DEEP LEVEL DELIGNE–LUSZTIG REPRESENTATIONS OF COXETER TYPE

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Abstract. In this article we study the cohomology of deep level Deligne–Lusztig varieties of Coxeter type, attached to a reductive group over a local non-archimedean field, which splits over an unramified extension. This allows to construct some new irreducible representations of parahoric subgroups of p-adic groups. Moreover, in the quasi-split case we prove that these compactly induce to finite direct sums of irreducible supercuspidal representations of the p-adic group. This extends previous results of [DI24], [CI23].

1. Introduction

Let k be a non-Archimedean local field with residue characteristic $p > 0$, integers $\mathcal{O}_k$, uniformizer ϖ and residue field $\mathbb{F}_q$. Let $\breve{k}$ be the completion of the maximal unramified extension of k, let $\mathcal{O}_{\breve{k}}$ denote the integers of $\breve{k}$. Let F denote the Frobenius automorphism of $\breve{k}$ over k.

Let G be a reductive group over k, which splits over $\breve{k}$. Let $T \subseteq B$ be a maximal torus and a Borel subgroup of G, such that T splits and B becomes rational over $\breve{k}$. Denote by W the Weyl group of T in G. Denote by U resp. U^- the unipotent radicals of B resp. the opposite Borel subgroup and assume that (T, U) is a Coxeter pair (see §2.1). Attached to (T, U) there is a p-adic Deligne–Lusztig space, on geometric points given by

$$X_{T,U} = \{g \in G(\breve{k}) \colon g^{-1}F(g) \in (U^- \cap FU)(\breve{k})\}.$$

It admits a continuous action of $G(k) \times T(k)$ given by $(g, t) \colon x \mapsto gxt$. See [Iva23a, §7 and §11] (and §5.2 below). Let $\theta \colon T(k) \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character. The θ-isotypic component $R_T^G(\theta)$ of the homology of $X_{T,U}$ is an object in the (derived) category of smooth $G(k)$-representations, cf. [IM]. The goal of this article is to further investigate properties of $R_T^G(\theta)$, extending and generalizing results from [DI24, CI23].

To describe our results we need more notation. The apartment of T in the reduced Bruhat–Tits building of G consists of one point. Bruhat–Tits theory attaches to this point a (connected) parahoric $\mathcal{O}_k$-model $\mathcal{G}$ of G. By [Iva23b, Nie24], $X_{T,U}$ admits a decomposition $X_{T,U} = \coprod_{\gamma \in G(k)/\mathcal{G}(\mathcal{O}_k)} \gamma X_{T,U}^{\mathcal{G}}$, where

$$X_{T,U}^{\mathcal{G}} = \{g \in \mathcal{G}(\mathcal{O}_{\breve{k}}) \colon g^{-1}F(g) \in (\mathcal{U}^- \cap F\mathcal{U})(\mathcal{O}_{\breve{k}})\}$$

is an affine $\overline{\mathbb{F}}_q$-scheme (here $\mathcal{U}$ denotes the closure of U in $\mathcal{G}$).

Fix some $r \leq \infty$. We can regard $\mathcal{G}(\mathcal{O}_{\breve{k}}/\varpi^r) = \mathbb{G}(\overline{\mathbb{F}}_q)$ as the geometric points of a perfect $\mathbb{F}_q$-scheme $\mathbb{G} = \mathbb{G}_r$. This is done via the (truncated, if $r < \infty$) positive loop functor, see e.g. [Zhu17, §1.1] (or [DI24, §2]) for details. For a subscheme $H \subseteq G$, we denote by $\mathcal{H}$ its closure in $\mathcal{G}$ and by $\mathbb{H} \subseteq \mathbb{G}$ the corresponding subscheme of $\mathbb{G}$. We denote by F the geometric Frobenius of $\mathbb{G}$, so that $\mathbb{G}^F = \mathbb{G}(\mathbb{F}_q)$. Then $X_{T,U}^{\mathcal{G}}$ is isomorphic to the inverse limit over r of its truncations in each $\mathbb{G}_r$. Each of these truncations is a perfectly smooth perfect $\overline{\mathbb{F}}_q$-scheme, and up to an $\mathbb{A}^n$-bundle (not affecting the cohomology), it equals

$$X = X_{T,U,r}^{\mathcal{G}} = \{x \in \mathbb{G} \colon x^{-1}F(x) \in F\mathbb{U}\}, \tag{1.1}$$

Note that X is equipped with the action of the finite group $\mathbb{G}^F \times \mathbb{T}^F$ given by $(g,t)\colon x \mapsto gxt$.

By these geometric considerations $(+\varepsilon)$, $R_T^G(\theta)$ admits the following more explicit description (which might, for the purposes of this article, also be considered as a definition). Let $Z \subseteq G$ denote the center of G. For any $\mathbb{T}^F$-module M, let $M[\chi]$ denote the χ-isotypic subspace. Then, if $\theta|_{\mathcal{T}(\mathcal{O}_k)}$ factors through a character χ of $\mathbb{T}^F$, then

$$R_T^G(\theta) = \mathrm{cInd}_{\mathcal{G}(\mathcal{O}_k)Z(k)}^{G(k)} H_c^*(X)[\chi],$$

where $H_c^*(X)[\theta] = \sum_{i\in\mathbb{Z}} H_c^i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ is the ℓ-adic equivariant Euler characteristic of X (regarded as a virtual $\mathbb{G}^F$-module), inflated to a $\mathcal{G}(\mathcal{O}_k)$-representation, and extended to $\mathcal{G}(\mathcal{O}_k)Z(k)$ in the unique way such that $Z(k)$ acts by $\theta|_{Z(k)}$. Our first main result concerns the representations in the cohomology of X.

Theorem 1.1. *Suppose that q satisfies condition* (2.1) *(this is always true when $q > 5$). Then there exists a Coxeter pair (T,U) such that*

$$\dim_{\overline{\mathbb{Q}}_\ell} \mathrm{Hom}_{\mathbb{G}^F}(H_c^*(X)[\chi], H_c^*(X)[\chi']) = \sharp\left\{w \in W_e^F; w(\chi) = \chi'\right\}$$

for any two smooth characters χ, χ' of $\mathbb{T}^F$, where W_e denotes the Weyl group of the special fiber of $\mathcal{T}$ in the reductive quotient of the special fiber of $\mathcal{G}$.

In particular, if $\{w \in W_e^F \colon w(\chi) = \chi\} = \{1\}$, then $H_c^*(X)[\chi]$ is up to sign an irreducible $\mathbb{G}^F$-representation. Note that Theorem 1.1 generalizes [DI24, Theorem 3.2.3] and [CI23, Theorem 4.1].

Remark 1.2. Recently, under a mild condition on p, Chan [Cha24] shows by a different approach that the inner product formula holds in a much more general case, which in particular includes the case that T is elliptic.

Our second main result concerns the cuspidality of the compactly induced $G(k)$-representation $R_T^G(\theta)$. It generalizes [CI23, Theorem 6.1].

Theorem 1.3. *Assume that G is unramified and that q satisfies condition* (2.1)*. Let $\theta\colon T(k) \to \overline{\mathbb{Q}}_\ell^\times$ be smooth with trivial stabilizer in W^F. Then $R_T^G(\theta)$ is up to sign a finite direct sum of irreducible supercuspidal representations of $G(k)$.*

Some comments on our results are in order. First, we explain why "it suffices" to establish Theorem 1.1 for a single Coxeter pair (T, U). Ultimately, we are interested in the smooth $G(k)$-representation $R_T^G(\theta)$. By [Iva23a, Corollary 7.25, Lemma 11.3], $X_{T,U}$ are mutually $G(k) \times T(k)$-equivariantly isomorphic, when (T, U) varies through all Coxeter pairs (T, U) with a fixed T.[1] Thus, $R_T^G(\theta)$ is independent of the choice of U. So, it suffices to know the statement of Theorem 1.1 for at least one Coxeter pair. In fact, our proof shows that for many groups G Theorem 1.1 holds for all pairs (T, U), see Remark 2.4.

Next, we explain why the condition on q in Theorems 1.1 and 1.3 is very mild, so that the theorems even gives rise to new supercuspidal representations of $G(k)$. Recall that by the work of Yu and Kaletha [Yu01, Kal19], one can attach a supercuspidal irreducible $G(k)$-representation $\pi_{(S,\theta)}$ to any regular elliptic pair (S, θ) consisting of a maximal elliptic torus $S \subseteq G$ and a sufficiently nice smooth character $\theta\colon S(k) \to \overline{\mathbb{Q}}_\ell^\times$. A crucial point for this to work is the existence of a Howe factorization of θ, cf. [Kal19, §3.6]. However, not all characters admit a Howe factorization, when the residue characteristic p is small and G is not an inner form of GL_n.

For instance, if $p \in \{2, 3, 5\}$, there exist many examples of pairs (T, θ) with T unramified Coxeter (hence covered by Theorem 1.1 when q satisfied condition (2.1) – in particular, whenever $q > 5$) such that $\mathrm{Stab}_{W_e^F}(\theta) = \{1\}$, but θ does not admit a Howe factorization. For examples of (T, θ) not admitting a Howe factorization we refer to the forthcoming work of Fintzen–Schwein [FS], where an algebraic approach to the extension of Yu's construction is pursued. As mentioned in [CO23], since $\mathrm{Stab}_{W_e^F}(\theta) = \{1\}$ one should expect an irreducible supercuspidal $G(k)$-representation attached to (T, θ), but Yu's construction does not apply as there is no Howe factorization. The point is now that our cohomological construction does not require any condition on p, but only a mild one on q. In particular, there are many examples of k, G, T, θ such that $\pm H_c^*(X)[\theta|_{\mathcal{T}(\mathcal{O}_k)}]$ is an irreducible $\mathcal{G}(\mathcal{O}_k)$-representation, which does not appear in Yu's construction. Moreover, Theorem 1.3 implies that its induction to $G(k)$ is supercuspidal.

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2. Preparations

2.1. **More notation.** We use the notation from the introduction. Moreover, we denote by $N_G(T)$ the normalizer of $T_{\breve{k}}$ in $G_{\breve{k}}$, so that $W = N_G(T)/T$

[1] This is not clear for the schemes $X_{T,U,r}$ at least if G is not quasi-split (for the quasi-split case, see [DI24, Corollary 4.1.4]).

is the Weyl group of T, by $X^*(T)$ (resp. $X_*(T)$) the group of characters (resp. cocharacters) of $T_{\breve{k}}$ and by $\langle\cdot,\cdot\rangle\colon X^*(T)\times X_*(T)\to\mathbb{Z}$ the natural pairing. We write Φ for the root system of $T_{\breve{k}}$ in $G_{\breve{k}}$, Φ^+ for the subset of positive roots determined by B, and $\Delta\subseteq\Phi^+$ for the subset of positive simple roots. We write $S\subseteq W$ for the corresponding set of simple reflections.

Let $c\in W$ be the unique element such that $FB={}^cB$. Then for any lift $\dot{c}$ of c, $\mathrm{Ad}(\dot{c})^{-1}\circ F : G(\breve{k})\to G(\breve{k})$ fixes the pinning (T,B), hence defines automorphisms, denoted by σ, of the based root system $\Delta\subseteq\Phi$ and of the Coxeter system (W,S). Note that σ does not depend on the choice of the lift $\dot{c}$. We call (T,B) (or (T,U)) a *Coxeter pair* if c is a σ-Coxeter element in the Coxeter triple (W,S,σ), that is, if a(ny) reduced expression of c contains precisely one element from each σ-orbit on S. *Moreover, we assume until the end of §4 that c is σ-Coxeter, and hence (T,U) is a Coxeter pair.*

Except for G, $\mathcal{G}$ and their subgroups (which are defined over $k,\breve{k}$ resp. $\mathcal{O}_k,\mathcal{O}_{\breve{k}}$), all schemes appearing below are perfect schemes perfectly of finite presentation and perfectly smooth over $\overline{\mathbb{F}}_q$. For a review of perfect geometry we refer to [Zhu17, Appendix A]. We freely make use of the 6-functor formalism of étale cohomology for such schemes with $\overline{\mathbb{Q}}_\ell$-coefficients. Moreover, we fix a prime number $\ell\neq p$, and for a perfect $\overline{\mathbb{F}}_q$-scheme we denote by $H^*(Y)=H^*_c(Y,\overline{\mathbb{Q}}_\ell)$ its ℓ-adic étale cohomology with compact support.

2.2. **Pinning.** We may express the action of the Frobenius F on $X_*(T)_{\mathbb{Q}}$ as $F=\mu c\sigma\colon x\mapsto\mu+c\sigma(x)$ for some $\mu\in X_*(T)$. There is a unique point $e\in\mathbb{Q}\Phi^\vee$ such that $F(e)\in e+X_*(Z)_{\mathbb{Q}}$, or equivalently, $\mu+c\sigma(e)-e\in X_*(Z)_{\mathbb{Q}}$. Let

$$\Phi_e=\{\alpha\in\Phi;\langle\alpha,e\rangle\in\mathbb{Z}\}.$$

We denote by Δ_e the set of simple roots of $\Phi_e^+=\Phi_e\cap\Phi^+$. Let $W_e\subseteq W$ be the Weyl group of Φ_e. Note that $\mathcal{G}$ from the introduction is the parahoric model attached to the image of e in the reduced building of G, and that Φ_e (resp. W_e) is the root system (resp. Weyl group) of the reductive quotient of the special fiber of $\mathcal{G}$.

Also, note that the action of F on W agrees with $\mathrm{Ad}(c)\circ\sigma$; we denote it by $F=c\sigma\colon W\to W$. This action stabilizes $W_e\subseteq W$. Finally, for an element $w\in W_e$ we denote by $\dot{w}\in\mathbb{G}(\overline{\mathbb{F}}_q)$ an arbitrary (fixed) lift of w.

2.3. **A condition on q.** Let $\omega_\alpha^\vee$ denotes the fundamental coweight of $\alpha\in\Delta$. For a σ-orbit $\mathcal{O}\subseteq\Delta$ of simple roots, we set $\omega_{\mathcal{O}}^\vee=\sum_{\alpha\in\mathcal{O}}\omega_\alpha^\vee$, where $\omega_\alpha^\vee$ denotes the fundamental coweight of $\alpha\in\Delta$. We prove our main result under the following condition on q:

$$q>M=\max\{\langle\gamma,\omega_{\mathcal{O}}^\vee\rangle;\gamma\in\Phi^+,\mathcal{O}\in\Delta/\langle\sigma\rangle\}. \tag{2.1}$$

Note that M only depends on the (relative) Dynkin diagram Δ of the quasi-split inner form of G over k. If Δ is connected then M takes the following values: $M=1$ for type A_n; $M=2$ for types $B_n,C_n,D_n,{}^2A_n,{}^2D_n$; $M=3$ for types $G_2,E_6,{}^3D_4$; $M=4$ for types $F_4,E_7,{}^2E_6$; $M=6$ for type E_8. If the quasi-split inner form of G is split, then M is the same as in [DI24, §2.7], and it differs otherwise. Just as in [DI24, §2.7], for arbitrary G the constant M equals the maximum of the values of M over all connected components of the Dynkin diagram of $G_{\breve{k}}$ (equipped with the smallest power of σ fixing the connected component). In particular, (2.1) holds whenever $q > 5$.

2.4. **A Coxeter element in W_e.** It turns out that c determines a (twisted) Coxeter element of W_e. Write $c = s_{\alpha_1} \cdots s_{\alpha_r}$, where $\{\alpha_1, \dots, \alpha_r\} \subseteq \Delta$ is a set of representatives of σ-orbits of Δ.

Let $I = (i_1 < i_2 < \cdots < i_m)$ be a subsequence of $[r] := (1 < 2 < \cdots < r)$, and let $I' = (j_1 < j_r < \cdots < j_{r-m})$ be the complement sequence of I in $[r]$. We define

$$\sigma_{I,c} = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_m}} \sigma;$$
$$c_I = s_{\beta_{j_1}} s_{\beta_{j_2}} \cdots s_{\beta_{j_{r-m}}};$$
$$\Delta_{I,c} = \{\beta_{j_l}; 1 \leqslant l \leqslant r-m\}$$

where $\beta_{j_l} = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_t}}(\alpha_{j_l})$ with $1 \leqslant t \leqslant m-1$ such that $i_t < j_l < i_{t+1}$. By definition, $c\sigma = c_I \sigma_{I,c}$.

Theorem 2.1. *Let c, μ and $e = e_{\mu,c}$ be as in §2.2. Then there exist a sequence $I = I_{\mu,c}$ of $1 < 2 < \cdots < r$ such that*

(1) $\sigma_{I,c}(\Delta_e) = \Delta_e$;

(2) $\Delta_{I,c} \subseteq \Delta_e$ is a representative set of $\sigma_{I,c}$-orbits of Δ_e;

(3) $\sigma_{I,c}^i = 1$ if and only if $\sigma_{I,c}^i$ fixes each root of Δ_e.

In particular, c_I is a $\sigma_{I,c}$-Coxeter element of W_e.

This theorem is proven in §4.

2.5. **Support.** For $\alpha \in \Phi$ we denote by $\mathrm{supp}(\alpha) \subseteq \Delta$ the minimal subset whose linear span contains α. For a subset $C \subseteq \Phi$ we set $\mathrm{supp}(C) = \cup_{\alpha \in C} \mathrm{supp}(\alpha)$. For $w \in W$ we denote by $\mathrm{supp}(w)$ the set of simple reflections which appear in some/any reduced expression of w.

Lemma 2.2. *Let $C \subseteq \Phi$ be a $c\sigma$-orbit. Then $\mathrm{supp}(C)$ is σ-stable.*

Proof. Let $c = s_{\alpha_1} \cdots s_{\alpha_r}$ be as in §2.4. Let $\alpha \in \mathrm{supp}(\gamma)$ for some $\gamma \in C$. It suffices to show that the σ-orbit $\mathcal{O}$ of α is contained in $\mathrm{supp}(C)$. Set $\delta = \sharp\mathcal{O}$. Let $1 \leqslant j \leqslant r$ be the unique integer such that $\alpha_j \in \mathcal{O}$. Let $0 \leqslant i_0 \leqslant \delta - 1$ such that

$$\alpha, \sigma^{-1}(\alpha), \dots, \sigma^{1-i_0}(\alpha) \neq \alpha_j \text{ and } \sigma^{-i_0}(\alpha) = \alpha_j.$$

Then one checks that $(c\sigma)^{-i_0} = \sigma^{-i_0} w$ for some $w \in W$ such that $\mathrm{supp}(w) \subseteq \Delta - \{\alpha\}$. Hence $\alpha \in \mathrm{supp}(w(\gamma))$ and $\alpha_j = \sigma^{-i_0}(\alpha) \in \mathrm{supp}(\sigma^{-i_0} w(\gamma)) = \mathrm{supp}((c\sigma)^{-i_0}(\gamma))$. So we can assume that $\alpha = \alpha_j$. Let $0 \leqslant i \leqslant \delta - 1$. Note that $(c\sigma)^i = u_i \sigma^i$ for some $u_i \in W$ with $\mathrm{supp}(u_i) \subseteq \Delta - \{\sigma^i(\alpha_j)\}$. It follows that $\sigma^i(\alpha) \in \mathrm{supp}((c\sigma)^i(\gamma))$. So the statement follows. ∎

Proposition 2.3. *Let C be a $c\sigma$-orbit of Φ. Then* $\mathrm{supp}(C) = \cup_{i\in\mathbb{Z}}\sigma^i(H)$, *where H is a connected component of Δ.*

Proof. Without loss of generality we may assume that $\Delta = \cup_{i\in\mathbb{Z}}\sigma^i(H)$. We argue by induction on $\sharp\Delta$. Assume the statement is false. Let $c = s_{\alpha_1}\cdots s_{\alpha_r}$ be as in §2.4. By Lemma 2.2 there exists $1 \leqslant j \leqslant r$ such that $C \subseteq \Phi_K$, where $K = \Delta - \mathcal{O}$ and $\mathcal{O}$ is the σ-orbit of α_j. By replacing c with its W_K-σ-conjugate $s_{\alpha_j}\cdots s_{\alpha_r}\sigma(s_{\alpha_1}\cdots s_{\alpha_{j-1}})$, we can assume that $j=1$ and $\alpha_1 \in \mathcal{O}$. Let $c' = s_{\alpha_1}c$, which is a σ-Coxeter element of W_K. As $C \subseteq \Phi_K$, C is also a $c'\sigma$-orbit of Φ_K. By induction hypothesis we have $\mathrm{supp}(C) = \cup_{i\in\mathbb{Z}}\sigma^i(D)$, where D is a connected component of $H - \{\alpha_1\}$. As H is connected, there exists $\gamma \in C$ and $\beta \in \mathrm{supp}(\gamma)$ such that

$$0 > \langle \alpha_1, \beta^\vee\rangle \geqslant \langle \alpha_1, \gamma^\vee\rangle.$$

Then we have $\sigma^{-1}(\alpha_1) \in \mathrm{supp}((c\sigma)^{-1}(\gamma))$, contradicting that $C \subseteq \Phi_K$. The proof is finished. □

2.6. **A condition on the σ-Coxeter element.** Let c, μ, $e = e_{\mu,c}$, $I = I_{\mu,c}$, c_I, $\sigma_I = \sigma_{I,c}$ and $\Delta_I = \Delta_{I,c} \subseteq \Delta_e$ be as in Theorem 2.1. Denote by $\ell\colon W \to \mathbb{Z}_{\geq 0}$ (resp. $\ell_e : W_e \to \mathbb{Z}_{\geqslant 0}$) the length function associated to the set Δ (resp. Δ_e) of simple roots. Let w_0 and w_e be the longest elements of W and W_e respectively. We consider the following condition on c, or, equivalently, on the pair (T, U):

(∗) There exists $N \in \mathbb{Z}_{\geqslant 1}$ such that $(c\sigma)^N = w_0\sigma^N, \quad N\ell(c) = \ell(w_0)$.

Remark 2.4. If Δ is connected, then there always exists a σ-Coxeter element $c \in W$ satisfying (∗), see [Bou68, Chap. V, Prop. 6.2]. Moreover, if the Coxeter number of G is even, then any c satisfies this condition.

Lemma 2.5. *Suppose c satisfies condition (∗). Then $(c_I\sigma_I)^N = w_e\sigma_I^N$ and $N\ell_e(c_I) = \ell_e(w_e)$.*

Proof. By Theorem 2.1, $c_I\sigma_I = c\sigma$ and $\sigma_I(\Delta_e) = \Delta_e$. As $(c_I\sigma_I)^N = w_0\sigma^N$, it follows that $(c_I\sigma_I)^N$ sends Φ_e^+ to $-\Phi_e^+$, that is, $(c_I\sigma_I)^N = w_e\sigma_I^N$.

It remains to show $\ell_e((c_I\sigma_I)^{i+1}) = \ell_e((c_I\sigma_I)^i) + \ell_e(c_I\sigma_I)$ for $1 \leqslant i \leqslant N-1$. Indeed, this is equivalent to that for any $\alpha \in \Phi_e^+$ with $(c_I\sigma_I)^{-1}(\alpha) < 0$ we have $(c_I\sigma_I)^i(\alpha) > 0$. This statement follows from that $c_I\sigma_I = c\sigma$ and $\ell((c\sigma)^{i+1}) = \ell((c\sigma)^i) + \ell(c\sigma)$ for $1 \leqslant i \leqslant N-1$. □

For $w \in W$ we denote by $\mathrm{supp}(w)$ the set of simple reflections in Δ that appears in some/any reduced expression of w. For $u \in W_e$, we can define $\mathrm{supp}_{\Delta_e}(u) \subseteq \Delta_e$ in a similar way.

Corollary 2.6. *Suppose c satisfies condition (∗). Let $K \subsetneq \Delta_e$ be a proper σ_I-stable subset. Then there exists a proper σ-stable subset $J \subsetneq \Delta$ such that $\sigma_I \in W_J\sigma$ and $w_eW_K \subseteq w_0W_J$.*

Proof. Let notation be as in §2.4. As $\Delta_I = \{\beta_j; j \in I'\}$ with $I' = [r] - I$ is a representative set of Δ_e, there exists $i \in I'$ such that $\beta_i \notin K$. Let $J = \Delta - \mathcal{O}_i$, where $\mathcal{O}_i$ is the σ-orbit of α_i. By construction, $\mathrm{supp}(s) \subseteq J$ for $s \in K$ and $\mathrm{supp}(\sigma_I \sigma^{-1}) \subseteq J$. By Lemma 2.5 we have

$$w_e = (c_I \sigma_I)^N \sigma_I^{-N} = (c\sigma)^N \sigma_I^{-N} = w_0 \sigma^N \sigma_I^{-N} \subseteq w_0 W_J.$$

Thus $w_e W_K \subseteq w_0 W_J$ as desired. □

Lemma 2.7. *Let $K_1, K_2 \subseteq \Delta_e$ be two σ_I-stable subsets. Let c_1 and c_2 be two σ_I-Coxeter elements of W_{K_1} and W_{K_2} respectively. Let $w \in W_e$ such that $c_1 \sigma_I(w) = w c_2$. Then there exists $x \in {}^{K_1}W_e{}^{K_2}$ such that ${}^xK_2 = K_1$ and $w \in x W_{K_2}$.*

Proof. By symmetry we may assume $\sharp K_1 \leqslant \sharp K_2$. Let $x \in {}^{K_1}W_e$ such that $w \in W_{K_1} x$. Then there exists $c_2' \leq c_2$ such that $x c_2 \in W_{K_1} x c_2'$ and $x c_2' \in {}^{K_1}W_e$. Hence we have $\sigma_I(x) = x c_2'$. Note that c_2' is a partial σ_I-Coxeter element, which is of minimal length (in the sense of ℓ_e) in its σ_I-conjugacy class. Thus $c_2' = 1$, $x = \sigma_I(x)$ and $x(\mathrm{supp}_{\Delta_e}(c_2)) \subseteq K_1$, which implies that $x(K_2) \subseteq K_1$. Hence $x(K_2) = K_1$ since $\sharp K_1 \leq \sharp K_2$. Thus $x \in {}^{K_1}W_e{}^{K_2}$ as desired. □

3. Cohomology of X

Recall the scheme X from (1.1) equipped with $\mathbb{G}^F \times \mathbb{T}^F$-action.

3.1. The schemes Σ^i.

Let $i \in \mathbb{Z}$. We define

$$\Sigma^i = \{(x, x', y) \in F\mathbb{U} \times F^{i+1}\mathbb{U} \times \mathbb{G}; xF(y) = yx'\}.$$

Let $\mathrm{pr}_3 : \Sigma^i \to \mathbb{G}$ be the natural projection. There is a locally closed decomposition

$$\Sigma^i = \bigsqcup_{w \in W_e} \Sigma^i_w,$$

where $\Sigma^i_w = \mathrm{pr}_3^{-1}(\mathbb{U} w \mathbb{T} \mathbb{G}^1 F^i \mathbb{U})$.

The group $\mathbb{T}^F \times \mathbb{T}^F$ acts on Σ^i and on each of the pieces Σ^i_w by

$$(t, t') : (x, x', y) \longmapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1}).$$

As in [DL76, p.137] there is a $\mathbb{T}^F \times \mathbb{T}^F$-equivariant isomorphism $X \times X/\mathbb{G}^F \xrightarrow{\sim} \Sigma^0$, and for characters χ, χ' of $\mathbb{T}^F$ we have

$$\dim_{\overline{\mathbb{Q}}_\ell} \mathrm{Hom}_{\mathbb{G}^F}(H_*(X)[\chi'], H_*(X)[\chi]) = \dim H_*(\Sigma^0)_{\chi', \chi^{-1}},$$

where $H_*(\Sigma^0)_{\chi', \chi}$ is the corresponding isotropic subspace of $H_c^*(\Sigma^0)$.

Let $Z \subseteq G$ denote the centre of G and consider the embedding $z \mapsto (z, z^{-1}) \colon Z \to T \times T$. Then the above $\mathbb{T}^F \times \mathbb{T}^F$-action on Σ^i factors through an action of the quotient $\mathbb{T}^F \times^{\mathbb{Z}^F} \mathbb{T}^F$. This latter action extends to the action of $\mathbb{T}^F \times^{\mathbb{Z}^F} \mathbb{T}^F \subseteq (\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$ on Σ^i (and Σ^i_w for $w \in W_e$) given by the same formula. By the discussion in [DI24, §4.2] which applies in our more general setting, Theorem 1.1 follows from the next result.

Theorem 3.1. *Suppose that q satisfies condition (2.1). Then there exists a Coxeter pair (T, U) such that*

$$H_*(\Sigma_w^0) = \begin{cases} H_*((\dot{w}\mathbb{T})^{c\sigma}) & \text{if } w \in W_e^{c\sigma}, \\ \{0\} & \text{otherwise.} \end{cases}$$

as virtual $(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$-modules.

As a first step towards the proof of Theorem 3.1 we observe that the whole discussion of [DI24, §4.3] applies *mutatis mutandis* in our setting. Thus it suffices to prove Theorem 3.1 in the case that Δ is connected. In particular, there exists some c satisfying condition $(*)$, cf. Remark 2.4. Now Theorem 3.1 follows from Corollary 3.7 and Proposition 3.12 below.

3.2. **An extension of action.** Let $w \in W_e$. We set $K_{w,i} = {}^{w^{-1}}U^- \cap F^i U^-$. Define

$$\hat{\Sigma}_w^i = \{(\tilde{x}, \tilde{x}', y_1, \tau, z, y_2) \in F\mathbb{U} \times F^{i+1}\mathbb{U} \times \mathbb{U} \times \dot{w}\mathbb{T} \times \mathbb{K}_{w,i}^1 \times F^i\mathbb{U}; \tilde{x}F(\tau z) = y_1 \tau z y_2 \tilde{x}'\}.$$

We define an action of $\mathbb{T}^F \times \mathbb{T}^F$ on $\hat{\Sigma}_w^i$ by

$$(t, t') : (\tilde{x}, \tilde{x}', y_1, \tau, z, y_2) \longmapsto (t\tilde{x}t^{-1}, t'\tilde{x}'t'^{-1}, ty_1t^{-1}, t\tau t'^{-1}, t'zt'^{-1}, t'y_2t'^{-1}).$$

Then there is an $\mathbb{T}^F \times \mathbb{T}^F$-equivariant affine space bundle

$$\pi_w^i : \hat{\Sigma}_w^i \longrightarrow \Sigma_w^i, \quad (\tilde{x}, \tilde{x}', y_1, \tau, z, y_2) \longmapsto (\tilde{x}F(y_1)^{-1}, \tilde{x}'F(y_2), y_1\tau z y_2).$$

Let $\chi \in X_*(T)$ which centralizes $K_{w,i}$. Define

$$H_{w,\chi} = \{(t, t') \in \mathbb{T} \times \mathbb{T}; w^{-1}t^{-1}F(t)w = t'^{-1}F(t') \in \operatorname{Im}(\chi)\}.$$

Then $H_{w,\chi}$ acts on $\hat{\Sigma}_w^i$ by

$$(t, t') : (\tilde{x}, \tilde{x}', y_1, \tau, z, y_2) \longmapsto ({}^{F(t)}\tilde{x}, {}^{F(t')}\tilde{x}', {}^{F(t)}y_1, t\tau t'^{-1}, {}^{t'}z, {}^{F(t')}y_2).$$

Lemma 3.2. *Let $w \in W_e \setminus W_e^{c\sigma}$ such that $\Sigma_w^i \neq \emptyset$. Then there exists a proper subset $K = \sigma_I(K) \subsetneq \Delta_e$ such that $w(c_I\sigma_I)^i\sigma_I^{-i} \in w_eW_K$.*

Proof. Let $w_i = w(c_I\sigma_I)^i\sigma_I^{-i} \in W_e$. By assumption we have

$$c\sigma\mathbb{B}w(c\sigma)^i\mathbb{B}\mathbb{G}^1(c\sigma)^{-i-1} \cap \mathbb{B}w(c\sigma)^i\mathbb{B}\mathbb{G}^1c\sigma\mathbb{B}(c\sigma)^{-i-1} \neq \emptyset.$$

As $c_I\sigma_I = c\sigma$, this implies that

$$c_I\sigma_I\mathbb{B}_1w(c_I\sigma_I)^i\mathbb{B}_1 \cap \mathbb{B}_1w(c_I\sigma_I)^i\mathbb{B}_1c_I\sigma_I \neq \emptyset,$$

that is,

$$c_I\mathbb{B}_1\sigma_I(w_i)\mathbb{B}_1 \cap \mathbb{B}_1w_i\mathbb{B}_1(\sigma_I)^i(c_I) \neq \emptyset.$$

In particular there are σ_I-Coxeter elements $v' \leq_e c_I$ and $v \leq_e (\sigma_I)^i(c_I)$ of some σ_I-stable subsets K' and K of Δ_e respectively (one of which is a proper subset of Δ_e since $w \in W_e \setminus W_e^{c\sigma}$) such that $v'\sigma_I(w_i) = w_iv$ and

(a) $$\mathbb{B}_1w_i\mathbb{B}_1(\sigma_I)^i(c_I) \cap \mathbb{B}_1w_iv\mathbb{B}_1 \neq \emptyset.$$

Applying Lemma 2.7, there exist $x = \sigma(x) \in {}^{K'}W_e{}^K$ such that $K' = {}^xK$ and $w_i \in xW_K$. Moreover, it follows from (a) that for any simple reflection

$s \in \mathrm{supp}_{\Delta_e}(\sigma_I^i(c_I)) \setminus K$ we have $xs \in W_{K'}x$ or $xs \leq_e x$. The former is impossible since $s \notin W_K = xW_{K'}x^{-1}$. So we have $xs \leq_e x$. Moreover, as $xsx^{-1} \notin W_{K'}$ we have $w_{K'}xs \leq_e w_{K'}x = xw_K$, where w_K and $w_{K'}$ are the maximal elements of W_K and $w_{K'}$ respectively. As xw_K is σ_I-stable, we have $xw_Ks \leq_e xw_K$ for all $s \in \Delta_e$, that is, $xw_K = w_e$. Hence $w_i \in w_eW_K$. □

Let $N_0 \in \mathbb{Z}_{\geqslant 0}$ be the order of $c\sigma \in W \rtimes \langle\sigma\rangle$. Define

$$N_F^{F^{N_0}} : \mathbb{T} \longrightarrow \mathbb{T}, \quad t \longmapsto tF(t)\cdots F^{N_0-1}(t).$$

Lemma 3.3. *Let $\chi \in X_*(T)$ and let C be a $c\sigma$-orbit of Φ. Assume χ is non-central on C and $|\langle\chi,\beta\rangle| < q$ for $\beta \in C$. Then $\sum_{i=0}^{N_0-1} q^i\langle\gamma,(c\sigma)^i(\chi)\rangle \neq 0$ for $\gamma \in C$. In particular, the action of $\mathbb{G}_m$ on $\mathbb{U}_\gamma$ for $\gamma \in C$, via the morphism $N_F^{F^{N_0}} \circ \chi$, is nontrivial.*

Proof. By assumption, $|\langle\gamma,(c\sigma)^i(\chi)\rangle| = |\langle(c\sigma)^{-i}(\gamma),\chi\rangle| < q$ for $0 \leqslant i \leqslant N_0-1$, and there exists $0 \leqslant i_0 \leqslant N_0-1$ such that $\langle(c\sigma)^{-i_0}(\gamma),\chi\rangle \neq 0$. Hence the statement follows. □

Let $\mathbb{G}_m \subseteq \mathcal{O}_{\breve{k}}^\times$ be the Teichmüller lift of the quotient map $\mathcal{O}_{\breve{k}}^\times \to \overline{\mathbb{F}}_q^\times$. Assume that $r \in \mathbb{Z}_{\geqslant 1}$.

Lemma 3.4. *Consider the homomorphism*

$$f_{w,\chi} : \mathbb{G}_m \longrightarrow \mathbb{T}\times\mathbb{T}, \quad x \longmapsto (N_F^{F^{N_0}}({}^w\chi(x)), N_F^{F^{N_0}}(\chi(x))).$$

Then $\mathrm{Im}(f_{w,\chi}) \subseteq H_{w,\chi}^\circ$.

Proof. By definition. $F^{N_0}(\lambda(x)) = \lambda(x^{q^{N_0}})$ for $x \in \breve{k}$. Hence

$$N_F(\chi(x))^{-1}F(N_F(\chi(x))) = \chi(x)^{-1}F^{N_0}(\chi(x)) = \chi(x^{-1}\sigma^{N_0}(x)).$$

So the statement follows. □

3.3. **Handling Σ_w^0 for $w \in W_e \setminus W_e^{c\sigma}$.** Let $i \in \mathbb{Z}$. Following [DI24, §5] we define an isomorphism of varieties

$$\alpha_i : \Sigma^i \longrightarrow \Sigma^{i+1}, \quad (x,x',y) \longmapsto (x,F(x'),yx').$$

For $w,u \in W_e$ we define

$$Y_{w,u}^i = \Sigma_w^i \cap (\alpha_i)^{-1}(\Sigma_u^{i+1});$$
$$Z_{w,u}^{i+1} = \alpha_i(\Sigma_w^i) \cap \Sigma_u^{i+1} = \alpha_i(Y_{w,u}^i).$$

Let $\hat{Y}_{w,u}^i = (\pi_w^i)^{-1}(Y_{w,u}^i)$ and $\hat{Z}_{w,u}^i = (\pi_u^{i+1})^{-1}(Z_{w,u}^{i+1})$.

Lemma 3.5. *Let $w,u \in W_e$. Let $\chi,\mu \in X_*(T)$ which centralizes $\mathbb{K}_{w,i}$ and $\mathbb{K}_{u,i+1}$ respectively. Then $H_{w,\chi}$ and $H_{u,\mu}$ preserve $\hat{Y}_{w,u}^i$ and $\hat{Z}_{w,u}^{i+1}$ respectively.*

Proof. This is proved in [DI24, §5]. □

Proposition 3.6. *Suppose that condition* ($*$) *holds and that* q *satisfies condition* (2.1). *Let* $i \in \mathbb{Z}$. *Then*

$$H_*(\hat{Y}^i_{w,u}) = H_*(Y^i_{w,u}) = H_*(Z^{i+1}_{w,u}) = H_*(\hat{Z}^{i+1}_{w,u}) = 0$$

if w *or* u *belongs to* $W_e \setminus W_e^{c\sigma}$.

Proof. Without loss of generality we can assume that $w \in W_e \setminus W_e^{c\sigma}$ and $\hat{Y}^i_{w,u} \neq \emptyset$. In particular, $\Sigma^i_w \neq \emptyset$. By Lemma 3.2 and Corollary 2.6, there are subsets $K = \sigma_I(K) \subsetneq \Delta_e$ and $J = \sigma(J) \subsetneq \Delta$ such that

$$w(c\sigma)^i \in w_e W_K (\sigma_I)^i \subseteq w_e W_J \sigma^i = w_0 W_J \sigma^i.$$

Thus

$$K_{w,i} \subseteq {}^{w^{-1}}(U^- \cap {}^{w(c\sigma)^i}U^-) \subseteq {}^{w^{-1}w_0}M_J,$$

where M_J is the Levi subgroup generated by T and U_γ for $\gamma \in \Phi_J$. Let $\mathcal{O} \in \Delta \backslash J$ be a σ-orbit. Then W_J fixes $\omega^\vee_{\mathcal{O}}$, and $K_w \subseteq {}^{w^{-1}w_0}M_J$ is centralized by

$$\chi := w^{-1}w_0(\omega^\vee_{\mathcal{O}}) = w^{-1}w(c\sigma)^i\sigma^{-i}(\omega^\vee_{\mathcal{O}}) = (c\sigma)^i(\omega^\vee_{\mathcal{O}}).$$

Moreover, $w(\chi) = w_0\sigma^N(\omega^\vee_{\mathcal{O}}) = (c\sigma)^N(\omega^\vee_{\mathcal{O}})$.

Let $f_{w,\chi} : \mathbb{G}_m \to H_{w,\chi}$ be the as in Lemma 3.4. In view of Lemma 3.5, via $f_{w,\chi}$ the action of $H_{w,\chi}$ on $\hat{Y}^i_{w,u}$ induces an action of $\mathbb{G}_m$ on $\hat{Y}^i_{w,u}$, which commutes with action of $\mathbb{T}^F \times \mathbb{T}^F$. Hence

$$H^*_c(Y_{w,u}) = H^*_c(\hat{Y}^i_{w,u}) = H^*_c((\hat{Y}^i_{w,u})^{\mathbb{G}_m}),$$

it suffices to show $(\hat{Y}^i_{w,u})^{\mathbb{G}_m} = \emptyset$. To this end, we can assume that $\Delta = \cup_{i\in\mathbb{Z}}\sigma^i(H)$ for some/any connected component H of Δ. Then by Proposition 2.3, $\chi, w(\chi) \in \{(c\sigma)^i(\omega^\vee_{\mathcal{O}}); i \in \mathbb{Z}\}$ are non-central on each $c\sigma$-orbit of Φ. As $q > M$, it follows from Lemma 3.3 that

$$(\hat{Y}^i_{w,u})^{\mathbb{G}_m} \subseteq \{1\} \times \{1\} \times \{1\} \times \mathbb{T} \times \{1\} \times \{1\}.$$

As $w \in W_e \setminus W_e^{c\sigma}$, we deduce that $(\hat{Y}^i_{w,u})^{\mathbb{G}_m} = \emptyset$ as desired. □

Corollary 3.7. *Let* $i \in \mathbb{Z}$ *and* $w \in W_e$. *If* $w \in W_e \setminus W_e^{c\sigma}$ *then* $H_*(\Sigma^i_w) = 0$. *Otherwise,*

$$H_*(\Sigma^i_w) = \sum_{u \in W_e^{c\sigma}} H_*(Y^i_{w,u}) = \sum_{u \in W_e^{c\sigma}} H_*(Z^i_{u,w}) = \sum_{u \in W_e^{c\sigma}} H_*(Y^{i-1}_{u,w}).$$

Proof. Note that $\Sigma^i_w = \sqcup_{u\in W_e} Y^i_{w,u} = \sqcup_{u \in W_e} Z^i_{u,w}$ and $Z^i_{u,w} \cong Y^{i-1}_{u,w}$. Then the statement follows from Proposition 3.6. □

3.4. Handling Σ^0_w for $w \in W_e^{c\sigma}$.

Lemma 3.8. *Suppose that Condition* ($*$) *holds. Let* $i \in \mathbb{Z}$ *and* $w, u \in W_e^{c\sigma}$ *such that* $Y^i_{w,u} \neq \emptyset$. *Then* $w = u$ *if either* $\sigma_I \neq 1$ *or* $\sigma_I = 1$ *and* $wc^i_I \neq w_e$.

Proof. By assumption, we have

$$\mathbb{B}_1 w(c_I\sigma_I)^i \mathbb{B}_1 c_I\sigma_I \mathbb{B}_1 (c_I\sigma_I)^{-i-1} \cap \mathbb{B}_1 u(c_I\sigma_I)^{i+1}\mathbb{B}_1(c_I\sigma_I)^{-i-1} \neq \emptyset,$$

that is, $\mathbb{B}_1 w(c_I\sigma_I)^i \mathbb{B}_1 c_I\sigma_I \mathbb{B}_1 \cap \mathbb{B}_1 u(c_I\sigma_I)^{i+1}\mathbb{B}_1 \neq \emptyset$. Thus there exists $v \leq_e c_I$ such that $w(c_I\sigma_I)^i v\sigma_I = u(c_I\sigma_I)^{i+1}$. Note that $w, u \in W_e^{c\sigma} \subseteq \langle c_I\sigma_I\rangle$. We have

$$v\sigma_I = (c_I\sigma_I)^{-i} w^{-1} u (c_I\sigma_I)^{i+1} = w^{-1}u(c_I\sigma_I) \in \langle c_I\sigma_I\rangle.$$

In particular, it follows from Lemma 2.5 that $\ell_e(v)$ is divided by $\ell_e(c_I)$.

Assume that either $\sigma_I \neq 1$ or $\sigma_I = 1$ and $wc_I^i \neq w_e$. If $v \neq 1$, then $\ell_e(v) = \ell_e(c_I)$ since $1 \neq v \leq c_I$. Hence we have $v = c_I$ and $w = u$ as desired. Suppose $v = 1$. Then $c_I = u^{-1}w \in W_e^{c\sigma}$, which means that $\sigma_I(c_I) = c_I$. Hence $\sigma_I = 1$ by Theorem 2.1 (3). By assumption we have $\sigma_I = 1$ and $w\sigma_I^i \neq w_e$. As $v = 1$, we have $wc_I^i s < wc_I^i$ for all $s \in \mathrm{supp}_{\Delta_e}(c_I) = \Delta_e$, that is, $wc_I^i = w_e$, a contradiction. □

Theorem 3.9 ([IN24], Theorem 3.1)**.** *The map $(u_1, u_2) \mapsto u_1^{-1}u_2F(u_1)$ gives an isomorphism*

$$\phi : (F\mathbb{U} \cap \mathbb{U}) \times (F\mathbb{U} \cap \mathbb{U}^-) \cong F\mathbb{U}.$$

In particular, ϕ restricts to an isomorphism

$$(F\mathbb{U}^1 \cap \mathbb{U}) \times (F\mathbb{U} \cap \mathbb{U}^-) \cong \mathbb{U}^1(F\mathbb{U} \cap \mathbb{U}^-).$$

For $i \in \mathbb{Z}$ and $w \in W_e$ we define

$${}^\flat\Sigma_w^i = \{(x, x', y) \in (F\mathbb{U} \cap \mathbb{U}^-) \times F^i(F\mathbb{U} \cap \mathbb{U}^-) \times (\mathbb{B}\dot{w}\mathbb{G}^1F^i\mathbb{B}); xF(y) = yx'\}.$$

Lemma 3.10. *The map $(x, x', y) \mapsto (x_2, x_2', x_1yF^i(x_1')^{-1}, x_1, x_1')$ gives an $\mathbb{T}^F \times \mathbb{T}^F$-equivariant isomorphism*

$$\Sigma_w^i \cong {}^\flat\Sigma_w^i \times (F\mathbb{U} \cap \mathbb{U}) \times (F\mathbb{U} \cap \mathbb{U}),$$

where $(x_1, x_2) = \phi^{-1}(x)$ and $(x_1', x_2') = \phi^{-1}(x')$. In particular, $H_c^*(\Sigma_w^i) \cong H_c^*({}^\flat\Sigma_w^i)$.

Proof. It follows by definition and Theorem 3.9. □

Lemma 3.11. *Suppose that c satisfies condition $(*)$. Let $w = (c\sigma)^m \in W_e^{c\sigma}$ for some $m \in \mathbb{Z}$. Then we have $H_*(\Sigma_w^{N-m}) = H_*(\dot{w}\mathbb{T}^F) = H_*({}^\flat\Sigma_w^{2N-m}) = H_*(\Sigma_w^{2N-m})$.*

Proof. The first equality is proved in [DI24]. We show the last two equalities. Let $(x, x', y) \in {}^\flat\Sigma_w^{2N-m}$. By definition,

$$y \in \mathbb{G}^1\mathbb{B}\dot{w}(c\sigma)^{2N-m}\mathbb{B}(c\sigma)^{m-2N} = \mathbb{U}\mathbb{T}\mathbb{U}^{-,1}\dot{w}.$$

So we may write $y = y_1\tau y_2 w$ uniquely with $y_1 \in \mathbb{U}$, $\tau \in \mathbb{T}$ and $y_2 \in \mathbb{U}^{-,1}$.

Then the equality $xF(y) = yx'$ is equivalent to

$$\tau^{-1}y_1^{-1}xF(y_1)F(\tau) = y_2\dot{w}x'\dot{w}^{-1}F(y_2^{-1}) = y_2x''F(y_2^{-1}),$$

where $x'' = {}^{\dot{w}}x' \in F\mathbb{U} \cap \mathbb{U}^-$ since $w = (c\sigma)^m$.

By Theorem 3.9, the map $(g_1, g_2) \mapsto g_1^{-1} g_2 F(g_1)$ gives isomorphisms

$$\mathbb{U} \times (F\mathbb{U} \cap \mathbb{U}^-) \cong \mathbb{U}(F\mathbb{U} \cap \mathbb{U});$$
$$\mathbb{U}^{-,1} \times (F\mathbb{U} \cap \mathbb{U}^-) \cong (F\mathbb{U} \cap \mathbb{U}^-) F\mathbb{U}^{-,1}.$$

So we can make changes of variables $(x, x'', y_1, y_2) \mapsto (z_1, z_2, z_3, z_4)$, where

$$(z_1, z_2, z_3, z_4) \in \mathbb{U} \times F\mathbb{U} \cap \mathbb{U}^- \times \mathbb{U}^{-,1}(F\mathbb{U} \cap \mathbb{U}^-) \times F\mathbb{U}^{-,1} \cap \mathbb{U}$$

such that $y_1^{-1} x F(y_1) = z_1 z_2$ and $y_2 x'' F(y_2)^{-1} = z_3 z_4$. Then we have

$$\tau^{-1} z_1 z_2 F(\tau) = {}^{\tau^{-1}} z_1 L(\tau)^{F(\tau)^{-1}} z_2 = z_3 z_4,$$

where $L(\tau) = \tau^{-1} F(\tau)$. As $z_4 \in \mathbb{U}^1$ we can have

$${}^{F(\tau)^{-1}} z_2 z_4^{-1} = h_+ h_0 h_- \in \mathbb{U}\mathbb{T}\mathbb{U}^-,$$

where $h_+ \in \mathbb{U}^1$, $h_0 \in \mathbb{T}^1$ and $h_- \in (F\mathbb{U} \cap \mathbb{U}^-)\mathbb{U}^{-,1} = F(\mathbb{U}\mathbb{U}^{-,1}) \cap \mathbb{U}^-$. Hence

$${}^{\tau^{-1}} z_1{}^{L(\tau)} h_+ L(\tau) h_0 h_- = z_3.$$

It follows that $z_1 = {}^{F(\tau)} h_+^{-1}$, $L(\tau) = h_0^{-1}$ and $z_3 = h_-$. Therefore,

$${}^\flat\Sigma_w^{2N-m} = \{(\tau, z_2, z_4) \in \mathbb{T} \times (F\mathbb{U} \cap \mathbb{U}^-) \times (F\mathbb{U}^{-,1} \cap \mathbb{U}); L(\tau) = \mathrm{pr}_0({}^{F(\tau)^{-1}} z_2 z_4^{-1})\},$$

where $\mathrm{pr}_0 : \mathbb{U}^1\mathbb{T}\mathbb{U}^- \to \mathbb{T}$ is the natural projection.

Note that $(t, t') \in \mathbb{T}^F \times \mathbb{T}^F$ acts on ${}^\flat\Sigma_w^i$ by $(\tau, z_2, z_4) \mapsto (t\tau w(t')^{-1}, {}^t z_2, {}^{w(t')} z_4)$. Now we define and action of $s \in \mathbb{T}$ on ${}^\flat\Sigma_w^i$ by $(\tau, z_2, z_4) \mapsto (\tau, {}^s z_2, {}^s z_4)$. Then the actions of $\mathbb{T}$ and $\mathbb{T}^F \times \mathbb{T}^F$ commutes with each other. Thus, by Lemma 3.10 we have

$$H_*(\Sigma_w^{2N-m}) = H_*({}^\flat\Sigma_w^{2N-m}) = H_*(({}^\flat\Sigma_w^{2N-m})^{\mathbb{T}}) = H_*(\dot{w}\mathbb{T}^F)$$

as desired. □

Proposition 3.12. *Suppose that Condition (∗) holds and that Δ is connected. Then $H_c^*(\Sigma_w^0) = H_c^*(\dot{w}\mathbb{T}^F)$ for $w \in W_e^{c\sigma}$.*

Proof. Let $w \in W_e^{c\sigma}$. As Δ is connected, we may write $w = (c\sigma)^m$ for some $m \in \mathbb{Z}$. By Corollary 3.7 we have

(a) $$H_*(\Sigma_w^i) = \sum_{u \in W_e^{c\sigma}} H_*(Y_{w,u}^i), \quad H_*(\Sigma_w^{i+1}) = \sum_{u \in W_e^{c\sigma}} H_*(Y_{u,w}^i).$$

First we assume $\sigma_I \neq 1$. By Lemma 3.8 for any $w', u' \in W_e^{c\sigma}$ we have $Y_{w',u'}^i \neq \emptyset$ if and only if $w' = u'$. It follows by (a) that

$$H_*(\Sigma_w^i) = H_*(Y_{w,w}^i) = H_*(\Sigma_w^{i+1}).$$

By Lemma 3.11 we have $H_*(\Sigma_w^0) = H_*(\Sigma_w^{N-m}) = H_*(w\mathbb{T}^F)$ as desired.

Now we assume $\sigma_I = 1$. Let notation be as in Lemma 2.5. We can assume that $w = c_I^m$ with $0 \leqslant m \leqslant 2N - 1$. If $0 \leqslant m \leqslant N$, it follows from (a), Lemma 3.8 and Lemma 3.11 that

$$H_*(\Sigma_w^0) = H_*(\Sigma_w^1) = \cdots = H_*(\Sigma_w^{N-m}) = H_*(\dot{w}\mathbb{T}^F).$$

If $N+1 \leqslant m \leqslant 2N-1$, similarly we have

$$H_*(\Sigma_w^0) = H_*(\Sigma_w^1) = \cdots = H_*(\Sigma_w^{2N-m}) = H_*(\dot{w}\mathbb{T}^F).$$

So the statement follows. □

4. Proof of Theorem 2.1

In this section, we fill in the proof for Theorem 2.1. First we show that it suffices to consider one particular Coxeter element.

Lemma 4.1. *Let $\alpha \in \{\alpha_1, \sigma^{-1}(\alpha_r)\}$ such that $c' = s_\alpha c \sigma(s_\alpha)$. Suppose Theorem 2.1 holds for (μ, c). Then it also holds for $(s_\alpha(\mu), c')$.*

Proof. Let μ, e, I be as in Theorem 2.1. Let $e' = e_{s_\alpha(\mu),c'} = s_\alpha(e)$ and $\Phi_{e'} = s_\alpha(\Phi_e)$. Assume that $I = (i_1 < \cdots < i_m)$. Without loss of generality we can assume $\alpha = \sigma^{-1}(\alpha_r)$ and $c' = s_{\alpha'_1} s_{\alpha'_2} \cdots s_{\alpha'_r}$ with $\alpha'_1 = \alpha_r$ and $\alpha'_i = \alpha_{i-1}$ for $2 \leqslant i \leqslant r$.

First we assume that $r \in I$. Then $r = i_m$ and $\sigma_{I,c}(\alpha) < 0$, which means that $\alpha \notin \Delta_e = \sigma_{I,c}(\Delta_e)$. Thus $\Phi_{e'}^+ = s_\alpha(\Phi_e^+)$ since $\alpha \in \Delta$ is a simple root. In particular, $\Delta_{e'} = s_\alpha(\Delta_e)$. We take

$$I' = (1 < i_1 + 1 < i_2 + 1 < \cdots < i_{m-1} + 1).$$

Then $\sigma_{I',c'} = s_\alpha \sigma_{I,c} s_\alpha$, $c'_{I'} = s_\alpha c_I s_\alpha$ and the statement follows.

Now we assume that $r \notin I$. Then $\sigma_{I,c}(\alpha) \in \Delta_{I,c} \subseteq \Delta_e = \sigma_{I,c}(\Delta_e)$. Thus $\alpha \in \Delta_e$ and $\Delta_{e'} = \Delta_e$. We take

$$I' = (i_1 + 1 < i_2 + 1 < \cdots < i_m + 1).$$

Then $\sigma_{I',c'} = \sigma_{I,c}$, $c'_{I'} = s_\alpha c_I \sigma_{I,c}(s_\alpha)$ and the statement also follows. □

To finish the proof, we will take a particular σ-Coxeter element c such that and verify the statement directly. Moreover, we can assume Δ is connected.

Let P be the coweight lattice of Φ. If $\mu = 0 \in P/(1 - c\sigma)P$, then $\Delta_e = \Delta$ and the statement is trivial. So we may assume that $P/(1-c)P \neq \{0\}$, which excludes the types ${}^2A_{n-1}$ (n odd), ${}^2D_n$, ${}^3D_4$, E_8, ${}^2E_6$, F_4, G_2. Then we will take a case-by-case analysis for the remaining types.

We adopt the labelling of Dynkin diagrams by positive integers as in [Hum72]. For $i \in \mathbb{Z}_{\geqslant 1}$. let s_i and $\omega_i^\vee$ denote the corresponding simple reflection and fundamental coweight, respectively.

Case (1): Δ is of type A_{n-1}. Take $c = s_1 s_2 \cdots s_{n-1}$. Then we have $P/(1 - c\sigma)P = \{0, \omega_1^\vee, \omega_2^\vee, \ldots \omega_{n-1}^\vee\}$. Assume $\mu = \omega_k^\vee$ with $k \in \mathbb{Z}$. Let $m = \gcd(k, n) \in \mathbb{Z}_{\geqslant 1}$. Then we take I to be the complement of the sequence $I' = (n/m, 2n/m, \cdots, (m-1)n/m)$.

Case (2): Δ is of type ${}^2A_{n-1}$ with $n \geqslant 4$ even. Take $c = s_1 s_2 \cdots s_{n/2}$. Then $P/(1 - c\sigma)P = \{0, \omega_1^\vee\}$. Assume $\mu = \omega_1^\vee$. Then we take $I = (n/2)$.

Case (3): Δ is of type B_n with $n \geqslant 2$. Take $c = s_1 s_2 \cdots s_n$. Then $P/(1 - c\sigma)P = \{0, \omega_1^\vee\}$. Assume $\mu = \omega_1^\vee$. Then we take $I = (n)$.

Case (4): Δ is of type C_n with $n \geqslant 3$. Take $c = s_1 s_2 \cdots s_n$. Then $P/(1 - c\sigma)P = \{0, \omega_n^\vee\}$. Assume $\mu = \omega_n^\vee$. Then we take $I = (1, 3, \dots, n - \frac{(-1)^n+1}{2})$.

Case (5): Δ is of type D_n with $n \geqslant 4$. Take $c = s_1 s_2 \cdots s_n$. Then $P/(1 - c\sigma)P = \{0, \omega_1^\vee, \omega_{n-1}^\vee, \omega_n^\vee\}$. If $\mu = \omega_1^\vee$, take $I = (n-1, n)$. It remains to handle the case $\mu = \omega_{n-1}^\vee$ by symmetry. If n is even, take $I = (1, 3, \dots, n-3, 4)$ if $4 \mid n$ and $I = (1, 3, \dots, n-3, n-1)$ if $4 \nmid n$. If n is odd, take $I = (1, 3, \dots, n-4, \dots, n-2, n)$.

Case (6): Δ is of type E_6. Take $c = s_1 s_3 s_4 s_2 s_5 s_6$. Then $P/(1 - c\sigma)P = \{0, \omega_1^\vee, \omega_6^\vee\}$. By symmetry we can assume $\mu = \omega_1^\vee$. Then take $I = (1, 3, 5, 6)$.

Case (7): Δ is of type E_7. Take $c = s_7 s_6 s_5 s_4 s_2 s_3 s_1$. Then $P/(1 - c\sigma)P = \{0, \omega_7^\vee\}$. If $\mu = \omega_7^\vee$, take $I = (7, 5, 2)$.

5. Quotients of the Coxeter variety

The goal of the rest of the article is to prove Theorem 1.3. Therefore, mainly following [Lus76a, §2], we investigate quotients of p-adic Deligne–Lusztig spaces of Coxeter type by the unipotent radical of a rational Borel subgroup resp. of a maximal parabolic subgroup. We apply this at the end of §6 to deduce a proof of Theorem 1.3.

5.1. **Notation.** We keep the notation from the introduction and §2.1, except for the following important change: *from now on we assume that G is unramified and that the Borel subgroup $B \subseteq G$ is k-rational.* We denote by $w_0 \in W$ the longest element (relative to S). If $v \in W$ is given, then by $\dot{v}$ we mean an arbitrary lift of v to $N_G(T)(\breve{k})$.

For $b \in G(\breve{k})$ and a subgroup $H \subseteq G$ we denote by $H_b(k)$ the F-centralizer of b in $H(\breve{k})$, that is $H_b(k) = \{h \in H : h^{-1} b F(h) = b\}$.

We use the setup from [Iva23a]. In particular, we denote by Perf the category of perfect $\overline{\mathbb{F}}_q$-algebras. For a $\breve{k}$-scheme X we write LX for the loop space of X, i.e., the functor $LX\colon \mathrm{Perf} \to \mathrm{Sets}$, $R \mapsto X(\mathbb{W}(R)[\varpi^{-1}])$, where $\mathbb{W}(R)$ is the unique ϖ-adically complete and separated $\mathcal{O}_k$-algebra in which ϖ is not a zero divisor and which satisfies $\mathbb{W}(R)/\varpi\mathbb{W}(R) = R$ (see [Iva23a, page 6] for details).

5.2. **Recollections on p-adic Deligne–Lusztig spaces.** To $w \in W$ and $b \in G(\breve{k})$, [Iva23a, Definition 7.3] attaches a p-adic Deligne–Lusztig space $X_w(b)$ equipped with a continuous $G_b(k)$-action ($G_b(k)$ is locally profinite and equals the group of k-points of an inner form of a Levi subgroup of G). The definition of $X_w(b)$ parallels the classical Deligne–Lusztig variety from [DL76]. Formally, $X_w(b)$ is an arc-sheaf on the category of perfect $\overline{\mathbb{F}}_q$-algebras; it is known to be ind-representatble in many cases. Moreover, for a lift $\dot{w} \in N_G(T)(\breve{k})$ one has a pro-étale $T_w(k)$-torsor $\dot{X}_{\dot{w}}(b)$ over a clopen subset of $X_w(b)$ [Iva23a, §10], where T_w is the $\breve{k}$-split form of the torus T given by twisting the Frobenius F by $\mathrm{Ad}(w)$.

To relate this with the previous part of the article, consider the case $w = c$ is a σ-Coxeter element and a lift $\dot{c} \in N_G(T)(\breve{k})$. If $T' \subseteq B' \subseteq G$ is a k-rational torus unramified and of type c and a Borel subgroup (rational over $\breve{k}$), such that $B', F(B')$ are in relative position c, then there is an $G_{\dot{c}}(k) \times T_c(k) \cong G_{\dot{c}}(k) \times T'(k)$-equivariant isomorphism $X_{T',U'} \cong \dot{X}_{\dot{c}}(\dot{c})$, where U' is the unipotent radical of B'.

5.3. **Embedding into the big cell.** Let $s_1, s_2, \dots, s_n$ be a sequence of pairwise distinct elements in S. Let $w = s_1 s_2 \dots s_n \in W$. Let $\alpha_i \in \Delta \subseteq \Phi^+$ be the simple root corresponding to s_i. Fix an isomorphism $\psi_i \colon U_{\alpha_i} \cong \mathbb{G}_a$ of the corresponding root subgroup with the additive group. We have the open subscheme $\mathbb{G}_m \subseteq \mathbb{G}_a$ and we put $U_{\alpha_i}^* := \psi_i^{-1}(\mathbb{G}_m)$.

Lemma 5.1 (Proposition 2.2 of [Lus76a])**.** *Let $\tau \in T(\breve{k})$. As locally closed subvarieties of G, we have*

$$(\tau U^-) \cap BwB = \{\tau v_1 v_2 \dots v_n \colon v_i \in (U_{-\alpha_i})^* \text{ for } 1 \leq i \leq n\}.$$

In particular, $U^- \cap BwB \cong \prod_{i=1}^n \mathbb{G}_m$.

Proof. As BwB is stable under left multiplication by τ, we may assume that $\tau = 1$. In this case the proof of [Lus76a, Prop. 2.2] for reductive groups over $\overline{\mathbb{F}}_q$ carries over to the present situation, using the geometric Bruhat decomposition for the split group $G_{\breve{k}}$. □

Lemma 5.2 (Lemma 2.3 of [Lus76a])**.** *Suppose any σ-orbit on S contains at most one s_i. Fix an algebraically closed field $\mathfrak{f} \supseteq \overline{\mathbb{F}}_q$, and let $\tilde{\mathfrak{f}} = \mathbb{W}(\mathfrak{f})[1/\varpi]$. Let $v \in W$, such that*

$$\dot{v}^{-1} u F(\dot{v}) = B(\tilde{\mathfrak{f}}) \dot{w} B(\tilde{\mathfrak{f}}) = (B\dot{w}B)(\tilde{\mathfrak{f}}),$$

for some $u \in U(\tilde{\mathfrak{f}})$. Then $F(v) = v$ and $v(\alpha_i) \in \Phi^-$ for $1 \leq i \leq n$.

Proof. The proofs of [Lus76a, Lemmas 2.3 and 2.4] carry over *verbatim.* □

For each $v \in W$, $U \cap {}^vU^- \to BvB/B$, $u \mapsto u\dot{v}B$ is an isomorphism of $\breve{k}$-schemes. In particular, $L(BvB/B) \cong L(U \cap {}^vU^-)$ is an ind-scheme. We can now show the analogue of [Lus76a, Cor. 2.5].

Proposition 5.3. *Suppose $b \in T(\breve{k})$, and w is a σ-Coxeter element. The natural inclusion $X_w(b) \hookrightarrow L(G/B)$ factors through the big cell $L(Bw_0B/B) \subseteq L(G/B)$.*

Proof. Let $g \in X_w(b)(R) \subseteq L(G/B)(R)$. We must show that $g \in L(B\dot{w}_0B/B)(R)$. By [Iva23a, Corollary 8.4], we may replace R by an arc-cover. Hence, by [Iva23a, Corollary 6.4] we may assume that g lifts to some $\dot{g} \in LG(R)$. It suffices to show that $\dot{g}\colon \operatorname{Spec} R \to LG$ factors through $L(Bw_0B) \subseteq LG$. Since $LG, L(Bw_0B)$ are perfect (hence reduced) ind-schemes, it suffices to do this on geometric points. Hence we may assume $R = \mathfrak{f}$ is an algebrically closed field; let $\tilde{\mathfrak{f}} = \mathbb{W}(\mathfrak{f})[1/\varpi]$. Note that we have $\dot{g}^{-1}b\sigma(\dot{g}) \in L(BwB)(\mathfrak{f}) = (BwB)(\tilde{\mathfrak{f}})$. We may argue as in the proof of [Lus76a, Corollary 2.5]: as $\tilde{\mathfrak{f}}$ is a field, and G is split over $\tilde{\mathfrak{f}}$, $G(\tilde{\mathfrak{f}})$ admits a Bruhat decomposition. Thus there is some $v \in W$ such that $\dot{g} = u\dot{v}\lambda$ for some $\lambda \in B(\tilde{\mathfrak{f}})$, $u \in U(\tilde{\mathfrak{f}})$. By the preceding paragraph we deduce $\dot{v}^{-1}u^{-1}bF(u)F(\dot{v}) = \dot{g}^{-1}bF(\dot{g}) \in (B\dot{w}B)(\tilde{\mathfrak{f}})$. By assumption $b \in T(\breve{k})$, and we deduce $\dot{v}^{-1}u'F(\dot{v}) \in (B\dot{w}B)(\tilde{\mathfrak{f}})$, where $u' = (u^b)^{-1}F(u) \in U(\tilde{\mathfrak{f}})$. Thus we may apply Lemma 5.2 (in the same way as in the proof of [Lus76a, Corollary 2.5]) to deduce that $v = w_0$. This shows our claim. □

5.4. **$U_b(k)$-quotient of $X_w(b)$.** We now show the analogue of [Lus76a, Theorem 2.6 and Corollary 2.7]. *Suppose $w = s_1 \dots s_n$ is a σ-Coxeter element and let $b \in T(\breve{k})$.* Consider the morphism

$$U^*_{-w_0(\alpha_1)} \times U^*_{-w_0(\alpha_2)} \times \cdots \times U^*_{-w_0(\alpha_n)} \longrightarrow bU, \quad (v_i)_{i=1}^n \longmapsto bv_1v_2\dots v_n,$$

with image the locally closed subscheme $b\cdot\prod_{i=1}^n U^*_{-w_0(\alpha_i)} \subseteq bU$. The product depends on the order of the factors. Define $X_w(b)'$ by the Cartesian diagram

$$\begin{array}{ccccc} X_w(b)' & \longrightarrow & L\left(b\cdot\prod_{i=1}^n U^*_{-w_0(\alpha_i)}\right) & = & b\cdot\prod_{i=1}^n LU^*_{-w_0(\alpha_i)} \\ \downarrow & & \downarrow & & \\ LU & \xrightarrow{u\longmapsto u^{-1}bF(u)} & b\cdot LU, & & \end{array} \tag{5.1}$$

where the lower map is well-defined as b normalizes LU. Note that $B_b(k)$ acts on $X_w(b)'$ by left multiplication.

Lemma 5.4. *Let $b \in T(\breve{k})$. Then $u \mapsto u^{-1}bF(u)\colon LU \to b\cdot LU$ is $U_b(k)$-torsor for the pro-étale topology. The upper map in (5.1) is a pro-étale $U_b(k)$-torsor. Moreover, $U_b(\breve{k})$ is the group generated by all $U_\alpha(\breve{k})$ with $\alpha \in \Phi^+$, $\mathrm{ord}_\varpi(\alpha(b)) = 0$.*

Proof. The second claim follows from the first. For the first claim, it is enough to show surjectivity of $\mathrm{La}_b\colon LU \to LU$, $u \mapsto (u^b)^{-1}F(u)$ for the pro-étale topology. Let the *hight* of a root $\alpha \in \Phi^+$ be the smallest integer $\mathrm{ht}(\alpha) \geq 1$, such that α can be written as a sum of $\mathrm{ht}(\alpha)$ simple roots. For $i \geq 1$, let $U_{\leq i}$ be the quotient of LU by the subsheaf generated by all LU_α with $\alpha \in \Phi^+$, $\mathrm{ht}(\alpha) > i$. Let $U_{=i}$ be the subsheaf of $U_{\leq i}$ generated by LU_α with $\mathrm{ht}(\alpha) = i$. Then $U_{=i} = \ker(U_{\leq i} \twoheadrightarrow U_{\leq i-1})$ is central in $U_{\leq i}$. Using this and induction on i, it suffices to show that La_b induces a surjection $U_{=i} \to U_{=i}$. But $U_i \cong \prod_{\alpha\colon \mathrm{ht}(\alpha)=i} L\mathbb{G}_a$, La_b stabilize all factors, and the result follows from Lemma 5.5 below. □

Lemma 5.5. *Let $\beta \in \breve{k}^\times$. Consider the map $\mathrm{La}_{\beta,\varphi}\colon L\mathbb{G}_a \to L\mathbb{G}_a$, $x \mapsto \beta x - \varphi(x)$. If $\mathrm{ord}_\varpi(\beta) \neq 0$, $\mathrm{La}_{\beta,\varphi}$ is an isomorphism. If $\mathrm{ord}_\varpi(\beta) = 0$, $\mathrm{La}_{\beta,\varphi}$ is a pro-étale torsor under the locally profinite group k.*

Proof. It suffices to show that if $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ is strictly henselian, and $x \in \mathbb{W}(R)[1/\varpi]$, then there exists an (unique if $\mathrm{ord}_\varpi(\beta) \neq 0$) element $y \in \mathbb{W}(R)[1/\varpi]$ with $\beta y - \varphi(y) = x$. This reduces to an explicit computation, using the (uniquely determined) ϖ-adic expansions $x = \sum_i [x_i]\varpi^i$, $y = \sum_i [y_i]\varpi^i$ in $\mathbb{W}(R)[1/\varpi]$. □

Proposition 5.6. *Suppose that $b \in T(\breve{k})$ and w is a σ-Coxeter element. Then*

$$X_w(b)' \xrightarrow{\sim} X_w(b), \quad u \longmapsto u\dot{w}_0 B$$

is an $B_b(k)$-equivariant isomorphism. Moreover, it induces a $T(k)$-equivariant isomorphism

$$X_w(b)/U_b(k) \cong \prod_{i=1}^n LU^*_{-w_0(\alpha_i)}.$$

Proof. By Proposition 5.3, we have the inclusion $X_w(b) \hookrightarrow L(B\dot{w}_0B/B) \xleftarrow{\sim} LU$, where the second isomorphism is $u \mapsto u\dot{w}_0$. This realizes $X_w(b)$ as a subsheaf of LU. To show that it agrees with $X_w(b)'$, we compute for any $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ and any $u \in LU(R)$:

$$\begin{aligned} u\dot{w}_0LB \in X_w(b)(R) \Leftrightarrow \dot{w}_0^{-1}u^{-1}bF(u)F(\dot{w}_0) \in L(b^{w_0} \cdot U^- \times_G B\dot{w}B)(R) &= \\ \overset{\text{Lm. 5.1}}{=} b^{w_0} \cdot \prod_{i=1}^n LU^*_{-\alpha_i}(R) & \\ \Leftrightarrow u^{-1}bF(u) \in \dot{w}_0 b^{w_0} \cdot \prod_{i=1}^n LU^*_{-\alpha_i}(R)\dot{w}_0^{-1} = b \cdot \prod_{i=1}^n LU^*_{-w_0(\alpha_i)}(R), & \end{aligned}$$

where we use that L commutes with finite products. The $B_b(k)$-equivariance is immediate, and the last claim is immediate from (5.1). □

We mention the following special case of Proposition 5.6, which is new to the p-adic setting due to the presence of regular σ-conjugacy classes. Recall that a σ-conjugacy class $[b] \subseteq G(\breve{k})$ is called *regular*, if there is some $\mu \in X_*(T)$ with $[b] = [\varpi^\mu]$ and $\langle \alpha, \mu \rangle \neq 0$. If b is regular, then $G_b = B_b = T$ and $U_b = 1$ (the latter also follows from Lemma 5.4).

Corollary 5.7. *Assume $b \in T(\breve{k})$ is regular. Then the map in (5.1) induces a $G_b(k) = T(k)$-equivariant isomorphism $X_w(b) \cong \prod_{i=1}^n LU^*_{-w_0(\alpha_i)}$.*

In particular, $X_w(b)$ is a disjoint union of affine schemes, there is a $T(k)$-equivariant isomorphism $\pi_0(X_w(b)) \cong X_*(T_{\mathrm{ad}})$ and any connected component of $X_w(b)$ is isomorphic to $L^+\mathbb{G}_m$.

Proof. This follows from Proposition 5.6 as $U_b = 1$ and $G_b = B_b = T$. □

5.5. $U_b(k)$**-quotient of** $\dot{X}_{\dot{w}}(b)$**.** Now we deduce an analogue of Proposition 5.6 for the spaces $\dot{X}_{\dot{w}}(b)$. Let $\widetilde{W} = N_G(T)(\breve{k})/T(\mathcal{O}_{\breve{k}})$ be the extended affine Weyl group and let

$$F_w = \text{fiber of } N_G(T) \longrightarrow W \text{ over } w \quad \text{ and } \quad \overline{F}_w = \text{fiber of } \widetilde{W} \longrightarrow W \text{ over } w$$

We regard F_w as a trivial T-torsor over $\breve{k}$ and $\overline{F}_w$ as a trivial $X_*(T)$-torsor over $\overline{\mathbb{F}}_q$ (in particular, $\pi_0(LF_w) = \overline{F}_w$). Recall from [Iva23a, §10] that we have maps $\kappa_w \colon LT \twoheadrightarrow X_*(T) \overset{\bar{\kappa}_w}{\twoheadrightarrow} X_*(T)_{\langle\sigma_w\rangle}$, where $\sigma = q^{-1}F$ is the automorphism of $X_*(T)$, and $\sigma_w = \mathrm{Ad}(w) \circ \sigma$ and that we have a natural map $\alpha_{w,b} \colon X_w(b) \to L\overline{F}_w/\ker\bar{\kappa}_w$ so that for any $\bar{w} \in L\overline{F}_w/\ker\bar{\kappa}_w$, $X_w(b)_{\bar{w}} = \alpha^{-1}(\bar{w})$ is clopen (possibly empty) $G_b(k)$-stable subset of $X_w(b)$ and $\dot{X}_{\dot{w}}(b) \to X_w(b)_{\bar{w}}$ is a pro-étale $T_w(k)$-torsor for any lift $\dot{w}$ of $\bar{w}$.

For a root $\alpha \in \Phi$, let $s_\alpha \in W$ denote the corresponding reflection. As in [BT72, 6.1.2(2)] we have the $\breve{k}$-subscheme $M_\alpha^\circ \subseteq F_{s_\alpha}$ and an isomorphism

$$m = m_\alpha \colon U_{-\alpha}^* \xrightarrow{\sim} M_\alpha^\circ \quad u \longmapsto m(u),$$

where for $u \in U_{-\alpha}(\breve{k})$, $m(u)$ is the unique element of $F_w(\breve{k})$ such that $u \in U_\alpha(\breve{k})m(u)U_\alpha(\breve{k})$.

Lemma 5.8. *Let $w = s_1 \dots s_n$ be as in the beginning of §5.3. Let $\dot{w} \in F_w(\breve{k})$. For $\tau \in T(\breve{k})$ we have*

$$\tau U^- \cap U\dot{w}U = \tau \cdot \left\{ \prod_{i=1}^n v_i \in \prod_{i=1}^n U_{-\alpha_i}^* \colon \prod_{i=1}^n m(v_i) = \tau^{-1}\dot{w} \right\}. \tag{5.2}$$

Proof. Multiplying both sides by τ^{-1} and using that τ^{-1} normalizes U, we may assume that $\tau = 1$. For better readability, we write U for $U(\breve{k})$ in the proof. As $v_i \in Um(v_i)U$ by construction of $m(\cdot)$, and as $U\dot{y}_1U\dot{y}_2U = U\dot{y}_1\dot{y}_2U$ whenever $y_1, y_2 \in W$ and $\dot{y}_i \in F_{y_i}$ with $\ell(y_1) + \ell(y_2) = \ell(y_1y_2)$, the right side of (5.2) is contained in the left side. For the converse, if $x \in U^- \cap U\dot{w}U$, then by Lemma 5.1, $x = \prod_{i=1}^n v_i$ with $v_i \in U_{-\alpha_i}^*$ and if $\prod_{i=1}^n m(v_i) = \dot{w}'$ for some $\dot{w}' \in F_w$, the above argument shows that $x \in U^- \cap U\dot{w}'U$, so we must have $\dot{w}' = \dot{w}$. ☐

From now on assume that $w = s_1 \dots s_n$ is a σ-Coxeter element and that $b \in T(\breve{k})$. Consider the closed sub-ind-scheme of $LT \times \prod_{i=1}^n LU_{-w_0(\alpha_i)}^*$,

$$Z_{\dot{w}}(b) = \left\{ (\tau, (v_i)_{i=1}^n) \in LT \times \prod_{i=1}^n LU_{-w_0(\alpha_i)}^* \colon \tau\dot{w}\sigma(\tau)^{-1} = (b^{w_0}) \cdot \prod_{i=1}^n m(v_i) \right\}.$$

The map

$$Z_{\dot{w}}(b) \longrightarrow b \cdot \prod_{i=1}^n LU_{-w_0(\alpha_i)}^* \overset{\text{Prop.5.6}}{\cong} X_w(b)/U_b(k), \quad (\tau, v_1, \dots, v_n) \longmapsto b \cdot \prod_{i=1}^n v_i$$

realizes $Z_{\dot{w}}(b)$ as a pro-étale $T_w(k)$-torsor over a clopen subset of the target.

Proposition 5.9. *Let w be a σ-Coxeter element and $b \in T(\breve{k})$. Then $\dot{X}_{\dot{w}}(b)/U_b(k) \cong Z_{\dot{w}}(b)$ and there is a cartesian diagram*

$$\begin{array}{ccc} \dot{X}_{\dot{w}}(b) & \longrightarrow & Z_{\dot{w}}(b) \\ \downarrow{\scriptstyle T_w(k)} & & \downarrow{\scriptstyle T_w(k)} \\ X_w(b) & \longrightarrow & X_w(b)/U_b(k) \end{array}$$

With other words, $\dot{X}_{\dot{w}}(b)$ is $B_b(k)\times T_w(k)$-equivariantly isomorphic to the set of all $(\tau,u) \in LT \times LU$ for which $u^{-1}b\sigma(u) = b\cdot\prod_{i=1}^n v_i \in \prod_{i=1}^n LU^*_{-w_0(\alpha_i)}$ and $\tau\dot{w}\sigma(\tau)^{-1} = b^{w_0}\cdot\prod_{i=1}^n m(v_i)$.

Proof. As $\sigma(w_0) = w_0$, we may (using Lang's theorem) choose a lift $\dot{w}_0 \in N_G(T)(\breve{k})$ with $\sigma(\dot{w}_0) = \dot{w}_0$. By Proposition 5.3, $\dot{X}_{\dot{w}}(b) \hookrightarrow L(G/U)$ factors through the preimage $L(U\dot{w}_0B/U) \subseteq L(G/U)$ of $L(U\dot{w}_0B/B)$. Now, $u\dot{w}_0\tau LU \in L(U\dot{w}_0B/U)$ lies in $\dot{X}_{\dot{w}}(b)$ if and only if $(u\dot{w}_0\tau)^{-1}b\sigma(u\dot{w}_0\tau) \in L(U\dot{w}U)$, or equivalently, if and only if $(u^{-1}b\sigma(u))^{\dot{w}_0} \in L(U\tau\dot{w}\sigma(\tau)^{-1}U)$. As $(u^{-1}b\sigma(u))^{\dot{w}_0} \in b^{w_0}\cdot LU^-$, the last condition is equivalent to

$$\left(u^{-1}b\sigma(u)\right)^{\dot{w}_0} \in L\left(b^{w_0}U^- \cap U\tau\dot{w}\sigma(\tau)^{-1}U\right)$$

which by Lemma 5.8 is equivalent to

$$\left(u^{-1}b\sigma(u)\right)^{\dot{w}_0} \in b^{w_0}\cdot\left\{\prod_i v_i \in \prod_{i=1}^n LU^*_{-\alpha_i} \colon \prod_{i=1}^R m(v_i) = (b^{w_0})^{-1}\tau\dot{w}\sigma(\tau)^{-1}\right\},$$

Conjugating both sides by $\dot{w}_0$, and renaming the variable ${}^{\dot{w}_0}v_i \in LU^*_{-w_0(\alpha_i)}$ to v_i, we obtain the proposition. ☐

As a corollary we obtain a description of the map $\alpha_{w,b}$ from [Iva23a, §10] in this case. Denote by ψ the map $\prod_{i=1}^n LU^*_{\alpha_i} \to F_w$, $(v_i)_{i=1}^n \mapsto \prod_{i=1}^n m(v_i)$.

Lemma 5.10. *Let $w = s_1 \dots s_n$ be σ-Coxeter. The image of the composed map*

$$\bar{\psi}\colon \prod_{i=1}^n LU^*_{-\alpha_i} \xrightarrow{L\psi} LF_w \longrightarrow L\overline{F}_w/\ker\overline{\kappa}_w, \quad (v_i)_{i=1}^n \longmapsto \prod_{i=1}^n m(v_i)$$

is equal to $\mathrm{im}(L\overline{F}^{\mathrm{sc}}_w/\ker\overline{\kappa}^{\mathrm{sc}}_w \to L\overline{F}_w/\ker\overline{\kappa}_w)$, *where* $(\cdot)^{\mathrm{sc}}$ *denote the object* $(\cdot)$ *for the simply connected cover of the derived group of* G.

Proof. We may assume that G is semisimple and simply connected. Then it suffices to show that $\prod_{i=1}^n LM^\circ_{\alpha_i} \to LF_w \to LF_w/\ker\kappa_w$ is surjective. For any coroot $\alpha^\vee \in \Phi^\vee$, let $T_{\alpha^\vee} \subseteq T$ denote its image. For each i choose some $\dot{s}_i \in M^\circ_{\alpha_i}(\breve{k})$, so that $M^\circ_{\alpha_i} = T_{\alpha_i^\vee}\dot{s}_i$. Then $\dot{w} := \dot{s}_1\dot{s}_2\dots\dot{s}_n \in F_w$. For $0 \le i < n$ let $\theta_i = s_1\dots s_i(\alpha_{i+1})$ (we have $\{\theta_i\}_{i=0}^{n-1} = \Phi \cap {}^{w^{-1}}\Phi^-$, cf. [Bou68, p. 158]). Trivializing all torsors reduces us to showing that the natural map

$$LT_{\theta_0^\vee} \times LT_{\theta_1^\vee} \times \cdots \times LT_{\theta_{n-1}^\vee} \longrightarrow LT/\ker\kappa_w \cong X_*(T)_{\langle\sigma_w\rangle}$$

is surjective. It suffices to show that the composition

$$X_*(T_{\theta_0^\vee}) \times X_*(T_{\theta_1^\vee}) \times \cdots \times X_*(T_{\theta_{n-1}^\vee}) \xrightarrow{\phi} X_*(T) \twoheadrightarrow X_*(T)_{\langle\sigma_w\rangle}$$

is surjective As $G^{\rm sc} = G$, we have $X_*(T) = \mathbb{Z}\Phi^\vee$ and the claim follows. ☐

Note that each $X_w(b)_{\bar{w}}$ is $G_b(k)$-stable, $X_w(b)/U_b(k)$ is the disjoint union of the clopen pieces $X_w(b)_{\bar{w}}/U_b(k)$. Let $\bar{\psi}$ be as in Lemma 5.10.

Corollary 5.11. *Let w be a σ-Coxeter element and $b \in T(\breve{k})$. Let $\bar{w} \in L\overline{F}_w/\ker\bar{\kappa}_w$. Under the isomorphism from Proposition 5.6, $X_w(b)_{\bar{w}}/U_b(k)$ corresponds to the subset of $\prod_{i=1}^n LU^*_{-w_0(\alpha_i)}$ cut out by the equation $\bar{w} = \bar{\kappa}_w(b)\bar{\psi}(\prod_i m(v_i)) \in L\overline{F}_w/\ker\bar{\kappa}_w$. In particular,*

$$\mathrm{im}(\alpha_{w,b}) = \bar{\kappa}_w(b)\cdot \mathrm{im}(L\overline{F}_w^{\rm sc}/\ker\overline{\kappa}_w^{\rm sc} \longrightarrow L\overline{F}_w/\ker\overline{\kappa}_w).$$

Proof. Passing to $LF_w/\ker\kappa_w$, the equation in the definition of $Z_{\dot{w}}(b)$ becomes $\bar{w} = \bar{\kappa}_w(b)\bar{\psi}(\prod_i m(v_i))$. By Proposition 5.9 all claims follow from this and Lemma 5.10. ☐

5.6. **Quotients by the unipotent radical of a parabolic.** Let $I \subseteq S/\langle\sigma\rangle$ be a subset, let $S_I \subseteq S$ be its preimage in S, $W_I \subseteq W$ be the corresponding parabolic subgroup; P_I the unique parabolic subgroup of G containing B and $U_{-\alpha}$ for all $\alpha \in I$; U_I the unipotent radical of P_I; G_I the unique Levi subgroup of P_I containing T. Then P_I, U_I, G_I are k-rational. Let $G'_I = P_I/U_I$ and denote the natural projection by $\pi\colon P_I \to G'_I$; the composition $G_I \to P_I \xrightarrow{\pi} G'_I$ is an isomorphism. Note that $\pi(B)$ is a Borel subgroup of G'_I. Let $\Pi_I \subseteq \Pi$ be the set of simple roots α corresponding to elements of S_I. Put $\Phi_I = \mathbb{Z}\Pi_I \cap \Phi$ and $\Phi_I^\pm = \Phi_I \cap \Phi^\pm$.

Write $n = |S/\langle\sigma\rangle|$ and assume that $|I| = n-1$. Let $w = s_1 \dots s_n \in W$ be a σ-Coxeter element. Then there is a unique index j, such that ${}^{w_0}s_j \notin S_I$. Let w_0^I denotes the longest element of W_I. Then ${}^{w_0^I w_0}s_i \in S_I$ for all $i \neq j$. Thus

$$w_I = {}^{w_0^I w_0}(s_1 \dots s_{j-1}s_{j+1}\dots s_n)$$

is a σ-Coxeter element of W_I. Denote by $X_{w_I}^{G'_I}(b)$ the corresponding p-adic Deligne–Lusztig space for the group G'_I.

Lemma 5.12.

(i) We have a well-defined map $G/B \supseteq Bw_0B/B \to G'_I/\pi(B)$ defined by sending $b_1\dot{w}_0B$ to $\pi(b_1)\dot{w}_0^I\pi(B)$.

(ii) Let $b \in T(\breve{k})$. The restriction of the map from (i) to $X_w(b)$ defines a $P_{I,b}(k)$-equivariant map $X_w(b) \to X_{w_I}^{G'_I}(b)$, where $P_{I,b}(k)$ acts on $X_{w_I}^{G'_I}(b)$ via its quotient in $G_I(k)$. This induces a $G'_b(k)$-equivariant map

$$\pi''\colon X_w(b)/U_{I,b}(k) \longrightarrow X_{w_I}^{G'_I}(b).$$

(iii) There exists a map $\alpha\colon X_w(b) \to LU^*_{-w_0(\alpha_j)}$, *such that* $\pi''\times\alpha\colon X_w(b)/U_{I,b}(k) \to X^{G'_I}_{w_I}(b) \times LU^*_{-w_0(\alpha_j)}$ *is an isomorphism.*

Proof. (i) is an immediate computation. Then (ii) follows from Proposition 5.6. The proof of (iii) is the same as that of the first claim of [Lus76a, Corollary 2.10]. □

We explicate the isomorphism of Lemma 5.12(iii). By Proposition 5.6, $X_w(b)$ identifies with the set of all $u \in LU$ satisfying $u^{-1}\dot{w}F(u) = b^{\dot{w}_0}\prod_{i=1}^n u_i$ with $u_i \in LU^*_{-w_0(\alpha_i)}$. There is a unique writing $u = u'u''$ with $u' \in U_I$ and $u'' \in U \cap G_I$. Then $u^{-1}bF(u) = u''^{-1}bgF(u'')$ where $u' \mapsto g = b^{-1}u'^{-1}bF(u')\colon LU \to LU$ defines a pro-étale $U_{I,b}(k)$-torsor. Thus $X_w(b)/U_{I,b}(k)$ identifies with the set of all (u'', g) such that

$$u''^{-1}bgF(u'') = \prod_{i=1}^n u_i \quad \text{with } u_i \in LU^*_{-w_0(\alpha_i)}. \tag{5.3}$$

Applying π and noting that it induces an isomorphism $G_I \to G'_I$, we see that all u_i for $i \neq j$ are uniquely determined by u'' and that $u'' \in X^{G'_I}_{w_I}(b)$ (under the identification of Proposition 5.6). It follows then that those $g \in U_I$ for which (5.3) holds, are in bijection with all $u_j \in LU^*_{-w_0(\alpha_j)}$, so that $(u'', g) \mapsto (u'', u_j)$ is an isomorphism as claimed in Lemma 5.12(iii).

Now we describe the quotient $\dot{X}_{\dot{w}}(b)/U_{I,b}(k)$. Consider

$$Z_{I,\dot{w}}(b) = \{(\tau, u'', u_j) \in LT \times L(U\cap G_I) \times LU_{-w_0(\alpha_j)} \colon \\ \tau^{-1}\dot{w}F(\tau) = b^{\dot{w}_0}\prod_{i=1}^n m(u_i),\ u''^{-1}bF(u'') = \prod_{\substack{i=1\\ i\neq j}}^n u_i \in \prod_{i\neq j} LU_i^*\},$$

where u_i $(i \neq j)$ are determined by u'' as above, and the last equality takes place in G_I. Then $Z_{I,\dot{w}}(b)$ is a pro-étale $T_w(k)$-torsor over $X^{G'_I}_{w_I}(b) \times LU^*_{-w_0(\alpha_j)}$.

Lemma 5.13. *There is an* $T(k)$*-equivariant isomorphism* $\dot{X}_{\dot{w}}(b)/U_{I,b}(k) \cong Z_{I,\dot{w}}(b)$, *and* $Z_{I,\dot{w}}(b)$ *fits into the diagram with cartesian squares,*

$$\begin{array}{ccccc} \dot{X}_{\dot{w}}(b) & \longrightarrow & Z_{I,\dot{w}}(b) & \longrightarrow & Z_{\dot{w}}(b) \\ \downarrow & & \downarrow & & \downarrow \\ X_w(b) & \longrightarrow & X_w(b)/U_{I,b}(k) & \longrightarrow & X_w(b)/U_b(k) \end{array}$$

where the outer square is as in Proposition 5.9, and all vertical maps are pro-étale $T_w(k)$*-torsors.*

Proof. It is clear from the definitions, that the right square is cartesian. This implies that there is a natural map $\dot{X}_{\dot{w}}(b) \to Z_{I,\dot{w}}(b)$. As the outer square is cartesian by Proposition 5.9, the left square has to be cartesian too. This implies the first claim of the lemma, and the $T_w(k)$-equivariance is clear. $\square$

6. Quotients on the integral/finite level

We investigate the analogues of the results from §5 for deep level Deligne–Lusztig varieties. We assume that $b = 1$ (only possibility with $b \in T(\breve{k})$ and basic). Let $\mathcal{G}$ be a hyperspecial model of G over $\mathcal{O}_k$. Let $w = s_1 \dots s_n \in W$ be σ-Coxeter element and let $\dot{w} \in \mathcal{G}(\mathcal{O}_{\breve{k}})$ be a lift of w. Let $\mathbb{G} = \mathbb{G}_r (= L_r^+\mathcal{G})$ with $r \leq \infty$ be as in the introduction. We have the Deligne–Lusztig variety $X_w = X_{w,r} \subseteq \mathbb{G}/\mathbb{B}$ and the $\mathbb{T}_r^F$-torsor $\dot{X}_{\dot{w}} = \dot{X}_{\dot{w},r} \subseteq \mathbb{G}/\mathbb{U}$ over it (as in [DI24, Definition 4.1.1]). Note that by [DI24, Lemma 4.1.2] there is an $\mathbb{G}^F \times \mathbb{T}^F$-equivariant isomorphism

$$\dot{X}_{\dot{w},r} \cong X = \{g \in \mathbb{G} \colon g^{-1}F(g) \in \overline{\mathbb{U}} \cap F\mathbb{U}\}. \tag{6.1}$$

Let $\pi \colon \mathbb{G} \to \mathbb{G}_1$ denote the natural projection map.

Lemma 6.1. *We have $X_{w,r} \subseteq \mathbb{B}\dot{w}_0\mathbb{B}$.*

Proof. $\pi(X_{w,r}) \subseteq X_{w,1}$ and $X_{w,1} \subseteq \mathbb{B}_1\dot{w}_0\mathbb{B}_1$ by [Lus76a, Cor. 2.5]. Thus $X_{w,r} \subseteq \pi^{-1}(\mathbb{B}_1\dot{w}_0\mathbb{B}_1) = \mathbb{B}\dot{w}_0\mathbb{B}$, the last equality being true since w_0 is the longest element of W. $\square$

Lemma 6.2. *Let $v, v' \in W$ with $\ell(vv') = \ell(v) + \ell(v')$. Then $\mathbb{B}\dot{v}\mathbb{B}\dot{v}'\mathbb{B} = \mathbb{B}\dot{v}\dot{v}'\mathbb{B}$.*

Proof. Let $\alpha \in \Phi^+$ be the simple root corresponding to s. For part (i), we are reduced by induction to the case that $v' = s \in S$ and $\ell(vs) = \ell(v) + 1$. Then we have $v(\alpha) \in \Phi^+$. It follows that $\mathbb{B}\dot{v}\mathbb{B}\dot{s}\mathbb{B} = \mathbb{B}\dot{v}\mathbb{U}_\alpha\dot{s}\mathbb{B} = \mathbb{B}\dot{v}\dot{s}\mathbb{B}$, where in the first step we move all $\mathbb{U}_\beta$ with $\beta \neq \alpha$ into the right $\mathbb{B}$, and in the second step we move $\mathbb{U}_\alpha$ into the left $\mathbb{B}$, using $v(\alpha) \in \Phi^+$. $\square$

For $\alpha \in \Phi$, we write $\mathbb{U}_\alpha^*$ for the open complement of $\mathbb{U}_\alpha^1$ in $\mathbb{U}_\alpha$. We have the following analogue of [Lus76b, Proposition 2.2].

Lemma 6.3. *We have*

$$\mathbb{U}^- \cap \mathbb{B}\dot{w}\mathbb{B} = \{v_1 \dots v_n \colon v_i \in (\mathbb{U}_{-\alpha_i})^*\},$$

Proof. Let $v_i \in \mathbb{U}_{-\alpha_i}^*$. Then we claim that $v_i \in \mathbb{B}s_i\mathbb{B}$. Indeed, let $G_{\alpha_i} \subseteq G$ be the subgroup generated by $U_{\alpha_i}, U_{-\alpha_i}, T$, and let $\mathbb{G}_{\alpha_i} \subseteq \mathbb{G}$ be the corresponding subgroup. Then $v_i \in \mathbb{G}_{\alpha_i}$, and it suffices to show the claim for $\mathbb{G}_{\alpha_i}$ instead of $\mathbb{G}$, which reduces to an explicit computation in $L_r^+\mathrm{SL}_2$, which uses the assumption that $v_i \notin \mathbb{U}_{\alpha_i}^1$. Using the claim,

$$v = \prod_i v_i \in \mathbb{B}\dot{s}_1\mathbb{B}\dot{s}_2 \dots \dot{s}_{n-1}\mathbb{B}s_n\mathbb{B} = \mathbb{B}\dot{w}\mathbb{B},$$

by Lemma 6.2. This shows one inclusion. For the converse, assume that $x \in \mathbb{U}^- \cap \mathbb{B}\dot{w}\mathbb{B}$. Then just as in [Lus76a, Proof of (2.2)], we may write $x = u_1\dot{s}_1u_2\dot{s}_2 \dots u_n\dot{s}_nb$ with $u_i \in \mathbb{U}_{\alpha_i}$ and $b \in \mathbb{B}$. Suppose that $u_1 \in \mathbb{U}_{\alpha_1}^1$.

Then consider the image $\bar{x} \in \mathbb{U}_{\alpha_1}^- \cap \mathbb{B}_1\dot{w}\mathbb{B}_1$ of x under $\pi\colon \mathbb{G} \to \mathbb{G}_1$, which is again of the form $\bar{x} = \bar{u}_1\dot{s}_1\bar{u}_2\dot{s}_2\dots\bar{u}_n\dot{s}_n\bar{b}$, with $\bar{u}_1 \neq 1$ in $(\mathbb{U}_{\alpha_1})_1$. As in *loc. cit.* this gives a contradiction. Thus we must have $u_1 \in \mathbb{U}_{\alpha_1}^*$. Similar as in *loc. cit.*, a computation in the group $\mathbb{G}_{\alpha_i}$ shows that there exist $u_1' \in \mathbb{U}_{\alpha_1}$, $v_1 \in (\mathbb{U}_{\alpha_1}^-)^*$, $t \in \mathbb{T}$ with $u_1 = v_1u_1't\dot{s}_1$. Using this, we see that

$$v_1^{-1}x = u_1't\dot{s}_1^2u_2\dot{s}_2\dots u_n\dot{s}_nb = u_1'tu_2\dot{s}_2\dots u_n\dot{s}_nb \in \mathbb{B}\dot{s}_2\mathbb{B}\dots\dot{s}_n\mathbb{B} = \mathbb{B}\dot{s}_2\dots\dot{s}_n\mathbb{B}.$$

Thus $v_1^{-1}x \in \mathbb{U}^- \cap \mathbb{B}\dot{s}_2\dots\dot{s}_n\mathbb{B}$, and we are done by induction. □

Note that Lemma 6.3 does not follow directly from [Lus76a, Prop. 2.2] as both sides of the equation are not equal to the preimages of their images under $\pi\colon \mathbb{G} \to \mathbb{G}_1$. Now we can generalize [Lus76a, Theorem 2.6].

Proposition 6.4. *We have the following isomorphisms:*

(i) $\{u \in \mathbb{U}\colon u^{-1}F(u) = u_1\dots u_n\colon u_i \in \mathbb{U}_{-w_0(\alpha_i)}^*\ \forall\, 1 \leq i \leq n\} \xrightarrow{\sim} X_{w,r}, \quad u \mapsto u\dot{w}_0\mathbb{B}$

(ii) $\mathbb{U}_{-w_0(\alpha_1)}^* \times \cdots \times \mathbb{U}_{-w_0(\alpha_n)}^* \xrightarrow{\sim} X_{w,r}/\mathbb{U}^F$.

Proof. This follows from Lemmas 6.1 and 6.3. □

Let $\alpha \in \Phi$. The map $m = m_\alpha\colon U_{-\alpha}^* \to M_\alpha^\circ$ from §5.5 induces an isomorphism

$$m_\alpha\colon \mathbb{U}_{-\alpha}^* \xrightarrow{\sim} \mathbb{M}_\alpha^\circ,$$

where $\mathbb{M}_\alpha^\circ$ is the preimage of w in $\mathbb{G}$. Then just as in §5.5 we have the scheme with $\mathbb{B}^F \times \mathbb{T}_w^F$-action

$$Z_{\dot{w},r} = \{(\tau, (v_i)_{i=1}^n) \in \mathbb{T} \times \prod_{i=1}^n \mathbb{U}_{-w_0(\alpha_i)}^*\colon \tau\dot{w}\sigma(\tau)^{-1} = \prod_{i=1}^n m(v_i)\}$$

equipped with the $\mathbb{B}^F$-equivariant map $(\tau, (v_i)_i) \mapsto (v_i)_i\colon Z_{\dot{w},r} \to X_{w,r}/\mathbb{U}^F$. With notation as in §5.6 we also have the scheme

$$Z_{I,\dot{w},r}(b) = \{(\tau,u'', u_j) \in \mathbb{T} \times (\mathbb{U} \cap \mathbb{G}_I) \times \mathbb{U}_{-w_0(\alpha_j)}\colon \tau^{-1}\dot{w}F(\tau) = b^{\dot{w}_0}\prod_{i=1}^n m(u_i),\ u''^{-1}bF(u'') = \prod_{\substack{i=1\\ i\neq j}}^n u_i \in \prod_{i\neq j}\mathbb{U}_i^*\},$$

and just as in §5.6 we have the following consequence of Proposition 6.4.

Corollary 6.5.

(1) There is $\mathbb{B}^F$*-equivariant isomorphism*

$$\dot{X}_{\dot{w},r}/\mathbb{U}^F \xrightarrow{\sim} Z_{\dot{w},r}$$

and $X_{\dot{w},r} = X_{w,r} \times_{X_{w,r}/\mathbb{U}^F} Z_{\dot{w},r}$.

(2) There is $\mathbb{T}_w^F$*-equivariant isomorphism*

$$\dot{X}_{\dot{w},r}/\mathbb{U}_I^F \xrightarrow{\sim} Z_{I,\dot{w},r}$$

and $\dot{X}_{\dot{w},r} = X_{w,r} \times_{X_{w,r}/\mathbb{U}_I^F} Z_{I,\dot{w},r}$.

6.1. **Extension of action.** Let the notation be as in §5.6. Let $\mu \in X_*(T)$ be be such that $\langle \alpha_j, \mu \rangle \neq 0$ (such μ exists). Put $\mu' = \langle \alpha_j, \mu \rangle \cdot {}^{s_1 \cdots s_{j-1}}(\alpha_j^\vee) \in X_*(T)$. As $w\sigma - 1\colon X_*(T^{\mathrm{sc}})_{\mathbb{Q}} \to X_*(T^{\mathrm{sc}})_{\mathbb{Q}}$ is bijective, we may (after replacing μ by an integral multiple, if necessary) assume that there is some $\lambda \in X_*(T^{\mathrm{sc}})$ with $\mu' = w\sigma(\lambda) - \lambda$.

Lemma 6.6. *With notation as above, there is an action of $\mathbb{G}_m$ on $Z_{I,\dot{w},r}$ given by the formula*

$$x\colon (\tau, u'', u_j) \longmapsto (\lambda(x)\tau, u'', {}^{\mu(x)}u_j)$$

for any $x \in \mathbb{G}_m$. Moreover, this $\mathbb{G}_m$-action commutes with the action of $\mathbb{T}_w^F$ and $Z_{I,\dot{w},r}^{\mathbb{G}_m} = \varnothing$.

Proof. Let $(\tau, u'', u_j) \in Z_{I,\dot{w},r}$ and let $u_i \in \mathbb{U}_{-w_0(\alpha_i)}$ (for $i \neq j$) be determined by u'' as above. The first sentence of the lemma follows from the computation

$$\begin{aligned}
(\lambda(x)\tau)^{-1}\dot{w}F(\lambda(x)\tau) &= \mu'(x)\tau^{-1}\dot{w}F(\tau) \\
&= \mu'(x)\prod_{i=1}^{n} m(u_i) \\
&= \prod_{i=1}^{j-1} m(u_i)(\alpha_j^\vee)^{\langle \alpha_j, \mu \rangle}(x) m(u_j) \prod_{i=j+1}^{n} m(u_i) \\
&= \prod_{i=1}^{j-1} m(u_i) m({}^{\mu(x)}u_j) \prod_{i=j+1}^{n} m(u_i)
\end{aligned}$$

where the last equality follows from a property of the map $m(\cdot)$, which can be checked by an explicit calculation after reducing to SL_2. The last sentence of the lemma is immediate. ☐

Proof of Theorem 1.3. By [DI24, Corollary 1.0.1] (or Theorem 1.1) and (6.1) we know that $H_c^*(X)[\chi] = H_c^*(\dot{X}_{\dot{w},r})[\chi]$ is up to sign an irreducible $\mathbb{G}^F$-representation. Thus, exploiting a theorem of Bushnell [Bus90, Theorem 1] as in the proof of [CI23, Theorem 6.1, Proposition 6.2], it suffices to show that for any maximal proper subset $I \subseteq S/\langle\sigma\rangle$, the virtual $\overline{\mathbb{Q}}_\ell$-vector space $H_c^*(X_{\dot{w},r}^{\mathcal{G}}/\mathbb{U}_I^F, \overline{\mathbb{Q}}_\ell)_\theta$ vanishes. But this follows directly from Lemma 6.6, as

$$H_c^*(X_{\dot{w},r}^{\mathcal{G}}/\mathbb{U}_I^F, \overline{\mathbb{Q}}_\ell)_\theta = H_c^*(Z_{I,\dot{w},r}, \overline{\mathbb{Q}}_\ell)_\theta = H_c^*(Z_{I,\dot{w},r}^{\mathbb{G}_m}, \overline{\mathbb{Q}}_\ell)_\theta = 0,$$

where the last equality follows from [DM91, 10.15]. ☐

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