On a generalization of the Neukirch-Uchida theorem

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Abstract

In this paper we generalize a part of Neukirch-Uchida theorem for number fields from the birational case to the case of curves $\operatorname{Spec}\mathcal{O}_{K,S}$ with S a stable set of primes of a number field K. In particular, such sets can have arbitrarily small (positive) Dirichlet density. The proof consists of two parts: first one establishes a local correspondence at the boundary S, which works as in the original proof of Neukirch. But then, in contrast to Neukirchs proof, a direct conclusion via Chebotarev density theorem is not possible, since stable sets are in general too small, and one has to use further arguments.

1 Introduction

Our goal is to generalize a part of the Neukirch-Uchida theorem ([Ne] Theorem 2) for number fields to schemes of the form $\operatorname{Spec}\mathcal{O}_{K,S}$ where K is a number field and S a stable set of primes. Stable sets were introduced in [Iv3], they have positive but arbitrarily small Dirichlet density and behave in many (but not all) aspects like sets of primes of density one. In particular, many Chebotarev sets are stable. Thus the case considered in this paper is somewhere between the birational case considering $\operatorname{Spec}K$ and the arithmetic case considering $\operatorname{Spec}\mathcal{O}_{K,S}$ with S a finite set of primes. Clearly, the arithmetic case is much harder. The same anabelian question for function fields in one variable over a finite field, was answered by Tamagawa [Ta] in the case of a finite set S. However, the number field analogue of his proof seems to be out of scope at the moment.

To state our main result, we recall briefly the definition of stability. Let $\lambda > 1$. Roughly speaking, a set S of primes of K is λ-stable, if there is a subset $S_0 \subseteq S$ and some $0 < a \leqslant 1$, such that the Dirichlet density of $S_0(L)$ for almost all finite subextensions of K_S/K lies in the interval $[a, \lambda a)$. The condition $(\dagger)_p$ in the theorem below should be understood as 'stable and good for p'. It is not very restrictive. In Section 2 we recall all necessary definitions and results about stable sets which we will use.

Theorem 1.1. *For $i = 1, 2$, let K_i be a number field and S_i a set of primes of K_i, such that*

(a) K_1 is totally imaginary and Galois over $\mathbb{Q}$,

(b) for $i = 1, 2$, the set S_i is 2-stable and satisfies $(\dagger)_p$ for almost all p,

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(c) there are two odd rational primes under S_1,

(d) there is an odd rational prime p *with* $S_p \subseteq S_2$ *and* S_i *satisfies* $(\dagger)_p$ *for* $i \in \{1,2\}$.

If $\mathrm{G}_{K_1,S_1} \cong \mathrm{G}_{K_2,S_2}$, *then* $K_1 \cong K_2$.

The important assumptions are that $K_1/\mathbb{Q}$ is Galois and that S_1, S_2 are stable. All other assumptions are of technical nature.

The first step towards a proof of this theorem is the "local correspondence at the boundary", which is very similar to the birational case. Roughly speaking, it is a bijection deduced out of σ, between primes of S_1 and S_2, which respects the residue characteristic and the absolute degree of primes. It is interesting that (in contrast to the original proof of Neukirch) we neither need to know that the decomposition groups of primes in S_i are the full local Galois groups, nor that there is a rational prime p invertible on $\operatorname{Spec}\mathcal{O}_{K,S}$. This last phenomenon can indeed happen for 'nice' stable sets S (but not for sets with density one; cf. [Iv3] Section 3.4).

Now the striking point is the following. In the birational case, the corresponding result of Neukirch follows directly from an established local correspondence at the boundary and an easy application of Chebotarev density theorem. The same would also work in the case of restricted ramification if $\delta_{K_i}(S_i) = 1$. But for general stable sets an analogous application of Chebotarev is impossible, since stable sets can have arbitrarily small (positive) density. Naively, this can be illustrated by the following easy consideration: the Chebotarev set $P_{M/K}(\sigma)$ do not determine the field M uniquely, i.e., there are different finite Galois extensions $M, N/K$ and elements σ, τ in the corresponding Galois groups with $P_{M/K}(\sigma) = P_{N/K}(\tau)$. Thus beside the local correspondence one needs some further arguments to prove the theorem. Those are given in Proposition 4.1 and Proposition 4.4. The first involves stability property once again and the second shows that certain Galois extensions of number fields do not come by base change from smaller Galois extensions. That are these extra arguments, which require the additional technical assumptions in the theorem. It is also not clear to the author how to adapt the Uchida part of the proof of the Neukirch-Uchida theorem [**?**], [Uc3] to the stable case.

Notation

In this paper we use the same notations as in [Iv2]. In particular, for a pro-finite group G we denote by $G(p)$ its maximal pro-p quotient and by G_p a p-Sylow subgroup. For a subgroup $H \subseteq G$, we denote by $\mathrm{N_G(H)}$ its normalizer in G.

For a Galois extension M/L of fields, $\mathrm{G}_{M/L}$ denotes its Galois group. By K we always denote an algebraic number field, that is a finite extension of $\mathbb{Q}$. If L/K is a Galois extension and $\bar{\mathfrak{p}}$ is a prime of L, then $D_{\bar{\mathfrak{p}},L/K} \subseteq \mathrm{G}_{L/K}$ denotes the decomposition subgroup of $\bar{\mathfrak{p}}$. If $\mathfrak{p} := \bar{\mathfrak{p}}|_K$ is the restriction of $\bar{\mathfrak{p}}$ to K, then we sometimes allow us to write $D_{\bar{\mathfrak{p}}}$ or $D_{\mathfrak{p}}$ instead of $D_{\bar{\mathfrak{p}},L/K}$, if no ambiguity can occur. We write Σ_K for the set of all primes of K and S, T will usually denote subsets of Σ_K. If L/K is an extension and S a set of primes of K, then we denote the pull-back of S to L by S_L, $S(L)$ or S (if no ambiguity can occur). We write K_S/K for the maximal extension of K, which is unramified outside S and $\mathrm{G}_S := \mathrm{G}_{K,S}$ for its Galois group. Further, for $p \leqslant \infty$ a (archimedean or non-archimedean) prime of $\mathbb{Q}$, $S_p = S_p(K)$ denotes the set of all primes of K lying over p and $S_f := S \smallsetminus S_\infty$.

An outline of the paper

After recalling necessary definitions and facts about stable sets in Section 2, we will in Section 3 establish the local correspondence at the boundary for a given isomorphism of two Galois groups of the form $\mathrm{G}_{K,S}$ with S stable. This is the first step towards a proof of Theorem 1.1. In Sections 4.1 and 4.2 we give two further arguments needed in its proof. Finally, in Section 4.3 we prove Theorem 1.1.

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2 Stable sets

We briefly recall the concept of stability from [Iv3].

Definition 2.1 (part of [Iv3] Definitions 2.4, 2.7)**.** Let S be a set of primes of K and $\mathscr{L}/K$ any extension.

(i) Let $\lambda > 1$. A finite subextension $\mathscr{L}/L_0/K$ is **λ-stabilizing for S for $\mathscr{L}/K$**, if there exists a subset $S_0 \subseteq S$ and some $a \in (0,1]$, such that $\lambda a > \delta_L(S_0) \geqslant a > 0$ for all finite subextensions $\mathscr{L}/L/L_0$. We say that S is **λ-stable**, if it has a λ-stabilizing extension for $\mathscr{L}/K$. We say that S is **stable for $\mathscr{L}/K$**, if it is λ-stable for $\mathscr{L}/K$ for some $\lambda > 1$. We say that S is (λ-)**stable**, if it is (λ-)stable for K_S/K.

(ii) Let p be a rational prime. We say that $(S, \mathscr{L}/K)$ satisfies $(\dagger)_p^{\mathrm{rel}}$, if $\mu_p \subseteq \mathscr{L}$ and S is p-stable for $\mathscr{L}/K$ or $\mu_p \nsubseteq \mathscr{L}$ and S is stable for $\mathscr{L}(\mu_p)/K$. We say that S satisfies $(\dagger)_p$, if $(S, K_S/K)$ satisfies $(\dagger)_p^{\mathrm{rel}}$.

We will need the two following results about stable sets, which we take from [Iv3].

Theorem 2.2 ([Iv3], Theorem 5.1(A))**.** *Let K be a number field, p a rational prime and $S \supseteq R$ sets of primes of K with R finite. Assume that $(S, K_S^R/K)$ is $(\dagger)_p^{\mathrm{rel}}$. Then*

$$K_{S,\mathfrak{p}}^R \supseteq \begin{cases} K_{\mathfrak{p}}(p), & \text{if } \mathfrak{p} \in S \smallsetminus R \\ K_{\mathfrak{p}}^{\mathrm{nr}}(p) & \text{if } \mathfrak{p} \notin S. \end{cases}$$

Proposition 2.3 ([Iv3], Proposition 5.13(ii))**.** *Let K be a number field, S a set of primes of K. Let p be a rational prime, $r > 0$ an integer. Assume that either p is odd or K_S is totally imaginary. Let $K_S/\mathscr{L}/K$ be a normal subextension. Assume $(S, \mathscr{L}/K)$ is $(\dagger)_p^{\mathrm{rel}}$ and $p^\infty | [\mathscr{L} : K]$. Then*

$$\varinjlim_{\mathscr{L}/L/K} Ш^2(K_S/L; \mathbb{Z}/p^r\mathbb{Z}) = 0.$$

Remarks 2.4.

1) Many results (e.g., such as the two quoted above, but also various Hasse-principles, Grunwald-Wang style results, finite cohomological dimension, etc.) holding for sets with density one also hold (with respect to a prime p) for stable sets of primes (satisfying $(\dagger)_p$). The proofs in the case of sets with density one rely heavily on the fact that various Tate-Shafarevich groups of $\mathrm{G}_{K,S}$ with finite resp. divisible coefficients vanish. This is in general not true for stable sets and the reason why many proofs still work, is that one can, using stability conditions, bound the size of Tate-Shafarevich groups, which in turn implies the vanishing of them in the limit taken over all finite subextensions of certain (infinite) subextensions $K_S/\mathscr{L}/K$.

2) Most natural examples of stable sets are almost Chebotarev sets: if M/K is a finite Galois extension and $\sigma \in \mathrm{G}_{M/K}$, then the associated Chebotarev set is defined as

   $$P_{M/K}(\sigma) := \{\mathfrak{p} \in \Sigma_K : \mathfrak{p} \text{ is unramified in } M/K \text{ and } \mathrm{Frob}_{\mathfrak{p},M/K} \text{ is the conjugacy class of } \sigma\}.$$

   We say that a set is almost Chebotarev, if it differs from a Chebotarev set only by a subset of density zero. They are stable in many cases (cf. [Iv3] Corollary 3.4), and even if not, they inherit many properties of stable sets. Also a stable almost Chebotarev set satisfies $(\dagger)_p$ for almost all rational primes p.

3) If S contains an almost Chebotarev set, then the global realization result Theorem 2.2 holds for S with respect to all rational primes p (i.e., $(K_S)_{\mathfrak{p}} = \overline{K_{\mathfrak{p}}}$), even if S does not satisfy $(\dagger)_p$ for some p's. The proof of this involves further arguments (cf. [Iv4]).

3 Local correspondence at the boundary

Generalizing results of Neukirch, we show that under certain conditions on the set S of primes of K, the decomposition groups of primes in S are intrinsically determined by $G_{K,S}$. Since we in general do not know, whether the decomposition groups are the full local groups, we can not characterize them as absolute Galois groups of local fields and thus we have to deal with p-Sylow subgroups, which are of particularly simple kind.

3.1 Some technical preparations

As in [Iv2] we use the following notational short-cut.

Definition 3.1 ([Iv2] Definition 2.1)**.** A group of *p-decomposition type* is a non-abelian pro-p Demushkin group of rank 2.

Thus a group of p-decomposition type is of the form $\mathbb{Z}_p \ltimes \mathbb{Z}_p$ with $\mathbb{Z}_p \hookrightarrow \mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^*$ injective (this follows from [NSW] 3.9.9, 3.9.11). Such groups have an easy structure theory and one easily describes all their subgroups explicitly (cf. [Iv2] Lemma 2.2).

Lemma 3.2. *Let K be a number field, S a set of primes of K, p a rational prime. Assume S is stable and satisfies $(\dagger)_p$. If $\bar{\mathfrak{p}}$ is a prime of K_S, let $D_{\bar{\mathfrak{p}},p} \subseteq D_{\bar{\mathfrak{p}}}$ denote a p-Sylow subgroup. For*

any $\bar{\mathfrak{p}}_1 \neq \bar{\mathfrak{p}}_2 \in S(K_S)$ *we have inside* $\mathrm{G}_{K,S}$*:*

$$(D_{\bar{\mathfrak{p}}_1,p} \colon D_{\bar{\mathfrak{p}}_1,p} \cap D_{\bar{\mathfrak{p}}_2,p}) = \infty$$

and

$$(D_{\bar{\mathfrak{p}}_1} \colon D_{\bar{\mathfrak{p}}_1} \cap D_{\bar{\mathfrak{p}}_2}) = \infty.$$

Proof. An easy index computation shows that one can go up to a finite extension of K inside K_S. Using this, we can assume that $\bar{\mathfrak{p}}_1|_K \neq \bar{\mathfrak{p}}_2|_K$ and (by [Iv3] Lemma 2.8) that for any subextensions $K_S/\mathscr{L}/L/K$ with L/K finite, the pair $(S, \mathscr{L}/L)$ is $(\dagger)_p^{\mathrm{rel}}$. Now an application of Theorem 2.2 (with $R = \{\bar{\mathfrak{p}}_2|_K\}$) shows that $D_{\bar{\mathfrak{p}}_2}$ lies in the kernel of the projection

$$\mathrm{G}_{K,S} \twoheadrightarrow \mathrm{G}_{K,S}^R,$$

whereas $D_{\bar{\mathfrak{p}}_1,p}$ has infinite image. Thus $(D_{\bar{\mathfrak{p}}_1,p} \colon D_{\bar{\mathfrak{p}}_1,p} \cap D_{\bar{\mathfrak{p}}_2,p}) \geqslant (D_{\bar{\mathfrak{p}}_1,p} \colon D_{\bar{\mathfrak{p}}_1,p} \cap V) = \infty$, where $V := \ker(\mathrm{G}_{K,S} \twoheadrightarrow \mathrm{G}_{K,S}^R)$. The second statement follows from the first. □

Lemma 3.3. *Let* κ *be a* p*-adic field. Let* κ'/κ *be a Galois extension containing the maximal pro-*p*-extension of any finite subfield* $\kappa'/\lambda/\kappa$*. Then* $\mathrm{G}_{\kappa'/\kappa}$ *do not contain any subgroup of* p*-decomposition type.*

Proof. Is the same as for [Iv2] Lemma 3.2. □

Lemma 3.4. *Let* K *be a number field,* S *a set of primes of* K *and* p *a rational prime. Assume* S *is stable and satisfies* $(\dagger)_p$*.*

(i) Let $H_0 \subseteq D_{\bar{\mathfrak{p}}}$ *be a subgroup of* p*-decomposition type, where* $\bar{\mathfrak{p}} \in S_f$*. Then* $\mathrm{N}_{\mathrm{G_{K,S}}}(\mathrm{H}_0) \subseteq \mathrm{D}_{\bar{\mathfrak{p}}}$*.*

(ii) Let H *be a subgroup of* $\mathrm{G}_{K,S}$ *of* p*-decomposition type. Assume there is an open subgroup* $H_0 \subseteq H$ *such that* $H_0 \subseteq D_{\bar{\mathfrak{p}},p}$ *for some prime* $\bar{\mathfrak{p}} \in S_f$*. Then* $H \subseteq D_{\bar{\mathfrak{p}}}$*.*

Proof. (i): Let $D_{\bar{\mathfrak{p}},p} \subseteq D_{\bar{\mathfrak{p}}}$ be a p-Sylow subgroup, containing H_0. First of all, by Lemma 3.3 (which assumptions are satisfied by Theorem 2.2), $\bar{\mathfrak{p}}$ does not lie over p. Consequently, $D_{\bar{\mathfrak{p}},p}$ is also of p-decomposition type (it can not be pro-cyclic, since it already contains H_0), and hence by [Iv2] Lemma 2.2, the inclusion $H_0 \subseteq D_{\bar{\mathfrak{p}},p}$ is open. Let $x \in \mathrm{N}_{\mathrm{G_{K,S}}}(\mathrm{H}_0)$. Then $H_0 = xH_0x^{-1} \subseteq xD_{\bar{\mathfrak{p}}}x^{-1} = D_{x\bar{\mathfrak{p}}}$. Thus $D_{\bar{\mathfrak{p}}} \cap D_{x\bar{\mathfrak{p}}} \supseteq H_0$ contains an open subgroup of $D_{\bar{\mathfrak{p}},p}$. Hence Lemma 3.2 implies $x\bar{\mathfrak{p}} = \bar{\mathfrak{p}}$, i.e., $x \in D_{\bar{\mathfrak{p}}}$.
(ii): Replacing H_0 by the intersection of all its H-conjugates, we can assume that H_0 is normal in H. Since H_0 is also of p-decomposition type, part (i) implies $H \subseteq D_{\bar{\mathfrak{p}}}$. □

3.2 Characterization of decomposition groups

Theorem 3.5. *Let* K *be a number field and* S *a set of primes of* K*. Assume there is a rational prime* p *such that* $S \supseteq S_\infty$ *is stable and satisfies* $(\dagger)_p$*. Let* $H \subseteq \mathrm{G}_{K,S}$ *be of* p*-decomposition type and assume that* H *satisfies the following technical condition: for any intermediate subgroup* $H \subseteq U \subseteq \mathrm{G}_{K,S}$ *with last inclusion open,* $p^\infty | (U : \langle\langle H \rangle\rangle_U)$ *where* $\langle\langle H \rangle\rangle_U$ *denotes the minimal closed normal subgroup of* U*, containing* H*. Then* H *is contained in a decomposition subgroup of a unique prime in* $(S_f \smallsetminus S_p)(K_S)$*.*

Proof. Uniqueness follows from Lemma 3.4 and [Iv2] Lemma 2.2(ii). Let $H \subseteq \mathrm{G}_{K,S}$ be of p-decomposition type. By Lemma 3.4(ii) it is enough to show only that an open subgroup of H is contained in a decomposition group of a prime in $S_f \smallsetminus S_p$. Hence by [Iv3] Lemma 2.8 we can assume that for all subextensions $K_S/\mathscr{L}/K$, the pair $(S, \mathscr{L}/K)$ is $(\dagger)_p^{\mathrm{rel}}$. Further, if $p = 2$, by Theorem 2.2 K_S is totally imaginary, hence we can assume that K is totally imaginary in this case. For any $H \subseteq U \subseteq \mathrm{G}_{K,S}$ with last inclusion open consider the restriction map $\mathrm{H}^2(\mathrm{U}, \mathbb{Z}/\mathrm{p}\mathbb{Z}) \to \bigoplus_{\mathfrak{p} \in \mathrm{S(U)}} \mathrm{H}^2(\mathrm{D}_{\mathfrak{p},\mathrm{K_S/K_S^U}}, \mathbb{Z}/\mathrm{p}\mathbb{Z})$. The main observation is that

$$\varinjlim_{H \subseteq U \subseteq \mathrm{G}_{K,S}} Ш^2(U, \mathbb{Z}/p\mathbb{Z}) = 0,$$

as by our assumption and by Proposition 2.3, for any class $\alpha \in Ш^2(U, \mathbb{Z}/p\mathbb{Z})$ there is a subgroup $U \supseteq V \supseteq \langle\langle H \rangle\rangle_U \supseteq H$ with first inclusion open, such that the image of α in $Ш^2(V, \mathbb{Z}/p\mathbb{Z})$ is zero. Thus passing to the direct limit over all open U containing H we obtain an injection:

$$\mathbb{Z}/p\mathbb{Z} \cong \mathrm{H}^2(\mathrm{H}, \mathbb{Z}/\mathrm{p}\mathbb{Z}) \hookrightarrow \prod_{\mathfrak{p} \in \mathrm{S(M)}} \mathrm{H}^2(\mathrm{D}_{\mathfrak{p},\mathrm{K_S/M}}, \mathbb{Z}/\mathrm{p}\mathbb{Z}).$$

where $M := K_S^H$. Hence there is a prime $\mathfrak{p} \in S(M)$ with $\mathrm{H}^2(\mathrm{D}_{\mathfrak{p},\mathrm{K_S/M}}, \mathbb{Z}/\mathrm{p}\mathbb{Z}) \neq 0$, this prime is non-archimedean, since K is totally imaginary if $p = 2$ and the proof can be finished as in the original paper of Neukirch [Ne] Theorem 1 (also cf. [NSW] 12.1.9). □

Corollary 3.6. *Let K be a number field and S a set of primes of K. Assume there is a rational prime p such that the following hold:*

(i) $S \supseteq S_\infty$ is stable and satisfies $(\dagger)_p$

(ii) for each $\mathfrak{p} \in S_f$, we have $\mu_p \subseteq K_{S,\mathfrak{p}}$.

Then the group $G_{K,S}$ given together with the prime p determines intrinsically the decomposition subgroups of primes in $S_f \smallsetminus S_p$.

Proof. The proof works exactly as in [Iv2] Section 3.4. For convenience we repeat it here (except for some technical details). First of all, since $\mu_p \subset K_{S,\mathfrak{p}}$ for any $\mathfrak{p} \in S_f$, it follows from Theorem 2.2 that for any $\bar{\mathfrak{p}} \in (S_f \smallsetminus S_p)(K_S)$, the composition

$$\mathscr{G}_{\mathfrak{p},p} \hookrightarrow \mathscr{G}_{\mathfrak{p}} \twoheadrightarrow D_{\bar{\mathfrak{p}}} \hookrightarrow \mathrm{G}_{K,S}$$

is injective, or with other words, the p-Sylow subgroup of $D_{\bar{\mathfrak{p}}}$ is of p-decomposition type. For any $U \subseteq \mathrm{G}_{K,S}$ open (and small enough) we claim the equality of the following subsets of the set of all subgroups of p-decomposition type inside U:

$$\mathrm{Syl}_p(U, S_f \smallsetminus S_p) = \left\{ H \subseteq U : \begin{array}{c} H \text{ is a subgroup of } p\text{-decomposition type satisfying} \\ \text{the technical condition in Theorem 3.5} \\ \text{and maximal of this type} \end{array} \right\},$$

where $\mathrm{Syl}_p(U, S_f \smallsetminus S_p)$ denotes the set of all p-Sylow subgroups of decomposition subgroups of primes in $S_f \smallsetminus S_p$ of the field K_S^U. Indeed, Theorem 3.5 assures that any group in the right set is contained in a decomposition group of a prime in $S_f \smallsetminus S_p$, and by maximality it has to be

a p-Sylow subgroup. Conversely, any group H lying in the left set is of p-decomposition type, satisfies the technical property from Theorem 3.5 (by Lemma 3.2) and is maximal with these properties. Indeed, to prove maximality assume $H \subseteq H'$ with H' of p-decomposition type. This inclusion has to be open by [Iv2] Lemma 2.2(ii) and thus by Lemma 3.4(ii), H' is contained in the same decomposition group as H. Since both are pro-p-groups and H is a p-Sylow subgroup, we get $H = H'$, proving the maximality of H.

Thus the data $(\mathrm{G}_{K,S}, p)$ determine intrinsically the set $\mathrm{Syl}_p(U, S_f \smallsetminus S_p)$ for any $U \subseteq \mathrm{G}_{K,S}$ open. U acts on this set by conjugation. We have an U-equivariant surjection

$$\psi\colon \mathrm{Syl}_p(U, S_f \smallsetminus S_p) \twoheadrightarrow (S_f \smallsetminus S_p)(U)$$

(U acts trivially on the right side), which sends H to the prime $\bar{\mathfrak{p}}|_L$, such that $H \subseteq D_{\bar{\mathfrak{p}},K_S/L}$, where $L := K_S^U$. By [Iv2] Lemma 3.9 we have a purely group theoretical criterion for two elements on the left side to lie in the same fiber of this surjection, which allows us to reconstruct the set $(S_f \smallsetminus S_p)(U)$. For any inclusion $V \hookrightarrow U$ of open subgroups of $\mathrm{G}_{K,S}$, we have (a priori non-canonical) maps $\mathrm{Syl}_p(V, S_f \smallsetminus S_p) \to \mathrm{Syl}_p(U, S_f \smallsetminus S_p)$, and via ψ they induce the restriction-of-primes maps $(S_f \smallsetminus S_p)(V) \to (S_f \smallsetminus S_p)(U)$. Finally, if $U \subseteq \mathrm{G}_{K,S}$ is normal, the $\mathrm{G}_{K,S}$-action by conjugation on $\mathrm{Syl}_p(U, S_f \smallsetminus S_p)$ induces via ψ the natural $\mathrm{G}_{K,S}$-action on $(S_f \smallsetminus S_p)(U)$ by permuting the primes. In this way we have reconstructed the projective system of $\mathrm{G}_{K,S}$-sets $\{(S_f \smallsetminus S_p)(U)\colon U \subseteq U_0, U \trianglelefteq \mathrm{G}_{K,S}\}$, where $U_0 \subseteq \mathrm{G}_{K,S}$ is some open subgroup. Now the decomposition subgroups of primes in $S_f \smallsetminus S_p$ are exactly the stabilizers in $\mathrm{G}_{K,S}$ of elements in the $\mathrm{G}_{K,S}$-set $\varprojlim_{U \subseteq U_0, U \trianglelefteq \mathrm{G}_{K,S}} (S_f \smallsetminus S_p)(U)$. □

Remark 3.7.

(i) It is remarkable that Theorem 3.5 and Corollary 3.6 hold even if $\mathbb{N}(S) = \{1\}$, i.e, if there is no rational prime invertible on $\operatorname{Spec} \mathcal{O}_{K,S}$. Indeed, there are examples of stable sets with arbitrarily small density satisfying $(\dagger)_p$ for all p and $\mathbb{N}(S) = \{1\}$ ([Iv3] Section 3.4)

(ii) Observe that $(\dagger)_2$ is equivalent to being 2-stable. In particular, the assumptions of Corollary 3.6 in the case $p = 2$ reduce to 'S is 2-stable and $S \supseteq S_\infty$'.

3.3 Local correspondence at the boundary

Definition 3.8. For $i = 1, 2$, let K_i be a number field, S_i a set of primes of K_i and let

$$\sigma\colon \mathrm{G}_{K_1,S_1} \xrightarrow{\sim} \mathrm{G}_{K_2,S_2}$$

be a (topological) isomorphism. If U_1 is a closed subgroup of G_{K_1,S_1} with fixed field L_1, we write U_2 for $\sigma(U_1)$ and L_2 for its fixed field, etc. We say that the **local correspondence at the boundary** holds for σ, if the following conditions are satisfied:

(i) For $i = 1, 2$, there is a finite exceptional set $S_i^{\mathrm{ex}} \subseteq S_i$, such that for any $\bar{\mathfrak{p}}_1 \in (S_{1,f} \smallsetminus S_1^{\mathrm{ex}})(K_{1,S_1})$, there is a unique prime $\sigma_*(\bar{\mathfrak{p}}_1) \in (S_{2,f} \smallsetminus S_2^{\mathrm{ex}})(K_{2,S_2})$, with $\sigma(D_{\bar{\mathfrak{p}}_1}) = D_{\sigma_*(\bar{\mathfrak{p}}_1)}$, such that σ induces a bijection

$$\sigma_*\colon (S_{1,f} \smallsetminus S_1^{\mathrm{ex}})(K_{1,S_1}) \xrightarrow{\sim} (S_{2,f} \smallsetminus S_2^{\mathrm{ex}})(K_{2,S_2})$$

which is Galois-equivariant, i.e.,

$$\sigma_*(g\bar{\mathfrak{p}}_1) = \sigma(g)\sigma_*(\bar{\mathfrak{p}}_1)$$

for each $g \in \mathrm{G}_{K_1,S_1}$ and $\bar{\mathfrak{p}}_1 \in S_{1,f}(K_{1,S_1})$. In particular, for any finite subextension L_1 of $K_{1,S_1}/K_1$ with corresponding open subgroup $U_1 \subseteq \mathrm{G}_{K_1,S_1}$, if two primes $\bar{\mathfrak{p}}_1, \bar{\mathfrak{q}}_1 \in S_{1,f}(K_{1,S_1})$ restrict to the same prime of L_1, then also $\sigma_*(\bar{\mathfrak{p}}_1), \sigma_*(\bar{\mathfrak{q}}_1)$ restrict to the same prime of L_2, and hence σ_* induces a bijection

$$\sigma_{*,U_1} \colon (S_{1,f} \smallsetminus S_1^{\mathrm{ex}})(L_1) \xrightarrow{\sim} (S_{2,f} \smallsetminus S_2^{\mathrm{ex}})(L_2).$$

(ii) For all $K_{1,S_1}/L_1/K_1$ finite with corresponding subgroup $U_1 \subseteq \mathrm{G}_{K_1,S_1}$ and for all but finitely many primes $\mathfrak{p}_1 \in (S_{1,f} \smallsetminus S_1^{\mathrm{ex}})(L_1)$, the residue characteristics and the local degrees of $\mathfrak{p}_1$ and $\sigma_{*,U_1}(\mathfrak{p}_1)$ are equal.

Corollary 3.9. *For $i = 1, 2$, let K_i be a number field and S_i a stable set of primes. Assume that K_{i,S_i} is totally imaginary and that S_i satisfies $(\dagger)_p$ for almost all rational primes p and in particular for $p = 2$. Let*

$$\sigma \colon \mathrm{G}_{K_1,S_1} \xrightarrow{\sim} \mathrm{G}_{K_2,S_2}$$

be an isomorphism. Then the local correspondence at the boundary holds for σ and moreover, one can choose S_i^{ex} to be the set of 2-adic primes in S_i. More precisely, for any $U_1 \subseteq \mathrm{G}_{K_1,S_1}$, σ_{*,U_1} preserves the residue characteristic and the absolute degree of all primes $\mathfrak{p} \in S_1(U_1)$, whose residue characteristic ℓ is odd and such that S_i satisfies $(\dagger)_\ell$ for $i = 1, 2$.

Proof. To avoid notational problems, let us exceptionally denote by $S_{2\text{-adic}}(L)$ the set of 2-adic primes of a number field L. We apply Corollary 3.6 to $(K_i, S_i, p = 2)$ for $i = 1, 2$. It shows that σ maps decomposition groups of primes in $S_{1,f} \smallsetminus S_{2-adic}$ to decomposition groups of primes in $S_{2,f} \smallsetminus S_{2-adic}$. Thus we can define $\sigma_*(\bar{\mathfrak{p}}_1)$ by the equality

$$D_{\sigma_*(\bar{\mathfrak{p}}_1)} = \sigma(D_{\bar{\mathfrak{p}}_1}).$$

The Galois-equivariance of σ_* is straightforward. It remains to show that for any finite subextension $K_{1,S_1}/L_1/K_1$ with corresponding open subgroup U_1 and for any $\mathfrak{p}_1 \in S_{1,f}(L_1)$ with residue characteristic ℓ, which is odd and such that S_i satisfies $(\dagger)_\ell$ for $i = 1, 2$, the map σ_{*,U_1} preserves the residue characteristic and the local absolute degree. For any such ℓ and any $\bar{\mathfrak{p}} \in S_{1,f}(K_{1,S_1})$ with residue characteristic ℓ, Theorem 2.2 implies that the maximal ℓ-extension of $K_{i,\mathfrak{p}}$ is realized by K_{i,S_i}. Let $\mathfrak{p} \in S_i(L_i)$ be a prime with residue characteristic ℓ and $\bar{\mathfrak{p}}$ an extension to K_{i,S_i}. Lemma 3.10 shows that $D_{\bar{\mathfrak{p}},K_S/L_i}$ encodes the information about the residue characteristic and the absolute degree of $\mathfrak{p}$. Thus σ_{*,L_1} preserves the residue characteristic and the absolute degree of all primes in $S_{1,f}$ with residue characteristic ℓ being odd and such that S_i satisfies $(\dagger)_\ell$ for $i = 1, 2$. □

Lemma 3.10. *Let κ be a local field with characteristic zero and some residue characteristic ℓ and λ/κ a Galois extension with Galois group D, which contains the maximal pro-ℓ-extension of κ. Then D encodes the information about ℓ and $[\kappa : \mathbb{Q}_\ell]$.*

Proof. Let $\mathscr{G}_\kappa$ be the absolute Galois group of κ. We have a surjection $\pi\colon \mathscr{G}_\kappa \twoheadrightarrow D$, and for any open $U \subseteq \mathscr{G}_\kappa$ with $\mathscr{G}_\kappa/U$ an ℓ-group, surjections $U \twoheadrightarrow \pi(U) \twoheadrightarrow U^{(\ell)}$, which for any rational prime p induce injections

$$\mathrm{H}^1(\mathrm{U}^{(\ell)}, \mathbb{Z}/\mathrm{p}\mathbb{Z}) \hookrightarrow \mathrm{H}^1(\pi(\mathrm{U}), \mathbb{Z}/\mathrm{p}\mathbb{Z}) \hookrightarrow \mathrm{H}^1(\mathrm{U}, \mathbb{Z}/\mathrm{p}\mathbb{Z}).$$

For all primes $p \neq \ell$, the $\mathbb{F}_p$-dimension of the space on the right (and hence also in the middle) is bounded by 2, and for $p = \ell$, the dimension of the space on the left gets arbitrary big, if U gets arbitrary small among all subgroups $U \subseteq \mathscr{G}_\kappa$ such that $\mathscr{G}_\kappa/U$ is an ℓ-group. Thus the residue characteristic ℓ is equal to the unique prime p, such that $\dim_{\mathbb{F}_p} \mathrm{H}^1(\mathrm{V}, \mathbb{Z}/\mathrm{p}\mathbb{Z})$ is unbounded as V varies over all open subgroups of D, such that D/V is a p-group. Further,

$$[\kappa : \mathbb{Q}_\ell] = \chi_\ell(\mathscr{G}_\kappa(\ell), \mathbb{Z}/\ell\mathbb{Z}) = \chi_\ell(D(\ell), \mathbb{Z}/\ell\mathbb{Z}). \qquad \square$$

Remark 3.11. The proofs (of Theorem 3.5, Corollary 3.6 and Corollary 3.9) would be less technical if one would assume the stronger condition: 'K_S realize the maximal local extension at each $\mathfrak{p} \in S_f$' on the involved stable sets S. It is satisfied in the following cases.

(i) If S is defined over a totally real subfield, (not necessarily stable), and $S \supseteq S_\infty \cup S_{p_1} \cup S_{p_2}$ for two different rational primes p_1, p_2 (by [CC] Remark 5.3(i)).

(ii) If S is stable and satisfies $(\dagger)_p$ for all rational p's (by [Iv3]).

(iii) If S contains an almost Chebotarev set (by [Iv4]).

We conjecture that it is true in general if S is stable.

4 Anabelian geometry of curves $\operatorname{Spec} \mathcal{O}_{K,S}$ with S stable

4.1 Uniform bound

Besides the local correspondence on the boundary, the following argument plays a central role in the proof of Theorem 1.1. From now on, we consider all occurring fields to be subfields of a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$.

Proposition 4.1 (Uniform bound)**.** *For $i = 1, 2$, let K_i be a number field, S_i a set of primes of K_i and let*

$$\sigma\colon \mathrm{G}_{K_1,S_1} \xrightarrow{\sim} \mathrm{G}_{K_2,S_2}$$

be an isomorphism. Assume that the local correspondence at the boundary holds. Assume that S_1 is stable. Then there is some $N > 0$, such that for all (not necessarily finite) intermediate subfields $K_{1,S_1}/M_1/K_1$, such that M_1 is normal over $\mathbb{Q}$, one has $[M_1 : M_1 \cap M_2] < N$, where M_2/K_2 corresponds to M_1/K_1 via σ.

Lemma 4.2. *Let κ be a field. If $(V_i)_{i\in I}$ is a cofiltered system of κ-vector spaces, such that $\dim_\kappa V_i < n$, and $V := \varinjlim_I V_i$, then $\dim_\kappa V < n$.*

Proof of Lemma 4.2. For any n vectors in V there is an $i \in I$, such that these vectors has preimages in V_i. These preimages are linearly dependent. Hence their images in V are linearly dependent. □

Proof of Proposition 4.1. Since S_1 is stable, by [Iv3] Proposition 2.6, there is some $N > 0$, such that $\delta_{L_1}(S_1) > N^{-1}$ for all finite subfields $K_{1,S_1}/L_1/K_1$. Let M_1 be a subextension of $K_{1,S_1}/K_1$, such that $M_1/\mathbb{Q}$ is normal. By Lemma 4.2 and since M_1 is a union of finite extensions of K_1, which are normal over $\mathbb{Q}$, we can assume that M_1/K_1 finite. Let

$$S_1' := S_1(M_1) \cap \mathrm{cs}(M_1/\mathbb{Q})(M_1).$$

Since $M_1/\mathbb{Q}$ is normal, $\delta_{M_1}(\mathrm{cs}(M_1/\mathbb{Q})(M_1)) = 1$ and hence

$$\delta_{M_1}(S_1') = \delta_{M_1}(S_1) > N^{-1}.$$

Lemma 4.3. *Let $S_2' := \sigma_*(S_1')$. Then*

(i) $\delta_{M_2}(S_2') = \delta_{M_1}(S_1')$.

(ii) $S_2' \overset{\subseteq}{\sim} \mathrm{cs}(M_1M_2/M_2)$.

Proof of Lemma 4.3. (i) follows from the local correspondence at the boundary by explicitly computing the density, since σ_* preserves the residue characteristic and the absolute degree of almost all primes in S_1'.

(ii): Let $\mathfrak{p}_1 \in S_1'$ be such that σ_* preserves the residue characteristic and the absolute degree of $\mathfrak{p}_1$. Let $\mathfrak{p}_2 := \sigma_*(\mathfrak{p}_1) \in S_2'$ and $\mathfrak{p} := \mathfrak{p}_2|_{M_1\cap M_2}$. The fiber $\mathcal{O}_{M_1M_2} \otimes_{\mathcal{O}_{M_2}} \kappa(\mathfrak{p}_2)$ over $\mathfrak{p}_2$ in $\operatorname{Spec} \mathcal{O}_{M_1M_2}$ is isomorphic to $(\mathcal{O}_{M_1} \otimes_{\mathcal{O}_{M_1\cap M_2}} \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}_2)$. By assumption, we have $\mathfrak{p}_2|_{\mathbb{Q}} = \mathfrak{p}_1|_{\mathbb{Q}} \in \mathrm{cs}(K_1/\mathbb{Q})$ and hence $\mathfrak{p} \in \mathrm{cs}(M_1/\mathbb{Q})(M_1 \cap M_2) \subseteq \mathrm{cs}(M_1/M_1 \cap M_2)$. This implies that $\mathcal{O}_{M_1} \otimes_{\mathcal{O}_{M_1\cap M_2}} \kappa(\mathfrak{p})$ is isomorphic to a product of copies of $\kappa(\mathfrak{p})$. Thus we obtain

$$\mathcal{O}_{M_1M_2} \otimes_{\mathcal{O}_{M_1}} \kappa(\mathfrak{p}_2) \cong \prod \kappa(\mathfrak{p}_2),$$

i.e., $\mathfrak{p}_2$ is completely decomposed in M_1M_2. □

Using Lemma 4.3 and the normality of M_1M_2/M_2, we obtain:

$$\begin{aligned} [M_1 : M_1 \cap M_2]^{-1} &= [M_1M_2 : M_2]^{-1} \\ &= \delta_{M_2}(\mathrm{cs}(M_1M_2/M_2)) \\ &\geqslant \delta_{M_2}(S_2') \\ &= \delta_{M_1}(S_1') \\ &> N^{-1}. \end{aligned}$$

This proves Proposition 4.1. □

4.2 Non-existence of lifts

Last but not least, Proposition 4.4 proven in this section provides the last argument which we need in the proof of Theorem 1.1. Let L/K be a Galois extension of global fields. We want

to study, under which conditions there is no Galois extension L_0/K_0, such that L/K is a base change of L_0/K_0, i.e., $K_0 = K \cap L_0$ and $L = KL_0$.

Proposition 4.4. *Let K, L_0 be two linearly disjoint Galois extensions of a global field K_0, and set $L = KL_0$. Assume one of the following holds:*

(a) – *K is a totally imaginary number field and*
– *$L = K_{S_p}(p)$ for some prime number p, or*

(b) There is a prime $\mathfrak{p}$ of K_0, which is completely split in K, such that for any $\bar{\mathfrak{p}}_1, \bar{\mathfrak{p}}_2 \in S_{\mathfrak{p}}(L)$ with $\bar{\mathfrak{p}}_1|_K \neq \bar{\mathfrak{p}}_2|_K$, we have $D_{\bar{\mathfrak{p}}_1, L/K} \neq D_{\bar{\mathfrak{p}}_2, L/K}$.

Then $K = K_0$.

We will only use part (a) of this proposition.

Proof. Assume (a) holds. Then L/K and L_0/K_0 are both Galois with Galois group isomorphic to $\mathrm{G}_{K,S_p}(p)$. By [NSW] 10.3.20, the number of independent $\mathbb{Z}_p$-extensions of K satisfies

$$\mathrm{rk}_{\mathbb{Z}_\mathrm{p}}\, \mathrm{G}^{\mathrm{ab}}_{\mathrm{K},\mathrm{S_p}}(\mathrm{p}) \geqslant \mathrm{r}_2(\mathrm{K}) + 1.$$

Since $\mathrm{G}_{L/K} \cong \mathrm{G}_{L_0/K_0}$, the field K_0 has at least $r_2(K) + 1$ independent $\mathbb{Z}_p$-extensions. Assume $K \neq K_0$. Then $[K : K_0] \geqslant 2$, and since K is totally imaginary, we obtain:

$$r_2(K) + 1 = \frac{[K : \mathbb{Q}]}{2} + 1 \geqslant [K_0 : \mathbb{Q}] + 1 > [K_0 : \mathbb{Q}].$$

But by [NSW] 10.3.20, the number of independent $\mathbb{Z}_p$-extensions of K_0 is $\leqslant [K_0 : \mathbb{Q}]$. This is a contradiction, hence $K = K_0$ (notice that we nowhere made use of Leopoldt's conjecture!).

Assume (b) holds. Let $\psi\colon \mathrm{G}_{L/K} \xrightarrow{\sim} \mathrm{G}_{L_0/K_0}$ denote the canonical isomorphism. Assume there are two different primes $\mathfrak{p}_1 \neq \mathfrak{p}_2$ in K over $\mathfrak{p}$. Let $\mathfrak{q}$ be some prime of L_0 over $\mathfrak{p}$. One can chose primes $\bar{\mathfrak{p}}_i \in S_{\mathfrak{p}}(L)$, such that $\bar{\mathfrak{p}}_i|_K = \mathfrak{p}_i$ and $\bar{\mathfrak{p}}_i|_{L_0} = \mathfrak{q}$. As $\mathfrak{p}_1, \mathfrak{p}_2$ are split over K_0, we obtain that ψ maps $D_{\bar{\mathfrak{p}}_i, L/K}$ isomorphically to $D_{\mathfrak{q}, L_0/K_0}$. But by assumption $D_{\bar{\mathfrak{p}}_1, L/K} \neq D_{\bar{\mathfrak{p}}_2, L/K}$, hence $D_{\mathfrak{q}, L_0/K_0} = \psi(D_{\bar{\mathfrak{p}}_1, L/K}) \neq \psi(D_{\bar{\mathfrak{p}}_2, L/K}) = D_{\mathfrak{q}, L_0/K_0}$, which is a contradiction. Thus there is only one prime over $\mathfrak{p}$ in K, and since $\mathfrak{p}$ is completely split, we obtain $[K : K_0] = 1$. □

4.3 Proof of Theorem 1.1

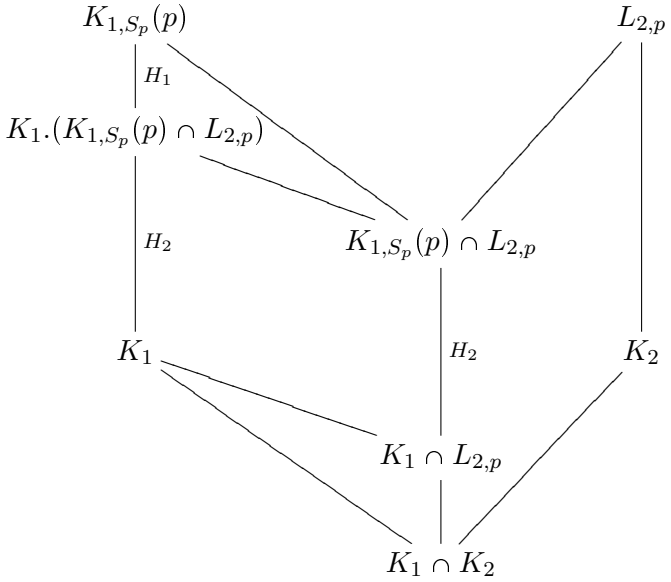
By assumption (b) in Theorem 1.1 and Corollary 3.9 the local correspondence at the boundary holds for σ: for any open subgroup $U_1 \subseteq \mathrm{G}_{K_1,S_1}$ with fixed field L_1, σ induces a functorial bijection

$$\sigma^*_{U_1}\colon (S_{1,f} \smallsetminus S_{2-adic})(L_1) \xrightarrow{\sim} (S_{2,f} \smallsetminus S_{2-adic})(L_2),$$

which preserves the residue characteristic and the absolute degree of all primes in $(S_{1,f} \smallsetminus (S_{2-adic} \cup T))(L_1)$, with $T := \{\ell\colon S_1 \text{ or } S_2 \text{ is not } (\dagger)_\ell\}$. We obtain $[K_2 : \mathbb{Q}] \leqslant [K_1 : \mathbb{Q}]$ from this. Indeed, by assumption (d) there is an odd prime p with $S_p \subseteq S_2$ and $p \notin T$, and hence σ_* preserves the residue characteristic and the absolute degree of primes in $S_p(K_1) \cap S_1$ by Corollary 3.9. Hence $\sigma_*(S_1 \cap S_p(K_1)) = S_p(K_2)$ and

$$
\begin{aligned}
[K_2:\mathbb{Q}] = \dim_{\mathbb{Q}_p} K_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p &= \sum_{\mathfrak{p}\in S_p(K_2)} [K_{2,\mathfrak{p}}:\mathbb{Q}_p] = \sum_{\mathfrak{p}\in (S_p\cap S_1)(K_1)} [K_{1,\sigma_{*,K_1}^{-1}(\mathfrak{p})}:\mathbb{Q}_p] \\
&\leqslant \sum_{\mathfrak{p}\in S_p(K_1)} [K_{1,\sigma_{*,K_1}^{-1}(\mathfrak{p})}:\mathbb{Q}_p] = [K_1:\mathbb{Q}].
\end{aligned} \tag{4.1}
$$

By (c) we have two rational primes p_1, p_2, such that $S_{p_j} \subseteq S_1$, $p_j > 2$. Let $p \in \{p_1, p_2\}$. The quotient $\mathrm{G}_{K_1,S_p}(p)$ of G_{K_1,S_1} is torsion-free (cf. [NSW] 8.3.18 and 10.4.8). Since K_1 is normal over $\mathbb{Q}$, S_p is defined over $\mathbb{Q}$ and the maximal pro-p-quotient of a profinite group is characteristic, we deduce that the field $K_{1,S_p}(p)$ is normal over $\mathbb{Q}$. Let $L_{2,p}$ be the field corresponding to $K_{1,S_p}(p)$ via σ (a priori, $L_{2,p}$ must not be equal $K_{2,S_p}(p)$). We have the following situation:



In this diagram the group H_1 is a subgroup of $\mathrm{G}_{K_{1,S_p}(p)/K_{1,S_p}(p)\cap L_{2,p}}$ and of $\mathrm{G}_{K_1,S_p}(p)$. But the first of these two groups is finite by Proposition 4.1 and the second is torsion-free. Hence $H_1 = 1$, i.e., $H_2 = \mathrm{G}_{K_1,S_p}(p)$. By Proposition 4.4(a) we get $K_1 = K_1 \cap L_{2,p}$, i.e., $K_1 \subseteq L_{2,p}$. Doing this for $p = p_1, p_2$, we get: $K_1 \subseteq L_{2,p_1} \cap L_{2,p_2} = K_2$, the last equality being true, since $L_{2,p_j}/K_2$ is a pro-p_j-extension for $j = 1, 2$. By (4.1) we conclude that $K_1 = K_2$. □

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