_V(y)$, where $\psi_* \colon B_* \to B_{U\cap V}$ is the natural map) is exact, as the natural maps $A \to B_U$ and $B_V \to B_{U\cap V}$ are isomorphisms. However, in a second step (see Proposition 12.2) we completed the above rings, to get the completion of (13.3):

$$0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V) \to 0, \tag{13.4}$$

where $\mathcal{O}_X(X) = A$ (as A was complete).

Completion can destroy exactness, but it does not under special circumstances. Our goal is to prove the following:

Proposition 13.14. *Suppose we are in case (3) or (4) of Theorem 13.11. Then the assumption of Proposition 13.12 is verified, i.e. the complex* (13.4) *is exact (for any rational open of X).*

First we handle the stably uniform case, where we use the notion of strictness for maps between topological groups.

Definition 13.15. A continuous map $f \colon M \to N$ of topological groups is called *strict* if the induced isomorphism $M/\ker f \xrightarrow{\sim} \operatorname{im} f$ (of abstract groups) is a homeomorphism. With other words, the quotient topology on $M/\ker f$ must coincide with the subspace topology on $\operatorname{im} f$.

Remark 13.16. Suppose we know that $\mathcal{O}_X$ is a sheaf of abstract abelian groups. Then Remark 12.5(1) tells us that $\mathcal{O}_X$ is a sheaf of topological abelian groups if and only if for any rational open cover $U = \bigcup_i U_i$ of any rational open $U \subseteq X$, the map $\mathcal{O}_X(U) \to \prod_i \mathcal{O}_X(U_i)$ is strict.

Completion of a complex of topological abelian groups with strict maps behaves well:

Proposition 13.17. *[[Bou], III,§2 N12, Lemme 2] Let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of continuous maps between abelian topological groups having countable fundamental systems of neighboorhoods of* 0*. Suppose that the sequence is exact and f, g are strict. Then the completion*

$$\widehat{L} \xrightarrow{\widehat{f}} \widehat{M} \xrightarrow{\widehat{g}} \widehat{N}$$

is exact and $\widehat{f}$, $\widehat{g}$ are strict.

This proposition encompasses our strategy: we will verify strictness of maps in the exact complex (13.3) to deduce exactness of (13.4). Recall that we are in the Tate case and that ϖ is a pseudo-uniformizer of A. Let $\alpha_U\colon A \to B_U$ and $\alpha_V\colon A \to B_V$ be the natural maps.

Lemma 13.18. *For the maps in (13.3) we have:*

(1) $B_U \oplus B_V \to B_{U\cap V}$ is strict.

(2) $A \to B_U \oplus B_V$ is strict if and only if there is some $n > 0$ with $\varpi^n(\alpha_U^{-1}(B_{U,0}) \cap \alpha_V^{-1}(B_{V,0})) \subseteq A_0$.

Proof. (1): As (13.3) is exact, $B_U \oplus B_V \to B_{U\cap V}$ is surjective. As an open continuous map is a homeomorphism, it suffices to show that $B_U \oplus B_V \to B_{U\cap V}$ is open. The sets $\varpi^n A_0[t] \oplus \varpi^n A_0[t^{-1}]$ for varying $n > 0$ for a system of open neighboorhoods of 0 in $B_U \oplus B_V$. It suffices to check that they are mapped to open subsets in $B_{U\cap V}$. But the image of such a set is $\varpi^n A_0[t, t^{-1}] = \varpi^n B_{U\cap V,0} \subseteq B_{U\cap V}$, which is open.

(2): Endow A (as an abelian group) with the topology in which $A_0' := \varpi^n(\alpha_U^{-1}(B_{U,0}) \cap \alpha_V^{-1}(B_{V,0}))$ is open (and bounded), and the topology on A_0' is ϖ-adic. Then $A \to B_U \oplus B_V$ is strict if and only if $(A, A_0, \varpi-\text{adic}) \to (A, A_0', \varpi-\text{adic})$ is a homeomorphism. This is the case if and only if A_0 is open in the right hand side. But this is precisely the condition in (2). □

Now, uniformicity ensures us the strictness in Lemma 13.18(2):

Proposition 13.19. *Suppose A is an uniform Tate algebra. Then for any $t \in A$, the map $A \to B_U \oplus B_V$ in (13.3) is strict. In particular, (13.4) is exact.*

Note that to apply Proposition 13.12 we need to know the conclusion of Proposition 13.19 for all rational opens in X, which means that we really need to assume stably uniform (and not just uniform) in Theorem 13.11.

Proof of Proposition 13.19 in the stably uniform case. The last claim follows by combining the first claim, Lemma 13.18(1) and Proposition 13.17. Let us prove the first claim.

Lemma 13.20 (Local criterion for power-boundedness)**.** *Let $t_1, \dots, t_n \in A$ generate the unit ideal. For each i, let $A \xrightarrow{\varphi_i} A_i := A[t_i^{-1}]$ with ring of definition $A_{i,0} = A_0\left[\frac{t_1,\dots,t_n}{t_i}\right]$. Let $a \in A$. If $\varphi_i(a) \in A_{i,0}$ for all i, then $a \in A^\circ$, i.e., a is power-bounded.*

Proof. This is slightly tricky, but not very hard, cf. [BV18, Lemma 3]. □

Apply Lemma 13.20 to the pair $t, 1$. It shows that any element in $(\alpha_U^{-1}(B_{U,0}) \cap \alpha_V^{-1}(B_{V,0}))$ is power-bounded, that is $(\alpha_U^{-1}(B_{U,0}) \cap \alpha_V^{-1}(B_{V,0})) \subseteq A^\circ$. As by assumption, A is uniform, A° is bounded, that is for some $n > 0$, $\varpi^n A^\circ \subseteq A_0$. Combining, we see that $\varpi^n(\alpha_U^{-1}(B_{U,0}) \cap \alpha_V^{-1}(B_{V,0})) \subseteq A_0$. Now we conclude by Lemma 13.18. □

Proof of Theorem 13.11 in the stably uniform case. Combining Propositions 13.14 (in the stably uniform case) and 13.12 we see that $\mathcal{O}_X$ is a sheaf of abelian groups and its higher cohomology vanishes. It remains to show that $\mathcal{O}_X$ is a sheaf of topological groups, which by Remark 13.16 amounts to say that for any standard rational cover $U = \bigcup_i U_i$ of any rational open $U \subseteq X$, the map $\mathcal{O}_X(U) \to \prod_i \mathcal{O}_X(U_i)$ is strict (as standard rational covers are cofinal). We may replace X by U. As strictness is preserved by completion (Proposition **??**) it suffices to show that the uncompleted map $\varphi\colon A \to \prod_i B_i$ is strict (where B_i are the localizations of A as constructed in Proposition 7.7). As A is uniform we may take $A_0 = A^\circ$. The subspace topology on $\varphi(A) \subseteq \prod_i B_i$ is given by the fundamental system of opens $\varphi(A) \cap \prod_i \varpi^n B_{i,0}$. The quotient topology on $\varphi(A)$ is given by $\varphi(\varpi^n A_0)$. As φ is continuous, the quotient topology is at least as fine as the subspace topology. Conversely, Lemma 13.20 tells us precisely that $\varphi(A) \cap \prod_i \varpi^n B_{i,0} \subseteq \varphi(\varpi^n A_0)$, and so both topologies coincide. □

13.5. **Appendix to §13.**

14. Banach open mapping theorem and consequences

We are aiming to prove Theorem 13.11 in the strongly Noetherian case. First we study some topological algebra, notably the Banach open mapping theorem, its consequences and the notion of maps topologically of finite type.

To make the statements less technical, let us rightaway assume that

all commutative topological groups have a countable fundamental system of neighboorhoods of 0.

This has as a consequence that the topology on all such objects is given by a translation invariant pseudo-metric (if the group is even Hausdorff, then the pseudo-metric is a metric). Thus, we always may fix such a metric and the topology is defined by the corresponding family of discs with arbitrary small radii. Also we restrict to the Tate case, the general statements being somewhat more general.

The hard direction in the following result is that a surjection between complete modules under a Tate ring is open.

Theorem 14.1 (Banach open mapping theorem)**.** *Let A be a Tate ring. Let M, N be Hausdorff topological A-modules. Assume M is complete. Let $f\colon M \to N$ be an A-linear map. Then the following are equivalent:*

(a) N is complete and f is surjective.
(b) N is complete and $f(M)$ is open.
(c) f is open.

Proof. (a) $\Rightarrow$ (b) is clear. Suppose (b). Let $\varpi \in A$ be a pseudo-uniformizer. Let $U \subseteq M$ be a neighboorhood of 0. Let V be an open disc centered at 0, with radius small enough such that $V - V \subseteq U$. For any $m \in M$, there is an $n > 0$ with $\varpi^n m \in V$. With other words, $M = \bigcup_{n \geq 0} \varpi^{-n} V$. But then

$$f(M) = \bigcup_{n\geq 0} \varpi^{-n} f(V) \subseteq \bigcup_{n\geq 0} \varpi^{-n} \overline{f(V)}.$$

As $\varpi^{-n} \in A$ are units, all sets $\varpi^{-n}\overline{f(V)}$ are closed. By assumption, $f(M) \subseteq N$ is open. In particular, $\bigcup_{n\geq 0} \varpi^{-n}\overline{f(V)}$ has a non-empty interior, and then Baire category theorem[35] implies that at least one of the closed subsets $\varpi^{-n}\overline{f(V)}$ is not nowhere-dense, that is has a non-empty interior. As multiplication by ϖ^n is a homeomorphism, also $\overline{f(V)}$ has non-empty interior. Let $x \in \overline{f(V)}$ be an interior point. Then

$$0 = x - x \in \overline{f(V)} - \overline{f(V)} \subseteq \overline{f(V) - f(V)} \subseteq \overline{f(U)}$$

[35] Let X be a non-empty complete metric space. Let $A \subseteq X$ be a subset with non-empty interior. Then A is not the union of nowhere-dense subsets. (A subset is nowhere-dense if its closure has empty interior.)

is an interior point (of all these sets). With other words, $\overline{f(U)}$ is a neighboorhood of 0 in N.

We thus have shown that for any neighborhood $0 \in U \subseteq M$, $\overline{f(U)}$ is a neighboorhood of 0 in N. Now we derive from this that f is open. Fixing a metric defining the topologies on M, N, the above says that for any $r > 0$ there is some $\rho(r) > 0$ such that

$$\overline{f(B_{M,r}(0))} \supseteq B_{N,\rho(r)}(0) \tag{14.1}$$

(where $B_{?,r}(0)$ is the open unit disc in ? centered at 0 with radius r).

Let $a > r > 0$ be arbitrary. Exploiting that the metrics are translation invariant, it is sufficient to show that

$$f(B_{M,a}(0)) \supseteq B_{N,\rho(r)}(0). \tag{14.2}$$

Choose a sequence of real numbers r_n ($n \geq 1$) such that $r_1 = r$, $\sum_{n\geq 1} r_n = a$. For any $n \geq 1$, choose some $0 < \rho_n < \rho(r_n)$, subject to the condition that $\lim_{n\to\infty} \rho_n = 0$. Let $y \in B_{N,\rho(r)}(0)$. We must show that $y \in f(B_{M,a}(0))$. We now define inductively a sequence of elements $x_n \in M$ ($n \geq 0$), which is subject to the two conditions:

- $x_n \in B_{M,r_n}(x_{n-1})$ for all $n > 0$, and
- $f(x_n) \in B_{N,\rho_{n+1}}(y)$.

Put $x_0 = 0$. Let $n > 0$ and suppose $x_0, \dots, x_{n-1}$ are already defined. Then $y \in B_{N,\rho_n}(f(x_{n-1})) \subseteq \overline{B_{M,r_n}(x_{n-1})}$ (where the inclusion follows from (14.1) by exploiting the translation invariance of the metric). Then $B_{N,\rho_{n+2}}(y) \cap f(B_{M,r_n}(x_{n-1})) \neq \varnothing$. We choose $x_n \in B_{M,r_n}(x_{n-1})$, such that $f(x)$ lies in this intersection.

Now the sequence $(x_n)_{n\geq 0}$ is Cauchy in M, as $\sum_{n\geq 1} r_n = a$ and so $\sum_{n\geq N} r_n \to 0$ for $N \to \infty$. As M is complete, $(x_n)_n$ has a limit $x \in M$. Then $x \in B_{M,a}(0)$ by the triangle inequality, and $f(x) = \lim_{n\to\infty} f(x_n) = y$ by continuity of f. This establishes (14.2), and hence we are done with (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a): By assumption, $f(M)$ is open in N, so is an open neighboorhood of 0. Then for any $x \in N$ we may find some $n \gg 0$ with $\varpi^n x \in f(M)$. As $f(M) \subseteq N$ is an A-submodule and $\varpi^n \in A^\times$, we must have $f(M) = N$, that is f is surjective. As f is open, also the induced continuous bijection $\bar{f}\colon M/\ker(f) \to N$ is open, hence homeomorphism. It remains to show that $M/\ker(f)$ is complete. As N is Hausdorff, $\ker(f) \subseteq M$ is a closed submodule, and the quotient of a complete abelian group by a closed subgroup is again complete. □

The Banach open mapping theorem has numerous important consequences, which we now discuss.

Corollary 14.2 (automatic continuity)**.** *Let A be a Tate ring, $f\colon M \to N$ an A-linear map between topological A-modules, where M is complete and finitely generated. Then f is continuous.*

Proof. There is some $n > 0$ and a surjective A-linear map $\pi\colon A^n \to M$. It is continuous (as the scalar multiplication and addition in M are continuous), and similarly $f \circ \pi$ is continuous. By Theorem 14.1, π is open. Let $U \subseteq N$ be an open subset. Then $f^{-1}(U) = \pi(\pi^{-1}(f^{-1}(U)))$ is open. □

Corollary 14.3 (automatic completness)**.** *Let A be a complete Tate ring. If the completion of a topological A-module M is finitely generated, then M is complete.*

Proof. Choose a surjection $\pi\colon A^n \twoheadrightarrow \widehat{M}$ to the completion of M. It is automatically continuous. By Theorem 14.1 it is open. Thus, as $(A^{\circ\circ})^n \subseteq A^n$ is a neighborhood of 0, its image $\sum_{i=1}^n A^{\circ\circ}\pi(e_i)$ in $\widehat{M}$ also is (where $(e_i)_{i=1}^n$ is the standard basis of A^n). In a commutative topological group, the sum of an open and a dense subset is the whole group. Thus

$$M + \sum_{i=1}^{n} A^{\circ\circ}\pi(e_i) = \widehat{M}.$$

Now the topological Nakayama lemma 14.4 shows that $M = \widehat{M}$. □

We have used a topological verison of the Nakayama lemma:

Lemma 14.4 (Nakayama lemma for complete Tate rings). *Let A be a complete Tate ring, M an A-module, and $N \subseteq M$ a submodule, such that there are some $m_1, \dots, m_r \in M$ with $M = N + \sum_{i=1}^r A^{\circ\circ} m_i$. Then $N = M$.*

Proof. For each $1 \leq i \leq r$, write $m_i = n_i + \sum_{j=1}^r a_{ij} m_j$ with $n_i \in N$ and $a_{ij} \in A^{\circ\circ}$. That is, if $a = (a_{ij})$, $m = (m_1, \dots, m_r)^T$ and similarly for n, we have $m = n + a \cdot m$ or with other words,

$$n = (1_r - a) \cdot m$$

in the ring of $r \times r$-matries over A. If we show that $1_r - a$ is invertible, we are done, as then it follows that $m_i \in N$ for each i. To show that $1_r - a$ is invertible it suffices to check that $\det(1_r - a) \in A^\times$. But all $a_{ij} \in A^{\circ\circ}$, and one checks that $\det(1_r - a) \in 1 + A^{\circ\circ} \subseteq A^\times$ (the last inclusion uses that A is complete). □

Corollary 14.5 (characterization of Noetherian modules). *Let A be a complete Tate ring and M a complete topological A-module. The following are equivalent:*

(1) M is Noetherian as an A-module.
(2) All A-submodules of M are closed.

In particular, A Noetherian if and only if all ideals of A are closed.

Proof. (1) ⇒ (2): If $N \subseteq M$ is any submodule. As M is complete (in particular, Hausdorff), we have $\widehat{N} \subseteq M$. As M is Noetherian, $\widehat{N}$ is finitely generated. Then Corollary 14.3 shows that $N = \widehat{N}$ is complete. A complete submodule is always closed.

(2) ⇒ (1): Let $M_1 \subseteq M_2 \subseteq \dots$ be an ascending chain of submodules of M. Let $M_\infty = \bigcup_i M_i$. This is again a submodule of M, and by assumption it is closed in M, hence complete. Thus, M_∞ is a complete A-module, hence in particular, a complete metric space and we can apply Baire category theorem to it and the union $M_\infty = \bigcup_i M_i$ of *closed* subsets. It shows that there is some i_0, such that M_{i_0} has a non-empty interior (within M_∞!). Then M_{i_0} contains an open neighborhood of $0 \in M_\infty$, hence for any $x \in M_\infty$, there is some $n > 0$ with $\varpi^n x \in M_{i_0}$, and hence hence $M_{i_0} = M_\infty$. □

Using the above we can show that any finitely generated (abstract) module over a complete Noetherian Tate ring has a canonical topology:

Corollary 14.6 (canonical topology). *Let A be a complete Noetherian Tate ring.*

(1) Every finitely generated A-module has a unique topology making it a complete topological A-module having a countable fundamental system of neighboorhoods of 0. We call it the canonical topology*.*
(2) Let $f\colon M \to N$ be an A-linear map between finitely generated A-modules. With respect to the canonical topologies on M, N, f is continuous, $\mathrm{im}(f) \subseteq N$ is closed and $M \twoheadrightarrow \mathrm{im}(f)$ is open.

Proof. (1): If $\mathcal{T}, \mathcal{T}'$ are two such topologies on M, then $\mathrm{id}\colon (M, \mathcal{T}) \to (M, \mathcal{T}')$ and its inverse are continuous by Corollary 14.2, that is, homeomorphisms. This shows uniqueness. For the existence, pick some $n > 0$ and a A-linear surjection $\pi\colon A^n \twoheadrightarrow M$. As A is Noetherian, A^n is a

Noetherian A-module. Hence, by Corollary 14.5, $\ker\pi$ is a closed submodule. Thus the quotient topology on $A^n/\ker\pi \cong M$ is Hausdorff. Clearly, as A^n was complete and metrizable, also the quotient topology on $A^n/\ker\pi$ is. Endowing M with this topology finishes the proof of (1).

(2): f is continuous by Corollary 14.2. Also, N is finitely generated A-module, hence Noetherian, hence by Corollary 14.5, $\operatorname{im} f$ is closed. As the canonical topology on N is complete, $\operatorname{im} f$ is therefore also complete with the subspace topology. The map $M \to \operatorname{im} f$ is then open Theorem 14.1. □

14.1. **Proof of sheafiness in the strongly Noetherian case.** Let A be a complete Noetherian Tate ring and let M be a finitely generated A-module, endowed with the canonical topology (Corollary 14.6). Consider

$$M\langle T\rangle = \left\{ \sum_{n\geq 0} m_n T^n \colon \text{ all } m_n \in M, \text{ and } m_n \to 0 \text{ for } n \to \infty \right\}.$$

This is a $A\langle T\rangle$-submodule of $M[\![T]\!]$.

Lemma 14.7. *The natural map*

$$A\langle T\rangle \otimes_A M \to M\langle T\rangle, \quad a, m \mapsto am$$

is an isomorphism (of abstract $A\langle T\rangle$-modules).

Proof. This is clear if $M \cong A^n$ is finite free. In the general case, we have an exact sequence of A-modules $A^m \xrightarrow{f} A^n \xrightarrow{g} M \to 0$. Tensoring it with $A\langle T\rangle$ we obtain the exact upper row of the following commutative diagram

$$\begin{array}{ccccccc} A\langle T\rangle \otimes A^m & \longrightarrow & A\langle T\rangle \otimes A^n & \longrightarrow & A\langle T\rangle \otimes M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A^m\langle T\rangle & \longrightarrow & A^n\langle T\rangle & \longrightarrow & M\langle T\rangle & \longrightarrow & 0 \end{array}$$

It follows from the properties of the canonical topology (Corollary 14.6(2)) that the lower row is exact too. As the first two vertical arrows are bijections, also the third is by the 5-lemma. □

Proposition 14.8 (Flatness of completed localization)**.** *Let A be a complete Noetheran Tate ring.*

(1) The ring $A\langle T\rangle$ is faithfully flat over A.
(2) Let $t \in A$. The rings $A\langle T\rangle/(t - X)$ and $A\langle T\rangle/(1 - tT)$ are flat over A.

Proof. If $M \hookrightarrow N$ is an injection of finitely generated A-modules, then $M\langle T\rangle \to N\langle T\rangle$ is obviously injective. Hence, by Lemma 14.7, $M \otimes_A A\langle T\rangle \to N \otimes_A A\langle T\rangle$ is also injective. This shows that $A \to A\langle T\rangle$ is flat. If $\mathfrak{p} \subseteq A$ is a prime ideal, then the subset $\{\sum_{n\geq 0} a_n T^n \colon a_0 \in \mathfrak{p}\} \subseteq A\langle T\rangle$ is a prime ideal whose intersection with A is $\mathfrak{p}$. Thus $\operatorname{Spec} A\langle T\rangle \to \operatorname{Spec} A$ is surjective and faithful flatness follows.

(2): Claim: Let $g \in A\langle T\rangle$. Assume that for any finitely generated A-module M, the multiplication map $x \mapsto gx \colon M\langle T\rangle \to M\langle T\rangle$ is injective. Then $B_g := A\langle T\rangle/g$ is flat over A.

Proof of Claim: The sequence

$$0 \to A\langle T\rangle \xrightarrow{g} A\langle T\rangle \to B_g \to 0$$

is exact by assumption applied to $M = A$. Applying $- \otimes_A M$ and using that $A\langle T\rangle$ is A-flat by (1), we get the exact sequence

$$0 \to \operatorname{Tor}^1_A(M, B_g) \to M \otimes_A A\langle T\rangle \xrightarrow{\operatorname{id}_M \otimes (g\cdot)} M \otimes_A A\langle T\rangle \to \dots$$

But the last map identifes under the bijection of Lemma 14.7 with the multiplication by g map $M\langle T\rangle \to M\langle T\rangle$, which is injective by assumption. Thus $\mathrm{Tor}_A^1(M, B_g) = 0$ and as M was arbitrary finitely generated, B_g is flat. This proves the claim.

To finish the proof of (2), we have to verify the assumption of the claim for $g = t - T$ and for $g = 1 - tT$. In the second case this is an immediate computation. In the first case, let $\sum_{n\geq 0} m_n T^n \in M\langle T\rangle$, such that $(t-T)\sum_n m_n T^n = 0$. This means that $tm_0 = 0$ and $tm_n = m_{n-1}$ for all $n \geq 1$. Let M' be the submodule of M generated by all the m_n's. It suffices to show that $M' = 0$. As M is finitely generated and A Noetherian, M' is finitely generated, hence generated by $m_0, \dots, m_r$ for some $r \gg 0$. Applying $tm_n = m_{n-1}$ succesively, we see that M' is generated by the single element m_r and that $t^{r+1}m_r = tm_0 = 0$. We can find some $a \in A$ with $m_{2r+1} = am_r$. Then $m_r = t^{r+1}m_{2r+1} = t^{r+1}am_r = 0$, and we are done. □

To prove sheafiness in the strongly Noetherian case, we first have to relate the (complete) A-algebra of sections $\mathcal{O}_X(U)$ to convergent power series:

Lemma 14.9. *Let A be a complete strongly Noetherian Tate algebra and let A^+ be any ring of integral elements. Let $X = \mathrm{Spa}(A, A^+)$ and let $t \in A$.*

(1) If $U = \{t \leq 1\} \subseteq X$, then $\mathcal{O}_X(U) \cong A\langle T\rangle/(T - t)$.
(2) If $V = \{t \geq 1\} \subseteq X$, then $\mathcal{O}_X(V) \cong A\langle T\rangle/(1 - tT)$.
(3) Let U, V are as in (1),(2). Then $\mathcal{O}_X(U\cap V) \cong A\langle T_1, T_2\rangle/(T_1 - t, 1 - T_1T_2) \cong A\langle T_1^{\pm 1}\rangle/(T_1 - t)$, where

$$A\langle T^{\pm 1}\rangle := \left\{ \sum_{n\in\mathbb{Z}} a_n T^n \colon a_n \to 0 \text{ as } |n| \to \infty \right\},$$

with multiplication defined by

$$\left(\sum_{n\in\mathbb{Z}} a_n T^n\right) \cdot \left(\sum_{n\in\mathbb{Z}} b_n T^n\right) = \left(\sum_{n\in\mathbb{Z}} c_n T^n\right),$$

where $c_n = \sum_{k+m=n} a_k b_m \in A$ is well-defined, as the infinite sum converges.

Proof. (1): As A is strongly Noetherian, $A\langle T\rangle$ is Noetherian. Thus, by Corollary 14.5, $(T - t)$ is closed, and so $A\langle T\rangle/(T - t)$ is Hausdorff and hence complete. It is clear that the map induced by $A \to A\langle T\rangle/(T - t)$ on adic spectra factors through the subset $\{t \leq 1\}$. Moreover, it is easy to see (compare Remark 13.7(2)) that $A\langle T\rangle/(T - t)$ satisfies the same universal property as $\mathcal{O}_X(U)$ (see Proposition 12.2).

(2),(3): Similar. Note that $A\langle T_1, T_2\rangle/(1 - T_1T_2) \cong A\langle T_1^{\pm 1}\rangle$ as both A-algebras are initial in the category of maps from A to complete Tate rings B together with a choice of a unit $s \in B$ such that s and s^{-1} are powerbounded (for $A\langle T_1, T_2\rangle/(1 - T_1T_2)$ to be complete we used that A is strongly Noetherian). □

Finally, we can approach Proposition 13.14 and Theorem 13.11 in the strongly Noetherian case.

Proof of Proposition 13.14 in the strongly Noetherian case. By Lemma 14.9, the sequence (13.4) (whose exactness we want to show) fits in as the last row of the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & (T-t)A\langle T\rangle \times (1-tT^{-1})A\langle T^{-1}\rangle & \longrightarrow & (T-t)A\langle T^{\pm 1}\rangle & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & A\langle T\rangle \times A\langle T^{-1}\rangle & \longrightarrow & A\langle T^{\pm 1}\rangle & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(X) & \longrightarrow & \mathcal{O}_X(U) \times \mathcal{O}_X(V) & \longrightarrow & \mathcal{O}_X(U\cap V) & \longrightarrow & 0
\end{array}$$

The second and third columns are exact by Lemma 14.9, and we also know that the lower left horizontal map is injective. Applying the snake lemma to the second and third columns, it becomes clear that the exactness of the lower row follows from the exactness of the two upper rows. One readily checks their exactness, cf. [Wed19, Proof of Lemma 8.33]. □

Remark 14.10. Note that in the above proof we are slightly cheating: in fact, when reducing sheafiness to exactness of (13.4), we replaced $X = \mathrm{Spa}(A, A^+)$ by a smaller rational open subset $U \subseteq X$. Now, the assumption in Theorem 13.11(3) only guarantees us that A is strongly Noetherian, but says nothing about $\mathcal{O}_X(U)$. However, if $\mathcal{O}_X(U)$ is also strongly Noetherian, as is sketched below, see Corollary 14.15.

Proof of Theorem 13.11 in the strongly Noetherian case. To finish this proof it just remains to show that $\mathcal{O}_X$ is a sheaf of topological groups (not only abstract ones). Therefore, it is enough to show that for a rational open $U \subseteq X$, $\varphi\colon \mathcal{O}_X(U) \to \prod_i \mathcal{O}_X(U_i)$ is strict for any covering $U = \bigcup_i U_i$ by rational opens. As U is quasi-compact, we may assume that the covering is finite. As $\mathcal{O}_X$ is a sheaf (of abstract groups), $\mathrm{im}(\varphi) = \ker(\prod_i \mathcal{O}_X(U_i) \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j))$, hence closed in $\prod_i \mathcal{O}_X(U_i)$, and hence complete as $\mathcal{O}_X(U)$-module. But then the Banach mapping theorem 14.1 implies that $\varphi\colon \mathcal{O}_X(U) \to \mathrm{im}(\varphi)$ is open. □

Remark 14.11. Suppose (A, A^+) is a complete Huber pair with A strictly Noetherian Tate ring, and let $X = \mathrm{Spa}(A, A^+)$. Then:

- For $U \subseteq X$ rational open, $\mathcal{O}_X(U)$ is a quotient of some free A-Tate algebra $A\langle T_1, \dots, T_r\rangle$ by a (necessarily closed) ideal.
- For $V \subseteq U \subseteq X$, $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is flat.

Both can be proven in a similar way as the special cases shown above.

14.2. **Homomorphisms topologically of finite type.** (The discussion in this section is quite sketchy.)

First, the A-Tate algebras $A\langle T\rangle$ generalize to the following construction: let A be a non-archimedean topological ring. Let $T = (T_i)_{i\in I}$ be a family of indeterminates and for each $i \in I$, let $S_i \subseteq A$ be a subset of A such that for each $n \geq 0$ and each open subgroup $U \subseteq A$, the subgroup $S_i^n \cdot U \subseteq A$ (generated by products of n elements of S_i and one of U) is open. Write $S = (S_i)_{i\in I}$. The we define

$$A\langle T\rangle_S := \left\{ \sum_{\nu\in\mathbb{N}^{(I)}} a_\nu T^\nu \in A[\![T]\!] \colon \text{ for all open subgroups } U \subseteq A,\ a_\nu \in S^\nu U \text{ for almost all } \nu \right\}.$$

This is a ring and it has a unique structure of a non-archimedean topological ring for which

$$U_{T,S} := \left\{ \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu T^\nu \in A\langle T\rangle_S \colon\ a_\nu \in S^\nu U \text{ for all } \nu \right\}$$

is a fundamental system of open neighboorhoods of 0, where U varies through all open subgroups of A.

Remark 14.12. If one assumes A to be complete and T_i to be bounded for each i, then the map $(A\langle T\rangle_S, T)$ is initial among all pairs $(f, (x_i)_{i\in I})$ consisting of a continuous ring map $f\colon A \to B$ into a complete non-archimedean ring B, and elements $x_i \in B$, such that for each $i \in I$, $\{f(t)x_i \colon t \in T_i\}$ is bounded. (Cf. [Mor19, Cor.II.3.3.4]).

Definition 14.13. Let A, B be complete Huber rings and let $f\colon A \to B$ be a (continuous) ring homomorphism.

(1) f is called *strictly topologically of finite type* (*strictly tft*), if there is some $n \geq 0$ and a surjective continuous open map

$$A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$$

of A-algebras.

(2) f is called *topologically of finite type* (*tft*) if f is adic, there is a finite set $M \subseteq B$ such that $f(A)[M] \subseteq B$ is dense, and there are rings of definition $A_0 \subseteq A$, $B_0 \subseteq B$ and a finite subset $N \subseteq B_0$ such that $f(A_0)[N] \subseteq B_0$ is dense. Equivalently, f is *tft* if there exist finite subsets $S_1, \dots, S_n \subseteq A$ with $S_i \cdot A \subseteq A$ open, and a continuous open surjection of A-algebras $A\langle T_1, \dots, T_n\rangle_{S_1,\dots,S_n} \twoheadrightarrow B$. (Cf. [Mor19, IV.1.2.1])

Looking at the construction in the proof of Proposition 12.2 we see that if (A, A^+) is a Huber pair, and $U \subseteq X = \operatorname{Spa}(A, A^+)$ is open, then $\mathcal{O}_X(U)$ is a tft A-algebra.

Clearly, tft over A implies strictly tft over A. For Tate rings the converse is true:

Proposition 14.14 ([Wed19], Proposition 6.34)**.** *Let A be a Tate ring. Then any tft A-algebra is strictly tft.*

Corollary 14.15. *Let A be a strongly Noetherian Tate ring. Then any tft algebra over A is strongly Noetherian.*

In particular, if A is a complete strictly Noetherian Tate ring, $X = \operatorname{Spa}(A, A^+)$ for some A^+, and $U \subseteq X$ rational open, then $\mathcal{O}_X(U)$ strongly Noetherian.

Proof. This follows from the definitions of strictly tft, strongly Noetherian and from Proposition 14.14. □

15. Formal schemes and adic spaces

15.1. **Formal schemes.** Let us first recall formal schemes. This is a brief discussion only. See [?, 0AHY] (or [Ans, §3], ...) for more details.

Formal schemes form a full subcategory of *locally topologically ringed spaces* (LTRS). An object in LTRS is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of topological rings, whose stalks (=colimits over open neighboorhoods) are (after forgetting topology) local rings. The morphisms are evident[36]. Now, an affine formal scheme is a object in LTRS, which corresponds to an *admissible* topological ring[37].

[36]Once one notes that if $Y \to X$ is a continuous map, then one can define f_* for sheaves of *topological* rings in the same way as for usual rings.

[37]Admissible is more general than adic. The main source of examples for us are adic rings with a finitely generated ideal of definition.

Definition 15.1. A topological ring A is called *admissible*, if it is

- linearly topologized, i.e., the topology is determined by a fundamental system $\{I_\alpha\}_\alpha$ of open neighboorhoods of 0, which are ideals.
- complete (as usual, this means Hausdorff+complete), i.e., $A \cong \varprojlim_\alpha A/I_\alpha$, and
- there is an ideal I of definition, i.e., I is open and any neighboorhood of 0 contains I^n for $n \gg 0$.

For an admissible ring A, let Adm_A denote the category of admissible A-algebras B (with $A \to B$ continuous) with continuous maps.

Note that any abstract ring may be regarded as an admissible one, equipped with the discrete topology. Note that I^n itself does not need to be open, and so not any admissible ring is adic. For an example, let $A = k[T_i : i \geq 0]$ and let $I = (T_i : i \geq 0)$ be the ideal generated by all T_i's. Then the I-adic completion $\widehat{A}_I$ is admissible, but not adic, see [Sta14, 05JA]. In this example, I is not finitely generated, but note that there are examples of Noethereian and admissible rings, which are not adic. However:

Lemma 15.2. *Let A be any ring and $I \subseteq A$ a finitely generated ideal. Then the I-adic completion $\widehat{A}_I$ is adic.*

Back to the definition of formal schemes. For an admissible ring A, the *formal spectrum* of A is

$$\mathrm{Spf} A := \{\mathfrak{p} \in \mathrm{Spec}\, A \colon \mathfrak{p} \text{ is open in } A\ \}.$$

with the topology induced from $\mathrm{Spec}\, A$. If $(I_\alpha)_\alpha$ is a fundamental system of open ideals as in the definition (wlog, contained in an ideal of definition), for any α, β with $I_\alpha \supseteq I_\beta$, one has homeomorphisms[38]

$$\mathrm{Spec}\, A/I_\alpha \xrightarrow{\sim} \mathrm{Spec}\, A/I_\beta \xrightarrow{\sim} \mathrm{Spf} A.$$

More canonically, we thus can write

$$\mathrm{Spf} A \cong \varprojlim_\alpha \mathrm{Spec}\, A/I_\alpha$$

(all transition maps homeomorphisms). For each α, we can regard the structure sheaf $\mathcal{O}_{\mathrm{Spec}\, A/I_\alpha}$ as a sheaf of (discrete) rings on $\mathrm{Spf} A$, which we denote by $\mathcal{O}_\alpha$.

If X is any topological space having a basis of qc opens, the forgetful functor from sheaves of *topological* rings (groups, modules, ...) on X to sheaves of rings (groups, modules, ...) admits a left(?) adjoint, given by equipping the value of the sheaf on each *qc* open with the discrete topology and topologizing the value on remaining opens appropriately. We call sheaves in the essential image of this adjoint *pseudo-discrete.*

Let A be an admissible ring. We equip $\mathrm{Spf} A$ with the sheaf of topological rings

$$\mathcal{O}_{\mathrm{Spf}\, A} := \varprojlim_\alpha \mathcal{O}_\alpha,$$

where $\mathcal{O}_\alpha$ is regarded as a pseudo-discrete sheaf of topological rings, and the limit is taken in topological rings.

Definition 15.3. An *affine formal scheme* is a locally topologically ringed space isomorphic to

$$(\mathrm{Spf}\, A, \mathcal{O}_{\mathrm{Spf}\, A})$$

for some adimssible ring A.

This is a consistent definition, due to the following facts (which we do not check here):

[38]Existence of these homeomorphisms uses the ideal of definition!

(a) For A admissible, $(\operatorname{Spf} A, \mathcal{O}_{\operatorname{Spf} A})$ is indeed in LTRS that is, the stalks are (abstract) local rings.
(b) For any $f \in A$ with corresponding principal open $D(f) \subseteq \operatorname{Spec} A$, one has
$$\Gamma(D(f) \cap \operatorname{Spf} A, \mathcal{O}_{\operatorname{Spf} A}) = A\langle f^{-1}\rangle := \varprojlim_{\alpha} A/I_\alpha[f^{-1}].$$
(c) Moreover, for $f \in A$, $A\langle f^{-1}\rangle$ is an admissible ring, and
$$(\operatorname{Spf} A\langle f^{-1}\rangle, \mathcal{O}_{\operatorname{Spf} A\langle f^{-1}\rangle}) \cong (D(f) \cap \operatorname{Spf} A, \mathcal{O}_{\operatorname{Spf} A}|_{D(f)\cap \operatorname{Spf} A}).$$
(d) $A \mapsto (\operatorname{Spf} A, \mathcal{O}_{\operatorname{Spf} A})$ defines a fully faithful contravariant embedding of the category of admissible rings (with continuous homomorphisms) into LTRS.

In what follows we write $\operatorname{Spf} A$ for $(\operatorname{Spf} A, \mathcal{O}_{\operatorname{Spf} A})$. These facts make the following definition natural:

Definition 15.4. A *formal scheme* is an object of LTRS, which is locally isomorphic to an affine formal scheme. A morphism of formal schemes is a morphism in LTRS.

Example 15.5. We have a fully faithfull (covariant) embedding of the category of schemes into the category of formal schemes. On affines, it is given by sending $\operatorname{Spec} A$ to $\operatorname{Spf} A$, where A is equipped with discrete topology. Note that then $\operatorname{Spf} A = \operatorname{Spec} A$ and $\mathcal{O}_{\operatorname{Spf} A}(U) = \mathcal{O}_{\operatorname{Spec} A}(U)$ as abstract rings, but on the LHS the topology might be non-discrete if U is not *qc* open.

Up to set-theoretic restrictions (omitted), a formal scheme is determined by its evaluation on schemes (regarded as formal schemes as in Example 15.5):

Lemma 15.6 ([Sta14], 0AI1)**.** *The functor*
$$\text{Formal schemes} \to \text{Fun}^{op}(\text{Schemes}, \text{Sets}), \quad \mathfrak{X} \mapsto (h_{\mathfrak{X}} \colon S \mapsto Mor_{form.\ sch.}(S, \mathfrak{X}))$$
is fully faithful.

Proof. Let us prove this in the affine case. Let A, B be admissible rings, and let $f \colon h_{\operatorname{Spf} B} \to h_{\operatorname{Spf} A}$ be a natural transformation of functors. Let J_β be a fundamental system of open ideals of B. For each β, we have the natural inclusion $\operatorname{Spec} B/J_\beta \hookrightarrow \operatorname{Spf} B$, and f maps it to a morphism $\operatorname{Spec} B/J_\beta \to \operatorname{Spf} A$ (in formal schemes). Then by Yoneda (item (d) above), this uniquely corresponds to a continuous homomorphism of rings $\varphi_\beta \colon A \to B/J_\beta$. Moreover, if $J'_\beta \subseteq J_\beta$, then $\varphi_{\beta'}$ and φ_β are compatible with the projection. Thus we obtain a map
$$\lim_\beta \varphi_\beta \colon A \to \varprojlim_\beta B/J_\beta = B,$$
which is continuous, since the preimage of each J_β is open in A (as $A \to B/J_\beta$ is continuous and B/J_β is discrete).

If $S = \operatorname{Spec} R$ is an affine scheme, regarded as pseudo-discrete formal affine scheme, any morphism $\operatorname{Spec} R \to \operatorname{Spf} B$ factors through $\operatorname{Spf} B/J_\alpha$ for some open ideal $J_\beta \subseteq A$. With other words, $h_{\operatorname{Spf} B} = \varinjlim_\beta h_{\operatorname{Spec} B/J_\beta}$. □

Let us also note that a formal scheme, regarded as a functor on schemes, is a sheaf for the fpqc topology, cf. [Sta14, 0AI2].

Example 15.7. Let R be a discrete ring and let $R[\![T]\!]$ be the R-algebra with T-adic topology. Then $R[\![T]\!]$ is admissible (even adic) and $\operatorname{Spf} R[\![T]\!] \cong \operatorname{Spec} R$ as topological spaces. Regarded as functors on (affine) R-schemes, we have
$$\operatorname{Spf}(R[\![T]\!])(R') = Hom_{R\text{-alg, cont}}(R[\![T]\!], R') = \operatorname{Nil}(R'), \tag{15.1}$$
the nilradical of R'.

For $f = \sum_{n\geq 0} f_n T^n \in R[\![T]\!]$, let us compute $R[\![T]\!]\langle f^{-1}\rangle$. For each $N > 0$, let $X_N = \operatorname{Spec}\operatorname{Spec} R[\![T]\!]/(T^N)$. We have

$$(R[\![T]\!]/(T^N))[f^{-1}] = \Gamma(D_{X_N}(f \bmod T^N), \mathcal{O}_{X_N}) = \Gamma(D_{X_N}(f_0), \mathcal{O}_{X_N}) = (R[\![T]\!]/(T^N))[f_0^{-1}]$$
$$= R_{f_0}[\![T]\!]/(T^N),$$

where we use that $(f \bmod T^N) - f_0$ is a nilpotent element of $R[\![T]\!]/(T^N)$. Passing to the limit over N, we then get

$$R[\![T]\!]\langle f^{-1}\rangle = R_{f_0}[\![T]\!].$$

Now $\operatorname{Spf} R_{f_0}[\![T]\!] \cong \operatorname{Spec} R_{f_0}$ as topological spaces. On the other side, the subset $D(f) \subseteq \operatorname{Spf} R[\![T]\!]$ consists of precisely those open idels of $R[\![T]\!]$, which do not contain f, and via $\operatorname{Spf} R[\![T]\!] \cong \operatorname{Spec} R$ this agrees with the subset $D(f_0) \subseteq \operatorname{Spec} R$. Note also that (say, if $f = f_0$), we have a proper containement $R[\![T]\!]_{f_0} \subseteq R_{f_0}[\![T]\!]$ and the RHS is the T-adic completion of the LHS.

Note also that the (pro-Yoneda) Lemma 15.6 allows us to compute the group of endomorphisms of the functor Nil on R-algbras. Namely,

$$\operatorname{End}(\mathrm{Nil}) = \operatorname{End}_{R-\mathrm{alg,cont}}(R[\![T]\!]) = \{f \in R[\![T]\!] \colon f(0) \in \mathrm{Nil}(R)\} = R[\![T]\!]^{\circ\circ}.$$

Completion along a subscheme. A very rich source for formal schemes are the completions of schemes along (closed) subschemes. The prototypical situation is that the *formal completion* of $\mathbb{A}^1_R = \operatorname{Spec} R[T]$ along the zero section $\operatorname{Spec} R \hookrightarrow \mathbb{A}^1_R$

$$\widehat{\mathbb{A}}^1_R = \operatorname{Spf} R[\![T]\!].$$

Note that both $\operatorname{Spec} R$ and $\widehat{\mathbb{A}}^1_R$ are subfunctors of $\mathbb{A}^1_R$ (the first via the zero section), and for any R-algebra R', the inclusions

$$(\operatorname{Spec} R)(R') \subseteq \widehat{\mathbb{A}}^1(R') \subseteq \mathbb{A}^1(R') \quad \text{are given by} \quad \{0\} \subseteq \mathrm{Nil}(R') \subseteq R'.$$

In a much bigger generality, let R be a ring, and let X be any functor on R-algebras, and let Y be a subfunctor. The *completion* $\widehat{X}_Y$ of X along Y is the subfunctor of X given by

$$\widehat{X}_Y(R') = \{f \in X(R') \colon \exists I \subseteq R' \text{ nilpotent ideal, with } \bar{f} \in Y(R'/I)\},$$

where $\bar{f} \in X(R'/I)$ denotes the image of f.

Let A a ring and $I \subseteq A$ an ideal. Let $\widehat{A}_I$ be the I-adic completion of A. Then one can check that the completion of $X = \operatorname{Spec} A$ along the closed subscheme $Y = \operatorname{Spec} A/I$ is canonically isomorphic to

$$\widehat{X}_Y \cong \operatorname{Spf} \widehat{A}_I.$$

Affine formal space. There is another important formal version of the affine line. Namely, let A be an adic ring with an ideal of definition I (you can safely think of $A = \mathbb{Z}_p$ equipped with the p-adic topology, but not $A = \mathbb{Q}_p$ as this is not adic). Then we have the inclusion of functors on (affine) schemes,

$$\operatorname{Spf} A \subseteq \operatorname{Spec} A.$$

What is the pull-back of $\mathbb{A}^1_A = \operatorname{Spec} A[T]$ along this inclusion? It is easy to check that it is represented (in the context of Lemma 15.6) by $\operatorname{Spf} A\langle T\rangle$, where

$$A\langle T\rangle = A[T]^\wedge_I = \left\{ \sum_{n\geq 0} a_n T^n \colon |a_n| \to 0 \text{ for } n \to \infty \right\}$$

is the I-adic completion of $A[T]$.

Question: What is the generic fiber? Now, we specialize the above setup and assume that $A = \mathbb{Z}_p$ with p-adic topology. Then $\mathbb{A}^1_{\mathbb{Z}_p}$ has a generic fiber, namely $\mathbb{A}^1_{\mathbb{Q}_p}$, but when we literally pullback the inclusion $\mathbb{A}^1_{\mathbb{Q}_p} \hookrightarrow \mathbb{A}^1_{\mathbb{Z}_p}$ to $\operatorname{Spf} \mathbb{Z}_p$, we get the empty functor. However, philosophically, this pullback should be (co)represented by the ring

$$\mathbb{Z}_p\langle T\rangle \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p\langle T\rangle$$

(check this equality). As this ring is not admissible, we cannot form its formal spectrum. However, it is a (strongly Noetherian and Tate) Huber ring, and so has a well-behaved affinoid adic space attached to it. This adic space will be the generic fiber of $\operatorname{Spf} \mathbb{Z}_p\langle T\rangle$. To make this precise, it is best to embed the category of (locally Noetherian) formal schemes into adic spaces.

Fiber products. Let us mention, that fiber products exist in the category of formal schemes, and that if A is admissible and B, C are admissible A-algebras, then

$$\operatorname{Spf} B \times_{\operatorname{Spf} A} \operatorname{Spf} C \cong \operatorname{Spf}(B \widehat{\otimes}_A C),$$

where $B\widehat{\otimes}_A C = \lim_{\beta,\gamma}(B/I_\beta \otimes_A C/I_\gamma)$ is the completed tensor product.

15.2. **Formal schemes as adic spaces.** The category of locally Noetherian formal schemes embeds fully faithfully into the category of adic spaces, cf. [Hub94, §4]. This embedding is induced by

$$\operatorname{Spf} A \mapsto \operatorname{Spa}(A, A)$$

for a Noetherian admissible topological ring A. Let us take a slightly closer look. Let $\mathscr{F}$ be the category of locally Noetherian formal schemes.

Theorem 15.8 (Formal schemes as adic spaces)**.** *There exists a fully faithful functor*

$$t\colon \mathscr{F} \to \{\textit{Adic spaces}\},$$

which is given by

$$t\colon \operatorname{Spf} A \mapsto \operatorname{Spa}(A, A)$$

on affines and is characterized by glueing in general. Moreover, when $X \in \mathscr{F}$, then there is a map in LTRS, $\pi\colon (t(X), \mathcal{O}^+_{t(X)}) \to (X, \mathcal{O}_X)$, which is universal for LTRS-maps from adic spaces to $(X, \mathcal{O}_X)$[39].

Moreover, a map $f\colon X \to Y$ in $\mathscr{F}$ is adic resp. locally of finite type if and only if $t(f)$ is.

Proof. We only construct the functor t. By universality of the construction it is sufficient to assume that $X = \operatorname{Spf} A$ is affine.

Claim. Let Y be any adic space. LTRS-maps $(Y, \mathcal{O}^+_Y) \to (X, \mathcal{O}_X)$ correspond bijectively to homomoprphisms $A \to \mathcal{O}^+_Y(Y)$.

Proof of claim. By universality, we may assume $Y = \operatorname{Spa}(B, B^+)$ with $(B, B+)$ complete Huber pair. Taking global sections gives the map in one direction. Conversely, assume a continuous map $\varphi\colon A \to B^+$ is given. Let $y \in Y$. Then

$$g(y) = \{a \in A\colon |\varphi(a)(y)| < 1\}$$

is a prime ideal as y is a valuation; as y and φ are continuous, $g(y)$ is open, with other words it is a point in $X = \operatorname{Spf} A$. For $s \in A$ we have $g^{-1}(D(s)) = \{y \in Y\colon |\varphi(s)(y)| \geq 1\}$, which is open in Y. Thus g is continuous. We have to define a morphism of sheaves $\mathcal{O}_X \to g_*\mathcal{O}^+_Y$. This can be done on the basis of principal opens in X. Let $s \in A$, so that $\Gamma(D(s), \mathcal{O}_X) = A\langle s^{-1}\rangle$ and $\Gamma(g^{-1}(D(s)), \mathcal{O}^+_Y) = B^+\langle \varphi(s)^{-1}\rangle$, and by the universal property of $A\langle s^{-1}\rangle$ the composition

[39]More precisely, the universal property is: for any adic space $(Z, \mathcal{O}_Z)$, any map $(Z, \mathcal{O}^+_Z) \to (X, \mathcal{O}_X)$ uniquely factors through π.

$A \to B^+ \to B^+\langle\varphi(s)^{-1}\rangle$ factors through a map $A\langle s^{-1}\rangle \to B^+\langle\varphi(s)^{-1}\rangle$. This defines the desired map of sheaves, and it remains to show that it is local, that is, if $\psi_y\colon \mathcal{O}_{X,g(y)} \to \mathcal{O}^+_{Y,y}$ is the map induced on stalks, then $\psi_y^{-1}(\mathfrak{m}_y) = \mathfrak{m}_{g(y)}$, if $\mathfrak{m}_y$, $\mathfrak{m}_{g(y)}$ denote the respective maximal ideals. Clearly, we have $\mathfrak{m}_{g(y)} \supseteq \psi_y^{-1}(\mathfrak{m}_y)$. Consider the localization map $h\colon A \to \mathcal{O}_{X,g(y)}$. Then we have the composition $\psi_y \circ h\colon A \to \mathcal{O}_{X,g(y)} \to \mathcal{O}_{Y,y}$, and a germ $\bar{t} \in \mathcal{O}_{Y,y}$ lies in $\mathfrak{m}_y$ if and only if $|t(y)| < 1$ by definition of $\mathfrak{m}_y$. Now by definition of $g(y)$, we see that $g(y) = (\psi_y \circ h)^{-1}(\mathfrak{m}_y)$. As also $g(y) = \psi_y^{-1}(\mathfrak{m}_{g(y)})$ and $\mathfrak{m}_{g(y)} = g(y)\mathcal{O}_{X,g(y)}$, the required equality follows. This proves the claim.

To finish the proof notice that the universal property follows now from the Yoneda embedding of complete (sheafy) Huber pairs into adic spaces. □

One also might determine the essential image of the functor t, cf. [Hub94, §4].

Example 15.9. Let V be a complete valuation ring of rank 1, equipped with its valuation topology. Let $\mathfrak{m}$ be its maximal ideal, $k = V/\mathfrak{m}$, $K = \operatorname{Frac} V$. Then A is adic, and we have $\operatorname{Spf} A = \{\mathfrak{m}\}$. Moreover, it is easy to see (use that rank 1 valuation rings are maximal wrt domination order inside their fraction field), that $\operatorname{Spa}(A, A) = \{|\cdot|_{\mathfrak{m},\mathrm{triv}}, x\}$ with $|\cdot|_{\mathfrak{m},\mathrm{triv}}$ the trivial valuatio with $\operatorname{supp}(x) = \mathfrak{m}$ and x the defining valuation of A. Under the map $\operatorname{Spa}(A, A) \to \operatorname{Spf}(A)$ from the proof of Theorem 15.8 both points go to $\mathfrak{m}$ (check this).

15.3. **The adic generic fiber of a formal scheme.** After having the functor t from locally Noetherian formal schemes to at our disposal, it is very easy to see what the adic generic fiber of a formal scheme is.

Definition 15.10. Let K be a discretely valued Non-archimedean field with ring of integers $\mathcal{O}_K$ and uniformizer ϖ. Let $\mathfrak{X}$ be a locally Noethereian formal scheme over $\mathcal{O}_K$. The generic fiber of $\mathfrak{X}$ is the the fiber product[40]

$$\mathfrak{X}_\eta = t(\mathfrak{X}) \times_{\operatorname{Spa}(\mathcal{O}_K,\mathcal{O}_K)} \operatorname{Spa}(K, \mathcal{O}_K).$$

in the category of adic spaces. More explicitly,

$$\mathfrak{X}_\eta = \{x \in t(\mathfrak{X})\colon |\varpi(x)| \neq 0\}$$

is an open (not necessarily qc) locus of $t(\mathfrak{X})$ on which ϖ is non-zero.

For the classical construction of Raynaud, see Berthelot

Example 15.11. For simplicity, fix $K = \mathbb{Q}_p$, $\mathcal{O}_K = \mathbb{Z}_p$.

(1) Let $\mathfrak{X} = \operatorname{Spf} \mathbb{Z}_p\langle T\rangle$ be as above. Then

$$\mathfrak{X}_\eta = \mathbb{B}^1_{\mathbb{Q}_p} = \operatorname{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$$

is the closed unit ball (e.g., equality can be checked as one of LTRS).

(2) Let $\mathfrak{X} = \operatorname{Spf} \mathbb{Z}_p[\![T]\!]$ with $\mathbb{Z}_p[\![T]\!]$ having its (p, T)-adic topology. Then

$$\mathfrak{X}_\eta = \mathbb{D}_{\mathbb{Q}_p} \subseteq \{x \in \mathbb{B}^1_{\mathbb{Q}_p}\colon |T| < 1\},$$

Note that the locus $|T| < 1$ is closed in $\mathbb{B}^1_{\mathbb{Q}_p}$, and that $\mathbb{D}_{\mathbb{Q}_p}$ is the locus within this, on which T is topologically nilpotent. Here $\{|T| < 1\} \setminus \mathbb{D}_{\mathbb{Q}_p} = \{\nu\}$ consists of the rank two valuation belonging to the closure of the Gaußpoint, for which $T < 1$, but not topologically nilpotent. Note that $\mathbb{D}^1_{\mathbb{Q}_p}$ is an open but non-qc subset of $\{|T| < 1\}$.

[40] Note that we do not in general know existence of fiber products in adic spaces. However, fiber products along open immersions exist, as can be shown easily.

16. Further topics

- Embedding of rigid-analytic spaces into adic spaces ([Hub94])
- Embedding of (f.t.) schemes into rigid-analytic adic spaces (cf. Berthelot, Coho rigide, (0.3))
- Relation of the rig/an space attached with a scheme with the generic fiber of the p-adic completion of the scheme. (cf. Berthelot, Coho rigide, (0.3.5))

References

[Ans] Johannes Anschütz, *Lecture notes for a course on Lubin-Tate spaces*, https://janschuetz.perso.math.cnrs.fr/skripte/lecture_notes_lubin_tate.pdf.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-archimedean analysis*, Grundlehren der Math. Wissenschaften, vol. 261, Springer, Berlin, 1984.

[Bha17] B. Bhatt, *Lecture notes for a class on perfectoid spaces*, http://www-personal.umich.edu/ bhattb/teaching/mat679w17/lectures.pdf.

[Bou] N. Bourbaki, *Algèbre commutative*, chap. I - VII, Springer (1989); chap. VIII- IX, Masson (1983); chap. X, Masson (1998).

[BV18] K. Buzzard and A. Verberkmoes, *Stably uniform affinoids are sheafy*, J. Reine Angew. Math. **740** (2018), 25–39.

[Con15] B. Conrad, *Number theory learning seminar 2014-2015*, http://virtualmath1.stanford.edu/~conrad/Perfseminar/.

[DG67] Jean Dieudonné and Alexander Grothendieck, *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1961–1967).

[Gle24] Ian Gleason, *Specialization maps for scholze's category of diamonds*, Mathematische Annalen (2024), 1–69.

[Har77] R. Hartshorne, *Algebraic geometry*, volume 52 of Graduate Texts in Mathematics, Springer, 1977.

[Hub93] R. Huber, *Continuous valuations*, Math. Z. **212** (1993), no. 3, 455–477.

[Hub94] ______, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551.

[Mat89] H. Matsumura, *Commutative ring theory*, volume 8 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1989.

[Mih16] T. Mihara, *On Tate's acyclicity and uniformity of Berkovich spectra*, Israel J. Math. **216** (2016), no. 1, 61–105.

[Mor19] S. Morel, *Adic spaces*, https://web.math.princeton.edu/~smorel/adic_notes.pdf.

[Sch] P. Scholze, *Algebraic Geometry I – WS 2016/17 (lecture notes)*, , typed by J. Davies.

[Sta14] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu, 2014.

[SW20] P. Scholze and J. Weinstein, *Berkeley lectures on p-adic geometry*, Preprint, to appear in Annals of Math Studies (2020+).

[Wed19] T. Wedhorn, *Adic spaces*, arXiv:1910.05934.

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany
Email address: a.ivanov@rub.de