MEROMORPHIC VECTOR BUNDLES ON THE FARGUES–FONTAINE CURVE

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Abstract. We introduce and study the stack of *meromorphic G-bundles* on the Fargues–Fontaine curve. This object defines a correspondence between the Kottwitz stack $\mathfrak{B}(G)$ and Bun_G. We expect it to play a crucial role in comparing the schematic and analytic versions of the geometric local Langlands categories. Our first main result is the identification of the generic Newton strata of $\mathrm{Bun}_G^{\mathrm{mer}}$ with the Fargues–Scholze charts $\mathcal{M}$. Our second main result is a generalization of Fargues' theorem in families. We call this the *meromorphic comparison theorem*. It plays a key role in proving that the analytification functor is fully faithful. Along the way, we give new proofs to what we call the *topological and schematic comparison theorems*. These say that the topologies of Bun_G and $\mathfrak{B}(G)$ are reversed and that the two stacks take the same values when evaluated on schemes.

Contents

1. Introduction

Let p be a prime number, let $E/\mathbb{Q}_p$ be a finite field extension, let ℓ be a prime with $\ell \neq p$ and $\Lambda = \overline{\mathbb{Q}}_\ell$. Let G be a connected reductive group over E. Let W_E be the Weil group and let ${}^LG = \hat{G} \rtimes W_E$ be the L-group.

1.1. **Motivation and context.** Let Π_G be the set of isomorphism classes of smooth irreducible representations of the locally profinite group $G(E)$ with values in Λ and let Φ_G be the set of $\hat{G}$-conjugacy classes of L-parameters. The basic form of the local Langlands correspondence gives a map

$$\mathrm{LLC}_G : \Pi_G \to \Phi_G$$

satisfying some properties [Kal16, Conjecture A], [SZ18]. For GL_n the map $\mathrm{LLC}_{\mathrm{GL}_n}$ is bijective [HT01, Hen00], but this does not hold more generally. Nevertheless, LLC_G has finite fibers that are called L-packets and understanding them is the subject of the refined local Langlands correspondence.

For quasi-split groups, one can fix a Whittaker datum $\mathfrak{w}$ to put the elements of an L-packet in canonical bijection with the set of isomorphisms classes of certain finite group constructed in terms of the L-parameter [Kal16, Conjecture B]. When G is not quasi-split Whittaker data do not exist. Vogan realized that to work with general G it is advantageous to consider its quasi-split inner form G^* and parametrize simultaneously the representations of all the pure inner twists of G^* [ABV92, Vog93].

Motivated by the study of special fibers of Shimura varieties, Kottwitz introduced the set $B(G)$ of isocrystals with G-structure [Kot85, Kot97]. The set of basic elements $B(G)_{\mathrm{bas}}$ gives rise to the so-called extended pure inner forms G_b of G. Kottwitz formulated a refined version of the local Langlands correspondence for non-Archimedean local fields using the inner forms that arise from $B(G)_{\mathrm{bas}}$ [Kal16, Conjecture F] [SZ18].

The set $B(G)$ can be realized as the underlying topological space of two geometric objects. One object is of analytic nature, Bun_G (the stack of G-bundles on the Fargues–Fontaine curve) and a second object is of schematic nature, $\mathfrak{B}(G)$ (the Kottwitz stack parametrizing isocrystals with G-structure). For every element $b \in B(G)$ one can define locally closed substrata $i_b : \mathfrak{B}(G)_b \to \mathfrak{B}(G)$ and $j_b : \mathrm{Bun}_G^b \to \mathrm{Bun}_G$. Interestingly, whenever $b \in B(G)_{\mathrm{bas}}$ both $\mathfrak{B}(G)_b$ and Bun_G^b agree with the classifying stack $[*/G_b(E)]$ for the extended pure inner form of G defined by b.[1] This leads to the hope that the refined local Langlands correspondence of Kottwitz has a categorical refinement that one can access by studying the geometry of the stacks Bun_G and/or $\mathfrak{B}(G)$.

Recent breakthroughs in p-adic and perfect geometry [SW20, FS21, Zhu17, XZ17, BS17, Zhu20] together with the introduction and study of the stack of L-parameters [DHKM20, Zhu20, FS21], have led experts to formulate precise conjectures that capture this hope. These efforts promote, in a precise way, the refined local Langlands correspondence mentioned above to a categorical statement [FS21, Zhu20, Hel23, BZCHN22].

There is widespread agreement of what to consider on the Galois side, namely a version of the derived category of coherent sheaves $\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda})$ of the stack $\mathcal{X}_{\hat{G},\Lambda}$ parametrizing L-parameters over Λ (see [FS21, Conjecture

[1]Even when b is not basic, the category of étale sheaves on Bun_G^b and $\mathfrak{B}(G)_b$ can be understood in terms of the representation category of a pure inner form of a Levi subgroup.

I.10.2], [AG15]). On the automorphic side, there are at least two reasonable constructions of the local Langlands category. The essential difference between them arises from the fact that $B(G)$ has two geometric incarnations.

Let G be quasi-split and let $W_{\mathfrak{w}}$ be the Whittaker representation associated to $\mathfrak{w}$. On the analytic side, Fargues–Scholze construct the category of lisse sheaves $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ [FS21, § VII.7] and prove it is compactly generated. Moreover, they endow this category with the so-called spectral action by the category of perfect complexes $\mathrm{Perf}(\mathcal{X}_{\hat{G},\Lambda})$. They conjecture that there is a unique $\mathrm{Perf}(\mathcal{X}_{\hat{G},\Lambda})$-linear equivalence of ∞-categories

$$\mathbb{L}_G^{\mathrm{an}} : \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\omega} \cong \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda})$$

that sends the analytic Whittaker sheaf $\mathcal{W}^{\mathrm{an}}_{\mathfrak{w}} = j_{1,!}W_{\mathfrak{w}}$ to the structure sheaf $\mathcal{O}_{\mathcal{X}_{\hat{G},\Lambda}}$ where both objects are treated as elements of their ind-completions.

On the schematic side, Xiao–Zhu consider the moduli stack of local shtukas $\mathrm{Sht}^{\mathrm{loc}}_k$ in the context of characteristic p perfect geometry. They attach their own candidates for the local Langlands category namely they construct a triangulated category of cohomological correspondences $\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_k)$ [XZ17, § 5.4] [Zhu20]. This approach is pushed further in the forthcoming work of Hemo–Zhu [HZ], where they construct an ∞-category $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$ whose homotopy category agrees with $\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_k)$. Zhu conjectures that there is an equivalence

$$\mathbb{L}_G^{\mathrm{sch}} : \mathrm{Shv}(\mathfrak{B}(G), \Lambda) \cong \mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda}))$$

sending $\mathcal{O}_{\mathcal{X}_{\hat{G},\Lambda}}$ to the schematic Whittaker sheaf $\mathcal{W}^{\mathrm{sch}}_{\mathfrak{w}} = i_{1,*}W_{\mathfrak{w}}$ [Zhu20, Conjecture 4.6.4]. Moreover, Hemo–Zhu have announced a proof of the unipotent part of the categorical local Langlands correspondence [Zhu20, Theorem 4.6.11]. Let us clarify. When Λ is of characteristic 0, the stack of L-parameters has an open and closed substack $\mathcal{X}^{\mathrm{unip}}_{\hat{G},\Lambda} \subseteq \mathcal{X}_{\hat{G},\Lambda}$ defining a full subcategory

$$\mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}^{\mathrm{unip}}_{\hat{G},\Lambda})) \subseteq \mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda})).$$

One can also define a full subcategory $\mathrm{Shv}^{\mathrm{unip}}(\mathfrak{B}(G), \Lambda) \subseteq \mathrm{Shv}(\mathfrak{B}(G), \Lambda)$ defined by the property that for all $b \in B(G)$ the restriction to $\mathfrak{B}(G)_b$ is given by a complex of G_b-representations that are unipotent in the sense of Lusztig [Lus95]. Using Bezrukavnikov's equivalence [Bez16], Hemo and Zhu prove that there is an equivalence of ∞-categories

$$\mathbb{L}_G^{\mathrm{sch}} : \mathrm{Shv}^{\mathrm{unip}}(\mathfrak{B}(G), \Lambda) \cong \mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}^{\mathrm{unip}}_{\hat{G},\Lambda})).$$

It is natural to expect that there exists an equivalence

$$\Psi : \mathrm{Shv}(\mathfrak{B}(G), \Lambda) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda),$$

satisfying $\Psi(\mathcal{W}^{\mathrm{sch}}_{\mathfrak{w}}) = \mathcal{W}^{\mathrm{an}}_{\mathfrak{w}}$. Indeed, the two local Langlands categories are conjectured to be equivalent to $\mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda}))$ and if the two conjectures are true one can simply define $\Psi = \mathbb{L}_G^{\mathrm{an},-1} \circ \mathbb{L}_G^{\mathrm{sch}}$.

A reasonable question the reader can ask is: why do we need two local Langlands categories? We believe that it is profitable to construct Ψ directly in order to better understand $\mathbb{L}_G^{\mathrm{sch}}$ and $\mathbb{L}_G^{\mathrm{an}}$. At a technical level, a direct construction of Ψ allows one to transfer Hemo–Zhu's results on unipotent categorical local Langlands correspondence to the Fargues–Scholze setup and conversely, endow $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$ with a spectral action. It would also allow us to formulate rigorously the eigensheaf property for the Deligne–Lusztig sheaves considered in [CI23, Conjecture 9.6]. More philosophically, the schematic perspective and the analytic perspective understand different phenomena. For example, the schematic perspective cannot witness the spectral action because "the paw" is fixed. On the other hand, $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$ is directly related to Bezrukavnikov's equivalence and its Frobenius-twisted categorical trace [Zhu18, §3] since, in contrast with Bun_G, both $\mathfrak{B}(G)$ and the Hecke stack are constructed in terms of Witt vector loop groups.

At the heart of the equivalence Ψ, there should be a geometric explanation. Namely, that the stacks $\mathfrak{B}(G)$ and Bun_G are incarnations of the same geometric object. In this paper, we reveal these geometric relations which we formulate in terms of three comparison theorems (see §10).

One of the achievements of this article is the construction of a third incarnation $\mathrm{Bun}_G^{\mathrm{mer}}$ that mediates between $\mathfrak{B}(G)$ and Bun_G. Roughly speaking, $\mathrm{Bun}_G^{\mathrm{mer}}$ is given by the same moduli problem as Bun_G, but we require a meromorphicity condition on the action of Frobenius (see Definition 5.8, Definition 7.4). This object defines a correspondence

$$\begin{tikzcd} \mathrm{Bun}_G^{\mathrm{mer}} \arrow[r, "\sigma"] \arrow[d, "\gamma"] & \mathrm{Bun}_G \\ \mathfrak{B}(G)^{\diamondsuit} & \end{tikzcd}$$

We call the map γ the *generic polygon map* and σ the *special polygon map* inspired by [KL13, Definition 7.4.1]. Morally, Ψ should be given by $\sigma_! \circ \gamma^* \circ c^*$ where

$$c^* : \mathrm{Shv}(\mathfrak{B}(G), \Lambda) \to \mathcal{D}(\mathfrak{B}(G)^{\diamondsuit}, \Lambda)$$

is an analytification functor [Sch17, §27]. Unfortunately, one can not define $\sigma_!$ naively since $\mathrm{Bun}_G^{\mathrm{mer}}$ is not an Artin v-stack (see §9.1).[2] In particular, the usual 6-functor formalisms considered in the analytic perspective [Sch17, GHW22, Man22] do not suffice to construct $\sigma_!$. For this reason, although we are convinced that our geometric considerations are the key to the construction and study of Ψ, we do not try to compare the local Langlands categories themselves on this work and we leave this comparison for a second article in which we justify the existence of $\sigma_!$ on an appropriate cohomological theory. Nevertheless, to orient the reader, we still provide some informal indication of the cohomological relevance that our main theorems have.

[2] One can work with $\sigma_\natural$ (i.e. the left adjoint of σ^*) which always exists, but this does not avoid the problem. Indeed, $\sigma_\natural \circ \gamma^*$ does not necessarily land in $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G)$.

1.2. **Main results.** For $b \in B(G)$ we let $\mathfrak{B}(G)_b \subseteq \mathfrak{B}(G)$ denote the locally closed substack determined by b. Then $\mathfrak{B}(G)_b^{\diamondsuit} \subseteq \mathfrak{B}(G)^{\diamondsuit}$ is also a locally closed substack and we have an identification

$$\mathfrak{B}(G)_b^{\diamondsuit} = [*/\underline{J_b(E)}].$$

Recall the moduli stack $\mathcal{M}$ of Fargues–Scholze [FS21, Definition V.3.2] that is used to define the smooth charts of Bun_G. It comes endowed with a map

$$q : \mathcal{M} \to \coprod_{b \in B(G)} [*/\underline{J_b(E)}] \cong \coprod_{b \in B(G)} \mathfrak{B}(G)_b^{\diamondsuit}$$

The following Theorem 1.1 is a relative and Tannakian version of Kedlaya's work on the slope filtration [Ked05, Section 5.4], and our first main result.

Theorem 1.1 (Theorem 7.13)**.** *We have a commutative diagram with Cartesian square*

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d, "q"] \arrow[rr, bend left, "\pi"] & \mathrm{Bun}_G^{\mathrm{mer}} \arrow[r, "\sigma"] \arrow[d, "\gamma"] & \mathrm{Bun}_G \\
\coprod_{b \in B(G)} \mathfrak{B}(G)_b^{\diamondsuit} \arrow[r] & \mathfrak{B}(G)^{\diamondsuit} &
\end{tikzcd}$$

In other words, the restriction of $\sigma\colon \mathrm{Bun}_G^{\mathrm{mer}} \to \mathrm{Bun}_G$ to $\gamma^{-1}([*/\underline{J_b(E)}])$ coincides with the Fargues–Scholze chart $\pi_b\colon \mathcal{M}_b \to \mathrm{Bun}_G$ [FS21, V.3].

Remark 1.2. To apply Tannakian formalism one has to take subtle care of the exact structure. We do this by justifying that a sequence is exact if and only if it is exact at every geometric point (see Proposition 5.10). Theorem 1.1 holds for an arbitrary non-Archimedean local field E.

This theorem has a cohomological consequence that we now discuss. Recall from [FS21, Chap. V] that we have identifications

$$\mathcal{D}(\mathrm{Rep}\, J_b(E), \Lambda) \cong \mathcal{D}_{\mathrm{lis}}([*/\underline{J_b(E)}], \Lambda) \cong \mathcal{D}_{\mathrm{lis}}(\mathfrak{B}(G)_b^{\diamondsuit}, \Lambda) \cong \mathcal{D}_{\mathrm{lis}}(\mathfrak{B}(G)_b, \Lambda).$$

Let $i_b : \mathfrak{B}(G)_b \to \mathfrak{B}(G)$ denote the inclusion maps of the Newton strata. The !-pushforward functors define full subcategories

$$\mathrm{Shv}_{b,!}(\mathfrak{B}(G), \Lambda) \subseteq \mathrm{Shv}(\mathfrak{B}(G), \Lambda).$$

On the analytic side we can consider a full subcategory

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\mathcal{M}_b} \subseteq \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$$

obtained as the essential image of the fully faithful functor

$$\pi_{b,!} \circ q_b^* : \mathcal{D}_{\mathrm{lis}}([*/\underline{J_b(E)}], \Lambda) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda).$$

The categories $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)^{\omega}$ and $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\omega}$ have semi-orthogonal decompositions by the subcategories $\mathrm{Shv}_{b,!}(\mathfrak{B}(G), \Lambda)^{\omega}$ and $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\omega}_{\mathcal{M}_b}$ respectively.

One can deduce from Theorem 1.1 that if

$$\sigma_! : \mathcal{D}(\mathrm{Bun}_G^{\mathrm{mer}}, \Lambda)^\omega \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega$$

exists and satisfies proper base change, then $\Psi = \sigma_! \circ \gamma^* \circ c^*$ restricts to an equivalence

$$\Psi : \mathrm{Shv}_{b,!}(\mathfrak{B}(G), \Lambda)^\omega \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega_{\mathcal{M}_b}$$

such that $\Psi(\mathcal{W}^{\mathrm{sch}}_{\mathfrak{w}}) = \mathcal{W}^{\mathrm{an}}_{\mathfrak{w}}$ as ind-objects. In particular, if $\sigma_! \circ \gamma^* \circ c^*$ is fully faithful it is also essentially surjective since every object in $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega$ is a finite colimit of objects in $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega_{\mathcal{M}_b}$ as b varies.

In summary, our first main geometric result would have as a cohomological consequence the essential surjectivity of Ψ.

Remark 1.3. Z. Wu made similar considerations using $\sigma_\natural$ instead of $\sigma_!$ (see Remark 7.14).

Our second main result is related to full faithfulness. Recall the analytification functor $X \mapsto X^\dagger$ obtained from sheafifying the formula

$$(R, R^+) \mapsto X(\mathrm{Spec}\, R^\circ).$$

For any small v-stack X we have a fully faithful map [GL22, Lemma 4.1]

$$\mathcal{D}_{\acute{e}t}(X, \mathbb{F}_\ell) \xrightarrow{c_X^*} \mathcal{D}_{\acute{e}t}(X^\diamondsuit, \mathbb{F}_\ell) \xrightarrow{b_X^*} \mathcal{D}_{\acute{e}t}(X^\dagger, \mathbb{F}_\ell).$$

Theorem 1.4 (Theorem 10.2)**.** *We have the following identification of small v-stacks*

$$\mathrm{Bun}_G^{\mathrm{mer}} \cong \mathfrak{B}(G)^\dagger$$

and an identification of maps $b^*_{\mathfrak{B}(G)} = \gamma^*$. A similar statement holds for the stack of $\mathcal{G}$-shtukas.

Remark 1.5. This is our deepest result and it can be regarded as a version of Fargues' theorem in families (see Remark 10.3). The proof of Theorem 1.4 does not generalize naively to local fields in equal characteristic.

Corollary 1.6. *We have a fully-faithful comparison map*

$$\gamma^* \circ c^* : \mathcal{D}_{\acute{e}t}(\mathfrak{B}(G), \mathbb{F}_\ell) \to \mathcal{D}_{\acute{e}t}(\mathrm{Bun}_G^{\mathrm{mer}}, \mathbb{F}_\ell).$$

Theorem 1.4 provides an approach to prove that Ψ is fully-faithful. Indeed, it suffices to prove that $\sigma_!$ is fully-faithful when restricted to those objects in the essential image of $\gamma^* \circ c^*$. The advantage being that the geometry of $\mathrm{Bun}_G^{\mathrm{mer}}$ is much closer to that of Bun_G than that of $\mathfrak{B}(G)$.

Remark 1.7. We warn the reader that $\mathcal{D}_{\acute{e}t}(\mathfrak{B}(G), \mathbb{F}_\ell)$ does not agree with $\mathrm{Shv}(\mathfrak{B}(G), \mathbb{F}_\ell)$. There is a fully faithful version of $\gamma^* \circ c^*$ for $\mathrm{Shv}(\mathfrak{B}(G), \mathbb{F}_\ell)$, but its target category is not $\mathcal{D}_{\acute{e}t}(\mathrm{Bun}_G^{\mathrm{mer}}, \mathbb{F}_\ell)$. This, among other cohomological subtleties, will be addressed in the follow up project.

Our third main theorem is of technical nature, but it has already found applications outside of the scope of this article. For example, it is a key technical ingredient in Zhang's proof of the integral version of Scholze's fiber product conjecture [Zha23].

Theorem 1.8 (Theorem 8.6). *Every vector bundle on the Fargues–Fontaine curve extends v-locally at* ∞*.*

As a direct consequence we obtain the classification of Corollary 1.9 below. We fix some notation. Let $S = \operatorname{Spa} R$ be a product of points with $R^\circ = \prod_{i\in I} O_{C_i}$ and family of pseudo-uniformizers $\varpi_\infty = (\varpi_i)_{i\in I}$ such that ϖ_∞ defines the topology on R°. Fix $S^\sharp$ an untilt given by a non-zero divisor $\xi_\infty = (\xi_i)_{i\in I}$, this induces for all $i \in I$ an untilt $C_i^\sharp$.

Corollary 1.9. *The following categories are equivalent:*

1. *The category of shtukas over S with paw at $S^\sharp$.*
2. *The category of Breuil–Kisin–Fargues modules over $\mathbb{A}_{\inf}(R^{\circ,\sharp})$.*
3. *The category of I-indexed families $\{(M_i, \Phi_i)\}_{i\in I}$ of Breuil–Kisin–Fargues modules over $\mathbb{A}_{\inf}(O_{C_i^\sharp})$ with uniformly bounded poles and zeroes at ξ_∞.*

1.3. **New proofs of two established results.** As a consequence of our considerations we found new approaches to previously proven theorems relating the geometry of $\mathfrak{B}(G)$ and Bun_G.

1.3.1. *The schematic comparison.* Recall the reduction functor introduced by the first author in [Gle21a] (see also §2). The following theorem is a reformulation of a result of Anschütz [Ans22a], generalized by Pappas–Rapoport [PR21, Theorem 2.3.5]. We clarify and strengthen their approach.

Theorem 1.10 (Theorem 10.4). *We have an identification of scheme theoretic v-stacks*

$$(\mathrm{Bun}_G)^{\mathrm{red}} \cong \mathfrak{B}(G).$$

A similar statement holds for the stack of $\mathcal{G}$-shtukas.

Remark 1.11. We regard Theorem 1.10 as a classicality statement. Anschütz proves the 0-dimensional case using the classification of vector bundles over the Fargues–Fontaine curve. Pappas–Rapoport prove this more generally using the 0-dimensional statement and in particular rely on the φ-structure. We give a uniform proof and work directly with the category of v-vector bundles over $Y_{(0,\infty)}$ showing that classicality is unrelated to the φ-structure. Güthge also realized this independently (see Remark 3.7).

1.3.2. *The topological comparison.* Recall that $B(G)$ comes endowed with a topology induced by its partial order. We can also consider $B(G)^{\mathrm{op}}$ endowed with the opposite topology. Viehmann [Vie23, Theorem 1.1] proves $|\mathrm{Bun}_G|^{\mathrm{op}} \cong B(G)$. Rapoport–Richartz [RR96] and He [He16, Theorem 2.12] prove $|\mathfrak{B}(G)| \cong B(G)$. We give a completely new proof of the following theorem.

Theorem 1.12 (Theorem 10.8). *The natural maps are homeomorphisms:*

$$|\mathrm{Bun}_G|^{\mathrm{op}} \cong |\mathfrak{B}(G)| \cong |\mathfrak{B}(G)^\diamondsuit|$$

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2. Notation, terminology and generalities

Let $\mathbb{F}_q$ be the field of $q = p^n$ elements. We let Perf denote the category of characteristic p perfectoid spaces over $\mathbb{F}_q$ endowed with the v-topology and let PSch the category of characteristic p perfect schemes over $\mathbb{F}_q$. We will consider several topologies on PSch, mainly the scheme theoretic v-topology, the arc-topology and the proétale topology.

We let Sets denote the category of sets, we let Grps denote the $(2,1)$-category of groupoids, we let $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ denote the $(2,1)$-category of closed symmetric monoidal exact categories, and we let $\mathrm{Cat}_1^{\otimes}$ denote the $(2,1)$-category of closed symmetric monoidal categories.

Let $\widetilde{\mathrm{Perf}}$ and $\widetilde{\mathrm{PSch}}$ be the categories of small v-sheaves and small scheme-theoretic v-sheaves respectively. There are several interesting constructions that go from one way to the other, that we will use below.

1. $\diamond : \widetilde{\mathrm{PSch}} \to \widetilde{\mathrm{Perf}}$: given by sheafifying the rule $\mathcal{S}^{\diamond_{\mathrm{pre}}}(R, R^+) = \mathcal{S}(\operatorname{Spec} R^+)$.
2. $\diamondsuit : \widetilde{\mathrm{PSch}} \to \widetilde{\mathrm{Perf}}$: given by sheafifying the rule $\mathcal{S}^{\diamondsuit_{\mathrm{pre}}}(R, R^+) = \mathcal{S}(\operatorname{Spec} R)$.
3. $\dagger : \widetilde{\mathrm{PSch}} \to \widetilde{\mathrm{Perf}}$: given by sheafifying the rule $\mathcal{S}^{\dagger_{\mathrm{pre}}}(R, R^+) = \mathcal{S}(\operatorname{Spec} R^\circ)$.
4. $\mathrm{red} : \widetilde{\mathrm{Perf}} \to \widetilde{\mathrm{PSch}}$: given by $\mathcal{F}^{\mathrm{red}}(\operatorname{Spec} A) = \mathcal{F}(\operatorname{Spd}(A, A))$, where (A, A) is given the discrete topology.
5. $\mathrm{mer} : \widetilde{\mathrm{Perf}} \to \widetilde{\mathrm{Perf}}$ given by sheafifying the rule $\mathcal{F}^{\mathrm{mer}}(R, R^+) = \mathcal{F}(\operatorname{Spd}(R^{\mathrm{dis}}, R^{\mathrm{dis},+}))$ where R^{dis} is R with its discrete topology.

Definition 2.1. We say that an affine scheme $\mathcal{S} = \operatorname{Spec} A$ is a *comb* if for all $x \in \pi_0(\mathcal{S})$ the closed subscheme attached to x is of the form $\operatorname{Spec} V_x$ for V_x a valuation ring. We say that $\mathcal{S}$ is a *strict comb* if the fraction field of all such V_x is algebraically closed. We say that a strict comb is an *extremally disconnected comb* if $\pi_0(\mathcal{S})$ is an extremally disconnected Hausdorff space.

If $A = \prod_{i \in I} V_i$ where V_i is a valuation ring, then we say that $\mathcal{S}$ is a *product comb.*

Observe that strict product combs are extremally disconnected combs.

Definition 2.2. Suppose that $\mathcal{S} \in \mathrm{PSch}$ is a product comb $\mathcal{S} = \operatorname{Spec} A$ and $\varpi \in A$ is a non-zero divisor. Let $R^+ = \widehat{A}_\varpi$ be the ϖ-adic completion of A and let $R = R^+[\frac{1}{\varpi}]$. Then $\operatorname{Spa} R$ is a totally disconnected space and we call any space obtained this way *a product of points.* If $\mathcal{S}$ is in addition strict, then $\operatorname{Spa} R$ is strictly totally disconnected and we call it a *strict product of points.*

Proposition 2.3. *Let $\mathcal{S} = \operatorname{Spec} A$ be a product comb with $A = \prod_{i \in I} V_i$ and $\varpi \in A$ a non-zero divisor. Let $R^+ = \widehat{A}_\varpi$ be the ϖ-adic completion. Let ϖ_i be the image of ϖ in V_i which is also a non-zero divisor. Let $K_i^+ = \widehat{V}_{i,\varpi_i}$ be the ϖ_i-adic completion. Then, the family of projection maps $R^+ \to K_i^+$ induces a ring isomorphism $R^+ = \prod_{i \in I} K_i^+$.*

Proof. Let $I \times \mathbb{N}$ be the partial order with $(i_1, n_1) \leq (i_2, n_2)$ if $i_1 = i_2$ and $n_1 \leq n_2$. We have a functor from $I \times \mathbb{N}$ to the category of rings sending (i, n) to V_i/ϖ_i^n. The constructions of R^+ and $\prod_{i \in I} K_i^+$ correspond to two different ways of computing the limit of this diagram. □

Proposition 2.4. *If* $\operatorname{Spa} R$ *is a (strict) product of points, then* $\operatorname{Spec} R$ *is a (strict) comb.*

Proof. By Proposition 2.3, $R^+ = \prod_{i \in I} C_i^+$ where C_i^+ are valuation rings. Since ultraproducts of valuation rings (with algebraically closed fraction field) are again valuation rings (with algebraically closed fraction field) $\operatorname{Spec} R^+$ is a (strict) comb. Now, since Zariski localizations of (strict) combs are (strict) combs again, $\operatorname{Spec} R$ is a (strict) comb. □

Proposition 2.5. *Let $\underline{G}$ be a locally profinite group then:*

$$[*/\underline{G}]^{\diamondsuit} = [*/\underline{G}]^{\diamond} = [*/\underline{G}].$$

Proof. Let $\operatorname{Spa} R \in \mathrm{Perf}$. Observe that $\underline{G}^{\diamond} = \underline{G}^{\diamondsuit} = \underline{G}$. Since $\diamondsuit$ (respectively $\diamond$) commutes with limits, it suffices to prove that the map $* \to [*/\underline{G}]^{\diamondsuit}$ (respectively $* \to [*/\underline{G}]^{\diamond}$) is surjective. This amounts to showing that if $\mathcal{F}$ is a $\underline{G}$-torsor for the schematic v-topology over $\operatorname{Spec} R$ (respectively $\operatorname{Spec} R^+$), then there is an analytic v-cover of $\operatorname{Spa} R' \to \operatorname{Spa} R$ such that $\mathcal{F}$ restricted to $\operatorname{Spec} R'$ is trivial. We can take $\operatorname{Spa} R'$ to be a strict product of points. Indeed, if follows from a theorem of Gabber [HS21, Theorem 1.5] that every $\underline{G}$-torsor is pro-étale locally trivial. Since $\operatorname{Spec} R'$ (respectively $\operatorname{Spec} R'^+$) are extremally disconnected combs, every pro-étale cover over them splits. □

We fix the following notation throughout the text. We let E be a mixed characteristic non-Archimedean local field, we let $O_E \subseteq E$ denote the ring of integers, we let $\pi \in O_E$ denote a choice of uniformizer, we assume that $\mathbb{F}_q = O_E/\pi$, we denote by $\mathbb{C}_p$ a fixed completed algebraic closure of E.

Let $\mathcal{S} \in \mathrm{PSch}$. If $\mathcal{S} = \operatorname{Spec} A$ for a perfect $\mathbb{F}_q$-algebra A, we let $\mathbb{W}A$ denote the topological ring of O_E-Witt vectors. More precisely, $\mathbb{W}A := \mathcal{W}(A) \otimes_{\mathbb{Z}_p} O_E$, where $\mathcal{W}(A)$ denotes the p-typical Witt vectors. We let $\mathcal{Y}_{\mathcal{S}}$ denote $\operatorname{Spa} \mathbb{W}A$. For general $\mathcal{S}$, we construct $\mathcal{Y}_{\mathcal{S}}$ by glueing on affine charts. We denote by $\varphi : \mathcal{Y}_{\mathcal{S}} \to \mathcal{Y}_{\mathcal{S}}$ the canonical lift of absolute Frobenius on $\mathcal{S}$. We let $Y_{\mathcal{S}} := \operatorname{Spa} \mathbb{W}A[\frac{1}{\pi}]$, this is an analytic sous-perfectoid adic space (indeed, after inverting π, $\mathbb{W}A \to \mathbb{W}A \widehat{\otimes}_{O_E} O_E[\pi^{1/p^\infty}]^\wedge_\pi$ becomes a perfectoid cover that splits by [KH21, Remark 7.2]). In the category of v-sheaves we have the identities $\mathcal{Y}_{\mathcal{S}}^\diamondsuit = \mathcal{S}^\diamond \times \operatorname{Spd} O_E$ and $Y_{\mathcal{S}}^\diamondsuit = \mathcal{S}^\diamond \times \operatorname{Spd} E$.

Similarly, let $S \in \mathrm{Perf}$. Recall that there is a unique sousperfectoid adic space $\mathcal{Y}_S$ (respectively Y_S) over $\operatorname{Spa} O_E$ (respectively $\operatorname{Spa} E$), such that $\mathcal{Y}_S^\diamondsuit = S \times \operatorname{Spd} O_E$ (respectively $Y_S^\diamondsuit = S \times \operatorname{Spd} E$), see for example [FS21, Proposition II.1.2]. If $S = \operatorname{Spa} R$ we let $\mathbb{A}_{\mathrm{inf}}(R^+)$ denote $\mathbb{W}(R^+)$ endowed with the $(\pi, [\varpi])$-adic topology. Then $\mathcal{Y}_S$ is the locus in $\operatorname{Spa} \mathbb{A}_{\mathrm{inf}}(R^+)$ where $[\varpi] \neq 0$ for some pseudo-uniformizer $\varpi \in R^+$, and Y_S is the locus in $\operatorname{Spa} \mathbb{A}_{\mathrm{inf}}(R^+)$ where $\pi \cdot [\varpi] \neq 0$.

3. Families of untilted vector bundles

If $\mathcal{F}$ is a small v-stack endowed with a map $f : \mathcal{F} \to \operatorname{Spd} O_E$, we can consider the ringed site $(\mathcal{F}_{\mathrm{v}}, \mathcal{O}^\sharp)$ whose objects are maps $m : \operatorname{Spa} R \to \mathcal{F}$ where $\operatorname{Spa} R \in \mathrm{Perf}$ and where $R^\sharp$ is the untilt defined by the composition $\operatorname{Spa} R \to \operatorname{Spd} O_E \to \operatorname{Spd} \mathbb{Z}_p$ [SW20, Definition 10.1.3]. By [Sch17, Lemma 15.1], $\mathcal{O}^\sharp$ is a v-sheaf of rings. We let $\mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(\mathcal{F})$ denote the category of vector bundles on this site (i.e. sheaves of $\mathcal{O}^\sharp$-modules that are v-locally isomorphic to a finite direct sum of $\mathcal{O}^\sharp$). Recall that if $\operatorname{Spa} R^\sharp$ is a perfectoid space over $\operatorname{Spa} O_E$ inducing a map $\operatorname{Spa} R \to \operatorname{Spd} O_E$, by v-descent of vector bundles [SW20, Lemma 17.1.8], we get an identity $\mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(\operatorname{Spd} R) = \mathrm{Vect}(\operatorname{Spa} R^\sharp)$. Moreover, we have a sheaf valued in closed exact symmetric monoidal categories

$$\mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp} : \{\text{small v-stacks}/\operatorname{Spd} \mathbb{Z}_p\} \to \mathrm{Cat}_1^{\otimes, \mathrm{ex}}.$$

Recall from [KH21, Definition 9.6] that an analytic Huber pair (A, A^+) over $\mathbb{Z}_p$ is said to be v-complete if $H^0(\operatorname{Spa}(A))_{\mathrm{v}}, \mathcal{O}^\sharp) = A$.

Proposition 3.1. *If (A, A^+) is sheafy and v-complete then the pullback functor $\mathrm{Vect}(\operatorname{Spa} A) \to \mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(\operatorname{Spd} A)$ is fully-faithful. In particular, if (A, A^+) is sous-perfectoid then $\mathrm{Vect}(\operatorname{Spa} A) \to \mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(\operatorname{Spd} A)$ is fully-faithful.*

Proof. Since A is sheafy, by [SW20, Theorem 5.2.8], any object in $\mathrm{Vect}(\operatorname{Spa} A)$ is given by $\widetilde{M}$ for a finite projective A-module M. Since we have internal $\underline{\mathrm{Hom}}$-objects it suffices to prove $H^0((\operatorname{Spd} A)_{\mathrm{v}}, \widetilde{M}_{\mathrm{v}}) = M$. Taking a two step pro-étale hypercover $\operatorname{Spa} R_2 \rightrightarrows \operatorname{Spa} R_1 \to \operatorname{Spa} A$ we see that:

$$H^0((\operatorname{Spd} A)_{\mathrm{v}}, \widetilde{M}_{\mathrm{v}}) = \mathrm{eq.}(M \otimes_A R_1 \rightrightarrows M \otimes_A R_2) = M \otimes_A A$$

Since by hypothesis $H^0((\operatorname{Spd} A)_{\mathrm{v}}, \mathcal{O}^\sharp) = A$.

The second claim follows from [KH21, Corollary 7.4, Lemma 11.4] □

We let $\mathrm{Vect}^{\mathrm{v}}_{\mathcal{Y}} : \mathrm{Perf} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ be given by the rule:

$$S \mapsto \mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(\mathcal{Y}_S^\diamondsuit),$$

where the map $\mathcal{Y}_S^\diamondsuit \to \mathrm{Spd}\, O_E \to \mathrm{Spd}\, \mathbb{Z}_p$ comes from the formula $\mathcal{Y}_S^\diamondsuit = \mathrm{Spd}\, R \times \mathrm{Spd}\, O_E \to \mathrm{Spd}\, O_E$. Similarly, we define $\mathrm{Vect}^{\mathrm{v}}_{Y} : \mathrm{Perf} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ by the rule:

$$S \mapsto \mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(Y_S^\diamondsuit).$$

Proposition 3.2. *The functors* $\mathrm{Vect}^{\mathrm{v}}_{Y}$ *and* $\mathrm{Vect}^{\mathrm{v}}_{\mathcal{Y}}$ *are v-sheaves.*

Proof. Given a v-cover $S_1 \to S$ with Čech nerve $S_\bullet \to S$. Now,

$$\mathrm{Vect}^{\mathrm{v}}_{Y}(S) = \mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(Y_S^\diamondsuit) = \varprojlim \mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(Y_{S_n}^\diamondsuit) = \varprojlim \mathrm{Vect}^{\mathrm{v}}_{Y}(S_n). \tag{3.1}$$

Here the second equality follows from the fact that $Y_{S_1}^\diamondsuit \to Y_S^\diamondsuit$ is a v-cover whose Cech nerve is $Y_{S_\bullet}^\diamondsuit$. □

Recall that by [SW20, Proposition 11.2.1] for any perfectoid S the space $\mathcal{Y}_S$ has a basis of open neighborhoods $\mathrm{Spa}\, A \subseteq \mathcal{Y}_S$ that are v-complete and sheafy. By Proposition 3.1 we obtain fully-faithful functor:

$$\mathrm{Vect}(Y_S) \to \mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(Y_S^\diamondsuit),\ \mathrm{Vect}(\mathcal{Y}_S) \to \mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(\mathcal{Y}_S^\diamondsuit).$$

We can define functors $\mathrm{Vect}^{\mathrm{cl}}_{Y}, \mathrm{Vect}^{\mathrm{cl}}_{\mathcal{Y}} : \mathrm{Perf} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$, given by the rule:

$$S \mapsto \mathrm{Vect}(Y_S),\ S \mapsto \mathrm{Vect}(\mathcal{Y}_S).$$

Proposition 3.3. *The natural maps* $\mathrm{Vect}^{\mathrm{cl}}_{Y} \to \mathrm{Vect}^{\mathrm{v}}_{Y}$ *and* $\mathrm{Vect}^{\mathrm{cl}}_{\mathcal{Y}} \to \mathrm{Vect}^{\mathrm{v}}_{\mathcal{Y}}$ *define subsheaves.*

Proof. It suffices to prove that if $\mathcal{E} \in \mathrm{Vect}^{\mathrm{v}}_{Y}(S)$ and there is a cover $S_1 \to S$ such that $\mathcal{E}_{S_1} \in \mathrm{Vect}^{\mathrm{cl}}_{Y}(S_1)$ then $\mathcal{E} \in \mathrm{Vect}^{\mathrm{cl}}_{Y}(S)$. But this follows from the proof of [SW20, Proposition 19.5.3]. □

Since $\mathrm{Vect}^{\mathrm{cl}}_{Y}$ is a v-sheaf we can evaluate it in any small v-stack $\mathcal{F}$, which gives a fully faithful embedding:

$$\mathrm{Vect}^{\mathrm{cl}}_{Y}(\mathcal{F}) \hookrightarrow \mathrm{Vect}^{\mathrm{v}}_{Y}(\mathcal{F}) \cong \mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(\mathcal{F} \times \mathrm{Spd}\, E).$$

As it turns out, testing classicality can be done at geometric points.

Proposition 3.4. *Let* $\mathcal{F}$ *be a small v-sheaf and let* $\mathcal{E} \in \mathrm{Vect}^{\mathrm{v}}_{Y}(\mathcal{F})$*. Suppose that for every geometric point* $c : \mathrm{Spa}\, C \to \mathcal{F}$ *the object* $c^*\mathcal{E}$ *lies in* $\mathrm{Vect}^{\mathrm{cl}}_{Y}(\mathrm{Spd}\, C)$*. Then* $\mathcal{E} \in \mathrm{Vect}^{\mathrm{cl}}_{Y}(\mathcal{F})$*.*

Proof. Since $\mathrm{Vect}^{\mathrm{cl}}_{Y}$ is a subsheaf, we may work v-locally and assume that the v-sheaf $\mathcal{F}$ is representable by an affinoid perfectoid $S := \mathrm{Spa}\, R$. In this case $\mathcal{E}_S \in \mathrm{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(Y_S^\diamondsuit)$, and we wish to show $\mathcal{E}_S \in \mathrm{Vect}(Y_S)$. This follows from Lemma 3.5, applied to $\mathcal{F} = \mathrm{Spd}\, R$. □

Lemma 3.5. *Let $\mathcal{F}$ be a small v-sheaf. Suppose that the second projection map $\mathcal{F}\times\operatorname{Spd}E\to\operatorname{Spd}E$ is represented by a map of adic spaces $Y_{\mathcal{F}}\to\operatorname{Spa}E$ such that $Y_{\mathcal{F}}$ is sous-perfectoid and $Y_{\mathcal{F}}\times_E\operatorname{Spa}\mathbb{C}_p$ is perfectoid. Suppose that $\mathcal{E}\in\operatorname{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(\mathcal{F}\times\operatorname{Spd}E)$ and that for every geometric point $c:\operatorname{Spa}C\to\mathcal{F}$ the pullback $c^*\mathcal{E}$ lies in $\operatorname{Vect}(Y_C)$, then $\mathcal{E}\in\operatorname{Vect}(Y_{\mathcal{F}})$.*

Proof. The category of sous-perfectoid spaces is stable under rational localization and has a well-behaved theory of vector bundles. Since vector bundles satisfy étale descent it suffices to construct an étale cover $f:U\to Y_{\mathcal{F}}$ for which $f^*\mathcal{E}\in\operatorname{Vect}(U)$. Indeed, such cover will again be sous-perfectoid by [KH21, Lemma 7.5, Remark 5.2], so that $\operatorname{Vect}(U)$ is a full subcategory of $\operatorname{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(U^\diamondsuit)$ and one can transfer all of the descent data. In particular, we can find an open cover $\coprod_{i\in I}U_i\to Y_{\mathcal{F}}$ by affinoid analytic adic spaces. Consider the universally open pro-étale Galois cover $\pi:\widetilde{Y_{\mathcal{F}}}:=\mathcal{F}\times\operatorname{Spd}\mathbb{C}_p\to Y_{\mathcal{F}}^\diamondsuit$, with Galois group $\Gamma:=\operatorname{Gal}(E)$. For every U_i, consider the restriction of $\mathcal{E}$ to $\operatorname{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(U_i^\diamondsuit)$ and let $\widetilde{U}_i:=\pi^{-1}(U_i)$. Since $\widetilde{U}_i$ is by hypothesis perfectoid, then $\pi^*\mathcal{E}\in\operatorname{Vect}(\widetilde{U}_i)$. Fix $y\in\widetilde{U}_i$ and take an affinoid open neighborhood $y\in\widetilde{U}_{y,i}\subseteq\widetilde{U}_i$, such that $\pi^*\mathcal{E}$ is free when restricted to $\widetilde{U}_{y,i}$. By shrinking $\widetilde{U}_{y,i}$ and choosing an open subgroup $\Gamma_y\subseteq\Gamma$ we may always assume that the action of Γ_y on $\widetilde{U}_i$ stabilizes $\widetilde{U}_{y,i}$. Let $U_{y,i}:=\widetilde{U}_{y,i}/\Gamma_y$, in this way $f_y:U_{y,i}\to U_i$ is an étale neighborhood, $\pi_y:\widetilde{U}_{y,i}\to U_{y,i}$ is a proétale Galois cover with Galois group Γ_y and the family of f_y form an étale cover of U_i. Let $K=\mathbb{C}_p^{\Gamma_y}$, let $\tilde{B}_{y,i}$ be the global sections of $\widetilde{U}_{y,i}$ and let $B_{y,i}=\tilde{B}_{y,i}^{\Gamma_y}$ which are also the global sections of $U_{y,i}$. Let n be the rank of $\mathcal{E}$ and let $\mathcal{B}_{y,i}:=M_{n\times n}(\tilde{B}_{y,i})$ which we treat as a p-adic Banach $\mathbb{C}_p$-algebra, by choosing a norm that induces its natural topology. Observe that $\mathcal{B}_{y,i}=M_{n\times n}(B_{y,i})\widehat{\otimes}_K\mathbb{C}_p$. After fixing a basis of $\pi_y^*f_y^*\mathcal{E}_S$ we may identify $\operatorname{End}(\pi_y^*f_y^*\mathcal{E}_S)$ with $\mathcal{B}_{y,i}$, and by transfer of structure the descent datum along π_y translates into a continuous semi-linear representation $\rho_{y,i}:\Gamma_y\to(\mathcal{B}_{y,i})^\times$ with $\rho_{y,i}(\gamma_1\cdot\gamma_2)=\rho_{y,i}(\gamma_1)\cdot[\rho(\gamma_2)^{\gamma_1}]$ as in [Sen93, §2.2]. Moreover, for every geometric point $c:\operatorname{Spa}C\to\mathcal{F}$ we can basechange all of our constructions to obtain a map $\mathcal{B}_{y,i}\to\mathcal{B}_{y,i,c}$ producing a semi-linear continuous representation $\rho_{y,i,c}:\Gamma_y\to\mathcal{B}_{y,i,c}$ that encodes the descent datum of $\mathcal{E}_c:=c^*\mathcal{E}$ along the map $\widetilde{U}_{y,i,c}\to U_{y,i,c}$ (assuming of course that $\widetilde{U}_{y,i,c}$ is non-empty, meaning the projection map $\widetilde{U}_{y,i}^\diamondsuit\to\mathcal{F}$ has non empty fibers along c).

In [Sen93], Sen attaches to any semi-linear continuous representations $\rho:\Gamma_y\to\mathcal{B}^\times$ with values on a $\mathbb{C}_p$-Banach algebra $\mathcal{B}$ an element $\varphi(\rho)\in\mathcal{B}$. This elements has the following properties:

1. (1) $\varphi(\rho)$ is functorial in $\mathcal{B}$. More precisely, given a map of $\mathbb{C}_p$ Banach algebras $f:\mathcal{B}_1\to\mathcal{B}_2$ and a representation $\rho_1:\Gamma_y\to\mathcal{B}_1$, if we let $\rho_2=f\circ\rho_1$ then $\varphi(\rho_2)=f(\varphi(\rho_1))$.
2. (2) $\varphi(\rho)$ detects "locally isomorphic classes" of continuous semi-linear representations. More precisely, if there exists an element $x\in\mathcal{B}^\times$

such that $x\varphi(\rho_1)x^{-1} = \varphi(\rho_2)$ then there exist an open subgroup $\Gamma' \subseteq \Gamma_y$ such that $(\rho_1)_{|\Gamma'}$ is isomorphic to $(\rho_2)_{|\Gamma'}$.

(3) $\varphi(\rho)$ is a topological invariant, i.e. it doesn't depend on the norm of $\mathcal{B}$ only on the topology induced by the norm.

(4) $\varphi(\rho) = 0$ if and only if ρ is trivial when restricted to an open subgroup.

We consider $\varphi(\rho_{y,i}) \in \mathcal{B}_{y,i}$, by hypothesis $\varphi(\rho_{y,i,c}) = 0$ for all $c : \operatorname{Spa} C \to \mathcal{F}$ since $\mathcal{E}_c \in \operatorname{Vect}(Y_C)$ and consequently the restriction to $U_{y,i,c}$ is also classical. This implies that $\varphi(\rho_{y,i}) = 0$. Indeed, it suffices to justify that if $a \in \tilde{\mathcal{B}}_{y,i}$ and $c(a) \in \tilde{\mathcal{B}}_{y,i,c}$ is equal to 0 for every geometric point $c : \operatorname{Spa} C \to S$ then $a = 0$. But $\tilde{\mathcal{B}}_{y,i}$ and each of the $\tilde{\mathcal{B}}_{y,i,c}$ are uniform (even perfectoid) so it suffices to prove that $|a|_x = 0$ for all points $x \in \operatorname{Spa}(\tilde{\mathcal{B}}_{y,i}) = \tilde{U}_{y,i}$. Now, the family of maps $|\operatorname{Spa}(\tilde{\mathcal{B}}_{y,i,c})| \to |\operatorname{Spa}(\tilde{\mathcal{B}}_{y,i})|$ is surjective which proves the claim. This implies that $\rho_{y,i}$ is locally isomorphic to the trivial semi-linear representation. In other words, $\pi_y^* f_y^* \mathcal{E}$ descends to the trivial bundle over the étale neighborhood of $U_{y,S}$ determined by some open subgroup $\Gamma'_y \subseteq \Gamma_y$. Let $U'_{y,i} = \widetilde{U}_{y,i}/\Gamma'_y$, the family of maps $\coprod_{i \in I,y} U'_{y,i} \to Y_{\mathcal{F}}$ is an étale cover over which $\mathcal{E}$ is classical as we needed to construct. □

Corollary 3.6. *Let $\mathcal{S} \in \operatorname{PSch}$ be a perfect scheme and let $\mathcal{F} \subseteq \mathcal{S}^\diamond$ be an open sub-v-sheaf. Then $\mathcal{F} \times \operatorname{Spd} E = Y_{\mathcal{F}}^\diamondsuit$ for a unique sous-perfectoid space $Y_{\mathcal{F}}$ and the natural functor $\operatorname{Vect}(Y_{\mathcal{F}}) \to \operatorname{Vect}_Y^{\mathrm{cl}}(\mathcal{F})$ is an equivalence in $\operatorname{Cat}_1^{\otimes,\mathrm{ex}}$.*

Proof. Recall that $\mathcal{S}^\diamond \times \operatorname{Spd} E$ is represented by $Y_{\mathcal{S}}$, which is sous-perfectoid by [SW20, Proposition 11.2.1]. Since $\mathcal{F} \times \operatorname{Spd} E$ is an open subsheaf of $Y_E^\diamond$, by [Sch17, Lemma 15.6] the former one is also represented by a corresponding sous-perfectoid open subspace of $Y_{\mathcal{S}}$.

Since $Y_{\mathcal{F}}$ is sous-perfectoid, we have a fully faithful embedding $\operatorname{Vect}(Y_{\mathcal{F}}) \to \operatorname{Vect}_{\mathrm{v}}^{\mathcal{O}^\sharp}(Y_{\mathcal{F}}^\diamondsuit) \cong \operatorname{Vect}_Y^{\mathrm{v}}(\mathcal{F})$. Moreover, this map is exact and reflects exactness since exactness can be tested on geometric points of both sides and $|Y_{\mathcal{F}}| = |Y_{\mathcal{F}}^\diamondsuit|$. Essential surjectivity follows from Lemma 3.5. □

We now wish to describe $\operatorname{Vect}_Y^{\mathrm{cl}}(\mathcal{F})$ for three specific types of v-sheaves corresponding to one of the following three setups:

(1) **The schematic setup:** $\mathcal{F} = \operatorname{Spd}(A, A)$ for A a perfect ring in characteristic p endowed with the discrete topology. In this case, $\operatorname{Vect}_Y^{\mathrm{cl}}(\mathcal{F})$ is the category of projective modules over $\mathbb{W}A[\frac{1}{\pi}]$.

(2) **The meromorphic setup:** $\mathcal{F} = \operatorname{Spd}(R, R^+)$ where R^+ is a perfect ring in characteristic p endowed with the discrete topology and $R = R^+[\frac{1}{\varpi}]$ for $\varpi \in R^+$ a non zero-divisor. Typically such a setup arises from considering the Huber pair obtained from a perfectoid Huber pair by replacing the usual topology by the discrete one. In this case, $\operatorname{Vect}_Y^{\mathrm{cl}}(\mathcal{F})$ agrees with vector bundles over the adic space $(\operatorname{Spa} \mathbb{W}(R^+)[\frac{1}{\pi}])_{[\varpi] \neq 0}$. This adic space is not quasi-compact.

(3) **The formal setup:** $\mathcal{F} = \mathrm{Spd}(R^+, R^+)$ where (R, R^+) is perfectoid and R^+ is endowed with the ϖ-adic topology for some pseudo-uniformizer $\varpi \in R^+$. In this case, $\mathrm{Vect}^{\mathrm{cl}}_Y(\mathcal{F})$ agrees with vector bundles over $Y^R_{(0,\infty]}$.

Remark 3.7. On his work comparing prismatic F-crystals to families of shtukas, Güthge also considered $\mathrm{Vect}^{\mathrm{cl}}_Y$ and $\mathrm{Vect}^{\mathrm{cl}}_{\mathcal{Y}}$ independently [Güt23, § 3]. In contrast with our work, he only considers the schematic and formal setups, but he proves classicality results for both $\mathrm{Vect}^{\mathrm{cl}}_Y$ and $\mathrm{Vect}^{\mathrm{cl}}_{\mathcal{Y}}$.

4. Dieudonné modules and Isocrystals

Let $\mathcal{S} \in \mathrm{PSch}$ be a qcqs perfect scheme over $\mathbb{F}_q$.

Definition 4.1. A *Dieudonné module* over $\mathcal{S}$ is a pair $(\mathcal{E}, \Phi_{\mathcal{E}})$ where $\mathcal{E}$ is a vector bundle over $\mathcal{Y}_{\mathcal{S}}$ and $\Phi_{\mathcal{E}}$ is an isomorphism:

$$\Phi_{\mathcal{E}} : \varphi^* \mathcal{E}_{Y_{\mathcal{S}}} \to \mathcal{E}_{Y_{\mathcal{S}}}.$$

An *isocrystal* over $\mathcal{S}$ is a pair $(\mathcal{F}, \Phi_{\mathcal{F}})$ where $\mathcal{F}$ is a vector bundle over $Y_{\mathcal{S}}$ and $\Phi_{\mathcal{F}}$ is an isomorphism:

$$\Phi_{\mathcal{F}} : \varphi^* \mathcal{F} \to \mathcal{F}.$$

Remark 4.2. The category we consider in Definition 4.1 is canonically equivalent (by the evident functor) to the one considered in [PR21, Definition 2.3.3]. Nevertheless, we prefer to work with the adic space $Y_{\mathcal{S}}$ because it is only on this space that one can apply the geometric reasoning used to prove Proposition 4.10.

Definition 4.3. A morphism of Dieudonné modules (respectively of isocrystals) over $\mathcal{S}$ is a φ-equivariant map. A sequence

$$\Sigma := [(\mathcal{E}_1, \varphi_1) \to (\mathcal{E}_2, \varphi_2) \to (\mathcal{E}_3, \varphi_3)]$$

of maps of Dieudonné modules (respectively of isocrystals) over $\mathcal{S}$ is exact if it is exact at the level of underlying vector bundles.

Definition 4.4. We denote by $\mathfrak{DM} : \mathrm{PSch} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$, respectively $\mathfrak{B} : \mathrm{PSch} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$, the presheaf that attaches to any scheme $\mathcal{S}$ the closed exact symmetric monoidal category of Dieudonné modules over $\mathcal{S}$, respectively isocrystals over $\mathcal{S}$.

Definition 4.5. A map $f : (\mathcal{E}_1, \varphi_1) \to (\mathcal{E}_2, \varphi_2)$ of Dieudonné modules is called an isogeny if there exists a map $g : (\mathcal{E}_2, \varphi_2) \to (\mathcal{E}_1, \varphi_2)$ and a locally constant function $N : |\mathcal{S}| \to \mathbb{N}$ such that $f \circ g = \pi^N$ and $g \circ f = \pi^N$ (the multiplication by $\pi^{N(s)}$ map). We denote by $\mathfrak{DM}(\mathcal{S})[\frac{1}{\pi}]$ the category obtained from $\mathfrak{DM}(\mathcal{S})$ by formally inverting isogenies.

The natural map $\mathfrak{DM}(\mathcal{S}) \to \mathfrak{B}(\mathcal{S})$ factors canonically through a fully faithful embedding $\mathfrak{DM}(\mathcal{S})[\frac{1}{\pi}] \hookrightarrow \mathfrak{B}(\mathcal{S})$ and $\mathfrak{DM}(\mathcal{S})[\frac{1}{\pi}]$ inherits the structure of an exact closed symmetric monoidal category from that of $\mathfrak{B}(\mathcal{S})$. We denote by $\mathfrak{DM}[\frac{1}{\pi}] : \mathrm{PSch} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ the presheaf with values in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ obtained by the rule $\mathcal{S} \mapsto \mathfrak{DM}(\mathcal{S})[\frac{1}{\pi}]$.

Proposition 4.6. *$\mathfrak{DM}$ and $\mathfrak{B}$ are scheme theoretic $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$-valued arc-sheaves (in particular, v-sheaves). Moreover, the v-sheafification of $\mathfrak{DM}[\frac{1}{\pi}]$ is $\mathfrak{B}$.*

Proof. See [FS21, Thm. I.2.1]. To be more precise, the first claim follows from [BS17, Theorem 4.1] for $\mathfrak{DM}$ and from [Iva23, Lemma 5.8] for $\mathfrak{B}$. For the second claim, it suffices to show that $\mathfrak{DM}[\frac{1}{\pi}](A) = \mathfrak{B}(A)$ for A ranging over a basis of the v-topology. But this holds when $\operatorname{Spec} A$ is a comb. Indeed, isocrystals over combs have a free underlying vector bundle by [Iva23, Theorem 6.1]. □

Proposition 4.7. *Let $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3]$ be a sequence in $\mathfrak{DM}(\mathcal{S})$. Suppose that for every geometric point $\bar{x} \to \mathcal{S}$ the sequence $\Sigma_{\bar{x}}$ is exact, then Σ is exact. Moreover, if Σ is assumed to be a complex then it suffices to check exactness of $\Sigma_{\bar{x}}$ for geometric points whose image in $\mathcal{S}$ is closed.*

Proof. Exactness can be verified on an open cover, so we may assume $\mathcal{S} = \operatorname{Spec} A$. A sequence in $\mathfrak{DM}(\mathcal{S})$ is exact if and only if its underlying vector bundle over $\operatorname{Spa}(\mathbb{W}A)$ is exact. By [KL13, Theorem 2.7.7] and [Ked19, Theorem 1.4.2] (see [SW20, Theorem 5.2.8]) it is equivalent to ask that the sequence is exact over $\operatorname{Spec}(\mathbb{W}A)$. Moreover any basis defined over $\operatorname{Spec} A$ deforms to a basis over $\operatorname{Spec}(\mathbb{W}A)$, so that shrinking A we may assume all of the bundles are free. The map $A \to \prod_{\bar{x}\to\mathcal{S}} C_x$ is injective, so we can check on geometric points that the sequence is a complex. Once we know the sequence is a complex whether it is exact or not can be checked on geometric points of $\operatorname{Spec}(\mathbb{W}A)$ whose image is closed in $\operatorname{Spec}(\mathbb{W}A)$. But every geometric point of that form can be lifted to a geometric point of $\operatorname{Spec}(\mathbb{W}C)$ where C ranges over geometric points $\bar{x} : \operatorname{Spec} C \to \mathcal{S}$ whose image is closed in $\mathcal{S}$. By hypothesis, over $\operatorname{Spec}(\mathbb{W}C)$ the sequence is exact. □

To prove an analogue of Proposition 4.7 for $\mathfrak{B}$, we first give a reinterpretation. Let $\mathrm{Bun}_{\mathrm{FF}} : \mathrm{Perf} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ denote the stack of vector bundles on the relative Fargues–Fontaine curve, we can reformulate it as the following Cartesian square of sheaves with values in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$:

$$\begin{tikzcd}
\mathrm{Bun}_{\mathrm{FF}} \arrow[r] \arrow[d] & \mathrm{Vect}_Y^{\mathrm{cl}} \arrow[d, "\Delta"] \\
\mathrm{Vect}_Y^{\mathrm{cl}} \arrow[r, "(\mathrm{id},\mathrm{Frob})"] & \mathrm{Vect}_Y^{\mathrm{cl}} \times \mathrm{Vect}_Y^{\mathrm{cl}}
\end{tikzcd}$$

Corollary 4.8. *For all $\mathcal{S} \in \mathrm{PSch}$, $\mathrm{Bun}_{\mathrm{FF}}(\mathcal{S}^\diamond) \cong \mathfrak{B}(\mathcal{S})$ in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$.*

Proof. By Corollary 3.6, $\mathrm{Vect}_Y^{\mathrm{cl}}(\mathcal{S}^\diamond) \cong \mathrm{Vect}(Y_{\mathcal{S}})$ in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$, and by the definition of $\mathfrak{B}(\mathcal{S})$ the following diagram is Cartesian in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$:

$$\begin{tikzcd} \mathfrak{B}(\mathcal{S}) \arrow[r] \arrow[d] & \mathrm{Vect}(Y_{\mathcal{S}}) \arrow[d, "\Delta"] \\ \mathrm{Vect}(Y_{\mathcal{S}}) \arrow[r, "(\mathrm{id},\mathrm{Frob})"] & \mathrm{Vect}(Y_{\mathcal{S}}) \times \mathrm{Vect}(Y_{\mathcal{S}}) \end{tikzcd}$$

□

Remark 4.9. The result of Corollary 4.8 is implicitly proved during the proof of [PR21, Theorem 2.3.5]. The key idea to approach the problem using Sen theory is already present in that work.

Proposition 4.10. *Let $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3] \in \mathfrak{B}(\mathcal{S})$ be a sequence with constant rank $\mathrm{rk.}(\mathcal{E}_i) = r_i$ and $r_1 + r_3 = r_2$. The following hold:*

(1) *The sequence is exact if and only if for every geometric point $\overline{x} \to \mathcal{S}$ the sequence $\Sigma_{\overline{x}} \in \mathfrak{B}(\overline{x})$ is exact.*
(2) *Moreover, if the sequence is already assumed to be a complex, then exactness can be checked on geometric points $\overline{x} \to \mathcal{S}$ whose image is a closed point.*

Proof. The forward implication is evident. Assume that for every geometric point of $\mathcal{S}$, the sequence is exact. Since a scheme-theoretic cover $\mathcal{S}' \to \mathcal{S}$ induces a v-cover $Y_{\mathcal{S}'} \to Y_{\mathcal{S}}$, we may test exactness locally. Thus, we may assume that $\mathcal{S} = \operatorname{Spec} R$ is a comb and by [Iva23, Theorem 6.1] that all the underlying vector bundles are free. We write $M_1 \in \mathrm{M}_{r_2 \times r_1}(\mathbb{W}(R)[\frac{1}{\pi}])$ and $M_2 \in \mathrm{M}_{r_3 \times r_2}(\mathbb{W}(R)[\frac{1}{\pi}])$ the matrices representing the maps $\mathcal{E}_1 \to \mathcal{E}_2$ and $\mathcal{E}_2 \to \mathcal{E}_3$ respectively. The induced map $\mathcal{E}_1 \to \mathcal{E}_3$ is the 0 map if and only if the matrix $M_2 \cdot M_1 = 0$. This can be tested on geometric points since R is perfect and in particular reduced. Exactness can now be expressed in terms of the rank of M_1 and M_2 at the different points of $\operatorname{Spa} \mathbb{W}(R)[\frac{1}{\pi}]$. The locus where M_1 has rank strictly smaller to r_1 is a Zariski closed subset (cut out by the minors of M_1) $Z \subseteq \operatorname{Spa} \mathbb{W}(R)$. Moreover, since the map $\mathcal{E}_1 \to \mathcal{E}_2$ is φ-equivariant we have $\varphi(Z) = Z$. Indeed, the rank of M_1 equals the rank of $\varphi(M_1)$. Suppose $Z \neq \emptyset$ and let $z \in Z$. Endow R with the discrete topology and consider the projection map $f : \operatorname{Spd} \mathbb{W}(R)[\frac{1}{\pi}] \to \operatorname{Spd}(R, R)$. Suppose that $f(z)$ is a d-analytic point [Gle22, Definition 2.2]. Then $f^{-1}(f(z))$ is of the form Y_T where $T = \operatorname{Spa}(K, K^+)$ is a perfectoid field. By φ-equivariance and because Z is closed we can conclude that Z contains the point at infinity in $\mathcal{Y}_{(0,\infty]} = \operatorname{Spd} E \times \operatorname{Spd}(K^+, K^+)$. If $k = O_K/K^{\circ\circ}$ this shows that $\operatorname{Spa} \mathbb{W}(k)[\frac{1}{\pi}] \subseteq Z$. Since $\mathbb{W}(k)[\frac{1}{\pi}]$ is a field, this shows that every r_1-minor in M_1 thought of as an element in $R^{\mathbb{N}} = \mathbb{W}(R)$ vanishes identically when restricted to $k^{\mathbb{N}}$. The same must be true for every point in the closure of $\operatorname{Spec}(k) \subseteq \operatorname{Spec}(R)$. In particular, we found a closed point $\overline{x} \to \operatorname{Spec}(R)$ for which $\mathcal{E}_{1,\overline{x}} \to \mathcal{E}_{2,\overline{x}}$ is not injective. This contradicts our assumption, so $Z = \emptyset$. A similar argument proves that $\mathcal{E}_2 \to \mathcal{E}_3$ is surjective and by rank considerations the sequence is also exact in the middle. □

We now consider analytic versions of the category of Dieudonné modules and the category of isocrystals.

Definition 4.11. If $S = \mathrm{Spa}\, R$ we let $\mathfrak{DM}^{\diamondsuit_{\mathrm{pre}}}(S) = \mathfrak{DM}(\mathrm{Spec}\, R)$ and we call elements of this category *analytic Dieudonné modules* over S.

Proposition 4.12. *Let* $S = \mathrm{Spa}\, R$*, then* $\mathfrak{DM}^{\diamondsuit_{\mathrm{pre}}}(S) \cong \mathfrak{DM}^{\diamondsuit}(S)$*, i.e.* $\mathfrak{DM}^{\diamondsuit_{\mathrm{pre}}}$ *is already a (*$\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$*-valued) v-sheaf. In particular, the equivalence is exact, reflects exactness and is compatible with the monoidal structure.*

Proof. This follows from the fact that the rule $S \mapsto \mathrm{Vect}(\mathbb{W}(R))$ is a v-stack [SW20, Corollary 17.1.9]. Exactness can be checked on maximal ideals of $\mathbb{W}(R)$. Moreover, the image of the map $\mathrm{Spec}\, R \to \mathrm{Spec}\, \mathbb{W}(R)$ contains every closed point, and if $\mathrm{Spa}\, R' \to \mathrm{Spa}\, R$ is a v-cover, the image of the map $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ also contains every closed point. □

Definition 4.13. We let $\mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S) = \mathfrak{DM}[\frac{1}{\pi}](\mathrm{Spec}\, R)$ and $\mathfrak{B}^{\diamondsuit_{\mathrm{pre}}}(S) = \mathfrak{B}(\mathrm{Spec}\, R)$ and consider them as functors from Perf to $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$. We let $\mathfrak{B}^{\diamondsuit}$ be the v-sheafification of $\mathfrak{B}^{\diamondsuit_{\mathrm{pre}}}$. We call objects of $\mathfrak{B}^{\diamondsuit}(S)$ *analytic isocrystals.*

Proposition 4.14. *Let* $S = \mathrm{Spa}\, R$*, the following hold:*

1. *The rule* $S \mapsto \mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S)$ *is a v-separated* $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$*-valued presheaf.*
2. *The v-sheafification of* $\mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}$ *is* $\mathfrak{B}^{\diamondsuit}$*.*
3. *We have a* $\otimes$*-exact fully-faithful embedding that reflects exactness:*
$$\mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S) \subseteq \mathfrak{B}^{\diamondsuit}(S).$$

Proof. Given $\mathcal{E} \in \mathfrak{DM}[\frac{1}{\pi}](\mathrm{Spec}\, R)$ we define a functor $\Gamma_{\mathcal{E}} : \mathrm{Perf} \to \mathrm{Sets}$ given by the rule:

$$\Gamma_{\mathcal{E}} : \mathrm{Spa}\, R' \mapsto \mathrm{H}^0(\mathrm{Spec}\, \mathbb{W}(R')[\frac{1}{\pi}], \mathcal{E})^{\varphi=\mathrm{Id}}$$

where we range over maps $\mathrm{Spa}\, R' \to \mathrm{Spa}\, R$. To prove the first claim, since all categories have internal $\underline{\mathrm{Hom}}$-objects, it suffices to prove that $\Gamma_{\mathcal{E}}$ is a v-sheaf. Consider now the functor $\mathcal{H}_{\mathcal{E}} : \mathrm{Perf} \to \mathrm{Sets}$ given by the rule:

$$\mathcal{H}_{\mathcal{E}} : \mathrm{Spa}\, R' \mapsto \mathrm{H}^0(\mathrm{Spec}\, \mathbb{W}(R'), \mathcal{E})$$

By [SW20, Corollary 17.1.9], $\mathcal{H}_{\mathcal{E}}$ is a v-sheaf. In particular, the filtered colimit

$$\mathcal{H}_{\mathcal{E}} \xrightarrow{\cdot \pi} \mathcal{H}_{\mathcal{E}} \xrightarrow{\cdot \pi} \dots$$

is also a v-sheaf which we denote by $\mathcal{H}_{\mathcal{E}}[\frac{1}{\pi}]$. Finally, we get a Cartesian diagram of presheaves:

$$\begin{tikzcd} \Gamma_{\mathcal{E}} \arrow[r] \arrow[d] & \mathcal{H}_{\mathcal{E}}[\frac{1}{\pi}] \arrow[d, "\Delta"] \\ \mathcal{H}_{\mathcal{E}}[\frac{1}{\pi}] \arrow[r, "(\mathrm{id},\Phi)"] & \mathcal{H}_{\mathcal{E}}[\frac{1}{\pi}] \times \mathcal{H}_{\mathcal{E}}[\frac{1}{\pi}] \end{tikzcd}$$

and we can conclude since the limit of sheaves is a sheaf.

The second claim follows from Proposition 2.4 and the proof of Proposition 4.6. The third claim is almost a reinterpretation of the first claims, it remains to prove that the functor reflects exactness. Let $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3] \in \mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S)$, be a sequence in $\mathfrak{DM}[\frac{1}{\pi}](\mathrm{Spec}\, R)$. Assume that Σ is exact in $\mathfrak{B}^{\diamondsuit}(S)$. By definition, this means that Σ is exact in $\mathfrak{DM}[\frac{1}{\pi}](\mathrm{Spec}\, R')$ for a v-cover $\mathrm{Spa}\, R' \to \mathrm{Spa}\, R$. Since the functor is fully-faithful, we deduce that the sequence is a complex. By the second part of Proposition 4.10, we can check exactness on closed points of $\mathrm{Spec}\, R$. Since $\mathrm{Spa}\, R' \to \mathrm{Spa}\, R$ is v-cover, every closed point of $\mathrm{Spec}\, R$ is in the image of $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$. In particular, if a sequence becomes exact over $\mathfrak{DM}[\frac{1}{\pi}](\mathrm{Spec}\, R')$ then it was already exact over $\mathfrak{DM}[\frac{1}{\pi}](\mathrm{Spec}\, R)$ as we needed to show. □

Proposition 4.15. *Let $S = \mathrm{Spa}\, R$. Let $\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \in \mathfrak{B}^{\diamondsuit}(S)$ be a sequence with constant rank $\mathrm{rk.}(\mathcal{E}_i) = r_i$ and $r_1 + r_3 = r_2$. The sequence is exact if and only if for every geometric point $\overline{x} \to S$ the sequence $\mathcal{E}_{1,\overline{x}} \to \mathcal{E}_{2,\overline{x}} \to \mathcal{E}_{3,\overline{x}} \in \mathfrak{B}^{\diamondsuit}(\bar{x})$ is exact.*

Proof. The forward implication is evident. Assume that for every geometric point of S the sequence is exact. By the definition of the exact structure on $\mathfrak{B}^{\diamondsuit}(S)$ via sheafification, we may test exactness locally. We can find a v-cover $S' := \mathrm{Spa}\, R' \to S$ such that each $\mathcal{E}_i \in \mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S')$. Since the map $R' \to \prod_{x \in \mathrm{Spa}\, R'} C_x$ is injective we can test on geometric points if the map is a complex. Once we know it is a complex, by Proposition 4.10 we can test exactness on closed points of $\mathrm{Spec}\, R'$. But every closed point of $\mathrm{Spec}\, R'$ supports a geometric point of $\mathrm{Spa}\, R'$. □

5. Shtukas, isoshtukas and meromorphic vector bundles

Definition 5.1. Let $S = \mathrm{Spa}\, R \in \mathrm{Perf}$. A *crystalline shtuka* over S is a pair $(\mathcal{E}, \Phi_{\mathcal{E}})$ where $\mathcal{E}$ is a vector bundle over $\mathcal{Y}_S$ and $\Phi_{\mathcal{E}}$ is an isomorphism

$$\Phi_{\mathcal{E}} : (\varphi^* \mathcal{E})_{Y_S} \to \mathcal{E}_{Y_S}$$

that is meromorphic (cf. [SW20, Definition 5.3.5]) along $\pi = 0$.

Definition 5.2. A morphism of crystalline shtukas is a φ-equivariant map. We say that a sequence of maps $\Sigma := [(\mathcal{E}_1, \varphi_1) \to (\mathcal{E}_2, \varphi_2) \to (\mathcal{E}_3, \varphi_3)]$ is *exact* if it is exact at the level of underlying vector bundles over $\mathcal{Y}_S$.

Definition 5.3. We let $\mathcal{SHT} : \mathrm{Perf} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ denote the presheaf that attaches to any perfectoid space S the closed exact symmetric monoidal category of crystalline shtukas over S.

Proposition 5.4. *$\mathcal{SHT}$ is a $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$-valued v-sheaf.*

Proof. This follows from the proof of [SW20, Proposition 19.5.3]. Indeed, we have a Cartesian diagram in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$:

$$\begin{tikzcd} \mathcal{SHT}(S) \arrow[r] \arrow[d] & \mathrm{Vect}(\mathcal{Y}_S)[\frac{1}{\pi}] \arrow[d, "\Delta"] \\ \mathrm{Vect}(\mathcal{Y}_S) \arrow[r, "(\mathrm{id},\varphi^*)"] & \mathrm{Vect}(\mathcal{Y}_S)[\frac{1}{\pi}] \times \mathrm{Vect}(\mathcal{Y}_S)[\frac{1}{\pi}] \end{tikzcd}$$

Although the rule $S \mapsto \mathrm{Vect}(\mathcal{Y}_S)[\frac{1}{\pi}]$ is not a v-sheaf, it is a v-separated presheaf. This already implies that $\mathcal{SHT}$ is a v-sheaf. $\square$

Definition 5.5. A map $f : (\mathcal{E}_1, \varphi_1) \to (\mathcal{E}_2, \varphi_2)$ of crystalline shtukas is called an isogeny if there exists a map $g : (\mathcal{E}_2, \varphi_2) \to (\mathcal{E}_1, \varphi_2)$ and $N : |S| \to \mathbb{N}$ a locally constant function such that $g \circ f = \pi^N$ and $g \circ f = \pi^N$. We denote by $\mathcal{SHT}[\frac{1}{\pi}](S)$ the category obtained from $\mathcal{SHT}(S)$ by formally inverting isogenies. We will call objects in this category isoshtukas.

For $\infty > r_2 \geq r_1 \geq 0$, let $B_{[r_1,r_2]} := H^0(\mathcal{Y}^{R^+}_{[r_1,r_2]}, \mathcal{O})$ be the global sections of an affinoid open of the relative Fargues–Fontaine curve (cf. [FS21, Chap. II.1]). We have the Frobenius map $\varphi \colon B_{[0,r]} \to B_{[0,\frac{r}{q}]}$. Then $\mathcal{SHT}(S)$ fits into the Cartesian square:

$$\begin{tikzcd} \mathcal{SHT}(S) \arrow[r] \arrow[d] & \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]) \arrow[d, "\Delta"] \\ \mathrm{Vect}(B_{[0,1]}) \arrow[r, "(\mathrm{res}^*,\varphi^*)"] & \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]) \times \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]), \end{tikzcd} \tag{5.1}$$

where $\mathrm{res}\colon B_{[0,1]} \to B_{[0,\frac{1}{q}]}$ is the natural restriction map, and $\mathrm{Vect}(B_{[0,r]})$ is the category of finite projective $B_{[0,r]}$-modules.

Analogously, we get a Cartesian diagram of categories

$$\begin{tikzcd} \mathcal{SHT}(S)[\frac{1}{\pi}] \arrow[r] \arrow[d] & \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]) \arrow[d, "\Delta"] \\ (\mathrm{Vect}(B_{[0,1]}))[\frac{1}{\pi}] \arrow[r, "(\mathrm{res}^*,\varphi^*)"] & \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]) \times \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]), \end{tikzcd} \tag{5.2}$$

since π is not a zero-divisor, we have a fully-faithful embedding of categories:

$$(\mathrm{Vect}(B_{[0,1]}))[\frac{1}{\pi}] \subseteq \mathrm{Vect}(B_{[0,1]}[\frac{1}{\pi}])$$

and we can endow $(\mathrm{Vect}(B_{[0,1]}))[\frac{1}{\pi}]$ with the exact structure it inherits from $\mathrm{Vect}(B_{[0,1]}[\frac{1}{\pi}])$. Moreover, we can endow $\mathcal{SHT}(S)[\frac{1}{\pi}]$ with the exact structure that makes this diagram a Cartesian square in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$. Consider the diagram of categories:

Proposition 5.6. *The following diagram is Cartesian in* $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$*:*

$$\begin{tikzcd} \mathcal{SHT}(S) \arrow[r] \arrow[d] & \mathcal{SHT}[\frac{1}{\pi}](S) \arrow[d] \\ \mathfrak{DM}^{\diamondsuit}(S) \arrow[r] & \mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S) \end{tikzcd} \tag{5.3}$$

Proof. The argument is a standard application of Beauville–Laszlo descent [SW20, Lemma 5.2.9]. We provide the details for the convenience of the reader. Recall the Cartesian diagram

$$\begin{tikzcd} \mathcal{SHT}(S) \arrow[r] \arrow[d] & \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]) \arrow[d, "\Delta"] \\ \mathrm{Vect}(B_{[0,1]}) \arrow[r, "(\mathrm{res}^*,\varphi^*)"] & \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]) \times \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]), \end{tikzcd}$$

Replacing the role of $B_{[0,1]}$ and $B_{[0,\frac{1}{q}]}$ by their π-completions, we obtain the Cartesian diagram:

$$\begin{tikzcd} \mathfrak{DM}^{\diamondsuit}(S) \arrow[r] \arrow[d] & \mathrm{Vect}(\mathbb{W}(R)[\frac{1}{\pi}]) \arrow[d, "\Delta"] \\ \mathrm{Vect}(\mathbb{W}(R)) \arrow[r, "(\mathrm{res}^*,\varphi^*)"] & \mathrm{Vect}(\mathbb{W}(R)[\frac{1}{\pi}]) \times \mathrm{Vect}(\mathbb{W}(R)[\frac{1}{\pi}]). \end{tikzcd} \tag{5.4}$$

Similarly, we obtain diagrams:

$$\begin{tikzcd} \mathcal{SHT}(S)[\frac{1}{\pi}] \arrow[r] \arrow[d] & \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]) \arrow[d, "\Delta"] \\ (\mathrm{Vect}(B_{[0,1]}))[\frac{1}{\pi}] \arrow[r, "(\mathrm{res}^*,\varphi^*)"] & \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]) \times \mathrm{Vect}(B_{[0,\frac{1}{q}]}[\frac{1}{\pi}]), \end{tikzcd} \tag{5.5}$$

$$\begin{tikzcd} \mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S) \arrow[r] \arrow[d] & \mathrm{Vect}(\mathbb{W}(R)[\frac{1}{\pi}]) \arrow[d, "\Delta"] \\ (\mathrm{Vect}(\mathbb{W}(R)))[\frac{1}{\pi}] \arrow[r, "(\mathrm{res}^*,\varphi^*)"] & \mathrm{Vect}(\mathbb{W}(R)[\frac{1}{\pi}]) \times \mathrm{Vect}(\mathbb{W}(R)[\frac{1}{\pi}]). \end{tikzcd} \tag{5.6}$$

Moreover, these four Cartesian diagrams can be organized in a commutative square of Cartesian diagrams. For any fixed $i \in \{left, right\}$ and $j \in \{upper, lower\}$, their (i,j)th corners form a commutative diagram, which we denote $C_{i,j}$. For example, $C_{left,upper}$ is the diagram that we wish to prove is Cartesian. Moreover, $C_{left,lower}$ is the left square of (5.7), where the horizontal arrows in the right square consist of fully-faithful embeddings.

$$\begin{tikzcd}
\mathrm{Vect}(B_{[0,1]}) \arrow[r] \arrow[d] & \mathrm{Vect}(B_{[0,1]})[\frac{1}{\pi}] \arrow[r] \arrow[d] & \mathrm{Vect}(B_{[0,1]}[\frac{1}{\pi}]) \arrow[d] \\
\mathrm{Vect}(\mathbb{W}(R)) \arrow[r] & \mathrm{Vect}(\mathbb{W}(R))[\frac{1}{\pi}] \arrow[r] & \mathrm{Vect}(\mathbb{W}(R)[\frac{1}{\pi}])
\end{tikzcd} \tag{5.7}$$

As $\pi \in B_{[0,1]}$ is not a zero-divisor and the π-adic completion of $B_{[0,1]}$ is $\mathbb{W}(R)$, the Beauville–Laszlo lemma [SW20, Lemma 5.2.9] implies that this diagram is Cartesian. For any j, $C_{right,j}$ is automatically Cartesian, as the horizontal maps in it are isomorphisms. From this it formally follows that $C_{upper,left}$ is also Cartesian. □

We now give a different presentation of $\mathcal{SHT}[\frac{1}{\pi}]$. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$, let $S^+ = \mathrm{Spd}(R^+, R^+)$, let $T = \mathrm{Spd}(R^{\mathrm{dis}}, R^{\mathrm{dis},+})$ where $(R^{\mathrm{dis}}, R^{\mathrm{dis},+})$ is (R, R^+) with its discrete topology and let $T^+ = \mathrm{Spd}(R^{\mathrm{dis},+}, R^{\mathrm{dis},+})$.

Proposition 5.7. *We have an equivalence* $\mathrm{Bun}_{\mathrm{FF}}(T) \xrightarrow{\sim} \mathcal{SHT}[\frac{1}{\pi}](S)$ *of categories in* $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$.

Proof. Pick $\varpi \in R^+$ a pseudo-uniformizer. We consider two different topologies on the ring $\mathbb{W}(R^+)$. On one hand we can endow it with the $(\pi, [\varpi])$-adic topology in which case we write $\mathbb{A}_{\mathrm{inf}}(R^+)$ for this topological ring. We can also endow it with its π-adic topology, in this case we simply write $\mathbb{W}(R^+)$.

Now, $\mathrm{Spd}(\mathbb{A}_{\mathrm{inf}}(R^+)) = \mathrm{Spd}\, O_E \times S^+$, whereas $\mathrm{Spd}\, \mathbb{W}(R^+) = \mathrm{Spd}\, O_E \times T^+$. Moreover, we have open immersion of v-sheaves $S \subseteq T \subseteq \mathrm{Spec}\,(R^+)^\diamond$. By Corollary 3.6, we have

$$\mathrm{Vect}^{\mathrm{cl}}_Y(T) = \mathrm{Vect}(\mathrm{Spa}\, \mathbb{W}(R^+)_{\pi\cdot[\varpi]\neq 0}) = \mathrm{Vect}(\mathrm{Spa}(\mathbb{W}(R^+)[\frac{1}{\pi}])_{[\varpi]\neq 0})$$

We can cover $\mathrm{Spa}(\mathbb{W}(R^+)[\frac{1}{\pi}])_{[\varpi]\neq 0}$ by sets of the form $\{\pi \leq [\varpi^{\frac{1}{q^n}}] \neq 0 \mid n \in \mathbb{N}\}$ and by φ-equivariance the value of $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(T)$ is determined by its value on $\{\pi \leq [\varpi] \neq 0\}$. More precisely, if $B^{\mathrm{disc}}_{[0,q^n]}$ is the ring of global sections of the locus $\{\pi \leq [\varpi^{\frac{1}{q^n}}] \neq 0\} \subseteq \mathrm{Spa}(\mathbb{W}(R^+))_{[\varpi]\neq 0}$, then $B^{\mathrm{disc}}_{[0,q^n]}[\frac{1}{\pi}]$ is the ring of global sections of $\{\pi \leq [\varpi^{\frac{1}{q^n}}] \neq 0,\, \pi \neq 0\}$ and we have the following Cartesian diagram in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$:

$$\begin{tikzcd}
\mathrm{Bun}_{\mathrm{FF}}(T) \arrow[r] \arrow[d] & \mathrm{Vect}(B^{\mathrm{disc}}_{[0,\frac{1}{q}]}[\frac{1}{\pi}]) \arrow[d, "\Delta"] \\
\mathrm{Vect}(B^{\mathrm{disc}}_{[0,1]}[\frac{1}{\pi}]) \arrow[r, "(\mathrm{id},\mathrm{Frob})"] & \mathrm{Vect}(B^{\mathrm{disc}}_{[0,\frac{1}{q}]}[\frac{1}{\pi}]) \times \mathrm{Vect}(B^{\mathrm{disc}}_{[0,\frac{1}{q}]}[\frac{1}{\pi}])
\end{tikzcd}$$

Let $B_{[0,q^n]}$ denote the ring of global sections of the locus $\{\pi \leq [\varpi^{\frac{1}{q^n}}] \neq 0\} \subseteq \mathrm{Spa}(\mathbb{A}_{\inf}(R^+))_{[\varpi]\neq 0}$. Then the natural map $B^{\mathrm{disc}}_{[0,q^n]} \to B_{[0,q^n]}$ is a continuous isomorphism of rings that is not a homeomorphism! [SW20, Lemma 14.3.1]. In particular, $\mathrm{Vect}(B^{\mathrm{disc}}_{[0,q^n]}[\frac{1}{\pi}]) \xrightarrow{\sim} \mathrm{Vect}(B_{[0,q^n]}[\frac{1}{\pi}])$ in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$, by a theorem of Kedlaya–Liu [SW20, Theorem 5.2.8]. $\square$

Definition 5.8. We define the stack of *meromorphic vector bundles on the relative Fargues–Fontaine curve*, which we denote by $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$, as the v-stackification of $\mathcal{SHT}[\frac{1}{\pi}]$, when this later is treated as a presheaf valued in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$.

There is a restriction map $\mathcal{SHT}(S) \to \mathrm{Bun}_{\mathrm{FF}}(S)$ which factors through the π-localization $\mathcal{SHT}(S)[\frac{1}{\pi}] \to \mathrm{Bun}_{\mathrm{FF}}(S)$. Since $\mathrm{Bun}_{\mathrm{FF}}$ is a v-sheaf this further extends to a $\otimes$-exact map $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S) \to \mathrm{Bun}_{\mathrm{FF}}(S)$. Analogously, we get a $\otimes$-exact map $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S) \to \mathfrak{B}^{\diamondsuit}(S)$.

Definition 5.9.
(1) We let $\sigma : \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}} \to \mathrm{Bun}_{\mathrm{FF}}$ denote the map constructed above, we call this map the *special polygon map*.
(2) We let $\gamma : \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}} \to \mathfrak{B}^{\diamondsuit}$ denote the map constructed above, we call this map the *generic polygon map*.

We now study basic properties of $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$.

Proposition 5.10. *The map $\mathcal{SHT}[\frac{1}{\pi}](S) \to \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ is $\otimes$-exact fully-faithful and reflects exactness. In other words, $\mathcal{SHT}[\frac{1}{\pi}]$ is a v-separated prestack in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$. Moreover, exactness in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ can be verified on geometric points of S.*

Proof. Since both categories have internal $\underline{\mathrm{Hom}}$-objects it suffices to prove that for $\mathcal{E} \in \mathcal{SHT}(S)$ the rule

$$T \mapsto \mathrm{H}^0(\mathcal{Y}_T, \mathcal{E}_{|T})^{\varphi=\mathrm{Id}}[\frac{1}{\pi}]$$

is a v-sheaf, where T is affinoid perfectoid over S. Since this is a filtered colimit of v-sheaves (by Proposition 5.4) full-faithfulness follows.

We now show that it reflects exactness. Let $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3]$ be a sequence in $\mathcal{SHT}[\frac{1}{\pi}](S)$, which becomes exact over $S' = \mathrm{Spa}\, R'$ for a v-cover $f : S' \to S$. By full-faithfulness above, we can already deduce that Σ is a complex. Let T' and T denote $\mathrm{Spd}(R'^{\,\mathrm{dis}}, R'^{\,\mathrm{dis},+})$ and $\mathrm{Spd}(R^{\mathrm{dis}}, R^{\mathrm{dis},+})$. By Proposition 5.7, we may interpret Σ as a sequence in $\mathrm{Bun}_{\mathrm{FF}}(T)$ that becomes exact in $\mathrm{Bun}_{\mathrm{FF}}(T')$. We can verify exactness of Σ on geometric points of T. We warn the reader that although the map $S' \to S$ is a v-cover the map $T' \to T$ might no longer be surjective even at the level of topological spaces. Nevertheless, it is surjective on the locus where ϖ is topologically nilpotent for a pseudo-uniformizer $\varpi \in R^+$. Indeed this locus agrees with S' and S respectively. So it suffices to prove exactness of Σ on the complement of S in T.

Let $U = \mathrm{Spd}(R, R)$. The complement of S in T is the closure $\overline{U}$ of U in T. Moreover, $\overline{U} \setminus U$ consists of vertical specializations of elements in U, and the same can be said of $\overline{U} \times \mathrm{Spd}\, E$ and $U \times \mathrm{Spd}\, E$. In particular, Σ is exact over $\overline{U}$ if and only if it is exact over U. We know that Σ is exact when restricted to $\mathrm{Spd}(R', R')$. By Corollary 4.8, we may interpret Σ restricted to U as a sequence in $\mathfrak{B}(\mathrm{Spec}\, R)$ that becomes exact over $\mathfrak{B}(\mathrm{Spec}\, R')$. By Proposition 4.10, we can check exactness on closed points. Fortunately, the map $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ covers all closed points. Indeed, every maximal ideal of R supports a valuation that is continuous for the ϖ-adic topology. The kernel of any lift of such valuation to R' maps to this maximal ideal.

Finally, we wish to prove that a sequence $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3]$ is exact $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ if and only if for every geometric point $\overline{x} \to S$ the sequence $\Sigma_{\overline{x}}$ is exact. By definition, exactness can be verified v-locally so we may assume that $S = \mathrm{Spa}\, R$ is a strict product of points with $R^+ = \prod_{i \in I} C_i^+$ and that each $\mathcal{E}_j \in \mathcal{SHT}[\frac{1}{\pi}]$ for $j \in \{1, 2, 3\}$. The map $R \to \prod_{i \in I} C_i$ is injective, so we deduce that Σ is a complex. We can now argue as above. Namely, we consider Σ as a sequence in $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(R^{\mathrm{dis}}, R^{\mathrm{dis},+}))$, and we show that Σ is exact on all points of $\mathrm{Spd}(R^{\mathrm{dis}}, R^{\mathrm{dis},+})$. This is clear on the locus where ϖ is topologically nilpotent by our assumption. To verify exactness on $\mathrm{Spd}(R^{\mathrm{dis}}, R^{\mathrm{dis}})$ we interpret this as an object in $\mathfrak{B}(\mathrm{Spec}\, R)$ and we may check exactness on closed points. For any closed point, the residue field map $\mathrm{Spec}\, C \to \mathrm{Spec}\, R$ can be promoted to a geometric point $\mathrm{Spa}\, C \to \mathrm{Spa}\, R$ and the induced sequence in $\mathfrak{B}(\mathrm{Spec}\, C)$ is induced from the corresponding one in $\mathcal{SHT}[\frac{1}{\pi}](C, O_C)$, which is exact by assumption. $\square$

Corollary 5.11. *All of the squares of the commutative diagram below are Cartesian in* $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$.

$$\begin{tikzcd}
\mathcal{SHT}(S) \arrow[r] \arrow[d] & \mathcal{SHT}[\frac{1}{\pi}](S) \arrow[r] \arrow[d] & \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S) \arrow[d] \\
\mathfrak{DM}^{\diamondsuit}(S) \arrow[r] & \mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S) \arrow[r] & \mathfrak{B}^{\diamondsuit}(S)
\end{tikzcd} \tag{5.8}$$

Proof. That the left square is Cartesian is Proposition 5.6. By Proposition 4.12 and Proposition 5.4, $\mathcal{SHT}$ and $\mathfrak{DM}^{\diamondsuit}$ are already v-sheaves with values in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$. From this and Proposition 5.6 it follows that the outer square is Cartesian by taking sheafification. Let $S = \mathrm{Spa}\, R$. We wish to show that the map

$$\mathcal{SHT}(S)[\frac{1}{\pi}] \to \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S) \times_{\mathfrak{B}^{\diamondsuit}(S)} \mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S)$$

is an exact equivalence that reflects exactness. By Proposition 5.10, the map is already fully-faithful and we must show it is essentially surjective. Suppose we are given objects $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ and $\mathcal{M} \in \mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S)$ together with an isomorphism $\alpha : \mathcal{E} \to \mathcal{M}$ on $\mathfrak{B}^{\diamondsuit}(S)$. We can lift $\mathcal{M}'$ to an object in $\mathfrak{DM}^{\diamondsuit}(S)$ and since the outer square is Cartesian this defines an object

$\mathcal{F} \in \mathcal{SHT}(S)$ inducing the triple $(\mathcal{E}, \mathcal{M}', \alpha)$. The image of $\mathcal{F}$ in $\mathcal{SHT}[\frac{1}{\pi}](S)$ induces the triple $(\mathcal{E}, \mathcal{M}, \alpha)$ as we needed to show. This shows that the right square is Cartesian in $\mathrm{Cat}_1^{\otimes}$. That it is even Cartesian in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ follows form part (3) of Proposition 4.14 and from Proposition 5.10. □

Proposition 5.12. *A sequence $\Sigma : [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3]$ in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ is exact if and only if its image in $\mathrm{Bun}_{\mathrm{FF}}(S)$ is exact.*

Proof. Since both can be checked at the level of geometric points we may assume $S = \mathrm{Spa}(C, O_C)$. In this case, $B^C_{[0,1]}$ is a principal ideal domain and the closed ideals correspond to untilts of C. The map of ringed topological spaces $f : \mathcal{Y}_{(0,1]} \to \mathrm{Spec}\, B^C_{[0,1]}[\frac{1}{\pi}]$ covers every maximal ideal of the target and $B^C_{[0,1]}[\frac{1}{\pi}] \to \mathrm{H}^0(\mathcal{Y}_{(0,1]}, \mathcal{O})$ is injective. Consequently f^* reflects exactness. □

6. Semi-stable filtrations

Convention 6.1. Given a $\lambda \in \mathbb{Q}$ with $\lambda = \frac{m}{n}$ and $(m, n) = 1$ we let $\mathcal{O}(\lambda) \in \mathfrak{B}(\mathbb{F}_q)$ be the simple standard isocrystal of slope λ given by the pair $(\mathbb{W}(\mathbb{F}_q)[\frac{1}{\pi}]^n, M)$ where M is the matrix operator with $M \cdot e_i = e_{i+1}$ for $1 \leq i \leq n-1$ and $M \cdot e_n = \pi^{-m} e_1$. We say that an isocrystal is standard if it has the form:

$$\bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}(\lambda)^{m_\lambda}$$

Where $m : \mathbb{Q} \to \mathbb{N}$ is a multiplicity function with finite support.

Remark 6.2. Notice that our convention for standard isocrystals reverses the signs in comparison to most classical conventions.

For us a Newton polygon is a function $f : \mathbb{Q} \to \mathbb{Z}_{\geq 0}$ with $f^{-1}(\mathbb{Z}_{>0})$ finite. Its slopes are the values $x \in \mathbb{Q}$ with $f(x) \neq 0$ and the multiplicity of the slope x is $f(x)$. We denote by $\mathcal{N}$ the set of all Newton polygons. Then $\mathcal{N}$ is endowed with the partial order $f \leq g$ if and only if $\sum_{x \in \mathbb{Q}} f(x)x = \sum_{x \in \mathbb{Q}} g(x)x$ and for all $x \in \mathbb{Q}$ one has $\sum_{y \geq x} f(y)y \leq \sum_{y \geq x} g(y)y$. We say a Newton polygon is semi-stable if it has a single slope. We let $\mathcal{N}^{\mathrm{ss}} \subseteq \mathcal{N}$ denote the subset of semi-stable polygons, this are the minimal elements in this set. Recall that on a geometric point, isomorphism classes of objects in $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spa}\, C)$ and $\mathfrak{B}^\diamondsuit(\mathrm{Spa}\, C)$ are both classified by elements in $\mathcal{N}$. If $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$ we define two functions $\gamma_\mathcal{E}, \sigma_\mathcal{E} : |S| \to \mathcal{N}$ which we call the *generic polygon* and *special polygon* respectively.

Remark 6.3. Using a different language, Kedlaya proves that for any $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$ we have $\gamma_\mathcal{E} \geq \sigma_\mathcal{E}$ [Ked05, Prop. 5.5.1]. This a key step in Kedlaya–Liu's proof of the semicontinuity theorem [KL13, Theorem 7.4.5].

Definition 6.4. Let $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ with constant rank and image $\mathcal{F} \in \mathfrak{B}^\diamondsuit(S)$ under the map $\gamma : \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S) \to \mathfrak{B}^\diamondsuit(S)$.

(1) We say that $\mathcal{F}$ is *locally standard* if its Newton polygon is locally constant.

(2) We say that it is *semi-stable* if it is locally standard and each of its Newton polygons has only one slope.
(3) We say $\mathcal{E}$ is *generically locally standard* if $\mathcal{F}$ is locally standard, equivalently if $\gamma_{\mathcal{E}}$ is locally constant.
(4) We say $\mathcal{E}$ is *semi-stable* if $\mathcal{F}$ is semi-stable.

We denote by $(\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}})^{\mathrm{loc}}$ and by $(\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}})^{\mathrm{ss}}$ denote the stacks of generically locally standard meromorphic vector bundles and semi-stable meromorphic vector bundles respectively.

Definition 6.5. Let $S = \mathrm{Spa}\, R$. We say that an object $(\mathcal{E}, \Phi) \in \mathfrak{DM}^{\diamondsuit}(S)$ is *anti-effective* if $\Phi^{-1} : \mathcal{E} \to \varphi^*\mathcal{E}$ extends to a map $\Psi : \mathcal{E} \to \varphi^*\mathcal{E}$ defined over $\mathrm{Spec}\, \mathbb{W}(R)$. An object in $\mathcal{F} \in \mathcal{SHT}(S)$ is *anti-effective* if its image in $\mathfrak{DM}^{\diamondsuit}(S)$ is anti-effective.

Proposition 6.6. *Let $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ such that $\gamma_{\mathcal{E}}$ is constant of smallest slope 0, then it lifts v-locally to an anti-effective crystalline shtuka.*

Proof. By Proposition 5.6 it suffices to prove that locally standard analytic isocrystals of smallest slope 0 lift v-locally to an anti-effective Dieudonné module. Working v-locally we may assume $\gamma(\mathcal{E}) \in \mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S)$, and since $\gamma(\mathcal{E})$ is locally standard, by [HK22, Theorem 2.11] we may even assume $\gamma(\mathcal{E}) \cong \oplus_{i=1}^n \mathcal{O}(\lambda_i)^{m_i}$ and by assumption $\lambda_i \geq 0$ for i. The standard models of $\mathcal{O}(\lambda_i)$ already define an anti-effective crystalline shtuka, by our sign Convention 6.1. ☐

Lemma 6.7. *Suppose that $S = \mathrm{Spa}\, R$ is a product of points. Let $(\mathcal{E}, \Phi) \in \mathcal{SHT}(S)$ be anti-effective, then*

$$\mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O}, \mathcal{E}) = \mathrm{Hom}_{\mathfrak{B}^{\diamondsuit}}(\mathcal{O}, \gamma(\mathcal{E})).$$

Moreover, if $f \in \mathrm{Hom}_{\mathfrak{B}^{\diamondsuit}}(\mathcal{O}, \gamma(\mathcal{E}))$ defines a sub-isocrystal $\mathcal{O} \subseteq \mathcal{E}$, then the corresponding lift also defines a sub-bundle $\mathcal{O} \subseteq \mathcal{E}$ in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$.

Proof. Since $B_{[0,r]}^R \subseteq \mathbb{W}R$ the map

$$\mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O}, \mathcal{E}) \to \mathrm{Hom}_{\mathfrak{B}^{\diamondsuit}}(\mathcal{O}, \gamma(\mathcal{E})).$$

is injective. To prove surjectivity we fix a basis of $\beta : \mathcal{O}^n \to \mathcal{E}$ over $\mathcal{Y}_{[0, \frac{q}{N}]}$ for some $N \in \mathbb{N}$, this induces a basis $\phi^*\beta : \mathcal{O}^n \to \varphi^*\mathcal{E}$ over $\mathcal{Y}_{[0, \frac{1}{N}]}$, let $r = \frac{1}{N}$. Since $(\mathcal{E}, \Phi)$ is anti-effective we can think of $(\mathcal{E}, \Phi)$ through β and $\varphi^*\beta$ as a matrix $M \in \mathrm{GL}_n(B_{[0,r]}^R)$ such that

$$M^{-1} \in \mathrm{GL}_n(B_{[0,r]}^R[\frac{1}{\pi}]) \cap M_{n\times n}(\mathbb{W}R).$$

A map $f \in \mathrm{Hom}_{\mathfrak{B}^{\diamondsuit}}(\mathcal{O}, \gamma(\mathcal{E}))$ can then be though of a vector $v \in \mathbb{W}(R)[\frac{1}{\pi}]^n$ satisfying the equation

$$M\varphi v = v.$$

On the other hand $v \in \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O}, \mathcal{E})$ if and only if $v \in B_{[0,s]}[\frac{1}{\pi}]$ for some $s > 0$. Indeed, we can use φ-equivariance to extend this map along $\mathcal{Y}_{[0,\infty)}$. Replacing v by $\pi^N \cdot v$ we may assume $v \in \mathbb{W}(R)^n$.

We fix a norm of $|\cdot|: R \to \mathbb{R}$ inducing the topology of R with $|\varpi| = \frac{1}{q}$ and define a function $|\cdot|_k : \mathbb{W}R \to \mathbb{R}$ by the formula:

$$\sum_{i=0}^{\infty} [a_i]\pi^i \mapsto \sup_{0\leq i\leq k} |a_i|.$$

This definition extends to $M_{n\times n}(\mathbb{W}R)$ and $(\mathbb{W}R)^n$ by taking supremum over the entries. By the strong triangle inequality, and because $M^{-1} \in M_{n\times n}(\mathbb{W}R)$, we have that for every $k \in \mathbb{N}$ the inequality $|M^{-1}\cdot v|_k \leq |M^{-1}|_k \cdot |v|_k$ holds and by inspection $|\varphi v|_k = |v|_k^q$. From this we deduce that $|v|_k^{q-1} \leq |M^{-1}|_k$. Let $m_{ij} \in B^R_{[0,r]}$ denote the (i,j) entry of M^{-1} and write $m_{ij} = \sum_{l=0}^{\infty}[m_{ijl}]\pi^l$. The sequences m_{ijl} all satisfy that $\lim_{l\mapsto\infty} |m_{ijl}| \cdot (\frac{1}{q})^{N\cdot l} = 0$. Now, Lemma 6.8 shows that $\lim_{l\mapsto\infty} |M^{-1}|_l \cdot (\frac{1}{q})^{N\cdot l} = 0$. In particular, $\lim_{l\mapsto\infty} |v|_l \cdot (\frac{1}{q})^{N\cdot(q-1)\cdot l} = 0$, which implies that $v \in (B^R_{[0,\frac{1}{N\cdot(q-1)}]})^n$ as we needed to show.

The last claim can be verified at the level of geometric points. Consider the ideal I in $B^C_{[0,1]}$ generated by the entries of v. Since $B^C_{[0,1]}$ is a principal ideal domain, the zero locus of I consists of finitely many closed points in $\operatorname{Spec} B^C_{[0,1]}$. Moreover, the zero locus is φ-equivariant so it is at worst the ideal cut by π, but then it avoids $\operatorname{Spec} B^C_{[0,1]}[\frac{1}{\pi}]$. □

Lemma 6.8. *Let I be a finite set and ρ a number with $0 < \rho < 1$. For each $i \in I$, let $(b_{i,j})_{j\geq 0}$ be a sequence in $\mathbb{R}_{\geq 0}$ such that $\lim_{j\mapsto\infty} b_{i,j}\cdot\rho^j = 0$. For each $j \geq 0$ let $B_j = \max_{i\in I, j'\leq j}\{b_{i,j'}\}$. Then $\lim_{j\mapsto\infty} B_j \cdot \rho^j = 0$.*

Proof. This reduces easily to the case $I = \{1\}$. Fix $\varepsilon > 0$. By assumption, there is some $j_{\varepsilon,0} > 0$ such that for all $j \geq j_{\varepsilon,0}$, $b_j\rho^j < \varepsilon$. Put

$$\lambda = \max_{j'<j_{\varepsilon,0}} b_{j'}\rho^{j'}.$$

Pick now j_ε big enough, such that $\rho^{j_\varepsilon - j_{\varepsilon,0}}\lambda < \varepsilon$. Then for any $j \geq j_\varepsilon$ we have

$$\begin{aligned} B_j\rho^j &= \max_{j'\leq j}\{b_{j'}\rho^j\} \\ &= \max\{\max_{j'<j_{\varepsilon,0}}\{b_{j'}\rho^{j'}\rho^{j-j'}\}, \max_{j_{\varepsilon,0}\leq j'\leq j}\{b_{j'}\rho^{j'}\rho^{j-j'}\}\} < \varepsilon \end{aligned}$$

Indeed, if $j' < j_{\varepsilon,0}$, then $b_{j'}\rho^{j'}\rho^{j-j'} \leq \lambda\rho^{j-j'} \leq \lambda\rho^{j_\varepsilon - j_{\varepsilon,0}} < \varepsilon$ (as $\rho < 1$ and $j - j' \geq j_\varepsilon - j_{\varepsilon,0}$); and if $j' > j_{\varepsilon,0}$, then $b_{j'}\rho^{j'} < \varepsilon$ and $\rho^{j-j'} < 1$. □

Proposition 6.9. *The maps $(\mathfrak{B}^\diamondsuit)^{\mathrm{ss}} \xleftarrow{\gamma} (\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{ss}} \xrightarrow{\sigma} (\mathrm{Bun}_{\mathrm{FF}})^{\mathrm{ss}}$ are $\otimes$-exact equivalences.*

Proof. To prove that γ is an equivalence it suffices to prove that it is fully-faithful. Indeed, essential surjectivity can then be verified locally and by [CS17, Proposition 4.3.13] (which is a special case of [HK22, Theorem 2.11]) every object of $\mathfrak{B}^\diamondsuit$ is pro-étale locally isomorphic to $\mathcal{O}(\lambda)^m$ which is already in $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(\operatorname{Spd}\mathbb{F}_q)$. Moreover, we may instead prove full-faithfulness of the maps $\mathcal{SHT}[\frac{1}{\pi}](S) \to \mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}(S)$ when restricted to the semi-stable locus, since this will pass to the sheafification. Let $(\mathcal{E}_i, \Phi_i) \in \mathcal{SHT}[\frac{1}{\pi}](S)$ with $i \in \{1, 2\}$.

We consider the internal $\underline{\mathrm{Hom}}$-object $\underline{\mathcal{H}}^{\mathrm{mer}} := \underline{\mathrm{Hom}}(\mathcal{E}_1, \mathcal{E}_2)$ and $\underline{\mathcal{H}} := \gamma(\underline{\mathcal{H}}^{\mathrm{mer}})$, and consider the functors:

$$\mathcal{H}^{\mathrm{mer}} : T \mapsto \mathrm{Hom}_{\mathrm{mer}}(\mathcal{O}, \underline{\mathcal{H}}^{\mathrm{mer}}_{|T}).$$

$$\mathcal{H} : T \mapsto \mathrm{Hom}(\mathcal{O}, \underline{\mathcal{H}}_{|T}).$$

We have an injective map of sheaves $\mathcal{H}^{\mathrm{mer}} \to \mathcal{H}$. Indeed, this can be checked on points where it follows from injectivity of $B^C_{[0,\infty)} \subseteq \mathbb{W}C$. It suffices to prove $\mathcal{H}^{\mathrm{mer}} \to \mathcal{H}$ is surjective. We may assume $\gamma_{\mathcal{E}_1} = \gamma_{\mathcal{E}_2}$ otherwise $\mathcal{H} = 0$.

Then $\underline{\mathcal{H}}^{\mathrm{mer}} \in \mathcal{SHT}[\frac{1}{\pi}](S)^{\mathrm{ss}}$ and $\underline{\mathcal{H}} \in \mathfrak{DM}[\frac{1}{\pi}](S)^{\mathrm{ss}}$ are semi-stable of slope 0. This case follows from Proposition 6.6 and Lemma 6.7.

Once we know that $(\mathfrak{B}^{\diamondsuit})^{\mathrm{ss}} \cong (\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{ss}}$ the equivalence $(\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{ss}} \cong (\mathrm{Bun}_{\mathrm{FF}})^{\mathrm{ss}}$ follows from [FS21, Theorem I.3.4]. Exactness of the equivalences can be checked on geometric points, but over points all categories are the category of finite modules over a central simple algebra over E. □

We extend the definition of semi-stable vector bundles to flags. For this we consider $\mathbb{Q}$-filtered meromorphic vector bundles (respectively vector bundles, respectively analytic isocrystals).

Let $S = \mathrm{Spa}\, R$. We consider sequences of the form $\{\mathcal{E}_r\}_{r\in\mathbb{Q}} \in \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S)$ (respectively $\{\mathcal{E}_r\}_{r\in\mathbb{Q}} \in \mathrm{Bun}_{\mathrm{FF}}(S)$, respectively $\{\mathcal{E}_r\}_{r\in\mathbb{Q}} \in \mathfrak{B}^{\diamondsuit}(S)$) with $\mathcal{E}_r \subseteq \mathcal{E}_s$ when $r < s$ and such that $\mathcal{E}_r/\mathcal{E}_{<r} = 0$ for all but finitely many $r \in \mathbb{Q}$. By hypothesis, there is $N >> 0$ such that $\mathcal{E}_s = \mathcal{E}_N$ for every $s > N$, we call $\mathcal{E}_N$ *the underlying vector bundle* of $\{\mathcal{E}_r\}_{r\in\mathbb{Q}}$.

Definition 6.10. We say that a $\mathbb{Q}$-filtered meromorphic vector bundle (respectively a vector bundle, respectively analytic isocrystal) is a *semi-stable filtration* if $\mathcal{E}_r/\mathcal{E}_{<r}$ is semi-stable of slope r. We let $\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}}(S)$ (respectively $\mathcal{F}il^{\sigma}_{\mathrm{ss}}(S)$, respectively $\mathcal{F}il^{\gamma}_{\mathrm{ss}}(S)$) denote the categories whose objects are semi-stable filtrations and whose morphisms are maps in $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S)$ (respectively $\mathrm{Bun}_{\mathrm{FF}}(S)$, respectively $\mathfrak{B}^{\diamondsuit}(S)$) that respect the filtration.

Proposition 6.11. *The natural map $\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}} \to \mathcal{F}il^{\sigma}_{\mathrm{ss}}$ is a $\otimes$-exact equivalence of v-stacks.*

Proof. Full-faithfulness: Let $\{\mathcal{E}_r\}_{r\in\mathbb{Q}}$ and $\{\mathcal{F}_r\}_{r\in\mathbb{Q}}$ be in $\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}}(S)$, with underlying meromorphic vector bundles $\mathcal{E}$ and $\mathcal{F}$. The internal $\underline{\mathrm{Hom}}$-bundle $\mathcal{H} := \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F})$ is naturally endowed with a $\mathbb{Q}$-filtration $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$. Now, it is not hard to verify that $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$ is a semi-stable filtration. Moreover, we have an identification:

$$\mathrm{Hom}_{\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}}}(\{\mathcal{E}_r\}_{r\in\mathbb{Q}}, \{\mathcal{F}_r\}_{r\in\mathbb{Q}}) = \mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}}(\mathcal{O}, \mathcal{H}_{\leq 0}).$$

Analogously,

$$\mathrm{Hom}_{\mathcal{F}il_{\mathrm{ss}}}(\{\mathcal{E}_r\}_{r\in\mathbb{Q}}, \{\mathcal{F}_r\}_{r\in\mathbb{Q}}) = \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}}(\mathcal{O}, \mathcal{H}_{\leq 0}).$$

Since $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$ is semistable, one can prove inductively on the support of $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$ that $\mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O},\mathcal{H}_{\leq r}) = 0 = \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}}(\mathcal{O},\mathcal{H}_{\leq r})$ for all $r<0$. To prove full-faithfulness it suffices to show:

$$\mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O},\mathcal{H}_{\leq 0}/\mathcal{H}_{<0}) \cong \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}}(\mathcal{O},\mathcal{H}_{\leq 0}/\mathcal{H}_{<0})$$

but $\mathcal{H}_{\leq 0}/\mathcal{H}_{<0}$ is semi-stable of slope 0, so the result follows directly from Proposition 6.9.

Essential surjectivity: Let $\{\mathcal{E}_r\} \in \mathcal{F}il^{\sigma}_{\mathrm{ss}}$ with underlying vector bundle $\mathcal{E}$ of rank n. If E_s is the degree s unramified extension of E then objects in $\mathrm{Bun}_{\mathrm{FF}}$ can be constructed by descent from objects in $\mathrm{Bun}_{\mathrm{FF},E_s}$, and by full-faithfulness a descent datum in $\mathcal{F}il^{\sigma}_{\mathrm{ss}}$ agrees with descent datum in $\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}}$. This allow us to assume that the support of the filtration is contained in $\mathbb{Z}$. Since essential surjectivity can now be proved v-locally we may think of every bundle $\mathcal{E}_r$ as a free module M_r over $B^R_{[1,q]}$ with φ-glueing data over $B^R_{[1,1]}$. We may even assume that the graded pieces $\mathcal{E}_N/\mathcal{E}_{<N}$ are isomorphic to $\mathcal{O}(N)^{m_N}$. We may choose basis for the M_r over $B^R_{[1,q]}$ compatible with the filtration and in such a way that after transferring the Frobenius structure to $\mathcal{O}^n$ the induced N-graded pieces are given by diagonal matrices of the form π^{-N}. This gives an upper block-diagonal matrix $A\in M_{n\times n}(B^R_{[1,1]})$, with diagonal blocks of the form $\pi^{-N}\cdot \mathrm{Id}_{m_N,m_N}$. To finish the argument, it suffices to show that there is a matrix $A_\infty \in P(B^R_{[0,1]}[\frac{1}{\pi}])$ and a matrix $U\in P(B^R_{[1,q]})$ with $U^{-1}A_\infty\varphi(U) = A$. This follows from Lemma 6.14 below. $\square$

Before proving the remaining Lemma 6.14, we need some preparations.

Lemma 6.12. *We have* $B^R_{[1,1]} = B^R_{[0,1]}[\frac{1}{\pi}] + [\varpi]B^R_{[1,\infty]}$.

Proof. Let $A_1 = W(R^+)[\frac{\pi}{[\varpi]}]$, $A_2 = W(R^+)[\frac{[\varpi]}{\pi}]$ and $A_{12} = W(R^+)[\frac{\pi}{[\varpi]},\frac{[\varpi]}{\pi}]$. We have $B^R_{[1,1]} = (A_{12})^\wedge_\pi[\frac{1}{\pi}]$, $B^R_{[0,1]} = (A_1)^\wedge_{[\varpi]}[\frac{1}{[\varpi]}]$ and $B^R_{[1,\infty]} = (A_2)^\wedge_\pi[\frac{1}{\pi}]$. After multiplication with a big enough power of π, it suffices to show that any element of $(A_{12})^\wedge_\pi$ can be written as a sum of an element of $(A_1)^\wedge_{[\varpi]}$ and an element of $\frac{[\varpi]}{\pi}\cdot(A_2)^\wedge_\pi$.

For any $n\geq 1$, let $I_n = \{(i,j)\in\mathbb{Z}^2\colon 0\leq i<n\}$ and let

$$S_n \subseteq \prod_{(i,j)\in I_n} R^+$$

be the subset of all sequences $a=(a_{ij})_{ij}$ for which $a_{ij}=0$ except for finitely many $(i,j)\in I_n$. Let also $S_n^+\subseteq S_n$ (resp. $S_n^-\subseteq S_n$) be the subset of all sequences for which $a_{ij}=0$ unless $j\geq 0$ (resp. $a_{ij}=0$ unless $j<0$). There is a commutative diagram, D_n, of sets

$$\begin{tikzcd} S_n^+ \arrow[r] \arrow[d] & S_n \arrow[d] & S_n^- \arrow[l] \arrow[d] \\ A_1/[\varpi]^n A_1 \arrow[r] & A_{12}/\pi^n A_{12} & \frac{[\varpi]}{\pi}\cdot(A_2/\pi^n A_2) \arrow[l] \end{tikzcd}$$

(note that $A_{12}/\pi^n A_{12} = A_{12}/[\varpi]^n A_{12}$), where the upper horizontal maps are the defining inclusions, the lower horizontal maps are induced by the natural ring maps $A_1 \to A_{12} \leftarrow A_2$ (and the inclusion of the ideal $\frac{[\varpi]}{\pi} A_2 \subseteq A_2$) and the vertical maps are given by sending $(a_{ij})_{ij}$ to $\sum_{ij} [a_{ij}] \pi^i \cdot (\frac{\pi}{[\varpi]})^j$.

We make three observations, which immediately follow from the explicit definition of the vertical maps: first, the middle vertical map is surjective. Second, there is an obvious map $D_{n+1} \to D_n$ of commutative diagrams and the resulting diagram is commutative. Third, when we define the map $+\colon S_n^+ \times S_n^- \to S_n$ by $(a+b)_{ij} = a_{ij}$ if $j \geq 0$ and $(a+b)_{ij} = b_{ij}$ if $j < 0$, then the resulting diagram

$$\begin{tikzcd}
S_n^+ \times S_n^- \arrow[r, "+"] \arrow[d] & S_n \arrow[d] \\
A_1/[\varpi]^n A_1 \times \frac{[\varpi]}{\pi} \cdot (A_2/\pi^n A_2) \arrow[r, "+"] & A_{12}/\pi^n A_{12}
\end{tikzcd}$$

is commutative.

Let now $S = \lim_n S_n$ and $S^{\pm} = \lim_n S_n^{\pm}$. Explicitly, $S \subseteq \prod_{(i,j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}} R^+$ is the subset of all sequences $(a_{ij})_{ij}$ satisfying the condition that for each i there is some $j(i) \geq 0$ such that $a_{ij} = 0$ unless $|j| < j(i)$ and S^+ and S^- are corresponding subsets of S. Passing to the limit over all $n > 0$, we obtain a commutative diagram

$$\begin{tikzcd}
S^+ \arrow[r] \arrow[d] & S \arrow[d] & S^- \arrow[l] \arrow[d] \\
(A_1)^\wedge_{[\varpi]} \arrow[r] & (A_{12})^\wedge_\pi & \frac{[\varpi]}{\pi} \cdot (A_2)^\wedge_\pi \arrow[l]
\end{tikzcd}$$

where the middle vertical arrow is still surjective. Moreover, we also get the commutative diagram

$$\begin{tikzcd}
S^+ \times S^- \arrow[r, "+"] \arrow[d] & S \arrow[d] \\
(A_1)^\wedge_{[\varpi]} \times \frac{[\varpi]}{\pi} \cdot (A_2)^\wedge_\pi \arrow[r, "+"] & (A_{12})^\wedge_\pi,
\end{tikzcd}$$

where the lower horizontal map is the restriction of the addition map $B_{[0,1]} \times \frac{[\varpi]}{\pi} \cdot B_{[1,\infty]} \to B_{[1,1]}$ and the upper horizontal map is defined in the same way as $S_n^+ \times S_n^- \to S_n$. Now, $S^+ \times S^- \to S$ and $S \to (A_{12})^\wedge_\pi$ are surjective, and hence also the lower horizontal map in the diagram is surjective, which is precisely what we had to show. $\square$

Recall that restriction of functions defines an inclusion $B^R_{[\frac{1}{q},\infty]} \subseteq B^R_{[1,\infty]}$ and Frobenius induces an isomorphism $\varphi\colon B^R_{[1,\infty]} \xrightarrow{\sim} B^R_{[\frac{1}{q},\infty]} \subseteq B^R_{[1,\infty]}$.

Lemma 6.13. *Let $k \in \mathbb{Z}_{\geq 0}$. The image of the map*

$$\psi_k \colon B^R_{[1,\infty]} \to B^R_{[1,\infty]}, \quad a \mapsto \pi^{-k}a - \varphi(a)$$

contains $[\varpi]B^R_{[1,\infty]}$. If $k > 0$, it contains $B^R_{[1,\infty]}$.

Proof. Let $A = W(R^+)[\frac{\varpi}{\pi}]$. Recall that $B^R_{[1,\infty]} = A^\wedge_\pi[\frac{1}{\pi}]$. Thus, as $\psi_k(\pi^n x) = \pi^n\psi_k(x)$, it suffices to show that the image contains $[\varpi]A^\wedge_\pi$ (resp. $A^\wedge_\pi$ if $k > 0$). Let $x \in [\varpi]A^\wedge_\pi$ if $k = 0$ (resp. $x \in A^\wedge_\pi$ if $k > 0$). Note that the sequence $(\pi^{i\cdot k}\varphi^{(i-1)}(x))_{i\geq 1}$ in $A^\wedge_\pi$ converges π-adically to 0. (Use that $\varphi(A^\wedge_\pi) \subseteq A^\wedge_\pi$ and $\varphi([\varpi]) = [\varpi]^q$.) Thus $y = \sum_{i=1}^\infty \pi^{ik}\varphi^{(i-1)}(x)$ exists in $A^\wedge_\pi$. By π-adic continuity of Frobenius and hence of ψ_k, it is immediate that $\psi_k(y) = x$. $\square$

Lemma 6.14. *Let $n \geq 1$ and let $A \in \mathrm{GL}_n(B^R_{[1,1]})$ be upper triangular with ith diagonal entry π^{s_i} for some $s_i \in \mathbb{Z}$ ($1 \leq i \leq n$). Assume that $s_1 \geq s_2 \geq \cdots \geq s_n$ holds. Then there exists a unipotent upper triangular matrix $U \in \mathrm{GL}_n(B^R_{[1,\infty]})$ such that $U^{-1}A\varphi(U)$ is upper triangular with entries in $B^R_{[0,1]}[\frac{1}{\pi}]$.*

Proof. We argue by induction on n. If $n = 1$, there is nothing to show. Assume n is fixed and we know the claim for all matrices of size $(n-1) \times (n-1)$. Let a_{ij} denote the (i,j)th entry of A. Exploiting the induction hypothesis for the lower right $(n-1) \times (n-1)$-minor of A, we may assume that $a_{ij} \in B^R_{[0,1]}[\frac{1}{\pi}]$ for all $i > 1$. Let now $1 < j \leq n$. Suppose, by induction, that for all $1 < j' < j$, one has $a_{1j'} \in B_{[0,1]}[\frac{1}{\pi}]$. It suffices to find, in this situation, a unipotent upper triangular matrix $U \in \mathrm{GL}_n(B^R_{[1,\infty]})$ such that $U^{-1}A\varphi(U)$ has all the above properties of A and additionally its $(1,j)$th entry lies in $B^R_{[0,1]}[\frac{1}{\pi}]$. Therefore, write $a_{1j} = a^{\mathrm{mer}}_{1j} + a'_{1j}$ with some $a^{\mathrm{mer}}_{1j} \in B^R_{[0,1]}[\frac{1}{\pi}]$ and $a'_{1j} \in [\varpi]B^R_{[1,\infty]}$, according to Lemma 6.12. By Lemma 6.13, there exists some $y \in B^R_{[1,\infty]}$ with $\psi_{s_1-s_j}(y) = a'_{1j}$ (we use $s_j \leq s_1$). Let $U = (U_{\ell m})_{\ell m} \in \mathrm{GL}_n(B^R_{[1,\infty]})$ be such that $U_{\ell m} = \delta_{\ell m}$ (Kronecker-delta), unless $(\ell, m) = (1,j)$ and $U_{1j} = y$. Then it is immediate to compute that $U^{-1}A\varphi(U)$ satisfies all the claimed conditions. $\square$

Proposition 6.15. *The forgetful functor $\mathcal{F}il^\gamma_{\mathrm{ss}} \to \mathfrak{B}^\diamondsuit$ factors through $(\mathfrak{B}^\diamondsuit)^{\mathrm{loc}}$ and defines a $\otimes$-exact equivalence:*

$$\mathcal{F}il^\gamma_{\mathrm{ss}} \to (\mathfrak{B}^\diamondsuit)^{\mathrm{loc}}.$$

Proof. On points, any filtration splits since the category of isocrystals is semi-simple. In particular, the Newton polygon can be computed on the graded pieces. By definition of semi-stable filtrations the Newton polygon is constant on the graded isocrystal.

We prove full-faithfulness, let $\{\mathcal{E}_r\}_{r\in\mathbb{Q}}$ and $\{\mathcal{F}_r\}_{r\in\mathbb{Q}}$ be two semi-stable filtrations with underlying analytic isocrystals $\mathcal{E}$ and $\mathcal{F}$. Let $\mathcal{H}$ denote the

$\underline{\mathrm{Hom}}$-bundle endowed with its induced semi-stable filtration $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$. We need to show that:

$$\mathrm{Hom}_{\mathfrak{B}^\diamond}(\mathcal{O},\mathcal{H}) = \mathrm{Hom}_{\mathfrak{B}^\diamond}(\mathcal{O},\mathcal{H}_{\leq 0}).$$

But, we can prove inductively on the support of $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$ that

$$\mathrm{Hom}_{\mathfrak{B}^\diamond}(\mathcal{O},\mathcal{H}_{\leq r}/\mathcal{H}_{\leq 0}) = 0,$$

for all $r > 0$ since the graded pieces all have slope larger than 0.

Since essential surjectivity can be proved v-locally it suffices to show that the standard objects can be endowed with a semi-stable filtration, but this is clear. □

Proposition 6.16. *The forgetful functor $\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}} \to \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$ factors through $(\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{loc}}$ and defines a $\otimes$-exact equivalence:*

$$\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}} \to (\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{loc}}.$$

Proof. That the map respects the monoidal structure and exactness is automatic, since it is defined in terms of those of $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$. That the map factors through $(\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{loc}}$ follows from Proposition 6.15. To show full-faithfulness we may pass again to a $\underline{\mathrm{Hom}}$-bundles $\mathcal{H}$ with semi-stable filtration $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$ as in the proof of Proposition 6.15. We need to show:

$$\mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}}(\mathcal{O},\mathcal{H}) = \mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}}(\mathcal{O},\mathcal{H}_{\leq 0}).$$

But as in the proof of Proposition 6.15, $\mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}}(\mathcal{O},\mathcal{H}_{\leq r}/\mathcal{H}_{\leq 0}) = 0$ for all $r > 0$.

Essential surjectivity can now be proved v-locally. So it suffices to show that every isoshtuka $\mathcal{E} \in (\mathcal{SHT}[\frac{1}{\pi}])^{\mathrm{loc}}(S)$ can be endowed with a semi-stable filtration. In other words, we must show that the unique semi-stable filtration of $\gamma(\mathcal{E})$ lifts to a filtration in $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$. Replacing E by its degree s field extension E_s, and since we have already proved full-faithfulness, we may assume that the generic Newton polygon only takes values in $\mathbb{Z}$. Twisting by a line bundle we may even assume that the smallest slope $\mathcal{E}$ is 0. We can now apply Proposition 6.6 and Lemma 6.7 to find a sub-bundle $\mathcal{O}^k \subseteq \mathcal{E}$, where k is the rank of $\gamma(\mathcal{E})_0$ and such that $\gamma(\mathcal{E})/\gamma(\mathcal{O}^k)$ has all slopes greater than 0. By induction on the rank, $\mathcal{E}/\mathcal{O}^k$ can be endowed with a semi-stable filtration $\{(\mathcal{E}/\mathcal{O}^k)_r\}_{r\in\mathbb{Q}}$, we can lift this filtration to $\mathcal{E}$. □

7. G-bundles with meromorphic structure

7.1. **$\mathcal{G}$-structure.** Let $\mathcal{G}$ be a smooth affine group scheme over $\mathrm{Spec}\, O_E$, and denote by G its generic fiber over $\mathrm{Spec}\, E$. Later on we will assume that $\mathcal{G}$ is parahoric and that G is reductive. We let $\mathrm{Rep}_{\mathcal{G}}$, respectively Rep_G, denote the Tannakian category of algebraic representations of $\mathcal{G}$ over O_E, respectively of G over E.

Definition 7.1. We let $\mathfrak{DM}_{\mathcal{G}} : \mathrm{PSch} \to \mathrm{Cat}_1^{\otimes}$ denote the presheaf in groupoids with

$$S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, \mathfrak{DM}(S)),$$

where $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}$ denotes the $\otimes$-compatible O_E-linear exact functors. Analogously, we let $\mathfrak{B}(G) : \mathrm{PSch} \to \mathrm{Cat}_1^{\otimes}$ denote the presheaf in groupoid with

$$S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathfrak{B}(S)).$$

Recall the loop group and positive loop group functors $LG, L^+\mathcal{G} : \mathrm{PSch} \to \mathrm{Sets}$ given on affine schemes $S = \mathrm{Spec}\, A$ by the formulas

$$LG(S) := G(\mathbb{W}(A)[\frac{1}{\pi}])$$

and

$$L^+\mathcal{G}(S) := \mathcal{G}(\mathbb{W}(A)).$$

We let LG and $L^+\mathcal{G}$ act on LG by φ-conjugation.

Proposition 7.2. *LG and $L^+\mathcal{G}$ are arc-sheaves.*

Proof. As both are ind-schemes and the arc-topology is subcanonical (in fact, canonical) on perfect $\mathbb{F}_p$-schemes by [BM21, Theorem 5.16], the claim follows. $\square$

Proposition 7.3. *The following statements hold:*

1. *$\mathfrak{DM}_{\mathcal{G}}$ and $\mathfrak{B}(G)$ are scheme theoretic small v-stacks.*
2. *The natural maps $LG \to \mathfrak{DM}_{\mathcal{G}}$ and $LG \to \mathfrak{B}(G)$ are v-covers.*
3. *We have identities $\mathfrak{DM}_{\mathcal{G}} = [LG//_{\varphi} L^+\mathcal{G}]$ and $\mathfrak{B}(G) = [LG//_{\varphi} LG]$.*

Proof. The first claim follows by Tannakian formalism from Proposition 4.6. The second claim holds as v-locally on R any $\mathcal{G}$-torsor resp. G-torsor is free. Indeed, this happens when R is a strict comb. For $\mathcal{G}$-torsors this is easy to see since étale locally in $\mathrm{Spec}\, \mathbb{W}R$ any $\mathcal{G}$-torsor is trivial, but if R is a strict comb (even a if it is only a w-contractible affine scheme [BS15, Definition 1.4]) any étale cover of $\mathrm{Spec}\, \mathbb{W}R$ has a section. Now, G-torsors are free on combs by [Ans22b, Theorem 11.4][3] (see [Iva23, Theorem 6.1] for the vector bundle case). The third claims follows directly from the second one by computing the fiber products $LG \times_{\mathfrak{DM}_{\mathcal{G}}} LG$ and $LG \times_{\mathfrak{B}(G)} LG$. $\square$

Definition 7.4. We define the following 4 presheaves over Perf with values in groupoids:

1. $\mathrm{Sht}_{\mathcal{G}}$ with: $S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, \mathcal{SHT}(S))$.
2. Isoc_G with: $S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, \mathfrak{B}^{\diamondsuit}(S))$.
3. $\mathrm{Bun}^{\mathrm{mer}}_G$ with: $S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S))$.
4. $\mathrm{DM}_{\mathcal{G}}$ with: $S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, \mathfrak{DM}^{\diamondsuit}(S))$.

Theorem 7.5. *The following statements hold:*

1. *$\mathrm{Sht}_{\mathcal{G}}$, $\mathrm{DM}_{\mathcal{G}}$, Isoc_G and $\mathrm{Bun}^{\mathrm{mer}}_G$ are small v-stacks.*
2. *We have a Cartesian diagram:*

[3] The running assumption on loc. cit. is that G is reductive, but the proof of Theorem 11.4 does not use this hypothesis.

$$\begin{tikzcd} \mathrm{Sht}_{\mathcal{G}} \arrow[r] \arrow[d] & \mathrm{Bun}_G^{\mathrm{mer}} \arrow[d] \\ \mathrm{DM}_{\mathcal{G}} \arrow[r] & \mathrm{Isoc}_G \end{tikzcd}$$

(3) *We have identifications*

$$\mathrm{DM}_{\mathcal{G}} = (\mathfrak{DM}_{\mathcal{G}})^{\diamondsuit} = [LG^{\diamondsuit}//_{\varphi} L^+\mathcal{G}^{\diamondsuit}]$$

and

$$\mathrm{Isoc}_G = (\mathfrak{B}(G))^{\diamondsuit} = [LG^{\diamondsuit}//_{\varphi} LG^{\diamondsuit}].$$

(4) *The maps* $\mathrm{DM}_{\mathcal{G}} \to \mathrm{Isoc}_G$ *and* $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Bun}_G^{\mathrm{mer}}$ *are v-covers.*

Proof. Since the application $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, -)$ commutes with 2-limits within $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ and all of $\mathcal{SHT}$, $\mathfrak{DM}^{\diamondsuit}$, $\mathfrak{B}^{\diamondsuit}$ and $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$ are v-stacks in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ all of the presheaves of Definition 7.4 are v-sheaves. For the same reason, the second claim follows directly from Corollary 5.11. Furthermore, $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, -)$ commutes with sheafification which implies directly $\mathrm{DM}_{\mathcal{G}} = (\mathfrak{DM}_{\mathcal{G}})^{\diamondsuit}$ and $\mathrm{Isoc}_G = \mathfrak{B}(G)^{\diamondsuit}$. Since the functor $(-)^{\diamondsuit}$ commutes with finite limits it suffices to prove that the maps $LG^{\diamondsuit} \to \mathrm{DM}_{\mathcal{G}}$ and $LG^{\diamondsuit} \to \mathrm{Isoc}_G$ are surjective to deduce the formulas from the third assertion. Let $\mathcal{F} \in \mathrm{Isoc}_G(S)$, the argument for $\mathrm{DM}_{\mathcal{G}}$ being analogous. Surjectivity can be shown v-locally so we may assume $S = \mathrm{Spa}\, R$ is a strict product of points and that for all objects $V \in \mathrm{Rep}_{\mathcal{G}}$ the object $\mathcal{F}(V) \in \mathfrak{B}^{\diamondsuit}(S)$ is isomorphic to one in $\mathfrak{DM}[\frac{1}{\pi}]^{\diamondsuit_{\mathrm{pre}}}$. We obtain $\otimes$-exact functor from $\mathrm{Rep}_{\mathcal{G}}$ to the category of projective $\mathbb{W}(R)[\frac{1}{\pi}]$-modules which we interpret as a G-torsor over $\mathrm{Spec}\, \mathbb{W}(R)[\frac{1}{\pi}]$. By [Ans22b, Theorem 11.4] such torsors are trivial over combs, and by Proposition 2.4 $\mathrm{Spec}\, R$ is a comb. After choosing a trivialization of $\mathcal{F}$, the φ-structure corresponds to an element $LG(\mathrm{Spec}\, R)$ which gives precisely a point $LG^{\diamondsuit}(S)$ lifting our original point. The final claim follows from basechange from the third claim and the second claim. □

7.2. **Newton strata on** Isoc_G**.** We now wish to study the geometry of Isoc_G and $\mathrm{Bun}_G^{\mathrm{mer}}$. Recall the Kottwitz set $B(G)$, which classifies isocrystals with G-structure over algebraically closed fields. The Newton point defines a map $B(G) \to \mathcal{N}(G)$, where the Newton cone $\mathcal{N}(G)$ of G is a partially ordered set (for $G = \mathrm{GL}_n$, $\mathcal{N}(G) = \mathcal{N}$ with $\mathcal{N}$ from Section 6). In particular, $B(G)$ inherits the partial order from $\mathcal{N}$. For more details on $B(G)$ see, for example, [Vie20, Section 3].

Definition 7.6. Let $\mathcal{S} = \mathrm{Spec}\, A \in \mathrm{PSch}$, and let $b \in B(G)$. We let $\mathfrak{B}(G)_{\leq b}(\mathcal{S}) \subseteq \mathfrak{B}(G)(\mathcal{S})$ denote the full subcategory of objects $\mathcal{E} \in \mathfrak{B}(G)(\mathcal{S})$ whose Newton polygon is bounded by b at geometric points of $\mathcal{S}$. We let $\mathfrak{B}(G)_b(\mathcal{S}) \subseteq \mathfrak{B}(G)_{\leq b}(\mathcal{S})$ denote the full subcategory of objects $\mathcal{E} \in \mathfrak{B}(G)_{\leq b}(\mathcal{S})$ whose Newton polygon is exactly b at geometric points of $\mathcal{S}$.

The following theorem due to work of various authors summarizes what we will need about the geometry of $\mathfrak{B}(G)$.

Theorem 7.7. *For any $b \in B(G)$ the map $\mathfrak{B}(G)_{\leq b} \to \mathfrak{B}(G)$ is a perfectly finitely presented closed immersion. Moreover, $\mathfrak{B}(G)_b = [*/\underline{J_b(\mathbb{Q}_p)}]$ as scheme-theoretic v-stacks.*

Proof. The first statement follows from [RR96, Theorem 3.6(ii)]. The last statement follows from [HK22, Theorem 2.11]. □

Proposition 7.8. *The elements of $|\operatorname{Isoc}_G|$ are in bijection with $B(G)$.*

Proof. Points in $|\operatorname{Isoc}_G|$ are in bijection with equivalence classes of C-valued points of Isoc_G, ranging over all v-covers of $\operatorname{Spa} C$. After replacing C by a v-cover they are of the form $\mathfrak{B}(G)(\operatorname{Spec} C)$, which is given by $B(G)$. □

Remark 7.9. It follows a posteriori from Proposition 7.11, that for C an algebraically closed non-Archimedean field the natural map is an equivalence of categories $\mathfrak{B}(G)(\operatorname{Spec} C) \cong \operatorname{Isoc}_G(\operatorname{Spa} C)$.

Definition 7.10. Let $S = \operatorname{Spa} R \in \operatorname{Perf}$. We let $\operatorname{Isoc}_G^{\leq b}(S) \subseteq \operatorname{Isoc}_G(S)$ denote the full subcategory of objects $\mathcal{E} \in \operatorname{Isoc}_G(S)$ whose Newton polygon is bounded by b at geometric points of S. We let $\operatorname{Isoc}_G^b(S) \subseteq \operatorname{Isoc}_G^{\leq b}(S)$ denote the full subcategory of objects $\mathcal{E} \in \operatorname{Isoc}_G^{\leq b}(\mathcal{S})$ whose Newton polygon is exactly b at geometric points of S.

Proposition 7.11. *For any $b \in B(G)$ the map $\operatorname{Isoc}_G^{\leq b} \to \operatorname{Isoc}_G$ is a closed immersion and agrees with $\mathfrak{B}(G)^\diamondsuit_{\leq b}$. The map $\operatorname{Isoc}_G^b \to \operatorname{Isoc}_G^{\leq b}$ is an open immersion. Moreover, $\operatorname{Isoc}_G^b = \mathfrak{B}(G)^\diamondsuit_b = [*/\underline{J_b(\mathbb{Q}_p)}]$ as v-stacks.*

Proof. Since $\diamondsuit$ preserves open and closed immersions, it suffices to identify $\operatorname{Isoc}_G^{\leq b}$ and Isoc_G^b with $\mathfrak{B}(G)^\diamondsuit_{\leq b}$ and $\mathfrak{B}(G)^\diamondsuit_b$ respectively. Let $S = \operatorname{Spa} R$, by definition $\operatorname{Isoc}_G^{\leq b}(S)$ is the subcategory of objects of $\mathcal{E} \in \operatorname{Isoc}_G(S)$ whose Newton polygon is pointwise bounded by b at every geometric point S. Whereas, $\mathfrak{B}(G)^{\diamondsuit \mathrm{pre}}_{\leq b}(S)$ correspond to isocrystals over $\operatorname{Spec} R$ whose polygon is bounded by b at every geometric point of $\operatorname{Spec} R$. To prove $\mathfrak{B}(G)^\diamondsuit_{\leq b} = \operatorname{Isoc}_G^{\leq b}$ it suffices to show that v-locally having Newton polygon be bounded by $\operatorname{Spa} R$ or by $\operatorname{Spec} R$ agree. Of course, the schematic condition is stronger than the analytic one, since on the analytic side a condition is imposed only on those ideals of $\operatorname{Spec} R$ that support a continuous valuation. Now, over product of points the two conditions agree. Indeed, principal connected components of a product of points support a continuous valuation. Moreover, these components are dense in $\operatorname{Spec} R$.

A similar argument shows $\mathfrak{B}(G)^\diamondsuit_b = \operatorname{Isoc}_G^b$. Indeed, if S is a product of points all of the maximal ideals of $\operatorname{Spec} R$ support a continuous valuation and the map $\mathfrak{B}(G)_b \to \mathfrak{B}(G)_{\leq b}$ is open.

The last claim follows directly from Proposition 2.5. □

7.3. **Newton strata on $\operatorname{Bun}_G^{\mathrm{mer}}$.** Recall the moduli stack $\mathcal{M}$ of Fargues–Scholze [FS21, Definition V.3.2]. The connected components of $\mathcal{M}$ are indexed by $b \in B(G)$ and the map $\mathcal{M}_b \to \operatorname{Bun}_G$ are the smooth charts.

Proposition 7.12. *The v-stack $\mathcal{M}$ is the moduli stack given by the formula*

$$\mathcal{M} : S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathcal{F}il^{\sigma}_{\mathrm{ss}}(S)).$$

Proof. It follows directly from the definition. □

Theorem 7.13. *The moduli stack $\mathcal{M}$ fits in the following Cartesian diagram of small v-stacks:*

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[d] & \mathrm{Bun}_G^{\mathrm{mer}} \arrow[d, "\gamma"] \\ \coprod_{b \in B(G)} \mathrm{Isoc}_G^b \arrow[r] & \mathrm{Isoc}_G \end{tikzcd}$$

Remark 7.14. While this article was in preparation we learned from a private communication with Z. Wu that he had proven independently a version of Theorem 7.13 in the language of relative Robba rings.

Proof. Observe that we have the following identification:

$$\coprod_{b \in B(G)} \mathrm{Isoc}_G^b(S) = \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, (\mathfrak{B}^{\diamondsuit})^{\mathrm{loc}}(S)).$$

Since $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, -)$ commutes with limits, it suffices to show that $\mathcal{F}il^{\sigma}_{\mathrm{ss}}(S)$ fits on the following Cartesian diagram:

$$\begin{tikzcd} \mathcal{F}il^{\sigma}_{\mathrm{ss}}(S) \arrow[r] \arrow[d] & \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S) \arrow[d] \\ (\mathfrak{B}^{\diamondsuit})^{\mathrm{loc}}(S) \arrow[r] & \mathfrak{B}^{\diamondsuit}(S) \end{tikzcd}$$

By definition, $(\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}})^{\mathrm{loc}}$ fits as the upper-left entry of the above Cartesian diagram. But by Proposition 6.11 and Proposition 6.16

$$(\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}})^{\mathrm{loc}}(S) \cong \mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}} \cong \mathcal{F}il^{\sigma}_{\mathrm{ss}}(S)$$

□

Corollary 7.15. *Let $S = \mathrm{Spa}\, R$ and let $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(S)$. The following hold:*

(1) *After replacing S by a v-cover, $\mathcal{E}$ can be lifted to $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$.*
(2) *After replacing S by a v-cover, $\mathcal{E}$ can be lifted to $\mathcal{SHT}(S)$.*
(3) *The map of small v-stacks $\mathrm{Bun}_G^{\mathrm{mer}} \to \mathrm{Bun}_G$ is surjective.*
(4) *The map of small v-stacks $\mathrm{Sht}_G^{\mathrm{mer}} \to \mathrm{Bun}_G$ is surjective.*

Proof. The first and second claims are particular instances of the third and fourth claim in the case where $G = \mathrm{GL}_n$. For the third claim, the map $\mathcal{M} \to \mathrm{Bun}_G$ is formally and ℓ-cohomologically smooth and surjects onto its image. In particular, it is a surjection of small v-stacks. The result follows since this map factors through $\mathrm{Bun}_G^{\mathrm{mer}} \to \mathrm{Bun}_G$. The fourth claim follows from Theorem 7.5 and from the third claim. □

Definition 7.16. Given two subsets $U_1, U_2 \subseteq B(G)$ We let $\mathcal{M}^{\sigma\in U_2}_{\gamma\in U_1}$ denote $\gamma^{-1}(\mathrm{Isoc}_G^{U_1}) \cap \sigma^{-1}(\mathrm{Bun}_G^{U_2})$. Whenever $U_i = B(G)$, we omit the subscript or superscript as an abbreviation.

We will mostly use Definition 7.16 when U_1 or U_2 are given by Newton polygon inequalities. In this case, we use more intuitive notation for example $\mathcal{M}^{\sigma=b}$ means $\sigma^{-1}(\mathrm{Bun}_G^b)$ and $\mathcal{M}_{\gamma=b} = \gamma^{-1}(\mathrm{Isoc}_G^b) = \mathcal{M}_b$.

8. Extending vector bundles at ∞

Let C be a non-Archimedean algebraically closed field. One interesting consequence of the classification theorem of vector bundles on the Fargues–Fontaine curve is that every such vector bundle extends at ∞ i.e. it is isomorphic to one obtained from a φ-module over $Y^C_{(0,\infty]}$. The purpose of this section is to prove that this statement holds in families when one is allowed to work v-locally.

Definition 8.1. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$ and $T = \mathrm{Spa}(R, R^\circ)$.

(1) We let $\mathrm{Bun}^+_{\mathrm{FF}} : \mathrm{Perf} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ denote presheaf given by the rule that attaches to S the category of pairs $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a vector bundle over $Y^{R^\circ}_{(0,\infty]}$ and $\Phi : \varphi^*\mathcal{E} \to \mathcal{E}$ is an isomorphism.
(2) We say that $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(S)$ *extends* at ∞ if it is in the essential image of the map $\mathrm{Bun}^+_{\mathrm{FF}}(S) \to \mathrm{Bun}_{\mathrm{FF}}(T) \cong \mathrm{Bun}_{\mathrm{FF}}(S)$.
(3) We denote $\mathfrak{DM}^{\dagger_{\mathrm{pre}}} : \mathrm{Perf} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ the presheaf given by the rule:
$$(R, R^+) \mapsto \mathfrak{DM}(\mathrm{Spec}\, R^\circ)$$
(4) We say that $\mathcal{E} \in \mathcal{SHT}(S)$ is a *BKF-shtuka* if it is in the essential image of the map $\mathfrak{DM}^{\dagger_{\mathrm{pre}}}(S) \to \mathcal{SHT}(T) \cong \mathcal{SHT}(S)$.
(5) We denote $\mathfrak{DM}[\frac{1}{\pi}]^{\dagger_{\mathrm{pre}}} : \mathrm{Perf} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ the presheaf given by the rule:
$$(R, R^+) \mapsto \mathfrak{DM}[\frac{1}{\pi}](\mathrm{Spec}\, R^\circ)$$
(6) We denote $\mathfrak{B}^{\dagger_{\mathrm{pre}}} : \mathrm{Perf} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ the presheaf given by the rule:
$$(R, R^+) \mapsto \mathfrak{B}(\mathrm{Spec}\, R^\circ)$$

Proposition 8.2. *Let $S = \mathrm{Spa}\, R \in \mathrm{Perf}$, the following hold:*

(1) *The map $\mathrm{Bun}^+_{\mathrm{FF}}(S) \to \mathrm{Bun}_{\mathrm{FF}}(S)$ is exact and fully-faithful.*
(2) *If S is a product of points then we have the following sequence of Cartesian diagrams in $\mathrm{Cat}_1^{\otimes}$:*
$$\begin{tikzcd} \mathfrak{DM}^{\dagger_{\mathrm{pre}}}(S) \arrow[r] \arrow[d] & \mathfrak{DM}[\frac{1}{\pi}]^{\dagger_{\mathrm{pre}}}(S) \arrow[r] \arrow[d] & \mathrm{Bun}^+_{\mathrm{FF}}(S) \arrow[d] \\ \mathcal{SHT}(S) \arrow[r] & \mathcal{SHT}[\frac{1}{\pi}](S) \arrow[r] & \mathrm{Bun}_{\mathrm{FF}}(S) \end{tikzcd}$$
(3) *If S is a product of points then $\mathfrak{DM}[\frac{1}{\pi}]^{\dagger_{\mathrm{pre}}}(S) \cong \mathfrak{B}^{\dagger_{\mathrm{pre}}}(S)$.*

(4) *The sheafification of* $\mathfrak{DM}[\frac{1}{\pi}]^{\dagger\mathrm{pre}}$ *is* $\mathfrak{B}^\dagger$.

Proof. The first claim is [PR21, Proposition 2.1.4]. For the second claim, note that by Kedlaya's GAGA [Ked20, Theorem 3.8] we can identify the category $\mathcal{SHT}(S) \times_{\mathrm{Bun}_{\mathrm{FF}}(S)} \mathrm{Bun}^+_{\mathrm{FF}}(S)$ with the category of vector bundles over $\operatorname{Spec} \mathbb{W}(R^\circ) \setminus (\{\pi = 0\} \cap \{[\varpi] = 0\})$ together with φ-action defined over $\operatorname{Spec} \mathbb{W}(R^\circ)[\frac{1}{\pi}]$. But as S is a product of points, by [Ked20] (or [Ans22b, Theorem 1.1]) and [Gle21b, Proposition 2.1.17], any such vector bundle extends uniquely to a vector bundle over $\operatorname{Spec} \mathbb{W}(R^\circ)$. This proves that the outer diagram is Cartesian. Moreover, the same argument also applies to the isogeny categories, proving that the right square is Cartesian. It then follows that the left square is Cartesian.

For the third claim, write $S = \mathrm{Spa}(R, R^+)$. We need to show that any isocrystal $\mathcal{E}$ over $\mathbb{W}(R^\circ)[\frac{1}{\pi}]$ contains a $\mathbb{W}(R^\circ)$-lattice. But as S is a product of points, Proposition 2.4 and [Iva23, Theorem 6.1] imply that $\mathcal{E}$ is free as a $\mathbb{W}(R^\circ)[\frac{1}{\pi}]$-module. But then an $\mathbb{W}(R^\circ)$-lattice obviously exists.

Fourth claim follows from the third. □

Remark 8.3. We warn the reader that the maps $\mathfrak{DM}^{\dagger\mathrm{pre}}(S) \to \mathcal{SHT}(S)$ and $\mathrm{Bun}^+_{\mathrm{FF}}(S) \to \mathrm{Bun}_{\mathrm{FF}}(S)$ do not reflect exactness.

The advantage of working with $\mathfrak{DM}^{\dagger\mathrm{pre}}$ is that its values on product of points are easy to describe.

Proposition 8.4. *Let* $S = \operatorname{Spa} R$ *be a product of points with* $R^+ = R^\circ = \prod_{i\in I} O_{C_i}$*, then the restriction functor*

$$\mathfrak{DM}^{\dagger\mathrm{pre}}(S) \to \prod_{i\in I} \mathfrak{DM}(\operatorname{Spec} O_{C_i})$$

is fully-faithful, and its essential image is the collection of families of $\{(\mathcal{E}_i, \Phi_i)\}_{i\in I}$ *with uniformly bounded zeros and poles on* π.

Proof. The fully-faithful functor is induced by the isomorphism $\mathbb{W}(\prod O_{C_i}) = \prod \mathbb{W}(O_{C_i})$. The pole (resp. zero) at each $i \in I$ of any object in the essential image is bounded by the pole (resp. zero) of its preimage. Conversely, if we have a uniform bound, then the Frobenius is represented by a matrix with entries in $\mathbb{W}(R^\circ)[\frac{1}{\pi}] = (\prod \mathbb{W}(O_{C_i}))[\frac{1}{\pi}] \subseteq \prod(\mathbb{W}(O_{C_i})[\frac{1}{\pi}])$, whose inverse also has entries in this subring. □

Moreover at the level of geometric points $\mathcal{SHT}$ is also easy to describe, this is the $\pi = \xi$ version of Fargues' theorem [SW20, Theorem 14.1.1].

Proposition 8.5. *Let* C *be a non-Archimedean field, then the following categories are equivalent:*

(1) *BKF-modules with* $\xi = \pi$*. In other words, the category pairs* (M, Φ) *where* M *is a free* $\mathbb{W}(O_C)$*-module and* $\Phi : M[\frac{1}{\pi}] \to \mathbb{W}(O_C)_\varphi \otimes_{\mathbb{W}(O_C)} M[\frac{1}{\pi}]$ *is an isomorphism.*
(2) $\mathfrak{DM}^{\dagger\mathrm{pre}}(C, C^+)$

(3) $\mathcal{SHT}(C, C^+)$.

Proof. By definition $\mathfrak{DM}^{\dagger \mathrm{pre}}(C, C^+) = \mathfrak{DM}(O_C)$, which is precisely a BKF-module with $\xi = \pi$, so the first two categories are the same category. The equivalence with the third category is given in [SW20, §12-14] when $\xi \neq \pi$. The same proof strategy applies. $\square$

In Proposition 9.3 we will extend Proposition 8.5 to the case of product of points.

Theorem 8.6. *Let $S = \operatorname{Spa} R \in \mathrm{Perf}$, the following hold:*

(1) *Given $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(S)$ there is a v-cover $S' \to S$ and a unique (up to isomorphism) $\mathcal{F} \in \mathrm{Bun}^+_{\mathrm{FF}}(S)$ with $\mathcal{F} \cong \mathcal{E}$ in $\mathrm{Bun}_{\mathrm{FF}}(S')$.*
(2) *Given $\mathcal{E} \in \mathcal{SHT}(S)$ there is a v-cover $S' \to S$ and a unique (up to isomorphism) $\mathcal{F} \in \mathfrak{DM}^{\dagger \mathrm{pre}}(S')$ with $\mathcal{F} \cong \mathcal{E}$ in $\mathcal{SHT}(S')$.*
(3) *The map $\mathfrak{DM}_n^\diamond \to \mathrm{Bun}_n$ is a v-cover.*

Proof. We reduce the first and second claim to the third as follows. Let $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(S)$ of rank n, since $\mathfrak{DM}_n^\diamond \to \mathrm{Bun}_n$ is surjective there is a cover $\mathrm{Spa}(R') = S' \to S$ and a map $\mathcal{F} \in \mathfrak{DM}_n^\diamond(S')$. Refining S' further, we may assume that $\mathcal{F}$ is given by an object $\mathcal{F} \in \mathfrak{DM}_n(\operatorname{Spec} R'^+)$, which we may think of as vector bundle over $\operatorname{Spd} \mathcal{O}_E \times \operatorname{Spec}(R'^+)^\diamond$ with φ-action defined over $\operatorname{Spd} E \times \operatorname{Spec}(R'^+)^\diamond$. We can consider the inclusion of v-sheaves

$$\mathrm{Spa}(R', R'^\circ) \subseteq \mathrm{Spd}(R'^\circ, R'^\circ) \subseteq \operatorname{Spec}(R'^\circ)^\diamond \subseteq \operatorname{Spec}(R'^+)^\diamond$$

where the pair (R'°, R'°) is given the ϖ-adic topology for some pseudo-uniformizer. The map $\mathfrak{DM}_n^\diamond \to \mathrm{Bun}_n$ is then obtained by restricting to open locus $\mathrm{Spa}(R', R'^\circ) \subseteq \mathrm{Spd}(R'^\circ, R'^\circ)$. The first claim then follows from Proposition 8.2 and the identity $(Y^{R'^\circ}_{(0,\infty]})^\diamondsuit = \mathrm{Spd}(R'^\circ, R'^\circ) \times \operatorname{Spd} E$.

The second claim follows from the first claim, from the second part of Proposition 8.2 and from the fact that product of points are basis for the v-topology.

We move on to prove the third claim. Let $T \subseteq \mathrm{GL}_n$ be the diagonal torus. Let $B(T)_{\mathrm{sr}}$ denote the set of strongly regular elements. This set classifies isomorphism classes of sums of n line bundles all of which have different slope. Observe that the map $\coprod_{b \in B(T)_{\mathrm{sr}}} \mathcal{M}_b \to \mathrm{Bun}_n$ is surjective. We will construct a perfect scheme Y_b together with a map $f_b : Y_b \to \mathfrak{DM}_n$ in such a way that $Y_b^\diamond$ contains an open subsheaf $S_b \subseteq Y_b^\diamond$ with the property that the map $S_b \to \mathrm{Bun}_n^{\mathrm{mer}}$ factors through $\mathcal{M}_b^\circ$ and surjects onto it.

Recall that $\mathrm{Isoc}_n^b \cong [*/\underline{T(E)}] \cong \mathrm{Isoc}_T^b$, whenever b is strongly regular. Moreover, $\mathrm{DM}_T^b = [*/\underline{T(O_E)}]$ and by Lemma 8.7 we get a closed immersion $\mathrm{DM}_T^b \to \mathrm{DM}_n^b$. We let $\mathrm{Sht}_n^{T,b} = \mathrm{DM}_T^b \times_{\mathrm{DM}_n} \mathrm{Sht}_n \subseteq \mathrm{Sht}_n$. We have an identification $\mathrm{Sht}_n^{T,b} \cong [\widetilde{\mathcal{M}}_b/\underline{T(O_E)}]$. Indeed, this follow from Theorem 7.5 and the following sequence of Cartesian diagrams:

$$\begin{tikzcd}
\mathrm{Sht}_n^{T,b} \arrow[r] \arrow[d] & \mathrm{Sht}_n^b \arrow[r] \arrow[d] & \mathcal{M}_b \arrow[r] \arrow[d] & \mathrm{Bun}_n^{\mathrm{mer}} \arrow[d] \\
\mathrm{DM}_T^b \arrow[r] & \mathrm{DM}_n^b \arrow[r] & \mathrm{Isoc}_n^b \arrow[r] & \mathrm{Isoc}_n
\end{tikzcd}$$

For every point $x \in |\,\mathrm{Sht}_n^{T,b}|$ we can find a non-Archimedean field C_x, an open bounded valuation ring $C_x^+ \subseteq C_x$ and $\tilde{x} \in \mathfrak{DM}_n(\mathrm{Spec}\, C_x^+)$ inducing x. More precisely, x is the underlying point obtained from the composition of maps:

$$\mathrm{Spa}(C_x, C_x^+) \subseteq (\mathrm{Spec}\, C_x^+)^\diamond \to \mathfrak{DM}_n^\diamond \to \mathrm{Sht}_n\,.$$

The product $\prod \tilde{x}$ produces a map $\prod \tilde{x} : Y_b = \mathrm{Spec} \prod C_x^+ \to \mathfrak{DM}_{\leq\mu}^{\leq b}$ where μ is the only cocharacter of T with $b \in B(T, \mu)$. In particular, it produces maps:

$$Y_b^\diamond \to (\mathfrak{DM}_{\leq\mu}^{\leq b})^\diamond \to \mathrm{Sht}_n^{\leq b} \to \mathcal{M}_{\gamma\leq b} \subseteq \mathrm{Bun}_n^{\mathrm{mer}}.$$

We let $S_b \subseteq Y_b^\diamond$ the locus that factors through $\mathcal{M}_b^\circ = \mathcal{M}_{\gamma=b}^{\sigma\neq b} \subseteq \mathcal{M}_{\leq b}$. By Lemma 8.8, S_b is a product of points and in particular qcqs. Moreover, by construction the map $S_b \to \mathrm{Sht}_n$ factors through Sht_n^b. Also, on principal components $S_b \to \mathrm{Sht}_n^b$ factors through $\mathrm{Sht}_n^{T,b}$ and since $\mathrm{Sht}_n^{T,b} \subseteq \mathrm{Sht}_n^b$ is a closed immersion all of S_b factors through $\mathrm{Sht}_n^{T,b}$. Recall from [FS21, Proposition V.3.6] that $\widetilde{\mathcal{M}}_b^\circ$ is a spatial diamond, this implies that the map $S_b \to [\widetilde{\mathcal{M}}_b^\circ/\underline{T(O_E)}]$ is qcqs. But by construction $|S_b| \to |[\widetilde{\mathcal{M}}_b^\circ/\underline{T(O_E)}]|$ is surjective so this map is a v-cover. □

Lemma 8.7. *With notation as in the proof of Theorem 8.6, the map* $\mathrm{DM}_T^b \to \mathrm{DM}_n^b$ *is a closed immersion.*

Proof. We have maps $\mathrm{DM}_T^b \to \mathrm{DM}_n^b \to \mathrm{Isoc}_n^b \cong [*/\underline{T(E)}]$. It suffices to prove this is a closed immersion after basechange by the v-cover $* \to [*/\underline{T(E)}]$. The resulting map is the inclusion of affine Grassmannians $\mathrm{Gr}_T \to \mathrm{Gr}_{\mathrm{GL}_n}$. □

Lemma 8.8. *We let the notation be as in the proof of Theorem 8.6. That is* $Y_b = \mathrm{Spec} \prod C_x^+$*, where the* C_x *are algebraically closed non-Archimedean fields and* $C_x^+ \subseteq C_x$ *are open and bounded valuation subrings. We are given a map* $Y_b \to \mathfrak{B}_n^{\leq b}$*, which induces a map* $Y_b^\diamond \to \mathcal{M}_{\gamma\leq b} \to \mathrm{Bun}_n^{\mathrm{mer}}$*. We let* $S_b \subseteq Y_b^\diamond$ *be the preimage of* $\mathcal{M}_b^\circ$ *in* $Y_b^\diamond$*. Then there exists a family of pseudo-uniformizers* $f_x \in C_x^+$ *defining an element* $f \in \prod C_x^+$ *such that* $S_b = \mathrm{Spa}\, R$ *where* $R^+ = \prod C_x^+$ *endowed with the* f*-adic topology and* $R = \prod C_x^+[\frac{1}{f}]$*.*

Proof. Recall that by Theorem 7.13, $\mathcal{M}_b \subseteq \mathrm{Bun}_n^{\mathrm{mer}}$ is $\gamma^{-1}(\mathrm{Isoc}_n^b)$. Moreover,

$$\mathcal{M}_b^\circ = \gamma^{-1}(\mathrm{Isoc}_n^b) \cap \sigma^{-1}(\mathrm{Bun}_n^{<b}).$$

For all $b' < b$ in $B(\mathrm{GL}_n)$ we get a perfectly finitely presented closed immersion $Z_{b'} \subseteq Y_b$ with open complement $U_{b'} \subseteq Y_b$ by Theorem 7.7. By finite presentation, and since all of C_x^+ are valuation rings, there is an element

$f_{b'} \in \prod C_x^+$ such that $Z_{b'}$ is the perfection of $\operatorname{Spec}(\prod C_x^+/f_{b'})$ and $U_{b'} = \operatorname{Spec}(\prod C_x^+)[\frac{1}{f_{b'}}]$. We get Cartesian diagrams:

$$\begin{tikzcd}
Z_{b'}^{\diamondsuit} \arrow[r] \arrow[d] & Y_b^{\diamondsuit} \arrow[d] \\
(\mathfrak{B}_n^{\leq b'})^{\diamondsuit} \arrow[r] & (\mathfrak{B}_n^{\leq b})^{\diamondsuit}
\end{tikzcd}$$

Moreover, if we let $f = \prod_{b'<b} f_b'$, then $U_b := Y_b \times_{\mathfrak{B}_n^{\leq b}} \mathfrak{B}_n^b$ can be obtained as $\operatorname{Spec}(\prod C_x^+)[\frac{1}{f}]$, and we get a Cartesian diagram:

$$\begin{tikzcd}
Y_b^{\diamond} \cap U_b^{\diamondsuit} \arrow[r] \arrow[d] & U_b^{\diamondsuit} \arrow[r] \arrow[d] & \mathrm{Isoc}_n^b \arrow[d] \\
Y_b^{\diamond} \arrow[r] & Y_b^{\diamondsuit} \arrow[r] & \mathrm{Isoc}_n^{\leq b}
\end{tikzcd}$$

On the other hand, we claim that the locus in $Y_b^{\diamond}$ that factors through $\mathrm{Bun}_n^{\leq b'}$ is the locus where $f_{b'}$ is topologically nilpotent. Indeed, since $\mathrm{Bun}_n^{\leq b} \subseteq \mathrm{Bun}_n$ is open and both are partially proper it suffices to verify this on rank 1 points. We take a map $x : \mathrm{Spa}(C, O_C) \to Y_b^{\diamond}$, which we can always promote to a map $\mathrm{Spd}(O_C, O_C) \to Y_b^{\diamond}$, and if k is the residue field of O_C we get a map $\operatorname{Spd} k \to Y_b^{\diamond}$ we denote the induced point $\mathrm{sp}(x)$. By construction, the composition $Z_{b'}^{\diamond} \subseteq Y_b^{\diamond} \to \mathrm{Bun}_n^{\leq b}$ factors through $\mathrm{Bun}_n^{\leq b'}$, and the locus where $f_{b'}$ is topologically nilpotent coincides with those points for which $\mathrm{sp}(x) \in Z_{b'}^{\diamond}$. On the other hand, for any map $\mathrm{Spd}(O_C, O_C) \to \mathrm{Bun}_n$ such that $\operatorname{Spd} k \to \mathrm{Bun}_n$ factors through $\mathrm{Bun}_n^{\leq b'}$ the whole map factors through $\mathrm{Bun}_n^{\leq b'}$.

Ranging over $b' < b$ we see that the locus in $Y_b^{\diamond}$ that factors through $\cup_{b'<b}\mathrm{Bun}_n^{\leq b'}$ is the locus where at least one of the $f_{b'}$ is topologically nilpotent. Since all of the $f_{b'} \in C_x^+$, this is equivalent to the locus where f is topologically nilpotent.

In this way, the locus in $Y_b^{\diamond}$ that factors through $\mathcal{M}_b^{\circ}$ is the locus where f is both topologically nilpotent and invertible. The description of S_b now follows. □

9. Meromorphic Banach–Colmez spaces

Recall that given a small v-stack S and an object $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(S)$ we can construct a Banach–Colmez space $\mathcal{BC}(\mathcal{E}) : \mathrm{Perf}/S \to \mathrm{Sets}$ by the formula:

$$[f : T \to S] \mapsto \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}(T)}(\mathcal{O}, f^*\mathcal{E})$$

The map $\mathcal{BC}(\mathcal{E}) \to S$ is partially proper and representable in locally spatial diamonds.

Definition 9.1. Let $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$, and let $\mathcal{F} \in \mathcal{SHT}(S)$.

(1) We define the *meromorphic Banach–Colmez space of* $\mathcal{E}$, that we denote by $\mathcal{BC}^{\mathrm{mer}}(\mathcal{E}) : \mathrm{Perf}/S \to \mathrm{Sets}$, as given by the formula:
$$[f : T \to S] \mapsto \mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(T)}(\mathcal{O}, f^*\mathcal{E}).$$
(2) We can treat $\mathcal{F}$ as an object in $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S)$ and write $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F})$. Then, we can consider the *canonical lattice*, that we denote by $\mathcal{BC}^{\mathrm{sht}}(\mathcal{F}) \subseteq \mathcal{BC}^{\mathrm{mer}}(\mathcal{F})$, as given by the formula:
$$[f : T \to S] \mapsto \mathrm{Hom}_{\mathcal{SHT}(T)}(\mathcal{O}, f^*\mathcal{F}).$$

Whenever $\mathcal{E} \in \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S)$, to ease the notation, we denote by $\mathcal{BC}(\mathcal{E})$ what strictly speaking should be written as $\mathcal{BC}(\sigma(\mathcal{E}))$.

Proposition 9.2. *Let S be a small v-stack, let $\mathcal{E} \in \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S)$, and let $\mathcal{F} \in \mathcal{SHT}(S)$. The following hold:*

(1) *The map $\mathcal{BC}^{\mathrm{mer}}(\mathcal{E}) \to S$ is representable in diamonds.*
(2) *The map $\mathcal{BC}^{\mathrm{sht}}(\mathcal{F}) \to S$ is proper, representable in spatial diamonds and quasi-pro-étale.*
(3) *The map $\mathcal{BC}^{\mathrm{sht}}(\mathcal{F}) \to \mathcal{BC}(\mathcal{F})$ is a closed immersion.*

Proof. Recall that $\mathcal{BC}^{\mathrm{mer}}(\mathcal{E}) \subseteq \mathcal{BC}(\mathcal{E})$ and $\mathcal{BC}^{\mathrm{sht}}(\mathcal{F}) \subseteq \mathcal{BC}(\mathcal{F})$, this implies that $\mathcal{BC}^{\mathrm{mer}}(\mathcal{E})$ and $\mathcal{BC}^{\mathrm{sht}}(\mathcal{F})$ are separated over S. By pro-étale descent, we may assume that S is a strictly totally disconnected perfectoid space. In this case, $\mathcal{BC}(\mathcal{E})$ and $\mathcal{BC}(\mathcal{F})$ are locally spatial diamonds and by [Sch17, Proposition 11.10] any subsheaf of them is again a diamond. This proves the first claim, and by [Sch17, Proposition 13.4.(v)] together with [Sch17, Proposition 10.11] we may work v-locally in S to prove the second claim. Thus, we may assume that $S = \mathrm{Spa}(R, R^+)$ is a strict product of points and that $\mathcal{F} \in \mathfrak{DM}^{\dagger\mathrm{pre}}(S)$. After choosing a basis for $\mathcal{F}$, we get a matrix $M_{\mathcal{F}} \in \mathrm{GL}_n(\mathbb{W}(R^\circ)[\frac{1}{\pi}])$ and we obtain the following Cartesian diagram for any $T \in \mathrm{Perf}/S$.

$$\begin{tikzcd}
\mathcal{BC}^{\mathrm{sht}}(\mathcal{F})[T] \arrow[r] \arrow[d] & \mathbb{W}(\mathcal{O}_T)^n \arrow[d, "\Delta"] \\
\mathbb{W}(\mathcal{O}_T)^n \arrow[r, "(\mathrm{id}, M_{\mathcal{F}}^T \cdot \varphi)"] & \mathbb{W}(\mathcal{O}_T)^n \times \mathbb{W}(\mathcal{O}_T)^n
\end{tikzcd} \tag{9.1}$$

The functor $T \mapsto \mathbb{W}(\mathcal{O}_T)^n$ is isomorphic to an infinite dimensional compact unit ball of radius 1, which is a spatial diamond proper over S. In particular, $\mathcal{BC}^{\mathrm{sht}}(\mathcal{F})$ is a spatial diamond proper over S, and the map $\mathcal{BC}^{\mathrm{sht}}(\mathcal{F}) \to \mathcal{BC}(\mathcal{F})$ is a closed immersion since it is injective and proper. Finally, to prove that the map is quasi-pro-étale we may by [Sch17, Proposition 13.6] assume that $S = \mathrm{Spa}(C, O_C)$ is a geometric point. In this case $\mathcal{BC}^{\mathrm{sht}}(\mathcal{F})$ is a closed subsheaf of $\mathrm{Hom}_{\mathfrak{DM}^\diamond}(\mathcal{O}, \mathcal{F}) \cong \underline{O_E^r}$ where r is the number of summands of $\mathcal{O}$ in $\mathcal{F}$ when we treat as an analytic isocrystal $\mathrm{Spa}(C, O_C)$. This is clearly quasi-pro-étale over S. $\square$

Proposition 9.3. *Let $S = \operatorname{Spa} R$ be a strict product of points, then the map $\mathfrak{DM}^{\dagger_{\mathrm{pre}}}(S) \to \mathcal{SHT}(S)$ is an equivalence in* $\mathrm{Cat}_1^{\otimes}$.

Proof. By the first and second parts of Proposition 8.2, the map $\mathfrak{DM}^{\dagger_{\mathrm{pre}}}(S) \to \mathcal{SHT}(S)$ is fully-faithful, and we wish to show essential surjectivity. Write $R^+ = \prod_{i\in I} C_i^+$, let $\mathcal{E} \in \mathcal{SHT}(S)$, let $S_i = \operatorname{Spa} C_i$ and let $\mathcal{E}_i$ denote the restriction of $\mathcal{E}$ to S_i. By Proposition 8.5, over points we have an equivalence $\mathfrak{DM}^{\dagger_{\mathrm{pre}}}(S_i) \cong \mathcal{SHT}(S_i)$, we let $\mathcal{F} \in \mathfrak{DM}^{\dagger_{\mathrm{pre}}}(S) = \prod_{i\in I} \mathcal{E}_i$.

Let $\mathcal{I}$ be the sheaf of isomorphisms between $\mathcal{E}$ and $\mathcal{F}$. We may regard $\mathcal{I}$ as closed subsheaf of $\mathcal{BC}^{\mathrm{sht}}(\underline{\mathrm{Hom}}(\mathcal{E},\mathcal{F}) \oplus \underline{\mathrm{Hom}}(\mathcal{F},\mathcal{E}))$. In particular, $\mathcal{I}$ is a spatial diamond and the map $\mathcal{I} \to S$ is proper and quasi-pro-étale. Moreover, the map $\mathcal{I}_i \to S_i$ has sections by the definition of $\mathcal{F}$, which implies that $\pi_0(\mathcal{I}) \to \pi_0(S)$ is surjective since principal components are dense and both spaces are compact Hausdorff. This says that $\mathcal{I} \to S$ is a pro-étale cover, and since S is extremally disconnected this map has a section. This proves that $\mathcal{E} \cong \mathcal{F}$. □

Remark 9.4. Combining Proposition 9.3 with Proposition 8.4 we get the following concrete description of $\mathcal{SHT}(S)$. Let $R^\circ = \prod_{i\in I} O_{C_i}$ for C_i a family of non-Archimedean fields, and let $\mathcal{E} \in \mathcal{SHT}(S)$ of constant rank n. Then there is a family of matrices $M_i \in \mathrm{GL}_n(\mathbb{W}(O_{C_i})[\frac{1}{\pi}])$ and a number $N \in \mathbb{N}$ with M_i and M_i^{-1} in $\frac{1}{\pi^N} \cdot M_{n\times n}(\mathbb{W}(O_{C_i}))$, such that $\mathcal{E}$ is isomorphic to $(\mathbb{W}(R^\circ)^n, M_{\mathcal{E}})$ where $M_{\mathcal{E}} = \prod_{i\in I} M_i$.

Proposition 9.5. *If $S = \operatorname{Spa} R$ is a strict product of points, then the map $\mathrm{Bun}^+_{\mathrm{FF}}(S) \to \mathrm{Bun}_{\mathrm{FF}}(S)$ is an equivalence in* $\mathrm{Cat}_1^{\otimes}$.

Proof. It suffices to show essential surjectivity. Let $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(S)$, let $T \subseteq \mathrm{GL}_n$ be the diagonal torus and take $b \in B(\mathrm{GL}_n)$ in the image of $B(T)$. Since $\mathcal{M}_b \to \mathrm{Bun}_n$ is formally smooth, by [FS21, Proposition IV.4.24, Theorem V.3.7], $\mathcal{E}$ lifts to $\mathcal{E} \in \mathcal{M}_{\gamma=b}(S)$. As in the proof of Theorem 8.6 we may interpret the map $\mathcal{M}_b \to [*/\underline{T(\mathbb{Q}_p)}]$ as a map to Isoc_T^b, and the map $[*/\underline{T(\mathbb{Z}_p)}] \to [*/\underline{T(\mathbb{Q}_p)}]$ as $\mathrm{DM}_T^b \to \mathrm{Isoc}_T$. In particular, étale locally we may lift and since S splits all étale covers we can choose an object $\mathcal{E} \in \mathcal{SHT}(S)$ lifting the original one. By Proposition 9.3 we may further find an object $\mathcal{E} \in \mathfrak{DM}^{\dagger_{\mathrm{pre}}}(S)$. This defines an object in $\mathrm{Bun}^+_{\mathrm{FF}}(S)$ lifting $\mathcal{E}$. □

9.1. **On the diagonal of $\mathrm{Bun}_G^{\mathrm{mer}}$.** Unfortunately, $\Delta_{\mathrm{Bun}_G^{\mathrm{mer}}}$ is not representable in locally spatial diamonds. As a consequence, $\mathrm{Bun}_G^{\mathrm{mer}}$ **cannot** be, by definition, an Artin v-stack.

In this subsection we provide an indication on how to prove that this diagonal map fails to be representable in locally spatial diamonds. The material on this subsection is irrelevant for the rest of the article and the reader can safely ignore it.

Example 9.6. Let $T = \{\frac{1}{n} > 0 \mid n > 0\} \cup \{0\}$ be the space of convergent sequences, and consider $S = \underline{T} \times \operatorname{Spa} C$. This is a strictly totally disconnected perfectoid spaces whose global sections is R the ring of continuous functions $f : T \to C$ which we may think of as convergent sequences. Fix a pseudo-uniformizer $\varpi \in C$ and let $r \in R$ denote the element with $r(\frac{1}{n}) = \varpi^n$ and $r(0) = 0$. We consider $\mathcal{F} \in \mathcal{SHT}(S)$ given by the matrix

$$M_{\mathcal{F}} := \left(\begin{smallmatrix} \frac{[r]}{\pi} & 1 \\ \frac{1}{\pi} & 0 \end{smallmatrix} \right) \in M_{2\times 2}(\mathbb{W}(R^\circ)[\frac{1}{\pi}]).$$

In this case, $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F})$ is **not** a locally spatial diamond as we discuss below.

Proposition 9.7. *Let the notation be as in Example 9.6. Then $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F})$ is a diamond that is not a locally spatial diamond.*

Proof. Suppose that $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F})$ is a locally spatial diamond. Since $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F}) \subseteq \mathcal{BC}(\mathcal{F})$, $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F})$ is quasiseparated and if $U \subseteq \mathcal{BC}^{\mathrm{mer}}(\mathcal{F})$ is a quasicompact open subset then U is a spatial diamond. Fix U quasicompact containing the zero section $0_T \in U$. The map $U \to \mathcal{BC}(\mathcal{F})$ is quasicompact, this implies that it is a point-wise subsheaf i.e. if $f : \operatorname{Spa} R \to \mathcal{BC}(\mathcal{F})$ is a map such that each of its geometric points factor through U then f factors through U. Since $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F}) = \varinjlim \frac{1}{\pi^n} U$ then $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F}) \subseteq \mathcal{BC}(\mathcal{F})$ is also a point-wise subsheaf. This contradicts the next paragraph.

We give some indication for why $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F}) \subseteq \mathcal{BC}(\mathcal{F})$ is not a point-wise subsheaf. The claim is that there are functions $a, b \in H^0(Y^S_{(0,\infty]}, \mathcal{O})$ such that

$$\left(\begin{smallmatrix} \frac{[r]}{\pi} & 1 \\ \frac{1}{\pi} & 0 \end{smallmatrix} \right) \left(\begin{smallmatrix} \varphi(a) \\ \varphi(b) \end{smallmatrix} \right) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix} \right),$$

such that $a(0) = b(0)$, and such that for all $n \in \mathbb{N}$ the function $a(\frac{1}{n})$ lies in $\mathbb{W}(O_C)[\frac{1}{\pi}]$ and has a pole of order $\lfloor \log(n) \rfloor$. In particular, this gives a map $S \to \mathcal{BC}(\mathcal{F})$ that point by point lies in $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F})$, but does not factor through $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F})$.

The elements a and b are roughly constructed as follows. The meromorphic bundle $\mathcal{F}$ is obtained from basechange by a meromorphic bundle $\mathcal{F}_t$ living over $\operatorname{Spd} \overline{\mathbb{F}}_q[\![t]\!]$, by the map $t \mapsto r$. One finds algebraic expressions in terms of t to construct elements $s_a(t), s_b(t) \in \mathbb{W}(\overline{\mathbb{F}_q[\![t]\!]})$ with Teichmüller expansion $s_a(t) = [s_0(t)] + \pi[s_1(t)] + \dots \pi^i[s_i(t)] \dots$, satisfying

$$\left(\begin{smallmatrix} \frac{[t]}{\pi} & 1 \\ \frac{1}{\pi} & 0 \end{smallmatrix} \right) \left(\begin{smallmatrix} \varphi(s_a(t)) \\ \varphi(s_b(t)) \end{smallmatrix} \right) = \left(\begin{smallmatrix} s_a(t) \\ s_b(t) \end{smallmatrix} \right).$$

For example, $s_0(t) = -t^{\frac{1}{q^2-q}}$. Then a is constructed as the sum $a = \Sigma_{k=1}^{\infty} a_k$ where a_k is the function on $Y^S_{(0,\infty]}$ with $a_k(1/k) = \frac{1}{\pi^{-\lfloor \log(k) \rfloor}} s_a(\varpi^k)$ and 0 in every other connected component of T. After a long computation one can show that the limit of the $\Sigma_{k=1}^n a_k$ exists in $H^0(Y^S_{(0,\infty]}, \mathcal{O})$. □

Example 9.8. We explain explicitly the case $\Delta_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{GL}_3}}$. Let the notation be as in Example 9.6, and consider the meromorphic vector bundle over S given by $\mathcal{E} = \mathcal{O} \oplus \mathcal{F}$. The group of meromorphic automorphisms $\mathrm{Aut}_{\mathrm{mer}}(\mathcal{E})$ arises as the basechange of the diagonal $\Delta_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{GL}_3}}$ by the map $S \to (\mathrm{Bun}_{\mathrm{GL}_3})^2$ given by $(\mathcal{E}, \mathcal{E})$. Moreover, $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F}) \subseteq \mathrm{Aut}_{\mathrm{mer}}(\mathcal{E})$ is a closed immersion corresponding to the unipotent radical of the Levi defined by the direct sum decomposition $\mathcal{O} \oplus \mathcal{F}$. In particular, if $\mathrm{Aut}_{\mathrm{mer}}(\mathcal{E})$ was a locally spatial diamond then $\mathcal{BC}^{\mathrm{mer}}(\mathcal{F})$ would also be, which contradicts Proposition 9.7.

10. Three comparison theorems

10.1. **The meromorphic comparison.** The following statement shows that extending at ∞ also holds for G-bundles.

Proposition 10.1. *Let G be a reductive group, $\mathcal{G}$ a parahoric model and $S \in \mathrm{Perf}$ of the form $S = \mathrm{Spa}(R, R^+)$. The following statements hold:*

(1) *We have fully-faithful embedding of groupoids $\mathrm{Bun}_G^+(S) \to \mathrm{Bun}_G(S)$.*
(2) *Given $\mathcal{E} \in \mathrm{Bun}_G(S)$ there is a v-cover $S' \to S$ and a unique up to isomorphism $\mathcal{F} \in \mathrm{Bun}_G^+(S')$ with $\mathcal{F} \cong \mathcal{E}$ in $\mathrm{Bun}_G(S')$.*
(3) *The v-sheafification of Bun_G^+ is Bun_G.*
(4) *If S is a strict product of points, we have a Cartesian diagram of groupoids:*

$$\begin{tikzcd} \mathfrak{DM}_{\mathcal{G}}^{\dagger_{\mathrm{pre}}}(S) \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{G}}(S) \arrow[d] \\ \mathrm{Bun}_G^+(S) \arrow[r] & \mathrm{Bun}_G(S). \end{tikzcd}$$

(5) *If S is a strict product of points, then $\mathfrak{DM}_{\mathcal{G}}^{\dagger_{\mathrm{pre}}}(S) \to \mathrm{Sht}_{\mathcal{G}}(S)$ and $\mathrm{Bun}_G^+(S) \to \mathrm{Bun}_G(S)$ are equivalences.*

Proof. The first claim follows from Proposition 8.2. Indeed, we can identify $\mathrm{Bun}_G^+(S)$ with the category $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathrm{Bun}^+_{\mathrm{FF}}(S))$.

The second claim follows from the first part of Theorem 8.6. Indeed, we regard $\mathcal{E} \in \mathrm{Bun}_G(S)$ as an object in $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathrm{Bun}_{\mathrm{FF}}(S))$. We know that if $V \in \mathrm{Rep}_G$ then there is a v-cover $S_V \to S$, and a unique (up to isomorphism) object in $\mathcal{F}_V \in \mathrm{Bun}^+_{\mathrm{FF}}(S)$ lifting $\mathcal{E}_V$. Taking the limit of v-covers $S' = \varprojlim_{V \in \mathrm{Rep}_G} S_V \to S$ we may promote $\mathcal{E}$ to an object in $\mathcal{F} \in \mathrm{Fun}^{\otimes}(\mathrm{Rep}_G, \mathrm{Bun}^+_{\mathrm{FF}}(S'))$. Now, since we assumed that G is reductive, the category Rep_G is semi-simple. Moreover, since the map $\mathrm{Bun}^+_{\mathrm{FF}}(S') \to \mathrm{Bun}_{\mathrm{FF}}(S')$ is fully-faithful, it reflects split-exact sequences. Since $\mathcal{E}$ is $\otimes$-exact, $\mathcal{F}$ must also be $\otimes$-exact.

The third claim follows directly from the first and second claims.

The fourth claim follows by applying the same arguments (GAGA and extending $\mathcal{G}$-torsors) as Proposition 8.2(2).

For the fifth claim, it suffices to prove that $\mathcal{E} \in \mathrm{Sht}_{\mathcal{G}}(S)$ is in the essential image of $\mathfrak{DM}_{\mathcal{G}}^{\dagger_{\mathrm{pre}}}(S)$, and by the fourth claim, it suffices to show that the induced object $\mathcal{E} \in \mathrm{Bun}_G(S)$ lifts to an object in $\mathrm{Bun}_G^+(S)$, but for every $V \in \mathrm{Rep}_G$ the corresponding bundle $\mathcal{E}_V \in \mathrm{Bun}_{\mathrm{FF}}(S)$ lifts uniquely to an object $\mathcal{E}_V^+ \in \mathrm{Bun}^+_{\mathrm{FF}}(S)$. By the argument given in the proof of the second claim the functor $V \mapsto \mathcal{E}_V^+$ is exact and defines a lift $\mathcal{E}^+ \in \mathrm{Bun}_G^+(S)$. $\square$

Theorem 10.2. *The following statements hold:*

(1) *We have an isomorphism of small v-stacks* $\mathfrak{DM}^{\dagger}_{\mathcal{G}} \cong \mathrm{Sht}_{\mathcal{G}}$.
(2) *We have an isomorphism of small v-stacks* $\mathfrak{B}(G)^{\dagger} \cong \mathrm{Bun}^{\mathrm{mer}}_{G}$.
(3) *The maps* $\mathfrak{DM}^{\diamond}_{\mathcal{G}} \to \mathrm{Sht}_{\mathcal{G}}$ *and* $\mathfrak{B}(G)^{\diamond} \to \mathrm{Bun}^{\mathrm{mer}}_{G}$ *are v-surjective.*

Remark 10.3. This result can be regarded as a version of Fargues' theorem [Far18, Theorem 1.12] in families. Recall that Fargues' theorem states that the category shtukas over (C, O_C) is equivalent to the category of BKF-modules of $\mathbb{W}(O_C)$. Although this statement is not true for general families, the theorem above shows that the statement is v-locally true. Indeed, $\mathrm{Sht}_{\mathcal{G}}(R, R^+)$ parametrizes $\mathcal{G}$-shtukas over $\mathrm{Spa}(R, R^+)$ while $\mathfrak{DM}^{\dagger}_{\mathcal{G}}$ is the sheafification of the functor attaching to (R, R^+) a BKF-modules over $\mathbb{W}(R^{\circ})$ with $\mathcal{G}$-structure.

Proof. Proposition 10.1 shows that $\mathfrak{DM}^{\dagger\mathrm{pre}}_{\mathcal{G}}(S) \to \mathrm{Sht}_{\mathcal{G}}(S)$ is fully-faithful and v-locally surjective, this proves the first claim.

For the second claim, consider the fully-faithful map

$$\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathfrak{DM}^{\dagger\mathrm{pre}}[\frac{1}{\pi}])(S) \to \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathcal{SHT}[\frac{1}{\pi}])(S).$$

from the second part of Proposition 8.2 and the second claim of Proposition 10.1 above this map is v-locally essentially surjective. In particular, after sheafification the map above becomes an isomorphism of sheaves of groupoids. The left hand side identifies with $\mathfrak{B}(G)^{\dagger}$ while the right hand side is $\mathrm{Bun}^{\mathrm{mer}}_{G}$.

For the third claim, it suffices to prove $\mathfrak{DM}^{\diamond}_{\mathcal{G}} \to \mathrm{Sht}_{\mathcal{G}}$ is surjective since $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Bun}^{\mathrm{mer}}_{G}$ is surjective and the map $\mathfrak{DM}^{\diamond}_{\mathcal{G}} \to \mathrm{Bun}^{\mathrm{mer}}_{G}$ factors through $\mathfrak{B}(G)^{\diamond}$. By the identity $\mathrm{Sht}_{\mathcal{G}} \cong \mathfrak{DM}^{\dagger}_{\mathcal{G}}$, it suffices to prove that $\mathcal{E} \in \mathfrak{DM}^{\dagger\mathrm{pre}}_{\mathcal{G}}(S)$ lifts to an object in $\mathfrak{DM}^{\diamond\mathrm{pre}}_{\mathcal{G}}(S')$ for some v-cover $S' \to S$. We can reduce this to the case where $S = \mathrm{Spa}\, R$ is a strict product of points with $R^+ = \prod_{i\in I} C_i^+$, and $\mathcal{E}$ is given by a matrix $M \in \mathcal{G}(\mathbb{W}(\prod_{i\in I} O_{C_i})[\frac{1}{\pi}])$. Any φ-conjugation by a matrix $N \in \mathcal{G}(\mathbb{W}(\prod_{i\in I} O_{C_i})) = \prod_{i\in I} \mathcal{G}(\mathbb{W}(O_{C_i}))$ defines an isomorphic object in $\mathfrak{DM}^{\dagger\mathrm{pre}}_{\mathcal{G}}(S)$. This allow us to reduce to the case where the set I is a singleton and we must show that M is φ-conjugate to a matrix defined over $M' \in \mathcal{G}(\mathbb{W}(C^+))$. We may do this at the level of residue rings $k = O_C/C^{\circ\circ}$ and $k^+ = C^+/C^{\circ\circ}$ where it follows from the ind-properness of affine Deligne–Lusztig varieties. □

10.2. **The schematic comparison.**

Theorem 10.4. *Let G be a reductive group and $\mathcal{G}$ be a parahoric model.*

(1) *The natural map* $\mathfrak{B}(G) \xrightarrow{\cong} (\mathrm{Bun}_G)^{\mathrm{red}}$ *is an isomorphism of scheme-theoretic v-sheaves valued in groupoids.*
(2) *The natural map* $\mathfrak{DM}_{\mathcal{G}} \xrightarrow{\cong} (\mathrm{Sht}_{\mathcal{G}})^{\mathrm{red}}$ *is an isomorphism of scheme-theoretic v-sheaves valued in groupoids.*

Proof. Let $X \in \mathrm{PSch}$, for the first claim we write:

$$\begin{aligned}\mathfrak{B}(G)(X) &\cong \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathfrak{B}(X)) \\ &\cong \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathrm{Bun}_{\mathrm{FF}}(X^\diamond)) \\ &\cong \mathrm{Bun}_G(X^\diamond) \\ &\cong (\mathrm{Bun}_G)^{\mathrm{red}}(X).\end{aligned}$$

Here, the second isomorphism is Corollary 4.8.

For the second claim, since $(\mathrm{Sht}_{\mathcal{G}})^{\mathrm{red}}$ and $\mathfrak{DM}_{\mathcal{G}}$ are v-sheaves (the latter by Proposition 7.3(1)) it suffices to prove $\mathfrak{DM}_{\mathcal{G}}(X) \xrightarrow{\cong} \mathrm{Sht}_{\mathcal{G}}(X^\diamond)$ when $X = \operatorname{Spec} A$ is a comb. In this case, $\mathfrak{DM}_{\mathcal{G}}(X)$ is equivalent to the category where the objects are elements $M_{\mathcal{E}} \in \mathcal{G}(\mathbb{W}(A)[\frac{1}{\pi}])$, and morphisms between $M_{\mathcal{E}_1}$ and $M_{\mathcal{E}_2}$ are element $N \in \mathcal{G}(\mathbb{W}(A))$ with $N^{-1} \cdot M_{\mathcal{E}_1}\varphi(N) = M_{\mathcal{E}_2}$. On the other hand by Theorem 10.2.(5), an isomorphism between $M_{\mathcal{E}_1}$ and $M_{\mathcal{E}_2}$ in $\mathrm{Sht}_{\mathcal{G}}(X^\diamond)$ corresponds to a functorial choice of elements $N_R \in \mathcal{G}(\mathbb{W}(R^\circ))$ with $N_R^{-1} \cdot M_{\mathcal{E}_1}\varphi(N_R) = M_{\mathcal{E}_2}$ ranging over maps $\operatorname{Spa} R \to X^\diamond$, with $\operatorname{Spa} R$ a product of points. Since $H^0(X^\diamond, \mathcal{O}^\circ) = A$ such collection of N_R come uniquely from an element $N \in \mathcal{G}(\mathbb{W}(A))$ which shows that $\mathfrak{DM}_{\mathcal{G}}(X) \to \mathrm{Sht}_{\mathcal{G}}(X^\diamond)$ is fully faithful. To prove essential surjectivity fix $\mathcal{E} \in \mathrm{Sht}_{\mathcal{G}}(X^\diamond)$ this induces elements $\mathcal{E}_{\mathrm{Bun}} \in \mathrm{Bun}_G(X^\diamond)$ and $\mathcal{E}_{\mathfrak{B}} \in \mathfrak{B}(G)(X)$ unique up to isomorphism. Objects in $\mathfrak{DM}_{\mathcal{G}}(X)$ lifting $\mathcal{E}_{\mathfrak{B}}$ correspond to sections of $\mathfrak{DM}_{\mathcal{G}} \times_{\mathfrak{B}(G)} X \to X$, whereas objects in $\mathrm{Sht}_{\mathcal{G}}(X^\diamond)$ lifting $\mathcal{E}_{\mathrm{Bun}}$ correspond to sections $\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_G} X^\diamond \to X^\diamond$. The result follows from Lemma 10.5 below. □

Lemma 10.5. *Let $X \in \mathrm{PSch}$ and $X \to \mathfrak{B}(G)$ be a map, then*

$$(\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_G} X^\diamond)^{\mathrm{red}} = \mathfrak{DM}_{\mathcal{G}} \times_{\mathfrak{B}(G)} X.$$

Proof. The argument given in [Gle21a, Proposition 2.30] works in this generality. □

10.3. **The topological comparison.** Recall that by results of Rapoport–Richartz [RR96] and He [He16, Theorem 2.12],

$$|\mathfrak{B}(G)| \cong B(G). \tag{10.1}$$

Here the latter is given the topology induced by the partial order defined by Kottwitz. Alternatively, by the results of Viehmann [Vie23, Theorem 1.1] we also have

$$|\mathrm{Bun}_G|^{\mathrm{op}} \cong B(G). \tag{10.2}$$

where $|\mathrm{Bun}_G|^{\mathrm{op}}$ is the topological space where a subset is open in $|\mathrm{Bun}_G|^{\mathrm{op}}$ if and only if it is closed in $|\mathrm{Bun}_G|$. Combining these two references we obtain that

$$|\mathfrak{B}(G)| \cong |\mathrm{Bun}_G|^{\mathrm{op}} \tag{10.3}$$

In this section we give a direct and new proof of the identity (10.3). As a consequence we prove that the identities (10.2) and (10.1) are equivalent statements. We set some notation.

Definition 10.6. Let $b_1, b_2 \in B(G)$.

(1) We say that $b_1 \preceq_{\mathfrak{B}(G)} b_2$ if $b_1 \in \overline{\{b_2\}}$ in $\mathfrak{B}(G)$.
(2) We say that $b_1 \preceq_{\mathrm{Isoc}_G} b_2$ if $b_1 \in \overline{\{b_2\}}$ in Isoc_G.
(3) We say that $b_1 \preceq_{\mathrm{Bun}_G^{\mathrm{op}}} b_2$ if $b_2 \in \overline{\{b_1\}}$ in Bun_G.

Moreover, we write $b_1 \precdot_{\mathfrak{B}(G)} b_2$, $b_1 \precdot_{\mathrm{Isoc}_G} b_2$ or $b_1 \precdot_{\mathrm{Bun}_G^{\mathrm{op}}} b_2$ whenever b_2 covers b_1 in the respective order.

Lemma 10.7. *Let $U \subseteq B(G)$. For $b \in B(G)$ we let $U_{\leq b} := U \cap B(G)_{\leq b}$.*

(1) *U is closed in $\mathfrak{B}(G)$ $\iff$ $U_{\leq b}$ is closed in $\mathfrak{B}(G)$ for all $b \in B(G)$.*
(2) *U is closed in Isoc_G $\iff$ $U_{\leq b}$ is closed in Isoc_G for all $b \in B(G)$.*
(3) *U is open in Bun_G $\iff$ $U_{\leq b}$ is open in Bun_G for all $b \in B(G)$.*
(4) *$|\mathrm{Bun}_G|^{\mathrm{op}}$ is a topological space.*
(5) *The topology on $\mathfrak{B}(G)$, Isoc_G and Bun_G is determined by their closure partial orders: $\preceq_{\mathfrak{B}(G)}$, $\preceq_{\mathrm{Isoc}_G}$, and $\preceq_{\mathrm{Bun}_G^{\mathrm{op}}}$.*

Proof. We prove the first claim, the second and third claim being analogous. The forward implication is evident since $\mathfrak{B}(G)_{\leq b} \subseteq \mathfrak{B}(G)$ is a closed immersion. For any map $f : \operatorname{Spec} R \to \mathfrak{B}(G)$, there are a finite number of elements $b_i^f \in B(G)$ such that f factors through $\bigcup_{i=1}^n \mathfrak{B}(G)_{\leq b_i^f}$. By assumption $U \cap \bigcup_{i=1}^n \mathfrak{B}(G)_{\leq b_i^f}$ is closed in $\mathfrak{B}(G)$. Since f factors through the set above, basechange of f along U defines a closed immersion.

The fourth claim follows from the third. Indeed, the only part that needs justification is that arbitrary union of open subsets in $|\mathrm{Bun}_G|^{\mathrm{op}}$ is open. This is equivalent to the preservation of open subsets of $|\mathrm{Bun}_G|$ under arbitrary intersections. But arbitrary intersections can be expressed as finite intersections when we restrict them to $\mathrm{Bun}_G^{\leq b}$.

The last claim follows from the first three. Indeed, $\mathfrak{B}(G)$, Isoc_G and Bun_G have the strong topology along the inclusion maps from $\coprod_{b\in B(G)} \mathfrak{B}(G)_{\leq b}$, $\coprod_{b\in B(G)} \mathrm{Isoc}_G^{\leq b}$ and $\coprod_{b\in B(G)} \mathrm{Bun}_G^{\leq b}$. Moreover, since these latter ones are finite topological spaces they are determined by their closure relations. ∎

Now, because $\mathfrak{B}(G)_{\leq b} \subseteq \mathfrak{B}(G)$ and $\mathrm{Isoc}_G^{\leq b} \subseteq \mathrm{Isoc}_G$ are closed immersions and $\mathrm{Bun}_G^{\leq b} \subseteq \mathrm{Bun}_G$ is an open immersion we know that:

(1) $b_1 \preceq_{\mathfrak{B}(G)} b_2 \implies b_1 \leq_{B(G)} b_2$
(2) $b_1 \preceq_{\mathrm{Isoc}_G} b_2 \implies b_1 \leq_{B(G)} b_2$
(3) $b_1 \preceq_{\mathrm{Bun}_G^{\mathrm{op}}} b_2 \implies b_1 \leq_{B(G)} b_2$

Theorem 10.8. *Let the notation be as above. The partial orders $\preceq_{\mathfrak{B}(G)}$, $\preceq_{\mathrm{Isoc}_G}$, $\preceq_{\mathrm{Bun}_G^{\mathrm{op}}}$ agree. In particular, $|\mathfrak{B}(G)| \cong |\mathrm{Isoc}_G| \cong |\mathrm{Bun}_G|^{\mathrm{op}}$.*

Proof. For the rest of the proof we fix $b_1, b_2 \in B(G)$ with $b_1 \leq_{B(G)} b_2$. We first prove $|\mathfrak{B}(G)| \cong |\mathrm{Isoc}_G|$. Recall that $\diamondsuit$ preserves closed immersion, consequently:

$$b_1 \preceq_{\mathrm{Isoc}_G} b_2 \implies b_1 \preceq_{\mathfrak{B}(G)} b_2.$$

Now, suppose that $b_1 \preceq_{\mathfrak{B}(G)} b_2$. We claim that there is a perfect rank 1 valuation ring V and a map $\operatorname{Spec} V \to \mathfrak{B}(G)$ such that the induced maps on $\operatorname{Spec} k_V$ (the residue field) and $\operatorname{Spec} K_V$ (the fraction field) factor through $\mathfrak{B}(G)_{b_1}$ and $\mathfrak{B}(G)_{b_2}$ respectively. Indeed, we may find a map $f : \operatorname{Spec} R \to \mathfrak{B}(G)$ with the property that for all $x \in \operatorname{Spec} R$ the induced map $\operatorname{Spec} k_x \to \mathfrak{B}(G)$ factors through either $\mathfrak{B}(G)_{b_1}$ or $\mathfrak{B}(G)_{b_2}$ and with the property that $\overline{f^{-1}(\mathfrak{B}(G)_{b_2})} \cap f^{-1}(\mathfrak{B}(G)_{b_1}) \neq \emptyset$. We may replace $\operatorname{Spec} R$ by a v-cover, so we may assume that $R = \prod_{i\in I} V_i$ is a product of valuation rings. Since the inclusion $\mathfrak{B}(G)_{\leq b_1} \to \mathfrak{B}(G)$ is perfectly finitely presented there is $r \in R$ such that $\operatorname{Spec} R/(r)^{\mathrm{perf}} \subseteq \operatorname{Spec} R$ is $f^{-1}(\mathfrak{B}(G)_{b_1})$. We may write $R = R_1 \times R_2$ where $R_1 = \prod_{\{i\in I | r_i = 0\}} V_i$ and $R_2 = \prod_{\{i\in I|r_i\neq 0\}} V_i$ and replace R by R_2. Let K_{V_i} denote the fraction field of V_i. Now, $\operatorname{Spec} \prod_{i\in I} K_{V_i} \subseteq \operatorname{Spec} R$ is a pro-open subset lying in $f^{-1}(\mathfrak{B}(G)_{b_2})$. Since $f^{-1}(\mathfrak{B}(G)_{b_1})$ is non-empty there is a connected component in $x \in \beta I$ with associated valuation ring V_x such that that the image of r in V_x, which we denote r_x, is not identically 0, but it is also not a unit. The largest prime ideal contained in $\langle r_x\rangle$ and the smallest prime ideal containing $\langle r_x\rangle$ define a rank 1 valuation ring with the desired properties.

The map $\operatorname{Spec} V \to \mathfrak{B}(G)$ induces a map $\operatorname{Spd}(V, V) \to \mathfrak{B}(G)^\diamond \to \operatorname{Isoc}_G$ such that the corresponding map on $\operatorname{Spd}(k_V, k_V)$ and $\operatorname{Spd}(K_V, K_V)$ factor through $\operatorname{Isoc}_G^{b_1}$ and $\operatorname{Isoc}_G^{b_2}$ respectively. This implies that $\operatorname{Spd}(K_V, V) \to \operatorname{Isoc}_G$ factors through $\operatorname{Isoc}_G^{b_2}$, but $\operatorname{Spd}(K, V) \subseteq \operatorname{Spd}(V, V)$ is dense. This proves:

$$b_1 \preceq_{\mathfrak{B}(G)} b_2 \implies b_1 \preceq_{\operatorname{Isoc}_G} b_2.$$

In the same fashion, the map $\operatorname{Spec} V \to \mathfrak{B}(G)$ induces a map $\operatorname{Spd}(V, V) \to \mathfrak{B}(G)^\diamond \to \operatorname{Bun}_G$ that restricted to $\operatorname{Spd}(k_V, k_V)$ and $\operatorname{Spd}(K_V, K_V)$ factors through $\operatorname{Bun}_G^{b_1}$ and $\operatorname{Bun}_G^{b_2}$ respectively. Let $\pi \in V$ be a pseudo-uniformizer, let $\hat{V}_\pi$ be the π-adic completion of V and let $K = V[\frac{1}{\pi}]$, then $\operatorname{Spa}(K, \hat{V}_\pi)$ is a perfectoid field. Also, $\operatorname{Spd}(\hat{V}_\pi, \hat{V}_\pi)$ has two points, one corresponding to $\operatorname{Spd}(K, \hat{V}_\pi)$ and one corresponding to $\operatorname{Spd}(k_V, k_V)$. By Corollary 3.6, the map $\operatorname{Spd}(\hat{V}_\pi, \hat{V}_\pi) \to \operatorname{Bun}_G$ corresponds to a $\otimes$-exact functor from Rep_G to the category of φ-equivariant objects in $\operatorname{Vect}(Y^K_{(0,\infty]})$. Using [SW20, Theorem 13.2.1, Theorem 13.4.1], we conclude that the map $\operatorname{Spd}(\hat{V}_\pi, \hat{V}_\pi) \to \operatorname{Bun}_G$ factors through $\operatorname{Bun}_G^{b_1}$ as $\operatorname{Spd}(k_V, k_V) \to \operatorname{Bun}_G$ does. Moreover, $\operatorname{Spd}(\hat{V}_\pi, \hat{V}_\pi) \subseteq \operatorname{Spd}(V, V)$ is an open subsheaf whose v-sheaf theoretic closure is $\operatorname{Spd}(V, V)$. This allows us to conclude:

$$b_1 \preceq_{\mathfrak{B}(G)} b_2 \implies b_1 \preceq_{\operatorname{Bun}_G^{\mathrm{op}}} b_2.$$

Finally, suppose that $b_1 \preceq_{\operatorname{Bun}_G^{\mathrm{op}}} b_2$. Using these assumptions we may find a map $\operatorname{Spa} R \to \operatorname{Bun}_G$ with the property that for all $x \in \operatorname{Spa} R$ the induced map $\operatorname{Spa} C_x \to \operatorname{Bun}_G$ factors through either $\operatorname{Bun}_G^{b_1}$ or $\operatorname{Bun}_G^{b_2}$ and with the property that $\overline{f^{-1}(\operatorname{Bun}_G^{b_1})} \cap f^{-1}(\operatorname{Bun}_G^{b_2}) \neq \emptyset$. Replacing $\operatorname{Spa} R$ by a v-cover we may assume that it is a product of points, with $R^+ = \prod_{i \in I} C_i^+$. By shrinking $\operatorname{Spa} R$ and ignoring some factors if necessary we may assume that the principal components of $\operatorname{Spa} R$ all factor through $\mathrm{Bun}_G^{b_1}$ without changing the condition that $\overline{f^{-1}(\mathrm{Bun}_G^{b_1})} \cap f^{-1}(\mathrm{Bun}_G^{b_2}) \neq \emptyset$. This forces at least one non-principal component to factor through $\mathrm{Bun}_G^{b_2}$. Moreover, we may assume $C_i^+ = O_{C_i}$ for all i so that $R^+ = R^\circ$. By Theorem 8.6 we may assume that our map $\operatorname{Spa} R \to \mathrm{Bun}_G$ is induced from a map $\operatorname{Spec} R^+ \to \mathfrak{B}(G)$. Let k_i denote the residue field of O_{C_i}. By assumption, the map $\operatorname{Spa}(C_i, O_{C_i}) \to \mathrm{Bun}_G$ factors through $\mathrm{Bun}_G^{b_1}$. In particular, $\operatorname{Spec} k_i \to \mathfrak{B}(G)$ factors through $\mathfrak{B}(G)_{b_1}$. Which implies that $\operatorname{Spec} \prod_{i \in I} k_i \to \mathfrak{B}(G)$ also factors through $\mathfrak{B}(G)_{b_1}$. Indeed, it certainly factors through $\mathfrak{B}(G)_{\leq b_1}$, and the locus where it factors through $\mathfrak{B}(G)_{b'}$ with $b' < b_1$ is finitely presented and contains no principal component of $\operatorname{Spec} \prod_{i \in I} k_i$ which implies that it is empty. We see that the closed point of every connected component of $\operatorname{Spec} R^+$ factors through $\mathfrak{B}(G)_{b_1}$. Furthermore, there is at least one point $x \in \operatorname{Spec} R^+$ mapping to $\mathfrak{B}(G)_{b_2}$. The connected component containing x defines a valuation ring V_x and a map $\operatorname{Spec} V_x \to \mathfrak{B}(G)$ such that the closed point factors through $\mathfrak{B}(G)_{b_1}$ and at least one point of $\operatorname{Spec} V_x$ factors through $\mathfrak{B}(G)_{b_2}$. This allows us to conclude that:

$$b_1 \preceq_{\mathrm{Bun}_G^{\mathrm{op}}} b_2 \implies b_1 \preceq_{\mathfrak{B}(G)} b_2. \qquad \square$$

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