THE DRINFELD STRATIFICATION FOR GL_n

CHARLOTTE CHAN AND ALEXANDER B. IVANOV

ABSTRACT. We define a stratification of Deligne–Lusztig varieties and their parahoric analogues which we call the Drinfeld stratification. In the setting of inner forms of GL_n, we study the cohomology of these strata and give a complete description of the unique closed stratum. We state precise conjectures on the representation-theoretic behavior of the stratification. We expect this stratification to play a central role in the investigation of geometric constructions of representations of p-adic groups.

CONTENTS

1. INTRODUCTION

Like the classical upper half plane, its nonarchimedean analogue—the Drinfeld upper half plane—appears naturally in a wide range of number theoretic, representation theoretic, and algebro-geometric contexts. For finite fields, the ℓ-adic étale cohomology of the Drinfeld upper half plane $\mathbb{P}^1(\overline{\mathbb{F}}_q) \smallsetminus \mathbb{P}^1(\mathbb{F}_q)$ with coefficients in nontrivial rank-1 local systems, is known to realize the cuspidal irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$. One can generalize this to $\mathrm{GL}_n(\mathbb{F}_q)$ by projectivizing the complement of all rational sub-vector spaces of $V = \overline{\mathbb{F}}_q^{\oplus n}$. This is the Drinfeld upper half space for $\mathbb{F}_q$. In this paper, we consider a stratification of the Drinfeld upper half space induced by "intermediate" Drinfeld upper half spaces of smaller dimension sitting inside $\mathbb{P}(V)$.

In earlier work [CI18], we proved that for inner forms of GL_n, Lusztig's *loop Deligne–Lusztig set* [Lus79] is closely related to a finite-ring analogue of the Drinfeld upper half space. This allowed us to endow this set with a scheme structure (a statement which is still conjectural for any group outside GL_n) and define its cohomology. Under a regularity condition, we prove in [CI18] that the cohomology of loop Deligne–Lusztig varieties for inner forms of GL_n realize certain irreducible supercuspidal representation and describe these within the context of the local Langlands and Jacquet–Langlands correspondences. In [CI19b], we are able to relax this regularity condition to something quite general by using highly nontrivial input obtained by studying the cohomology of a stratification—the *Drinfeld stratification*—which comes from the aforementioned stratification of the Drinfeld upper half space.

In [CI19a], we studied a class of varieties X_h associated to parahoric subgroups of a(ny) connected reductive group G which splits over an unramified extension. We define a stratification in this general context as well; the Drinfeld stratification is indexed by twisted Levi subgroups of G. The purpose of this paper is to initiate the study of these strata and, in due course, supply the necessary input for the final step in [CI19b].

We focus on the setting of inner forms of GL_n and prove the first foundational representation-theoretic traits of the cohomology of the Drinfeld stratification: irreducibility (Theorem 5.2.1) and a special character formula (Proposition 5.3.1). Using Theorem 5.2.1, in Section 6 we prove that the torus eigenspaces in the cohomology of the unique closed Drinfeld stratum is supported in a single (possibly non-middle) degree. Furthermore, this stratum is a *maximal variety* in the sense of Boyarchenko–Weinstein [BW16]: the number of rational points on the closed Drinfeld stratum attains its Weil–Deligne bound. Our analysis relies on techniques developed in [Cha19] in the special case of division algebras and gives some context for what we expect to be the role of maximal varieties in these Deligne–Lusztig varieties for p-adic groups.

In practice, it is sometimes only possible to work directly with the Drinfeld stratification of the parahoric Deligne–Lusztig varieties X_h instead of with the entire X_h. In this paper, for example, the maximality of the closed stratum allows us to give an exact formula (Corollary 6.6.1) for the formal degree of the associated representation of the p-adic group. We prove a comparison theorem in [CI19b] relating the Euler characteristic of this stratum to that of X_h. This formal degree input, by comparison with Corwin–Moy–Sally [CMS90], allows us to obtain a geometric supercuspidality result in [CI19b].

We finish the paper with a precise formulation of some conjectures (Conjecture 7.1.1 and 7.2.1) which describe what we expect to be the shape of the cohomology of the Drinfeld stratification and its relation to the cohomology of loop Deligne–Lusztig varieties. In the Appendix, we present an analysis of the fibers of the natural projection maps $X_h \to X_{h-1}$; we believe this could be a possible approach to proving Conjecture 7.2.1 and may be of independent interest. It would be interesting to see if the Drinfeld stratification plays a role in connections to orbits in finite Lie algebras, à la work of Chen [Che19].

2. Notation

Let k be a nonarchimedean local field with residue field $\mathbb{F}_q$ and let $\breve{k}$ denote the completion of the maximal unramified extension of k. We write $\mathcal{O}_{\breve{k}}$ and $\mathcal{O}_k$ for the rings of integers of $\breve{k}$ and k, respectively. For any positive integer m, let k_m denote the degree-m unramified extension of k. Additionally, we define $L := k_n$.

With the exception of Section 3, in this paper, G will be an inner form of GL_n defined over k. Let $\sigma \in \mathrm{Gal}(\breve{k}/k)$ denote the Frobenius which induces the qth power automorphism on the residue field $\overline{\mathbb{F}}_q$. Abusing notation, we also let σ denote the map $\mathrm{GL}_n(\breve{k}) \to \mathrm{GL}_n(\breve{k})$ by applying σ to each matrix entry. The inner forms of GL_n are indexed by an integer $0 \leq \kappa \leq n-1$; fix such an integer. Throughout the paper, we write $\kappa/n = k_0/n_0$, $(k_0, n_0) = 1$, and $\kappa = k_0 n'$. We define an element b_{cox} with $\mathrm{val}\det(b_{\mathrm{cox}}) = \kappa$ and set $G = J_{b_{\mathrm{cox}}}$ (the σ-stabilizer of b_{cox}) with k-rational structure induced by the Frobenius

$$F \colon \mathrm{GL}_n(\breve{k}) \to \mathrm{GL}_n(\breve{k}), \qquad g \mapsto b_{\mathrm{cox}}\sigma(g)b_{\mathrm{cox}}^{-1}.$$

Note that $G \cong \mathrm{GL}_{n'}(D_{k_0/n_0})$, where D_{k_0/n_0} denotes the division algebra over k of dimension n_0^2 with Hasse invariant k_0/n_0. Let T denote the set of diagonal matrices in G. Let x be the unique point in the intersection $\mathcal{A}(T) \cap \mathscr{B}(\mathrm{GL}_n, \breve{k})^F$. Note that $T(k) \cong L^\times$.

If k has characteristic p, we let $\mathbb{W}(A) = A[\![\pi]\!]$ for any $\mathbb{F}_q$-algebra A and write $[a_i]_{i \geq 0}$ to denote the element $\sum_{i \geq 0} a_i \pi^i \in \mathbb{W}(A)$. If k has characteristic zero, we let $\mathbb{W} = W_{\mathcal{O}_k} \times_{\mathrm{Spec}\,\mathcal{O}_k} \mathrm{Spec}\,\mathbb{F}_q$, where $W_{\mathcal{O}_k}$ is the $\mathcal{O}_k$-ring scheme of $\mathcal{O}_k$-Witt vectors [FF18, Section 1.2]. Following the notation of *op. cit.* we write the elements of $\mathbb{W}(A)$ as $[a_i]_{i \geq 0}$ where $a_i \in A$. We may now talk about $\mathbb{W}$ uniformly, regardless of the characteristic of k. As usual, we have the Frobenius and Verschiebung morphisms

$$\sigma \colon \mathbb{W} \to \mathbb{W}, \qquad [a_i]_{i \geq 0} \mapsto [a_i^q]_{i \geq 0},$$
$$V \colon \mathbb{W} \to \mathbb{W}, \qquad [a_i]_{i \geq 0} \mapsto [0, a_0, a_1, \ldots].$$

For any $h \in \mathbb{Z}_{\geq 0}$, let $\mathbb{W}_h = \mathbb{W}/V^h\mathbb{W}$ denote the corresponding truncated ring scheme.

For the benefit of the reader, we present a summary of the relationship between the various schemes appearing in this paper. Let $r \mid n'$ and let h be a positive integer. We have

$$
\begin{array}{ccc}
S_h^{(r)} & \lhook\joinrel\longrightarrow & S_h \\
\downarrow & & \downarrow \\
X_h^{(r)} & \lhook\joinrel\longrightarrow & X_h
\end{array}
$$

where the vertical maps are quotients by an affine space. We have $X_h = X_h(b_{\mathrm{cox}}, b_{\mathrm{cox}})$ and when $X_h(b, w) \cong X_h$, then $X_h(b, w)^{(r)} \cong X_h^{(r)}$ by definition. We have

$$X_h^{(r)} = \bigsqcup_{g \in \mathbb{G}_h(\mathbb{F}_q)/(\mathbb{L}_h^{(r)}(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q))} g \cdot (X_h \cap \mathbb{L}_h^{(r)}\mathbb{G}_h^1).$$

The rth Drinfeld stratum is

$$X_{h,r} = X_h^{(r)} \smallsetminus \bigcup_{r \mid s \mid n', \, r < s} X_h^{(s)}$$

and the closure of $X_{h,r}$ in X_h is $X_h^{(r)}$. The unique closed Drinfeld stratum is the setting $r = n'$; in this case,

$$X_{h,n'} = X_h^{(n')} = \bigsqcup_{g \in \mathbb{G}_h(\mathbb{F}_q)} g \cdot X_h^1, \qquad \text{where } X_h^1 = X_h \cap \mathbb{G}_h^1.$$

For any positive integer m, we let $[l]_m$ denote the unique representative of $l\mathbb{Z}/m\mathbb{Z}$ in the set $\{1, \ldots, m\}$.

3. THE DRINFELD STRATIFICATION

In this section only, we let G be any reductive group over k which splits over $\breve{k}$. Let F denote a Frobenius associated to the k-rational structure on G. Fix a k-rational, $\breve{k}$-split maximal torus $T \subset G$, let $x \in \mathcal{A}(T) \cap \mathscr{B}(G, \breve{k})^F$, and let $G_{x,0}$ be the attached parahoric model. Pick a $\breve{k}$-rational Borel subgroup $B \subset G_{\breve{k}}$ containing T and let U be the unipotent radical of B. Let $h \geq 1$ be an integer. There is a smooth affine group scheme $\mathbb{G}_h$ over $\mathbb{F}_q$ such that

$$\mathbb{G}_h(\mathbb{F}_q) = G_{x,0}(\mathcal{O}_k)/G_{x,(h-1)+}(\mathcal{O}_k), \quad \mathbb{G}_h(\overline{\mathbb{F}}_q) = G_{x,0}(\mathcal{O})/G_{x,(h-1)+}(\mathcal{O})$$

(see [CI19a, Section 2.5, 2.6] for more details). The subgroups T, U have associated subgroup schemes $\mathbb{T}_h$, $\mathbb{U}_h$ of $\mathbb{G}_h$ such that

$$\mathbb{T}_h(\mathbb{F}_q) = (T(k) \cap G_{x,0}(\mathcal{O}_k))/G_{x,(h-1)+}(\mathcal{O}_k), \quad \mathbb{T}_h(\overline{\mathbb{F}}_q) = (T(\breve{k}) \cap G_{x,0}(\mathcal{O}))/G_{x,(h-1)+}(\mathcal{O}),$$

and $\mathbb{U}_h(\overline{\mathbb{F}}_q) = (U(\breve{k}) \cap G_{x,0}(\mathcal{O}))/G_{x,(h-1)+}(\mathcal{O})$ (note here that $\mathbb{U}_h$ is defined over $\overline{\mathbb{F}}_q$ but may not be defined over $\mathbb{F}_q$ as U may not be k-rational).

3.1. **The schemes S_h and X_h.** The central object of study is X_h:

Definition 3.1.1. Define $\overline{\mathbb{F}}_q$-scheme

$$X_h := \{x \in \mathbb{G}_h : x^{-1}F(x) \in \mathbb{U}_h\}/(\mathbb{U}_h \cap F^{-1}(\mathbb{U}_h)).$$

X_h comes with a natural action of $\mathbb{G}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$ by left- and right-multiplication:

$$(g, t) \cdot x = gxt, \qquad \text{for } (g, t) \in \mathbb{G}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q), \, x \in X_h.$$

In some contexts, it will be more convenient to study S_h:

Definition 3.1.2. Define $\overline{\mathbb{F}}_q$-scheme

$$S_h := \{x \in \mathbb{G}_h : x^{-1}F(x) \in \mathbb{U}_h\}.$$

So, S_h is the closed subscheme of $\mathbb{G}_h$ obtained by pulling back $\mathbb{U}_h$ along the (finite étale) Lang map $\mathbb{G}_h \to \mathbb{G}_h$, $g \mapsto g^{-1}F(g)$. Note that S_h comes with the same natural action of $\mathbb{G}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$ as X_h.

Observe that since $\mathbb{U}_h \cap F^{-1}(\mathbb{U}_h)$ is an affine space, the cohomology of X_h and S_h differs only by a shift, and in particular, for any $\theta \colon \mathbb{T}_h(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$, we have

$$H_c^*(X_h, \overline{\mathbb{Q}}_\ell)[\theta] = H_c^*(S_h, \overline{\mathbb{Q}}_\ell)[\theta]$$

as elements of the Grothendieck group of $\mathbb{G}_h(\mathbb{F}_q)$.

3.2. **The scheme $X_h(b, w)$.** In this section, we further assume that G is quasisplit over k and $B \subset G$ is k-rational. In this section, we write $\sigma = F$ for our q-Frobenius associated to the k-rational structure on G.

Definition 3.2.1. Let $b, w \in G(\breve{k})$. Assume that b, w normalize both subgroups $G_{x,0}(\mathcal{O}_{\breve{k}})$, $G_{x,(h-1)+}(\mathcal{O}_{\breve{k}})$ of $G(\breve{k})$, and additionally assume that w normalizes $T(\breve{k})$. Define the $\overline{\mathbb{F}}_q$-scheme

$$X_h(b, w) := \{x \in \mathbb{G}_h : x^{-1}b\sigma(x) \in \mathbb{U}_h w \mathbb{U}_h\}/\mathbb{U}_h,$$

where the condition $x^{-1}b\sigma(x) \in \mathbb{U}_h w \mathbb{U}_h$ means the following: For any lift $\widetilde{x} \in G$ of $x \in \mathbb{G}_h$, the element $\widetilde{x}^{-1}b\sigma(\widetilde{x})$ is an element of $(U \cap G_{x,0})w(U \cap G_{x,0})G_{x,(h-1)+} \subset G$. More precisely, $X_h(b, w) = S_h(b, w)/\mathbb{U}_h$, where $S_h(b, w)$ is the reduced $\overline{\mathbb{F}}_q$-subscheme of $\mathbb{G}_h$ such that $S_h(b, w)(\overline{\mathbb{F}}_q)$ is equal to the image of $\{x \in G_{x,0}(\mathcal{O}_{\breve{k}}) : x^{-1}b\sigma(x) \in (U(\breve{k}) \cap G_{x,0}(\mathcal{O}_{\breve{k}}))w(U(\breve{k}) \cap G_{x,0}(\mathcal{O}_{\breve{k}}))G_{x,(h-1)+}(\mathcal{O}_{\breve{k}})\}$ in $\mathbb{G}_h(\overline{\mathbb{F}}_q)$. Note that $X_h(b, w)$ comes with a natural action by left- and right-multiplication of $G_h(b)$ and $T_h(w)$, where $G_h(b) \subset \mathbb{G}_h(\overline{\mathbb{F}}_q)$ is the image of $\{g \in G_{x,0}(\mathcal{O}_{\breve{k}}) : b\sigma(g)b^{-1} = g\}$ and $T_h(w) \subset \mathbb{T}_h(\overline{\mathbb{F}}_q)$ is the image of $\{t \in T(\breve{k}) \cap G_{x,0}(\mathcal{O}_{\breve{k}}) : w\sigma(t)w^{-1} = t\}$.

The next lemma is a one-line computation; we record it for easy reference.

Lemma 3.2.2. *Let $\gamma \in G_{x,0}(\mathcal{O}_{\breve{k}})$. Then we have an isomorphism*

$$X_h(b, w) \to X_h(\gamma^{-1}b\sigma(\gamma), w), \qquad x \mapsto \overline{\gamma}x,$$

where $\overline{\gamma}$ is the image of γ in the quotient $\mathbb{G}_h(\overline{\mathbb{F}}_q)$.

Lemma 3.2.3. *Consider the morphism* $F \colon (\mathbb{G}_h)_{\overline{\mathbb{F}}_q} \to (\mathbb{G}_h)_{\overline{\mathbb{F}}_q}$ *given by* $g \mapsto b\sigma(g)b^{-1}$. *If* $wG_{x,0}b^{-1} = G_{x,0}$ *and* $F(\mathbb{U}_h) = w\mathbb{U}_h b^{-1}$, *then*

$$X_h(b, w) = X_h,$$

where X_h *is the* $\overline{\mathbb{F}}_q$*-scheme in Definition 3.1.1 associated to the group scheme* $(\mathbb{G}_h)_{\overline{\mathbb{F}}_q}$ *endowed with the* $\mathbb{F}_q$*-rational structure associated to the q-Frobenius* F.

Proof. We have

$$
\begin{aligned}
X_h(b, w) &= \{x \in \mathbb{G}_h : x^{-1}F(x) \in \mathbb{U}_h w\mathbb{U}_h b^{-1}\}/\mathbb{U}_h \\
&= \{x \in \mathbb{G}_h : x^{-1}F(x) \in \mathbb{U}_h F(\mathbb{U}_h)\}/\mathbb{U}_h \\
&= \{x \in \mathbb{G}_h : x^{-1}F(x) \in \mathbb{U}_h\}/(\mathbb{U}_h \cap F^{-1}\mathbb{U}_h) = X_h. \qquad \square
\end{aligned}
$$

3.3. **The Drinfeld stratification for** S_h**.** Let L be a k-rational twisted Levi subgroup of G and assume that L contains T. Recall a k-rational subgroup $L \subset G$ is a *twisted Levi* if $L_{\overline{k}}$ is a Levi subgroup of $G_{\overline{k}}$. Note also that the condition that L contains T forces L to be split over $\breve{k}$. Following [CI19a, Section 2.6], the schematic closure L_x in $G_{x,0}$ is a closed subgroup scheme defined over $\mathcal{O}_k$. Applying the "positive loop" functor to L_x, for $h \in \mathbb{Z}_{>0}$, we can define a $\overline{\mathbb{F}}_q$-scheme $\mathbb{L}_h$ such that $\mathbb{L}_h(\overline{\mathbb{F}}_q)$ is the image of $L_x(\mathcal{O}_{\breve{k}})$ in $\mathbb{G}_h(\overline{\mathbb{F}}_q)$.

Definition 3.3.1 (Drinfeld stratification for S_h). Define

$$S_h^{(L)} := \{x \in \mathbb{G}_h : x^{-1}F(x) \in (\mathbb{L}_h \cap \mathbb{U}_h)\mathbb{U}_h^1\},$$

where $(\mathbb{L}_h \cap \mathbb{U}_h)\mathbb{U}_h^1 \subset \mathbb{U}_h$ is the subgroup generated by $\mathbb{L}_h \cap \mathbb{U}_h$ and $\mathbb{U}_h^1$ (which is normalized by $\mathbb{L}_h \cap \mathbb{U}_h$). Note that the subscheme $S_h^{(L)}$ of S_h is closed and stable under the action of $\mathbb{G}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$.

Definition 3.3.2 (Drinfeld stratification for X_h, $X_h(b, w)$). Define $X_h^{(L)}$ to be the image of $S_h^{(L)}$ under the surjection $S_h \to X_h$. Recall that for any $\gamma \in G_{x,0}(\mathcal{O}_{\breve{k}})$, we have $X_h(b, w) \cong X_h(\gamma^{-1}b\sigma(\gamma), w)$ via $x \mapsto \overline{\gamma}x$. If $F(\mathbb{U}_h) = w\mathbb{U}_h b^{-1}$, then $X_h = X_h(b, w)$; in this setting, let $X_h(\gamma^{-1}b\sigma(\gamma), w)^{(L)}$ denote the image of $X_h^{(L)}$.

Another subscheme of S_h which we may associate to the twisted Levi subgroup $L \subset G$ is the intersection

$$
\begin{aligned}
S_h \cap \mathbb{L}_h\mathbb{G}_h^1 &= \{x \in \mathbb{L}_h\mathbb{G}_h^1 : x^{-1}F(x) \in \mathbb{U}_h\} \\
&= \{x \in \mathbb{L}_h\mathbb{G}_h^1 : x^{-1}F(x) \in (\mathbb{L}_h \cap \mathbb{U}_h)\mathbb{U}_h^1\},
\end{aligned}
$$

where $\mathbb{L}_h\mathbb{G}_h^1$ denotes the subgroup scheme of $\mathbb{G}_h$ generated by $\mathbb{L}_h$ and $\mathbb{G}_h^1$ (which is normalized by $\mathbb{L}_h$). Note that $S_h \cap \mathbb{L}_h\mathbb{G}_h^1$ is stable under the action of $\mathbb{L}_h(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$.

Lemma 3.3.3. *Let L be a k-rational twisted Levi subgroup of G containing T. Then*

$$S_h^{(L)} = \bigsqcup_{\gamma \in \mathbb{G}_h(\mathbb{F}_q)/(\mathbb{L}_h(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q))} \gamma \cdot (S_h \cap \mathbb{L}_h\mathbb{G}_h^1).$$

Proof. Pick any $u \in \mathbb{U}_h(\overline{\mathbb{F}}_q)\mathbb{U}_h^1(\overline{\mathbb{F}}_q)$. By surjectivity of the Lang map, there exists $x \in \mathbb{L}_h(\overline{\mathbb{F}}_q)\mathbb{G}_h^1(\overline{\mathbb{F}}_q)$ and $y \in \mathbb{G}_h(\overline{\mathbb{F}}_q)$ such that $x^{-1}F(x) = u$ and $y^{-1}F(y) = u$. Then

$$(xy^{-1})^{-1}F(xy^{-1}) = yx^{-1}F(x) = F(y)^{-1} = yuF(y)^{-1} = yuu^{-1}y^{-1} = 1.$$

Therefore $xy^{-1} \in \mathbb{G}_h(\mathbb{F}_q)$. The assertion now follows from the fact that the stabilizer of $S_h \cap \mathbb{L}_h\mathbb{G}_h^1$ in $\mathbb{G}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$ is $\mathbb{L}_h(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$. $\qquad \square$

By Lemma 3.3.3, we see:

Lemma 3.3.4. *If L is a twisted Levi subgroup of G containing T, then for any character $\theta \colon \mathbb{T}_h(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ and for all $i \geq 0$,*

$$H_c^i(S_h^{(L)}, \overline{\mathbb{Q}}_\ell)[\theta] \cong \mathrm{Ind}_{\mathbb{L}_h(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h(\mathbb{F}_q)} \left(H_c^i(S_h \cap \mathbb{L}_h\mathbb{G}_h^1, \overline{\mathbb{Q}}_\ell)[\theta] \right).$$

4. The case of GL_n

In this paper, study the varieties introduced in Section 3 in the special case when G is an inner form of GL_n. We emphasize that these varieties $S_h, X_h, X_h(b,w)$—at least *a priori*—depend on a choice of Borel subgroup containing the torus at hand. From now until the end of the paper, we work with the varieties associated with the Borel subgroup explicitly chosen in Section 4.2. We explicate the Drinfeld stratification for S_h, X_h, and certain $X_h(b,w)$, and give a description in terms of Drinfeld upper half-spaces and $\mathscr{L}_h \subset \mathbb{W}_h^{\oplus n}$, a finite-ring analogue of an isocrystal.

Let $\sigma \in \mathrm{Gal}(\breve{k}/k)$ denote the q-Frobenius induces $x \mapsto x^q$ on the residue field $\overline{\mathbb{F}}_q$. Abusing notation, also let

$$\sigma \colon \mathrm{GL}_n(\breve{k}) \to \mathrm{GL}_n(\breve{k}), \qquad (M_{i,j})_{i,j=1,\ldots,n} \mapsto (\sigma(M_{i,j}))_{i,j=1,\ldots,n}.$$

For $b \in \mathrm{GL}_n(\breve{k})$, let J_b be the σ-stabilizer of b: for any k-algebra R,

$$J_b(R) := \{ g \in \mathrm{GL}_n(R \otimes_k \breve{k}) : g^{-1}b\sigma(g) = b \}.$$

J_b is an inner form of the centralizer of the Newton point of b (which is a Levi subgroup of GL_n), and we may consider

$$\mathrm{GL}_n(\breve{k}) \to \mathrm{GL}_n(\breve{k}), \qquad g \mapsto b\sigma(g)b^{-1}$$

to be an associated q-Frobenius for the k-rational structure on J_b. If b is *basic* (i.e. the Newton point of b is central), then J_b is an inner form of GL_n and every inner form arises in this way. If $\kappa = \kappa_{\mathrm{GL}_n}(b) := \mathrm{val}(\det(b))$, then then $J_b(k) \cong \mathrm{GL}_{n'}(D_{k_0/n_0})$ where $\kappa/n = k_0/n_0$, $(k_0, n_0) = 1$, and $\kappa = k_0 n'$. Note that the isomorphism class of J_b only depends on the σ-conjugacy class $[b] := \{ g^{-1}b\sigma(g) : g \in \mathrm{GL}_n(\breve{k}) \}$.

Fix an integer $0 \leq \kappa \leq n-1$. In the next sections, we will focus on representatives b revolving around the *Coxeter representative* (Def 4.1.1) and give explicit descriptions of the varieties $X_h, X_h(b,w)$, and their Drinfeld stratifications $\{X_h^{(r)}\}, \{X_h(b,w)^{(r)}\}$, where r runs over the divisors of n'. The $X_h^{(r)}, X_h(b,w)^{(r)}$ are closed subvarieties of $X_h, X_h(b,w)$; we call the rth Drinfeld stratum

$$X_h^{(r)} \smallsetminus \left(\bigcup_{\substack{r < r' \leq n' \\ r \mid r' \mid n'}} X_h^{(r')} \right), \qquad X_h(b,w)^{(r)} \smallsetminus \left(\bigcup_{\substack{r < r' \leq n' \\ r \mid r' \mid n'}} X_h(b,w)^{(r')} \right) \tag{4.1}$$

so that the closure of the rth Drinfeld stratum is $X_h^{(r)}, X_h(b,w)^{(r)}$.

4.1. Explicit parahoric subgroups of G. Set

$$b_0 := \begin{pmatrix} 0 & 1 \\ 1_{n-1} & 0 \end{pmatrix}, \quad \text{and} \quad t_{\kappa,n} := \begin{cases} \mathrm{diag}(\underbrace{1,\ldots,1}_{n-\kappa}, \underbrace{\varpi,\ldots,\varpi}_{\kappa}) & \text{if } (\kappa,n) = 1, \\ \mathrm{diag}(\underbrace{t_{k_0,n_0},\ldots,t_{k_0,n_0}}_{n'}) & \text{otherwise.} \end{cases}$$

Fix an integer $e_{\kappa,n}$ such that $(e_{\kappa,n}, n) = 1$ and $e_{\kappa,n} \equiv k_0 \bmod n_0$. If κ divides n (i.e. $k_0 = 1$), we always take $e_{\kappa,n} = 1$.

Definition 4.1.1. The *Coxeter-type representative* attached to κ is $b_{\mathrm{cox}} := b_0^{e_{\kappa,n}} \cdot t_{\kappa,n}$.

Define $G := J_{b_{\mathrm{cox}}}$ with Frobenius

$$F \colon \mathrm{GL}_n(\check{k}) \to \mathrm{GL}_n(\check{k}), \qquad g \mapsto b_{\mathrm{cox}}\sigma(g)b_{\mathrm{cox}}^{-1}$$

and define T to be the set of diagonal matrices in G. Observe that T is F-stable and that $T(k) \cong L^\times$. Since T is elliptic, the intersection $\mathcal{A}(T) \cap \mathcal{B}(G, \check{k})^F$ consists of a single point x and $G_{x,0}$ consists of invertible matrices $(A_{i,j})_{1 \le i,j \le n}$ where

$$A_{i,j} \in \begin{cases} \mathbb{W} & \text{if } [i]_{n_0} \ge [j]_{n_0}, \\ V\mathbb{W} & \text{if } [i]_{n_0} < [j]_{n_0}. \end{cases}$$

For technical reasons, we will need to write down the relationship between the Coxeter element $b_0^{e_{\kappa,n}}$ and the Coxeter element b_0. Define γ to be the unique permutation matrix which a) fixes the first elementary column vector and b) has the property that

$$\gamma b_0^{e_{\kappa,n}} \gamma^{-1} = b_0. \tag{4.2}$$

Note that one can express γ explicitly as well: it corresponds to the permutation of $\{1, \ldots, n\}$ given by

$$i \mapsto [(i-1)e_{\kappa,n} + 1]_n.$$

4.2. **An explicit description of X_h.** The choices in this section are the same as those from [CI18, Section 7.7]. In the setting of division algebras, these choices also appear in [Cha18, Cha19].

Let $U_{\mathrm{up}}, U_{\mathrm{low}} \subset G_{\check{k}}$ denote the subgroups of unipotent upper- and lower-triangular matrices. Define

$$U := \gamma^{-1} U_{\mathrm{low}} \gamma, \qquad U^- := \gamma^{-1} U_{\mathrm{up}} \gamma. \tag{4.3}$$

Let $\mathbb{U}_h, \mathbb{U}_h^-$ be the associate subgroup schemes of $\mathbb{G}_h$. By [CI18, Lemma 7.12], we have an isomorphism of $\overline{\mathbb{F}}_q$-schemes

$$(\mathbb{U}_h \cap F\mathbb{U}_h^-) \times (\mathbb{U}_h \cap F^{-1}\mathbb{U}_h) \to \mathbb{U}_h, \qquad (g, x) \mapsto x^{-1}gF(x). \tag{4.4}$$

We will need a refinement of this isomorphism later (see Lemma 4.3.1). Define

$$\mathscr{L}_h := \left(\mathbb{W}_h \oplus (\mathbb{W}_{h-1})^{\oplus n_0 - 1} \right)^{\oplus n'}.$$

Write $t_{\kappa,n} = \mathrm{diag}\{t_1, \ldots, t_n\}$. Viewing any $v \in \mathscr{L}_h$ as a column vector, consider the associated matrix

$$\lambda(v) := \left(v_1 \,\Big|\, v_2 \,\Big|\, v_3 \,\Big|\, \cdots \,\Big|\, v_n \right), \tag{4.5}$$

$$\text{where } v_{[ie_{\kappa,n}+1]_n} := \varpi^{-\lfloor ik_0/n_0 \rfloor} \cdot (b\sigma)^i(v) \text{ for } 0 \le i \le n-1. \tag{4.6}$$

Lemma 4.2.1. *We have*

$$X_h = \{x \in \mathbb{G}_h : x^{-1}F(x) \in \mathbb{U}_h \cap F(\mathbb{U}_h^-)\}$$

$$= \{\lambda(v) \in \mathbb{G}_h : v \in \mathscr{L}_h \text{ and } \sigma(\det \lambda(v)) = \det \lambda(v)\}.$$

Proof. The first equality holds by (4.4). The second equality is an explicit computation: in the division algebra setting, see [Lus79, Equation (2.2)], [Boy12, Lemma 4.4], [Cha19, Section 2.1]; in the present setting of arbitrary inner forms of GL_n, see [CI18, Section 6]. We give an exposition of these works here.

By direct computation, $\mathbb{U}_h \cap F(\mathbb{U}_h^-)$ is the subgroup of $\mathbb{G}_h$ consisting of unipotent lower-triangular matrices whose entries outside the first column vanish:

$$\mathbb{U}_h \cap F(\mathbb{U}_h^-) = \left\{ \begin{pmatrix} 1 & & & \\ * & 1 & & \\ \vdots & & \ddots & \\ * & & & 1 \end{pmatrix} \right\}.$$

Suppose that $x \in \mathbb{G}_h$ is such that $x^{-1}F(x) \in \mathbb{U}_h \cap F(\mathbb{U}_h^-)$ and let x_i denote the ith column of x. Then recalling that $b = b_0^{e_{\kappa,n}} t_{\kappa,n}$ and writing $t_{\kappa,n} = \mathrm{diag}\{t_1, \ldots, t_n\}$, we have

$$F(x) = \left(b\sigma(x_1) \,\middle|\, b\sigma(x_2) \,\middle|\, \cdots \,\middle|\, b\sigma(x_n) \right) b^{-1}$$
$$= \left(t_{[1-e_{\kappa,n}]_n}^{-1} b\sigma(x_{[1-e_{\kappa,n}]_n}) \,\middle|\, t_{[2-e_{\kappa,n}]_n}^{-1} b\sigma(x_{[2-e_{\kappa,n}]_n}) \,\middle|\, \cdots \,\middle|\, t_{[n-e_{\kappa,n}]_n}^{-1} b\sigma(x_{[n-e_{\kappa,n}]_n}) \right).$$

On the other hand, we have

$$x(\mathbb{U}_h \cap F(\mathbb{U}_h^-)) = \left(* \,\middle|\, x_2 \,\middle|\, x_3 \,\middle|\, \cdots \,\middle|\, x_n \right).$$

Comparing columns, we see that each x_i is uniquely determined by x_1 and that we have

$$x_{[(n-1)e_{\kappa,n}+1]_n} = t_{[(n-2)e_{\kappa,n}+1]_n}^{-1} b\sigma(x_{[(n-2)e_{\kappa,n}+1]_n}) \tag{4.7}$$
$$= t_{[(n-2)e_{\kappa,n}+1]_n}^{-1} t_{[(n-3)e_{\kappa,n}+1]_n}^{-1} b\sigma(b\sigma(x_{[(n-3)e_{\kappa,n}+1]_n})) \tag{4.8}$$
$$= t_{[(n-2)e_{\kappa,n}+1]_n}^{-1} t_{[(n-3)e_{\kappa,n}+1]_n}^{-1} \cdots t_1^{-1} (b\sigma)^{n-1}(x_1). \tag{4.9}$$

Using Lemma 4.2.2, we now see that $x = \lambda(x_1)$, and finally, the condition $\sigma(\det \lambda(x)) = \det \lambda(x)$ comes from observation that $x^{-1}F(x)$ must have determinant 1. $\qquad\square$

Lemma 4.2.2. *For $1 \leq i \leq n-1$,*

$$\prod_{j=0}^{i-1} t_{[je_{\kappa,n}+1]_n} = \varpi^{\lfloor ik_0/n_0 \rfloor}.$$

Proof. We prove this by induction on i. If $i = 1$, then by definition we have $t_1 = 1$, so this proves the base case. Now assume that the lemma holds for i. We would like to prove that it holds for $i+1$. This means we need to prove two assertions:

    (a) If $\lfloor (i+1)k_0/n_0 \rfloor > \lfloor ik_0/n_0 \rfloor$, then $t_{[ie_{\kappa,n}+1]_n} = \varpi$.
    (b) If $\lfloor (i+1)k_0/n_0 \rfloor = \lfloor ik_0/n_0 \rfloor$, then $t_{[ie_{\kappa,n}+1]_n} = \varpi$.

The arguments are very similar. For (a): Observe that $\lfloor (i+1)k_0/n_0 \rfloor > \lfloor ik_0/n_0 \rfloor$ if and only if $n_0 > [ie_{\kappa,n}]_{n_0} \geq n_0 - k_0$ since $e_{\kappa,n} \equiv k_0 \bmod n_0$. But this happens if and only if $[ie_{\kappa,n} + 1]_{n_0} > n_0 - k_0$, which means $t_{[ie_{\kappa,n}+1]_n} = \varpi$ by definition. For (b): Observe that $\lfloor (i+1)k_0/n_0 \rfloor = \lfloor ik_0/n_0 \rfloor$ if and only if $[ie_{\kappa,n}]_{n_0} = n_0$ or $[ie_{\kappa,n}]_{n_0} < n_0 - k_0$. But this happens if and only if $[ie_{\kappa,n} + 1]_{n_0} \leq n_0 - k_0$, which means that $t_{[ie_{\kappa,n}+1]_n} = 1$ by definition. $\qquad\square$

4.3. The Drinfeld stratification of X_h.

For any divisor $r \mid n'$, define $L^{(r)}$ to be the twisted Levi subgroup of G consisting of matrices $(A_{i,j})_{1 \leq i,j \leq n}$ such that $A_{i,j} = 0$ unless $i - j \equiv 0$ modulo rn_0. Note that $L^{(r)} \cong \mathrm{Res}_{k_{\frac{n}{r}}/k}(\mathrm{GL}_r)$ and that every k-rational twisted Levi subgroup of G containing T is conjugate to $L^{(r)}$ for some $r \mid n'$. Let $\mathbb{L}_h^{(r)}$ denote subgroup of $\mathbb{G}_h$ associated to $L^{(r)}$ and define

$$\mathbb{U}_{h,r} := \mathbb{L}_h^{(r)} \mathbb{U}_h^1 \cap \mathbb{U}_h, \qquad \mathbb{U}_{h,r}^- := \mathbb{L}_h^{(r)} \mathbb{U}_h^{-,1} \cap \mathbb{U}_h^-.$$

Lemma 4.3.1. *The isomorphism of $\overline{\mathbb{F}}_q$-schemes (4.4)*

$$(\mathbb{U}_h \cap F\mathbb{U}_h^-) \times (\mathbb{U}_h \cap F^{-1}\mathbb{U}_h) \to \mathbb{U}_h, \qquad (g, x) \mapsto x^{-1}gF(x)$$

restricts to an isomorphism

$$(\mathbb{U}_{h,r} \cap F\mathbb{U}_{h,r}^-) \times (\mathbb{U}_{h,r} \cap F^{-1}\mathbb{U}_{h,r}) \to \mathbb{U}_{h,r}.$$

Proof. This lemma is a refinement of [CI18, Lemma 7.12]. Recall that $\gamma \mathbb{U}_h \gamma^{-1}$ and $\gamma \mathbb{U}_h^- \gamma^{-1}$ are the subgroups consisting of unipotent lower- and upper-triangular matrices in $\mathbb{G}_h$. Recall

also that $F(g) = b_0^{e_{\kappa,n}} t_{\kappa,n} \sigma(g) t_{\kappa,n}^{-1} b_0^{e_{\kappa,n}}$. Conjugating (4.4), which is proven in *op. cit.*, we have

$$(\gamma \mathbb{U}_h \gamma^{-1} \cap F_0(\gamma \mathbb{U}_h^- \gamma^{-1})) \times (\gamma \mathbb{U}_h \gamma^{-1} \cap F_0^{-1}(\gamma \mathbb{U}_h \gamma^{-1})) \to \gamma \mathbb{U}_h \gamma^{-1},$$

where $F_0(g) = (b_0 \gamma t_{\kappa,n} \gamma^{-1}) \sigma(g) (b_0 \gamma t_{\kappa,n} \gamma^{-1})^{-1}$. Since $\gamma L^{(r)} \gamma^{-1} = L^{(r)}$, to prove the lemma, it suffices to show that if $(g, x) \in (\gamma \mathbb{U}_h \gamma^{-1} \cap F_0(\gamma \mathbb{U}_h^- \gamma^{-1})) \times (\gamma \mathbb{U}_h \gamma^{-1} \cap F_0^{-1}(\gamma \mathbb{U}_h \gamma^{-1}))$ is such that $A = x^{-1} g F(x) \in \gamma \mathbb{U}_{h,r} \gamma^{-1}$, then

$$(g, x) \in (\gamma \mathbb{U}_{h,r} \gamma^{-1} \cap F_0(\gamma \mathbb{U}_{h,r}^- \gamma^{-1})) \times (\gamma \mathbb{U}_{h,r} \gamma^{-1} \cap F_0^{-1}(\gamma \mathbb{U}_{h,r} \gamma^{-1})). \tag{4.10}$$

Keeping the same notation as in [CI18, Lemma 7.12], write

$$x = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ b_{21} & 1 & 0 & \cdots & \cdots & 0 \\ b_{31} & b_{32} & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,n-2} & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ c_{n-1} & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let $\gamma t_{\kappa,n} \gamma^{-1} = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ so that we have

$$F_0(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma(b_{21})s_2/s_1 & 1 & 0 & & 0 \\ 0 & \sigma(b_{31})s_3/s_1 & \sigma(b_{32})s_3/s_2 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \sigma(b_{n-1,1})s_{n-1}/s_1 & \sigma(b_{n-1,2})s_{n-1}/s_2 & \cdots & \sigma(b_{n-1,n-2})s_{n-1}/s_{n-2} & 1 \end{pmatrix}.$$

As in [CI18, Lemma 7.12], we see that the (i,j)th entry of $gF_0(x)$ is

$$(gF_0(x))_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i < j, \\ c_{i-1} & \text{if } i > j = 1, \\ \sigma(b_{i-1,j-1})s_{i-1}/s_{j-1} & \text{if } i > j > 1. \end{cases} \tag{4.11}$$

We also compute the (i,j)th entry of xA when $A = (a_{i,j})_{i,j} \in \gamma \mathbb{U}_h \gamma^{-1}$:

$$(xA)_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i < j, \\ b_{ij} + \sum_{k=j+1}^{i-1} b_{ik} a_{kj} + a_{ij} & \text{if } j < i \leq n - 1, \\ a_{nj} & \text{if } j < i = n. \end{cases} \tag{4.12}$$

We now have n^2 equations given by (4.11) = (4.12), viewed as equations in the variables $b_{i,j}$ and c_i. Let $\bar{b}_{i,j}$, $\bar{c}_i$, $\bar{a}_{i,j}$ denote the images of $b_{i,j}$, c_i, $a_{i,j}$ in $\mathbb{W}_1$. In particular, we have the following:

$$\bar{b}_{n-1,j-1} = 0 \qquad \Longleftrightarrow \qquad \bar{a}_{n,j} = 0, \tag{4.13}$$

and for $1 < j < i < n$,

$$\bar{b}_{i-1,j-1} = 0 \qquad \Longleftrightarrow \qquad \bar{b}_{i,j} + \sum_{k=j+1}^{i-1} \bar{b}_{i,k} \bar{a}_{k,j} + \bar{a}_{i,j} = 0. \tag{4.14}$$

Assume now that $A \in \gamma \mathbb{U}_{h,r} \gamma^{-1} = \gamma(\mathbb{L}_h^{(r)} \mathbb{U}_h^1 \cap \mathbb{U}_h) \gamma^{-1}$. Then $\bar{a}_{i,j} = 0$ if $rn_0 \nmid i-j$. From (4.13) we see that $\bar{b}_{n-1,j-1} = 0$ if $rn_0 \nmid n - j = (n-1) - (j-1)$. We now proceed by (decreasing) induction on i. If i,j are such that $1 < j < i < n$ and $rn_0 \nmid i - j$, then necessarily either $rn_0 \nmid i - k$ or $rn_0 \nmid k - j$, and therefore each term in the sum on the right-hand side of (4.14) is zero, and so $\bar{b}_{i-1,j-1} = 0$.

We have therefore shown that $x \in \gamma(\mathbb{L}_h^{(r)} \mathbb{U}_h^1 \cap \mathbb{U}_h) \gamma^{-1} \cap F^{-1}(\gamma \mathbb{U}_h \gamma^{-1})$. In particular, $F(x) \in \gamma \mathbb{U}_h \gamma^{-1}$. Since $\mathbb{L}_h^{(r)}$ is F-stable, we have that $F(\bar{x}) \in \mathbb{L}_1^{(r)}$ and therefore $F(x) \in \gamma(\mathbb{U}_h \cap \mathbb{L}_h^{(r)} \mathbb{U}_h^1) \gamma^{-1}$. Hence $x \in \gamma(\mathbb{U}_{h,r} \cap F^{-1}\mathbb{U}_{h,r}) \gamma^{-1}$.

Now since $\overline{A}, \overline{x} \in \mathbb{L}_1^{(r)}$, we must have $\overline{g} \in \mathbb{L}_1^{(r)}$. Since $g \in \gamma \mathbb{U}_h \gamma^{-1}$, we must have $g \in \gamma(\mathbb{L}_h^{(r)} \mathbb{U}_h^1 \cap \mathbb{U}_h) \gamma^{-1} = \gamma \mathbb{U}_{h,r} \gamma^{-1}$, and since $g \in F(\gamma \mathbb{U}_h^- \gamma^{-1})$, we must have $g \in F(\gamma(\mathbb{L}_h^{(r)} \mathbb{U}_h^{-,1} \cap \mathbb{U}_h^-) \gamma^{-1})$. Hence $g \in \gamma \mathbb{U}_{h,r} \gamma^{-1} \cap F(\gamma \mathbb{U}_{h,r}^- \gamma^{-1})$. This establishes (4.10) and finishes the proof of the lemma. $\qquad \square$

Definition 4.3.2 (Drinfeld stratification for X_h). For each divisor $r \mid n'$, we define

$$S_h^{(r)} := \{x \in \mathbb{G}_h : x^{-1} F(x) \in \mathbb{U}_{h,r}\},$$

$$X_h^{(r)} := \{x \in \mathbb{G}_h : x^{-1} F(x) \in \mathbb{U}_{h,r}\} / (\mathbb{U}_{h,r} \cap F^{-1} \mathbb{U}_{h,r})$$
$$= \{x \in \mathbb{G}_h : x^{-1} F(x) \in \mathbb{U}_{h,r} \cap F \mathbb{U}_{h,r}^-\},$$

where the second equality in $X_h^{(r)}$ holds by Lemma 3.3.

Note that $S_h^{(r)}$ is the variety $S_h^{(L)}$ defined in Section 3.3 in the special case that G is an inner form of GL_n, the twisted Levi L is $L^{(r)}$, and U is the unipotent radical of the Borel subgroup specified in Section 4.2. By Lemma 4.3.1, we can change the quotient in the definition of $X_h^{(r)}$ from $\mathbb{U}_{h,r} \cap F^{-1} \mathbb{U}_{h,r}$ to $\mathbb{U}_h \cap F^{-1} \mathbb{U}_h$ so that

$$X_h^{(r)} = \{x \in \mathbb{G}_h : x^{-1} F(x) \in \mathbb{U}_{h,r}\} / (\mathbb{U}_h \cap F^{-1} \mathbb{U}_h) \subset X_h.$$

Hence we have the picture:

$$
\begin{array}{ccc}
S_h^{(r)} & \hookrightarrow & S_h \\
\downarrow & & \downarrow \\
X_h^{(r)} & \hookrightarrow & X_h
\end{array}
$$

4.4. **The Drinfeld stratification for the Drinfeld upper half-space.** Consider the twisted Frobenius $b_{\mathrm{cox}}\sigma \colon \breve{k}^{\oplus n} \to \breve{k}^{\oplus n}$. Then $G(k)$ is equal to the subgroup consisting of all elements of $\mathrm{GL}_n(\breve{k})$ which commute with $b_{\mathrm{cox}}\sigma$. Now consider the subquotient of $\breve{k}^{\oplus n}$ given by

$$\mathscr{L}_h := \left(\mathbb{W}_h(\overline{\mathbb{F}}_q) \oplus (\mathrm{V}\,\mathbb{W}_{h-1}(\overline{\mathbb{F}}_q))^{\oplus n_0 - 1} \right)^{\oplus n'} \subset \mathbb{W}_h(\overline{\mathbb{F}}_q)^{\oplus n}$$

and write $\mathscr{L} = \varprojlim_h \mathscr{L}_h$. The action of $G(k)$ on $\breve{k}^{\oplus n}$ restricts to an action of $G_{x,0}(\mathcal{O}_k)$ on $\mathscr{L}$ which induces an action of $\mathbb{G}_h(\mathbb{F}_q)$ on $\mathscr{L}_h$.

Now consider the n'-dimensional $\overline{\mathbb{F}}_q$-vector space $V := \mathscr{L}_1 \subset \overline{\mathbb{F}}_q^{\oplus n}$. The morphism $\varpi^{-k_0}(b_{\mathrm{cox}}\sigma)^{n_0}$ is a Frobenius automorphism of V and defines a $\mathbb{F}_{q^{n_0}}$-rational structure on V. Observe that $\mathbb{G}_1(\mathbb{F}_q)$ is isomorphic to the subgroup of $\mathrm{GL}(V)$ consisting of elements which commute with $\varpi^{-k_0}(b_{\mathrm{cox}}\sigma)^{n_0}$. For any divisor $r \mid n'$ and any $\mathbb{F}_{q^{n_0 r}}$-rational subspace W of V, consider

$$\Omega_{W,q^{n_0 r}} := \{[x] \in \mathbb{P}(V) : W \text{ is the smallest } \mathbb{F}_{q^{n_0 r}}\text{-rational subspace of } V \text{ containing } x\}.$$

Note that $\Omega_{W,q^{n_0 r}} \subset \mathbb{P}(V)$ is isomorphic to the Drinfeld upper half-space for W with respect to $\mathbb{F}_{q^{n_0 r}}$. For any divisor $r \mid n'$, define

$$\mathscr{S}_r := \bigcup_W \Omega_{W,q^{n_0 r}},$$

where the union ranges over all $\mathbb{F}_{q^{n_0 r}}$-rational subspaces W of dimension n'/r in V. The following lemma records some easy facts.

Lemma 4.4.1. *We have*

(i) $\mathscr{S}_1 = \Omega_{V,q^{n_0}}$ *and* $\mathscr{S}_{n'} = \mathbb{P}(V)(\mathbb{F}_{q^n})$.

(ii) *If* $r \mid r' \mid n'$ *and* W *is a* $\mathbb{F}_{q^{n_0 r}}$*-rational subspace of* V, *then* $\Omega_{W,q^{n_0 r'}} \subseteq \Omega_{W,q^{n_0 r}}$.

(iii) *If* $r \mid r' \mid n'$, *then* $\mathscr{S}_1 \cap \mathscr{S}_{r'} \subseteq \mathscr{S}_1 \cap \mathscr{S}_r$.

Note that $\mathscr{S}_1$ is the classical Deligne–Lusztig variety for $\mathbb{G}_1(\mathbb{F}_q) \cong \mathrm{GL}_{n'}(\mathbb{F}_{q^{n_0}})$ with respect to the nonsplit maximal torus $\mathbb{T}_1(\mathbb{F}_q) \cong \mathbb{F}_{q^n}^\times$ [DL76, Section 2.2] and the variety X_h when $h = 1$ is a $\mathbb{F}_{q^n}^\times$-cover of $\mathscr{S}_1$. Hence for any $h \geq 1$, we have a map

$$X_h \to X_1 \to \mathscr{S}_1.$$

Lemma 4.4.2. *For any divisor* $r \mid n'$, *the variety* $X_h^{(r)}$ *is the preimage of* $\mathscr{S}_1 \cap \mathscr{S}_r$ *under the composition map* $X_h \to X_1 \to \mathscr{S}_1$.

Proof. To prove this, we use the explicit description of X_h coming from Lemma 4.2.1:

$$X_h = \{\lambda(v) \in \mathbb{G}_h : v \in \mathscr{L}_h \text{ and } \sigma(\det \lambda(v)) = \det \lambda(v)\}.$$

By Definition 4.3.2, if $v \in \mathscr{L}_h$ is such that $\lambda(v) \in X_h^{(r)}$, then $\lambda(v)^{-1} F(\lambda(v)) \in \mathbb{U}_{h,r} \cap F\mathbb{U}_{h,r}^-$, which is equivalent to

$$F(\lambda(v)) = \lambda(v)A, \qquad \text{for some } A \in \mathbb{U}_{h,r} \cap F\mathbb{U}_{h,r}^-.$$

Note that $A = (a_{i,j})_{1 \leq i,j \leq n}$ has the property that

$$
\begin{aligned}
&a_{i,i} = 1, &&\text{for } i = 1, \ldots, n, \\
&a_{i,1} \in \mathbb{W}_h, &&\text{if } i \equiv 1 \bmod rn_0, \\
&a_{i,1} \in V\mathbb{W}_{h-1} \subset \mathbb{W}_h, &&\text{if } i \not\equiv 1 \bmod rn_0, \\
&a_{i,j} = 0 &&\text{otherwise.}
\end{aligned}
$$

The first column of $F(\lambda(v))$ is the vector $\sigma^n(v)$. Therefore (4.4) implies that

$$\sigma^n(v) = \sum_{i=1}^n a_{i,1}\lambda(v)_i = v + \sum_{i=2}^n a_{i,1}\lambda(v)_i,$$

where $\lambda(v)_i$ denotes the ith column of $\lambda(v)$. Recall from (4.6) that $\lambda(v)_{[ie_{\kappa,n}+1]_n} = \prod_{j=0}^{i-1} t_{[je_{\kappa,n}+1]}^{-1} \cdot (b\sigma)^i(v)$. If $[ie_{\kappa,n} + 1]_n \equiv 1$ modulo rn_0, then $i \equiv 0$ modulo rn_0. Therefore, if $\mathfrak{v}$ denotes the image of v in $\mathscr{L}_1$, we have (using (4.6)),

$$\sigma^n(\mathfrak{v}) \in \mathrm{span}\{\mathfrak{v}, \varpi^{-rk_0}(b\sigma)^{rn_0}(\mathfrak{v}), \varpi^{-2rk_0}(b\sigma)^{2rn_0}(\mathfrak{v}), \ldots, \varpi^{-(n'-1)rk_0}(b\sigma)^{(n'-1)rn_0}(\mathfrak{v})\}.$$

Since $\lambda(v) \in \mathbb{G}_h$, necessarily $\mathfrak{v}, \varpi^{-rk_0}(b\sigma)^{rn_0}(\mathfrak{v}), \ldots, \varpi^{-(n'-1)rk_0}(b\sigma)^{(n'-1)rn_0}(\mathfrak{v})$ are linearly independent and therefore span a n'/r-dimensional subspace of $\mathscr{L}_1$. This exactly means that $\mathfrak{v} \in \mathscr{S}_1 \cap \mathscr{S}_r$, so the proof is complete. $\square$

Remark 4.4.3. By Lemma 4.4.2, we see that for GL_n and its inner forms, the Drinfeld stratification of X_h is induced by considering intermediate Drinfeld upper half-spaces of smaller dimension embedding in $\mathbb{P}_{\mathbb{F}_{q^{n_0}}}^{n'}$.

4.5. **The Drinfeld stratification of** $X_h(b,w)$**.** In this section, we consider the varieties $X_h(b,w)$ in the special case

$$b = g_0 b_{\mathrm{cox}} \sigma(g_0)^{-1} \text{ for some } g_0 \in G_{x,0}(\mathcal{O}_{\breve{k}}), \qquad \text{and} \qquad w = b_{\mathrm{cox}}.$$

For any such b, recall from Lemmas 3.2.2 and 3.2.3 that

$$X_h = X_h(b_{\mathrm{cox}}, b_{\mathrm{cox}}) \cong X_h(b, b_{\mathrm{cox}}), \tag{4.15}$$

where the second isomorphism is given by $x \mapsto \bar{g}_0 x$, where $\bar{g}_0$ is the image of g_0 in $\mathbb{G}_h(\overline{\mathbb{F}}_q)$. Therefore the Drinfeld stratification $\{X_h^{(r)}\}$ of X_h gives rise to a stratification $\{X_h(b, b_{\mathrm{cox}})^{(r)}\}$ for $X_h(b, b_{\mathrm{cox}})$. The proof of Lemma 4.5.3 shows that if $\sigma^n(\bar{g}_0) = \bar{g}_0$, then the Drinfeld stratification of $X_h(b, b_{\mathrm{cox}})$ does not depend on the choice of g_0.

Definition 4.5.1. Let $b = g_0 b_{\mathrm{cox}} \sigma(g_0)^{-1} \in G(\breve{k})$ for some $g_0 \in G_{x,0}(\mathcal{O}_{\breve{k}})$. To each $v \in \mathscr{L}_h$, define

$$g_b(v) := \left(v_1 \,\middle|\, v_2 \,\middle|\, v_3 \,\middle|\, \cdots \,\middle|\, v_n \right)$$
$$\text{where } v_i := \varpi^{\lfloor (i-1)k_0/n_0 \rfloor} \cdot (b\sigma)^{i-1}(v) \text{ for } 1 \le i \le n-1,$$

where we abuse notation by writing $\varpi^{\lfloor (i-1)k_0/n_0 \rfloor} \cdot (b\sigma)^{i-1}$ for the map $\mathscr{L}_h \to \mathscr{L}_h$ which takes v to the image $\varpi^{\lfloor (i-1)k_0/n_0 \rfloor} \cdot (b\sigma)^{i-1}(\tilde{v})$ in the subquotient $\mathscr{L}_h$ of $\breve{k}^{\oplus n}$, where $\tilde{v}$ is any lift of v in $\mathscr{L} \subset \breve{k}^{\oplus n}$.

Lemma 4.5.2. If $b = g_0 b_{\mathrm{cox}} \sigma(g_0)^{-1}$ for some $g_0 \in G_{x,0}(\mathcal{O}_{\breve{k}})$, then

$$X_h(b, b_{\mathrm{cox}}) \cong \left\{ v \in \mathscr{L}_h : \sigma(\det g_b(v)) = \frac{\det b_{\mathrm{cox}}}{\det b} \cdot \det g_b(v) \in \mathbb{W}_h^\times \right\}.$$

Proof. First note that one can obtain $g_{b_{\mathrm{cox}}}(v)$ from $\lambda(v)$ by permuting columns. In particular,

$$X_h(b_{\mathrm{cox}}, b_{\mathrm{cox}}) = X_h \cong \{ v \in \mathscr{L}_h : \sigma(\det g_{b_{\mathrm{cox}}}(v)) = \det g_{b_{\mathrm{cox}}}(v) \in \mathbb{W}_h^\times \}.$$

Since $X_h(b_{\mathrm{cox}}, b_{\mathrm{cox}}) \cong X_h(b, b_{\mathrm{cox}})$ is given by $x \mapsto \bar{g}_0 x$ where $\bar{g}_0$ denotes the image of g_0 in $\mathbb{G}_h(\overline{\mathbb{F}}_q)$, we have that $X_h(b, b_{\mathrm{cox}})$ is isomorphic to the set of $\bar{g}_0 \cdot g_{b_{\mathrm{cox}}}(v)$ where $v \in \mathscr{L}_h$ satisfies the above criterion. By direct computation,

$$\bar{g}_0 \cdot g_{b_{\mathrm{cox}}}(v) = g_b(\bar{g}_0 \cdot v),$$

and hence if $\sigma(\det g_{b_{\mathrm{cox}}}(v)) = \det g_{b_{\mathrm{cox}}}(v)$, then

$$\sigma(\det g_b(\bar{g}_0 \cdot v)) = \sigma(\overline{\det} g_0) \cdot \sigma(\det g_{b_{\mathrm{cox}}}(v)) = \sigma(\det \bar{g}_0) \cdot \det g_{b_{\mathrm{cox}}}(v)$$

$$= \frac{\sigma(\det \bar{g}_0)}{\det \bar{g}_0} \cdot \det g_b(\bar{g}_0 \cdot v) = \frac{\det b_{\mathrm{cox}}}{\det b} \cdot \det g_b(\bar{g}_0 \cdot v). \qquad \square$$

Lemma 4.5.3. Let $b = g_0 b_{\mathrm{cox}} \sigma(g_0)^{-1}$ for some $g_0 \in G_{x,0}(\mathcal{O}_{\breve{k}})$ and assume that the image $\bar{g}_0 \in \mathbb{G}_h(\overline{\mathbb{F}}_q)$ of g_0 has the property that $\sigma^n(\bar{g}_0) = \bar{g}_0$. Let $r \mid n'$ be any divisor. For $v \in \mathscr{L}_h$, let $\mathfrak{v}$ denote its image in $\mathscr{L}_1$. Then

$$X_h(b, b_{\mathrm{cox}})^{(r)} \cong \left\{ v \in \mathscr{L}_h : \begin{array}{c} \sigma(\det g_{b_{\mathrm{sp}}}(v)) = \dfrac{\det b_{\mathrm{cox}}}{\det b} \cdot \det g_b(v) \in \mathbb{W}_h^\times \\ \sigma^n(\mathfrak{v}) \in \mathrm{span}\{\varpi^{-ik_0 r}(b\sigma)^{irn_0}(\mathfrak{v}) : 0 \le i \le n'-1\} \end{array} \right\}.$$

In particular, the Drinfeld stratification of $X_h(b, b_{\mathrm{cox}})$ does not depend on the choice of g_0.

Proof. Recall that

$$X_h(b_{\mathrm{cox}}, b_{\mathrm{cox}})^{(r)} \cong \left\{ v \in \mathscr{L}_h : \begin{array}{c} \sigma(\det g_{b_{\mathrm{cox}}}(v)) = \det g_{b_{\mathrm{cox}}}(v) \in \mathbb{W}_h^\times \\ \sigma^n(\mathfrak{v}) \in \mathrm{span}\{\varpi^{-ik_0 r}(b_{\mathrm{cox}}\sigma)^{irn_0}(\mathfrak{v}) : 0 \le i \le n'-1\} \end{array} \right\}.$$

By definition, every element in $X_h(b, b_{\text{cox}})^{(r)}$ is of the form $\overline{g}_0 g_{b_{\text{cox}}}(v)$ for some $v \in \mathscr{L}_h$ satisfying the above criteria. Since $\overline{g}_0 g_{b_{\text{cox}}}(v) = g_b(\overline{g}_0 v)$ and since $\sigma^n(\overline{g}_0) = \overline{g}_0$, we have

$$\overline{g}_0 \sigma^n(\mathfrak{v}) \in \text{span}\{\overline{g}_0 \varpi^{-ik_0 r}(b_{\text{cox}}\sigma)^{irn_0}(\mathfrak{v}) : 0 \leq i \leq n' - 1\}.$$

But now $\overline{g}_0 \varpi^{-ik_0 r}(b_{\text{cox}}\sigma)^{irn_0}(\mathfrak{v}) = \varpi^{-ik_0 r}(b\sigma)^{irn_0}(\mathfrak{v})$ and therefore the desired conclusion follows. $\qquad\square$

Remark 4.5.4. In Appendix A, we will work directly with a particular b called the *special representative* in [CI18] (see Definition A.1.1 of the present paper). The special representative satisfies the hypotheses of Lemma 4.5.3.

5. Torus eigenspaces in the cohomology

We prove an irreducibility result for torus eigenspaces in the alternating sum of the cohomology of $X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1$.

5.1. **Howe factorizations.** Let $\mathscr{T}_{n,h}$ denote the set of characters $\theta \colon \mathbb{W}_h^\times(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$. Recall that if $h \geq 2$, we have natural surjections $\text{pr} \colon \mathbb{W}_h^\times \to \mathbb{W}_{h-1}^\times$ and injections $\mathbb{G}_a \to \mathbb{W}_h^\times$ given by $x \mapsto [1, 0, \ldots, 0, x]$. Furthermore, for any subfield $F \subset L$, the norm map $L^\times \to F^\times$ induces a map $\text{Nm} \colon \mathbb{W}_h^\times(k_L) \to \mathbb{W}_h^\times(k_F)$. These maps induce

$$\text{pr}^* \colon \mathscr{T}_{n,h'} \to \mathscr{T}_{n,h}, \qquad\qquad \text{for } h' < h,$$
$$\text{Nm}^* \colon \mathscr{T}_{m,h} \to \mathscr{T}_{n,h}, \qquad\qquad \text{for } m \mid n.$$

First consider the setting $h \geq 2$. By pulling back along $\mathbb{G}_a \to \mathbb{W}_h^\times, x \mapsto [1, 0, \ldots, 0, x]$, we may restrict characters of $\mathbb{W}_h^\times(\mathbb{F}_{q^n})$ to characters of $\mathbb{F}_{q^n}$. We say that $\theta \in \mathscr{T}_{n,h}$ is *primitive* if $\theta|_{\mathbb{F}_{q^n}}$ has trivial stabilizer in $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. If $h = 1$, then $\theta \in \mathscr{T}_{n,h}$ is a character $\theta \colon \mathbb{F}_{q^n}^\times \to \overline{\mathbb{Q}}_\ell^\times$, and we say it is *primitive* if θ has trivial stabilizer in $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. For any $h \geq 1$, we write $\mathscr{T}_{n,h}^0 \subset \mathscr{T}_{n,h}$ to denote the subset of primitive characters.

We can decompose $\theta \in \mathscr{T}_{n,h}$ into primitive components in the sense of Howe [How77, Corollary after Lemma 11].

Definition 5.1.1. A *Howe factorization* of a character $\theta \in \mathscr{T}_{n,h}$ is a decomposition

$$\theta = \prod_{i=1}^d \theta_i, \qquad \text{where } \theta_i = \text{pr}^* \text{Nm}^* \theta_i^0 \text{ and } \theta_i^0 \in \mathscr{T}_{m_i, h_i}^0,$$

such that $m_i < m_{i+1}$, $m_i \mid m_{i+1}$, and $h_i > h_{i+1}$. It is automatic that $m_i \leq n$ and $h \geq h_i$. For any integer $0 \leq t \leq d$, set θ_0 to be the trivial character and define

$$\theta_{\geq t} := \prod_{i=t}^d \theta_i \in \mathscr{T}_{n, h_t}.$$

Observe that the choice of θ_i in a Howe factorization $\theta = \prod_{i=1}^r \theta_i$ is not unique, but the m_i and h_i only depend on θ. Hence the Howe factorization attaches to each character $\theta \in \mathscr{T}_{n,h}$ a pair of well-defined sequences

$$1 =: m_0 \leq m_1 < m_2 < \cdots < m_d \leq m_{d+1} := n$$
$$h =: h_0 \geq h_1 > h_2 > \cdots > h_d \geq h_{d+1} := 1$$

satisfying the divisibility $m_i \mid m_{i+1}$ for $0 \leq i \leq d$.

Example 5.1.2. We give some examples of the sequences associated to characters $\theta \in \mathscr{T}_{n,h}$.

(a) If θ is the trivial character, then $d = 1$ and the associated sequences are

$$\{m_0, m_1, m_2\} = \{1, 1, n\}, \qquad \{h_0, h_1, h_2\} = \{h, 1, 1\},$$

where we note that $\mathscr{T}_{1,1} = \mathscr{T}_{1,1}^0$ since any character of $\mathbb{F}_q^\times$ has trivial $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-stabilizer.

(b) Say $h \geq h'$. We say that θ is a primitive character of level $h' \geq 2$ if $\theta|_{U_L^{h'}} = 1$ and $\theta|_{U_L^{h'-1}/U_L^{h'}}$ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer. Then $d = 1$ and the associated sequences are

$$\{m_0, m_1, m_2\} = \{1, n, n\}, \qquad \{h_0, h_1, h_2\} = \{h, h', 1\}.$$

In the division algebra setting, this case is studied in [Cha16, Cha18]. For arbitrary inner forms of GL_n over K, we considered *minimal admissible* θ, which are exactly the characters $\theta \in \mathscr{T}_{n,h}$ which are either primitive or have $d = 2$ with associated sequences

$$\{m_0, m_1, m_2, m_3\} = \{1, 1, n, n\}, \qquad \{h_0, h_1, h_2, h_3\} = \{h, h_1, h_2, 1\}.$$

This is a very slight generalization over the primitive case.

(c) Say $h \geq 2$. If $\theta|_{U_L^2} = 1$ and the stabilizer of $\theta|_{U_L^1/U_L^2}$ in $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_{q^m})$, then $d = 1$ and the associated sequences are

$$\{m_0, m_1, m_2\} = \{1, m, n\}, \qquad \{h_0, h_1, h_2\} = \{h, 2, 1\}.$$

In the division algebra setting, the case $h = 2$ is studied in [Boy12, BW16].

(d) Say $h \geq 1$. If $\theta|_{U_L^1} = 1$ and the stabilizer of $\theta \colon \mathbb{F}_{q^n}^\times \to \overline{\mathbb{Q}}_\ell^\times$ is $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_{q^m})$, then $d = 1$ and the associated sequences are

$$\{m_0, m_1, m_2\} = \{1, m, n\}, \qquad \{h_0, h_1, h_2\} = \{h, 1, 1\}.$$

This is the so-called "depth zero" case.

5.2. **Irreducibility.** Recall that the intersection $X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1$ has an action by the subgroup $\mathbb{L}_h^{(r)}(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q) \subset \mathbb{G}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$. In this section, we study the irreducibility of the virtual $\mathbb{L}_h^{(r)}(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q)$-representation $H_c^*(X_h \cap \mathbb{L}_h^{(r)}\mathbb{G}_h^1)[\theta]$, where $\theta \colon \mathbb{T}_h(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ is arbitrary.

We follow a technique of Lusztig which has appeared in the literature in many incarnations, the closest analogues being [Lus04, Sta09, CI19a]. In these works, the strategy is to translate the problem of calculating an inner product between two representations to calculating the cohomology of a third variety Σ. This is done by first writing $\Sigma = \Sigma' \sqcup \Sigma''$, proving the cohomology of Σ'' gives the expected outcome, and then putting a lot of work into showing that the cohomology of Σ' does not contribute. In the three works cited, one can only prove the vanishing of (certain eigenspaces of) the Euler characteristic of Σ' under a strong *regularity* condition on the characters θ, θ'. The key new idea here is adapted from [CI19b, Section 3.2], which allows us to relax this regularity assumption by working directly with Σ throughout the proof. We give only a sketch of the proof of Theorem 5.2.1 here, as the proof of [CI19b, Theorem 3.1] is very similar.

Theorem 5.2.1. *Let $\theta, \theta' \colon \mathbb{T}_h(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ be any two characters. Then*

$$\left\langle H_c^*(X_h \cap \mathbb{L}_h^{(r)}\mathbb{G}_h^1)[\theta], H_c^*(X_h \cap \mathbb{L}_h^{(r)}\mathbb{G}_h^1)[\theta'] \right\rangle_{\mathbb{L}_h^{(r)}(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q)} = \#\{w \in W_{\mathbb{L}_h^{(r)}}^F : \theta' = \theta \circ \mathrm{Ad}(w)\},$$

where $W_{\mathbb{L}_h^{(r)}}^F = N_{\mathbb{L}_h^{(r)}(\mathbb{F}_q)}(\mathbb{T}_h(\mathbb{F}_q))/\mathbb{T}_h(\mathbb{F}_q)$.

Since $W^F_{\mathbb{L}^{(r)}_h} \cong \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0 r}})$, we obtain the following theorem as a direct corollary of Theorem 5.2.1.

Corollary 5.2.2. *Let $\theta\colon \mathbb{T}_h(\mathbb{F}_q) \cong \mathbb{W}^\times_h(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}^\times_\ell$ be any character. Then the virtual $\mathbb{L}^{(r)}_h(\mathbb{F}_q)\mathbb{G}^1_h(\mathbb{F}_q)$-representation $H^*_c(X_h \cap \mathbb{L}^{(r)}_h\mathbb{G}^1_h)[\theta]$ is (up to sign) irreducible if and only if θ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0 r}})$-stabilizer.*

In the special case that $r = n'$, we have $\mathbb{L}^{(n')}_h = \mathbb{T}_h$ and using Lemma 4.2.1 and Definition 4.3.2, we have that $S_h \cap \mathbb{T}_h\mathbb{G}^1_h$ is an affine fibration over

$$\{x \in \mathbb{T}_h\mathbb{G}^1_h : x^{-1}F(x) \in \mathbb{U}^1_h \cap F\mathbb{U}^{-,1}_h\}.$$

and that

$$X_h \cap \mathbb{T}_h\mathbb{G}^1_h = \bigsqcup_{t \in \mathbb{T}_h(\mathbb{F}_q)} t \cdot X^1_h, \qquad \text{where } X^1_h = X_h \cap \mathbb{G}^1_h.$$

Here we have

$$X^1_h = \{x \in \mathbb{G}^1_h : x^{-1}F(x) \in \mathbb{U}^1_h \cap F\mathbb{U}^{-,1}_h\}. \tag{5.1}$$

Corollary 5.2.3. *Let $\chi\colon \mathbb{T}^1_h(\mathbb{F}_q) \to \overline{\mathbb{Q}}^\times_\ell$ be any character. Then $H^*_c(X^1_h, \overline{\mathbb{Q}}_\ell)[\chi]$ is an irreducible representation of $\mathbb{G}^1_h(\mathbb{F}_q)$. Moreover, if χ, χ' are any two characters of $\mathbb{T}^1_h(\mathbb{F}_q)$, then $H^*_c(X^1_h, \overline{\mathbb{Q}}_\ell)[\chi] \cong H^*_c(X^1_h, \overline{\mathbb{Q}}_\ell)[\chi']$ if and only if $\chi = \chi'$.*

Corollary 5.2.3 follows from Corollary 5.2.2 (by arguing the relationship between the cohomology of X^1_h and the cohomology of $X_h \cap \mathbb{T}_h\mathbb{G}^1_h$), but one can give an alternate proof using [Cha19, Section 6.1], which is based on [Lus79]. We do this in Section 5.2.2.

Remark 5.2.4. Recall that specializing Lemma 3.3.4 yields that

$$H^*_c(X^{(r)}_h, \overline{\mathbb{Q}}_\ell)[\theta] \cong \mathrm{Ind}^{\mathbb{G}_h(\mathbb{F}_q)}_{\mathbb{L}^{(r)}_h(\mathbb{F}_q)\mathbb{G}^1_h(\mathbb{F}_q)} \left(H^*_c(X_h \cap \mathbb{L}^{(r)}_h\mathbb{G}^1_h, \overline{\mathbb{Q}}_\ell)[\theta]\right).$$

We note that one needs a separate argument to study the irreducibility of $H^*_c(X^{(r)}_h, \overline{\mathbb{Q}}_\ell)[\theta]$. In the case that $r = n'$, this is done in [CI19b, Theorem 4.1(b)].

5.2.1. *Proof of Theorem 5.2.1.* Recall that by definition

$$S_h \cap \mathbb{L}^{(r)}_h\mathbb{G}^1_h = \{g \in \mathbb{L}^{(r)}_h\mathbb{G}^1_h : g^{-1}F(g) \in \mathbb{U}_{h,r}\}, \qquad \text{where } \mathbb{U}_{h,r} = \mathbb{L}^{(r)}_h\mathbb{U}^1_h \cap \mathbb{U}_h.$$

Consider the variety

$$\Sigma^{(r)} = \{(x, x', y) \in F(\mathbb{U}_{h,r}) \times F(\mathbb{U}_{h,r}) \times \mathbb{L}^{(r)}_h\mathbb{G}^1_h : xF(y) = yx'\}$$

endowed with the $\mathbb{T}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$-action given by $(t, t')\colon (x, x', y) \mapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$. Then we have an isomorphism

$$\mathbb{L}^{(r)}_h(\mathbb{F}_q)\mathbb{G}^1_h(\mathbb{F}_q)\backslash \left((S_h \cap \mathbb{L}^{(r)}_h\mathbb{G}^1_h) \times (S_h \cap \mathbb{L}^{(r)}_h\mathbb{G}^1_h)\right) \to \Sigma^{(r)},$$

$$(g, g') \mapsto (g^{-1}F(g), g'^{-1}F(g'), g^{-1}g'),$$

equivariant with respect to $\mathbb{T}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$. To prove Theorem 5.2.1, we need to establish

$$\sum_i (-1)^i \dim H^i_c(\Sigma^{(r)}, \overline{\mathbb{Q}}_\ell)_{\theta,\theta'} = \#\{w \in W^F_{\mathbb{L}^{(r)}_h} : \theta' = \theta \circ \mathrm{Ad}(w)\}. \tag{5.2}$$

The Bruhat decomposition of the reductive quotient $\mathbb{G}_1$ lifts to a decomposition $\mathbb{G}_h = \bigsqcup_{w \in W_{\mathbb{G}_h}} \mathbb{G}_{h,w}$, where $\mathbb{G}_{h,w} = \mathbb{U}_h\mathbb{T}_h\dot{w}\mathbb{K}^1_h\mathbb{U}_h$ and $\mathbb{K}^1_h = (\mathbb{U}^-_h)^1 \cap \dot{w}^{-1}\mathbb{U}^{-,1}_h\dot{w}$ [CI19a, Lemma 8.6]. This induces the decomposition

$$\mathbb{L}^{(r)}_h\mathbb{G}^1_h = \bigsqcup_{w \in W^F_{\mathbb{L}^{(r)}_h}} \mathbb{G}^{(r)}_{h,w}, \qquad \text{where } \mathbb{G}^{(r)}_{h,w} = \mathbb{G}_{h,w} \cap \mathbb{L}^{(r)}_h\mathbb{G}^1_h.$$

and also the locally closed decomposition

$$\Sigma^{(r)} = \bigsqcup_{w \in W_{\mathcal{O}}} \Sigma_w^{(r)}, \qquad \text{where } \Sigma_w^{(r)} = \Sigma \cap (F(\mathbb{U}_{h,r}) \times F(\mathbb{U}_{h,r}) \times \mathbb{G}_{h,w}^{(r)}).$$

We will calculate (5.2) by analyzing the cohomology of

$$\widehat{\Sigma}_w^{(r)} = \{(x, x', y_1, \tau, z, y_2) \in F(\mathbb{U}_{h,r}) \times F(\mathbb{U}_{h,r}) \times \mathbb{U}_{h,r} \times \mathbb{T}_h \times \mathbb{K}_h^1 \times \mathbb{U}_h :$$
$$xF(y_1\tau\dot{w}zy_2) = y_1\tau\dot{w}zy_2x'\}.$$

Since $\widehat{\Sigma}_w^{(r)} \to \Sigma_w^{(r)}, (x, x', y_1, \tau, z, y_2) \mapsto (x, x', y_1\tau z y_2)$ is a locally trivial fibration, showing (5.2) is equivalent to showing

$$\sum_i (-1)^i \dim H_c^i(\widehat{\Sigma}_w^{(r)}, \overline{\mathbb{Q}}_\ell)_{\theta,\theta'} = \begin{cases} 1 & \text{if } w \in W_{\mathbb{L}_h^{(r)}}^F \text{ and } \theta' = \theta \circ \mathrm{Ad}(w), \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

As in [Lus04, 1.9], we can simplify the formulation of $\widehat{\Sigma}_w$ by replacing x by $xF(y_1)$ and replacing x' by $x'F(y_2)^{-1}$. We then obtain

$$\widehat{\Sigma}_w^{(r)} = \{(x, y_1, \tau, z, y_2) \in F\mathbb{U}_{h,r} \times \mathbb{U}_{h,r} \times \mathbb{T}_h \times \mathbb{K}_h^1 \times \mathbb{U}_{h,r} : xF(\tau\dot{w}z) \in y_1\tau\dot{w}zy_2F\mathbb{U}_{h,r}\}.$$

Lemma 5.2.5. *Assume that there exists some $2 \le i \le n$ which satisfies the string of inequalities $[\gamma\dot{w}\gamma^{-1}(i)] > [\gamma\dot{w}\gamma^{-1}(i-1)+1] > 1$. Then $\widehat{\Sigma}_w = \varnothing$.*

Proof. By the same argument as in [CI19b, Lemma 3.4], we may assume $h = 1$ and come to the statement that $\widehat{\Sigma}_w = \varnothing$ if there does not exist $(x, y_{12}, y_{21}, \tau) \in F\mathbb{U}_{1,r} \times (\mathbb{U}_{1,r} \cap F\mathbb{U}_{1,r}^-) \times (\mathbb{U}_{1,r} \cap F\mathbb{U}_{1,r}^-) \times \mathbb{T}_1$ such that

$$\dot{w}^{-1}\tau y_{12}xF(\dot{w}) \in y_{21}F(\mathbb{U}_1 \cap \mathbb{L}_1^{(r)}).$$

Therefore to prove the lemma, it is enough to analyze the intersection

$$\left[\dot{w}^{-1}(\mathbb{U}_{1,r} \cap F\mathbb{U}_{1,r}^-) \cdot F\mathbb{U}_{1,r}F(\dot{w})\right] \cap \left[(\mathbb{U}_{1,r} \cap F\mathbb{U}_{1,r}^-) \cdot F(\mathbb{U}_1 \cap \mathbb{L}_1^{(r)})\right].$$

By construction (see (4.2), (4.3), and write $F_0(g) = b_0\gamma t_{\kappa,n}\gamma^{-1}\sigma(g)t_{\kappa,n}^{-1}\gamma b_0\gamma^{-1})$, we have

$$\dot{w}^{-1}(\mathbb{T}_1 \cap (\mathbb{U}_{1,r} \cap F\mathbb{U}_{1,r}^-) \cdot F\mathbb{U}_{1,r})F(\dot{w}) \cap ((\mathbb{U}_{1,r} \cap F\mathbb{U}_{1,r}^-) \cdot F\mathbb{U}_{1,r})$$
$$= \gamma^{-1}(\gamma\dot{w}^{-1}\gamma^{-1})(\mathbb{T}_1 \cdot (\mathbb{U}_{\mathrm{low},1,r} \cap F_0\mathbb{U}_{\mathrm{up},1,r}) \cdot F_0\mathbb{U}_{\mathrm{low},1,r})F_0(\gamma\dot{w}^{-1}\gamma^{-1})\gamma$$
$$\cap \gamma^{-1}((\mathbb{U}_{\mathrm{low},1,r} \cap F_0\mathbb{U}_{\mathrm{up},1,r}) \cdot F_0\mathbb{U}_{\mathrm{low},1,r})\gamma.$$

Now the desired result holds by [CI19b, Lemma 3.5]. $\square$

The rest of the proof now proceeds exactly as in [CI19b, Section 3.3, 3.4], which we summarize now. By [CI19b, Lemma 3.5], if $1 \neq w \in W_{\mathbb{L}_h^{(r)}}$ is such that $\widehat{\Sigma}_w \neq \varnothing$, then $\mathbb{U}_h \cap \dot{w}^{-1}U_h\dot{w}$ is centralized by a subtorus of $\mathbb{T}_h$ which properly contains the center of $\mathbb{G}_h$. In particular, the group

$$H_w = \{(t, t') \in \mathbb{T}_h \times \mathbb{T}_h : \dot{w}^{-1}t^{-1}F(t)\dot{w} = t'^{-1}F(t') \text{ centralizes } \mathbb{K}_h = \mathbb{U}_{\cap}\dot{w}^{-1}\mathbb{U}_h\dot{w}\}$$

has the property that its image under the projections $\pi_1, \pi_2 : \mathbb{T}_h \times \mathbb{T}_h \to \mathbb{T}_1 \times \mathbb{T}_1 \to \mathbb{T}_1$ contains a rank-1 regular[1] torus. Crucially, H_w acts on $\widehat{\Sigma}_w^{(r)}$ via

$$(t, t') : (x, y_1, \tau, z, y_2) \mapsto (F(t)xF(t)^{-1}, F(t)y_1F(t)^{-1}, t\tau\dot{w}t'^{-1}\dot{w}^{-1}, t'zt'^{-1}, F(t')y_2F(t')^{-1}),$$

---

[1]We mean here that this torus is not contained in $\ker(\alpha)$ for any root α of $\mathbb{T}_1$ the reductive group $\mathbb{G}_1$. See [CI19b, Lemma 3.7].

and this action extends the action of $\mathbb{T}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$. Then $H_c^*(\widehat{\Sigma}_w, \overline{\mathbb{Q}}_\ell) = H_c^*(\widehat{\Sigma}_w^{H_{w,\mathrm{red}}^0}, \overline{\mathbb{Q}}_\ell)$ and using [CI19b, Lemma 3.6], we can calculate:

$$\widehat{\Sigma}_w^{H_{w,\mathrm{red}}^0} = \begin{cases} (\mathbb{T}_h \dot{w})^F & \text{if } F(\dot{w}) = \dot{w}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Now (5.3) holds for all $w \neq 1$. To obtain (5.3) for $w = 1$, we may apply [CI19b, Section 3.4] directly. We have now finished the proof of Theorem 5.2.1

5.2.2. *Proof of Corollary 5.2.3.* Consider

$$\Sigma^1 = \{(x, x', y) \in (\mathbb{U}_h^1 \cap F\mathbb{U}_h^{-,1}) \times (\mathbb{U}_h^1 \cap F\mathbb{U}_h^{-,1}) \times \mathbb{G}_h^1 : xF(y) = yx'\}.$$

Then we have an isomorphism

$$\mathbb{G}_h(\mathbb{F}_q) \backslash \big((X_h \cap \mathbb{T}_h \mathbb{G}_h^1) \times (X_h \cap \mathbb{T}_h \mathbb{G}_h^1)\big) \to \Sigma^1, \qquad (g, g') \mapsto (g^{-1}F(g), g'^{-1}F(g'), g^{-1}g').$$

Since $\mathbb{G}_h^1$ has an Iwahori factorization, any $y \in \mathbb{G}_h^1$ can be written uniquely in the form

$$y = y_1' y_2' y_1'' y_2'', \qquad y_1' \in \mathbb{U}_h^1 \cap F^{-1}(\mathbb{U}_h^1), \qquad\qquad y_2' \in \mathbb{U}_h^1 \cap F^{-1}(\mathbb{U}_h^{-,1}),$$
$$y_1'' \in \mathbb{T}_h \cdot (\mathbb{U}_h^{-,1} \cap F^{-1}\mathbb{U}_h^{-,1}), \qquad y_2'' \in \mathbb{U}_h^{-,1} \cap F^{-1}\mathbb{U}_h^1.$$

Then our definition equation becomes

$$xF(y_1' y_2' y_1'' y_2'') = y_1' y_2' y_1'' y_2'' x'.$$

By (4.4), every element of $\mathbb{U}_h$ can be written uniquely in the form $y_1'-1xF(y_1')$. We also have $F(y_2''), x' \in \mathbb{U}_h^1 \cap F\mathbb{U}_h^{-,1}$ and we can replace x' by $x'F(y_2'')^{-1}$. Therefore Σ^1 is the set of tuples $(x', y_2', y_1'', y_2'') \in (\mathbb{U}_h^1 \cap F\mathbb{U}_h^{-,1}) \times (\mathbb{U}_h^1 \cap F^{-1}\mathbb{U}_h^{-,1}) \times (\mathbb{T}_h \cdot (\mathbb{U}_h^{-,1} \cap F^{-1}\mathbb{U}_h^{-,1})) \times (\mathbb{U}_h^{-,1} \cap F^{-1}\mathbb{U}_h^1)$ which satisfy

$$y_1'' y_2'' x' \in y_2'^{-1} \mathbb{U}_h F(y_2') F(y_1'') = \mathbb{U}_h F(y_2') F(y_1'').$$

Now consider the subgroup

$$H := \{(t, t') \in \mathbb{T}_h \times \mathbb{T}_h : t^{-1}F(t) = t'^{-1}F(t') \text{ centralizes } \mathbb{T}_h \cdot (\mathbb{U}_h^{-,1} \cap F^{-1}\mathbb{U}_h^{-,1})\}.$$

It is a straightforward check that for any $(t, t') \in H$, the map

$$(x', y_2', y_1'', y_2'') \mapsto (F(t')^{-1}x'F(t'), t^{-1}y_2't, t^{-1}y_1''t', F(t')^{-1}y_2''F(t'))$$

defines an action of H on Σ^1. By explicit calculation, one can check that H contains an algebraic torus $\mathcal{T}$ over $\overline{\mathbb{F}}_q$ and that the fixed points of Σ^1 under $\mathcal{T}$ is equal to $\mathbb{T}_h^1(\mathbb{F}_q)$. We therefore have

$$\dim H_c^*(\Sigma^1, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'} = \begin{cases} 1 & \text{if } \chi = \chi', \\ 0 & \text{otherwise,} \end{cases}$$

and this completes the proof.

5.3. **Very regular elements.** Recall that we say that an element $g \in \mathbb{T}_h(\mathbb{F}_q) \cong \mathbb{W}_h(\mathbb{F}_{q^n})^\times$ is *very regular* if its image in $\mathbb{F}_{q^n}^\times$ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer.

Proposition 5.3.1. *Let* $\theta \colon \mathbb{T}_h(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ *be any character. If* $g \in \mathbb{T}_h(\mathbb{F}_q) \subset \mathbb{L}_h^{(r)}(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q)$ *is a very regular element, then*

$$\mathrm{Tr}(g; H_c^*(X_h \cap \mathbb{L}_h^{(r)}\mathbb{G}_h^1)[\theta]) = \sum_{\gamma \in \mathrm{Gal}(L/k)[n'/r]} \theta^\gamma(x),$$

where $\mathrm{Gal}(L/k)[n'/r]$ *is the unique order-n'/r subgroup of* $\mathrm{Gal}(L/k)$.

Proof. Let $g \in \mathbb{T}_h(\mathbb{F}_q)$ be a very regular element and let $t \in \mathbb{T}_h(\mathbb{F}_q)$ be any element. Since the action of (g, t) on $X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1$ is a finite-order automorphism of a separated, finite-type scheme over $\mathbb{F}_{q^n}$, by the Deligne–Lusztig fixed point formula,

$$\mathrm{Tr}\left((g, t)^*; H_c^*(X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1)[\theta]\right) = \mathrm{Tr}\left((g_u, t_u)^*; H_c^*((X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1)^{(g_s, t_s)})[\theta]\right),$$

where $g = g_s g_u$ and $t = t_s t_u$ are decompositions such that g_s, t_s is a power of g, t of p-power order and g_u, t_u is a power of g, t of prime-to-p order.

Recall from Section 4.2 that every element x of $X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1$ is a matrix that is uniquely determined by its first column $(x_1, x_2, \ldots, x_n)$. Furthermore, we have an isomorphism

$$\mathbb{W}_h(\mathbb{F}_{q^n})^\times \to \mathbb{T}_h(\mathbb{F}_q), \qquad t \mapsto \mathrm{diag}(t, \sigma^l(t), \sigma^{2l}(t), \ldots, \sigma^{(n-1)l}(t)).$$

Under this identification, for $g, t \in \mathbb{T}_h(\mathbb{F}_q)$, the element $gxt \in X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1$ corresponds to the vector $(gtx_1, \sigma^l(g)tx_2, \sigma^{2l}(g)tx_3, \ldots, \sigma^{(n-1)l}(g)tx_n)$. In particular, we see that if $x \in (X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1)^{(g,t)}$, then (for any $i = 1, \ldots, n$) $x_i \neq 0$ implies $t = \sigma^{(i-1)l}(g)^{-1}$. Using the assumption that g is very regular and therefore g_s has trivial $\mathrm{Gal}(L/k)$-stabilizer, this implies that $(X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1)^{(g,t)}$ exactly consists of elements corresponding to vectors with a single nonzero entry x_i. Now, if $i \not\equiv 1$ modulo n_0, then the corresponding x cannot lie in X_h as then $\det(x) \notin \mathbb{W}_h(\overline{\mathbb{F}}_q)^\times$. On the other hand, if $i \equiv 1$ modulo n_0 and $i \not\equiv 1$ modulo $n_0 r$, then the corresponding x cannot lie in $\mathbb{L}_h^{(r)} \mathbb{G}_h^1$. If $x \in X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1$ corresponds to $(0, \ldots, 0, x_i, 0, \ldots, 0)$ for some $i \equiv 1$ modulo $n_0 r$, then x_i can be any element of $\mathbb{W}_h^\times(\mathbb{F}_{q^n})$. Hence:

$$(X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1)^{(g_s, t_s)} = \begin{cases} b_0^i \mathbb{T}_h(\mathbb{F}_q) & \text{if } t = \sigma^{(i-1)l}(g)^{-1} \text{ for some } i \equiv 1 \bmod n_0 r, \\ \varnothing & \text{otherwise.} \end{cases}$$

Furthermore, for $g_u, t_u \in \mathbb{T}_h(\mathbb{F}_q)$ and $b_0^i x \in (X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1)^{(g_s, t_s)}$,

$$g_u \cdot b_0^i x \cdot t_u = b_0^i (b_0^{-i} g_u b_0^i) x t_u = b_0^i (\sigma^{(i-1)l}(g_u) x t_u).$$

We are now ready to put all the above together. We have

$\mathrm{Tr}(g; H_c^*(X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1)[\theta])$

$$= \frac{1}{\#\mathbb{T}_h(\mathbb{F}_q)} \sum_{t \in \mathbb{T}_h(\mathbb{F}_q)} \theta(t)^{-1} \mathrm{Tr}((g, t); H_c^*(X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1))$$

$$= \frac{1}{\#\mathbb{T}_h(\mathbb{F}_q)} \sum_{t \in \mathbb{T}_h(\mathbb{F}_q)} \theta(t)^{-1} \mathrm{Tr}((g_u, t_u); H_c^*((X_h \cap \mathbb{L}_h^{(r)} \mathbb{G}_h^1)^{(g_s, t_s)}))$$

$$= \frac{1}{\#\mathbb{T}_h(\mathbb{F}_q)} \sum_{\substack{1 \leq i \leq n \\ i \equiv 1 \,(\mathrm{mod}\, n_0 r)}} \theta(\sigma^{(i-1)l}(g_s)) \sum_{t_u \in \mathbb{T}_h^1(\mathbb{F}_q)} \theta(t_u)^{-1} \mathrm{Tr}((g_u, t_u); H_c^*(b_0^i \mathbb{T}_h(\mathbb{F}_q)))$$

$$= \frac{1}{\#\mathbb{T}_h(\mathbb{F}_q)} \sum_{\substack{1 \leq i \leq n \\ i \equiv 1 \,(\mathrm{mod}\, n_0 r)}} \theta(\sigma^{(i-1)l}(g_s)) \sum_{t_u \in \mathbb{T}_h^1(\mathbb{F}_q)} \theta(t_u)^{-1} \sum_{\theta': \mathbb{T}_h(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell} \theta'(\sigma^{(i-1)l}(g_u)) \theta'(t_u)$$

$$= \sum_{\substack{1 \leq i \leq n \\ i \equiv 1 \,(\mathrm{mod}\, n_0 r)}} \theta(\sigma^{(i-1)l}(g_s)) \theta(\sigma^{(i-1)l}(g_u)) = \sum_{\gamma \in \mathrm{Gal}(L/k)[n'/r]} \theta^\gamma(g). \qquad \square$$

6. The closed stratum is a maximal variety

Recall that $X_h^{(r)}$ is the closure of the rth Drinfeld stratum and that the unique closed Drinfeld stratum is the n'th Drinfeld stratum

$$X_h^{(n')} := \{x \in \mathbb{G}_h : x^{-1}\sigma(x) \in \mathbb{U}_r^1\}.$$

Recall that $X_h^{(n')}$ is a finite disjoint union of copies of $X_h^1 := X_h^{(n')} \cap \mathbb{G}_h^1$:

$$X_h^{(n')} = \bigsqcup_{g \in \mathbb{G}_1(\mathbb{F}_q)} [g] \cdot X_h^1,$$

where $[g]$ denotes a coset representative in $\mathbb{G}_h(\mathbb{F}_q)$ for $g \in \mathbb{G}_1(\mathbb{F}_q) = \mathbb{G}_h(\mathbb{F}_q)/\mathbb{G}_h^1(\mathbb{F}_q)$. For any character $\theta \colon \mathbb{T}_h(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$, we have an isomorphism of $\mathbb{G}_h(\mathbb{F}_q)$-representations

$$H_c^i(X_h^{(n')}, \overline{\mathbb{Q}}_\ell)[\theta] \cong \mathrm{Ind}_{\mathbb{T}_h(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h(\mathbb{F}_q)} \left(H_c^i(X_h^{(n')} \cap \mathbb{T}_h\mathbb{G}_h^1, \overline{\mathbb{Q}}_\ell)[\theta] \right), \qquad \text{for all } i \geq 0.$$

Let $\chi := \theta|_{\mathbb{T}_h^1(\mathbb{F}_q)}$. As $\mathbb{G}_h^1(\mathbb{F}_q)$-representations,

$$H_c^i\left(X_h^{(n')} \cap \mathbb{T}_h\mathbb{G}_h^1, \overline{\mathbb{Q}}_\ell\right)[\theta] \cong H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi], \qquad \text{for all } i \geq 0.$$

The subvariety $X_h^1 \subset X_h$ is stable under the action of $\Gamma_h := \{(\alpha, \alpha^{-1}) : \alpha \in \mathbb{T}_h(\mathbb{F}_q)\} \cdot (\mathbb{G}_h^1(\mathbb{F}_q) \times \mathbb{T}_h^1(\mathbb{F}_q))$, where the product is viewed as a product of subgroups of $\mathbb{G}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$. Observe that $\Gamma_h \cong \mathbb{F}_{q^n}^\times \ltimes (\mathbb{G}_h^1(\mathbb{F}_q) \times \mathbb{T}_h^1(\mathbb{F}_q))$ and note that $\Gamma_h \cdot (\{1\} \times \mathbb{T}_h(\mathbb{F}_q)) = \mathbb{G}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)$. Therefore

$$\mathrm{Ind}_{\Gamma_h}^{\mathbb{G}_h(\mathbb{F}_q) \times \mathbb{T}_h(\mathbb{F}_q)}(H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi]) \cong \bigoplus_{\theta'} H_c^i(X_h \cap \mathbb{T}_h\mathbb{G}_h^1)[\theta'],$$

where θ' ranges over all characters of $\mathbb{T}_h(\mathbb{F}_q)$ which restrict to χ on $\mathbb{T}_h^1(\mathbb{F}_q)$. The action of $(\zeta, g, t) \in \mathbb{F}_{q^n}^\times \ltimes (\mathbb{G}_h^1(\mathbb{F}_q) \times \mathbb{T}_h^1(\mathbb{F}_q)) \cong \Gamma_h$ on $x \in X_h^1$ is given by

$$(\zeta, g, t) * x = \zeta(gxt)\zeta^{-1},$$

where we view $\zeta \in \mathbb{F}_{q^n}^\times$ as an element of $\mathbb{W}_h(\mathbb{F}_{q^n})^\times \cong \mathbb{T}_h(\mathbb{F}_q)$.

6.1. The nonvanishing cohomological degree.

Recall from Section 5.1 that any character $\theta \colon \mathbb{T}_h(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ has a Howe factorization. For any Howe factorization $\theta = \prod_{i=1}^d \theta_i$ of θ, define a Howe factorization for $\chi := \theta|_{\mathbb{T}_h^1(\mathbb{F}_q)}$ by

$$\chi = \prod_{i=1}^{d'} \chi_i, \qquad \text{where } \chi_i := \theta_i|_{\mathbb{T}_h^1(\mathbb{F}_q)} \text{ and } d' := \begin{cases} d & \text{if } h_d \geq 2, \\ d-1 & \text{if } h_d = 1. \end{cases}$$

As in Section 5.1, although the characters χ_i are not uniquely determined, we have two well-defined sequences of integers

$$1 =: m_0 \leq m_1 < m_2 < \cdots < m_{d'} \leq m_{d'+1} \leq m_{d+1} := n$$
$$h =: h_0 \geq h_1 > h_2 > \cdots > h_{d'} > h_{d'+1} = h_{d+1} := 1$$

satisfying the divisibility $m_i \mid m_{i+1}$ for $0 \leq i \leq d$.

We state the main result of this section.

Theorem 6.1.1. Let $\chi \colon \mathbb{T}_h^1(\mathbb{F}_q) \cong \mathbb{W}_h^1(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ be any character. Then

$$H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi] = \begin{cases} \text{irreducible } \mathbb{G}_h^1(\mathbb{F}_q)\text{-representation} & \text{if } i = r_\chi, \\ 0 & \text{if } i \neq r_\chi, \end{cases}$$

where

$$r_\chi = 2(n' - 1) + 2e_\chi + f_\chi$$

$$e_\chi = \Big(\frac{n}{m_{d'}} - 1\Big)(h_{d'} - 1) - \Big(\frac{n}{\mathrm{lcm}(m_{d'}, n_0)} - 1\Big) - (h_0 - h_{d'}) + \sum_{t=0}^{d'-1} \frac{n}{m_t}(h_t - h_{t+1})$$

$$f_\chi = \Big(n - \frac{n}{m_{d'}}\Big) - \Big(n' - \frac{n}{\mathrm{lcm}(m_{d'}, n_0)}\Big) + \sum_{t=0}^{d'-1} \Big(\frac{n}{m_t} - \frac{n}{m_{t+1}}\Big)h_{t+1}$$

Moreover, Fr_{q^n} *acts on* $H_c^{r_\chi}(X_h^1, \overline{\mathbb{Q}}_\ell)$ *as multiplication by* $(-1)^i q^{ni/2}$.

The assertion about the action of Fr_{q^n} on $H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)[\theta]$ is equivalent to saying that X_h^1 is a *maximal variety* in the sense of Boyarchenko–Weinstein [BW16]; that is, $\#X_h^1(\mathbb{F}_{q^n})$ attains its Weil–Deligne bound

$$\#X_h^1(\mathbb{F}_{q^n}) = \sum_{i \geq 0}(-1)^i \mathrm{Tr}(\mathrm{Fr}_{q^n}; H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)) \leq \sum_{i \geq 0} q^{ni/2} \dim H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell).$$

For easy reference later, we record the following special case of Theorem 6.1.1.

Corollary 6.1.2. *Let* $\chi\colon \mathbb{T}_h^1(\mathbb{F}_q) \cong \mathbb{W}_h^1(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ *be any character with trivial* $\mathrm{Gal}(L/k)$-*stabilizer. Then*

$$H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi] = \begin{cases} irreducible & if\ i = r_\chi, \\ 0 & if\ i \neq r_\chi, \end{cases}$$

where

$$r_\chi = n(h - h_1) + h(n - 2) + h_{d'} - (n - n') + \sum_{t=1}^{d'-1} \frac{n}{m_t}(h_t - h_{t+1}).$$

Proof. The assumption that χ has trivial $\mathrm{Gal}(L/k)$-stabilizer is equivalent to the assumption that $m_{d'} = n$. We see then that the formula for r_χ given in Theorem 6.1.1 simplifies as follows:

$$r_\chi = 2(n' - 1) + \sum_{t=0}^{d'-1} 2\Big(\frac{n}{m_t} - 1\Big)(h_t - h_{t+1})$$

$$+ \sum_{t=0}^{d'-1}\Big(\Big(\frac{n}{m_t} - \frac{n}{m_{t+1}}\Big)(h_{t+1} - 1) - \Big(\frac{n}{\mathrm{lcm}(m_t, n_0)} - \frac{n}{\mathrm{lcm}(m_{t+1}, n_0)}\Big)\Big)$$

$$= 2(n' - 1) - 2(h_0 - h_{d'}) - \Big(\frac{n}{m_0} - \frac{n}{m_{d'}}\Big) - \Big(\frac{n}{\mathrm{lcm}(m_0, n_0)} - \frac{n}{\mathrm{lcm}(m_{d'}, n_0)}\Big)$$

$$+ \frac{n}{m_0}(2h_0 - h_1) - \frac{n}{m_{d'}}(h_{d'}) + \sum_{t=1}^{d'-1} \frac{n}{m_t}(h_t - h_{t+1}).$$

Using the fact that $h_0 = h$ and $m_0 = 1$ by construction, the above expression simplifies to the one given in the statement of the corollary. $\square$

6.2. **Ramified Witt vectors.** We give a brief summary of ramified Witt vectors, following [Cha19, Section 3.1]. In this section, we assume k has characteristic 0. We first define a "simplified version" of the ramified Witt ring $\mathbb{W}$.

Definition 6.2.1. For any $\mathbb{F}_q$-algebra A, let $W(A)$ be the set $A^{\mathbb{N}}$ endowed with the following coordinatewise addition and multiplication rule:

$$[a_i]_{i \geq 0} +_W [b_i]_{i \geq 0} = [a_i + b_i]_{i \geq 0},$$

$$[a_i]_{i \geq 0} *_W [b_i]_{i \geq 0} = \left[\sum_{j=0}^{i} a_j^{q^{i-j}} b_{i-j}^{q^i} \right]_{i \geq 0}.$$

It is a straightforward check that W is a commutative ring scheme over $\mathbb{F}_q$. It comes with Frobenius and Verschiebung morphisms φ and V.

The relationship between the ring scheme W and the ring scheme $\mathbb{W}$ of ramified Witt vectors is captured by the following lemma. The key point here is the notion of "major contribution" and "minor contribution"; this will appear in Lemma 6.3.3 and (implicitly) in Proposition 6.4.4.

Lemma 6.2.2. *Let A be an $\mathbb{F}_q$-algebra.*

(a) *For any $[a_1], \ldots, [a_n] \in A^{\mathbb{N}}$ where $[a_j] = [a_{j,i}]_{i \geq 0}$,*

$$\prod_{\substack{1 \leq j \leq n \\ w.r.t. \ \mathbb{W}}} [a_j] = \left(\prod_{\substack{1 \leq j \leq n \\ w.r.t. \ W}} [a_j] \right) +_W [c],$$

where $[c] = [c_i]_{i \geq 0}$ for some $c_i \in A[a_{1,i_1}^{e_1} \cdots a_{n,i_n}^{e_n} : i_1 + \cdots + i_n < i, e_1, \ldots, e_n \in \mathbb{Z}_{\geq 0}]$.

(b) *For any $[a_1], \ldots, [a_n] \in A^{\mathbb{N}}$ where $[a_j] = [a_{j,i}]_{i \geq 0}$,*

$$\sum_{\substack{1 \leq j \leq n \\ w.r.t. \ \mathbb{W}}} [a_j] = \left(\sum_{\substack{1 \leq j \leq n \\ w.r.t. \ W}} [a_j] \right) +_W [c],$$

where $[c] = [c_i]_{i \geq 0}$ for some $c_i \in A[a_{1,j}, \ldots, a_{n,j} : j < i]$.

We call the portion coming from W the "major contribution" and $[c]$ the "minor contribution."

6.3. **Normed indexing sets.** The group $\mathbb{G}_h^1$ is an affine space of dimension $n^2(h-1)$. To prove Theorem 6.1.1, we will need to coordinatize $\mathbb{G}_h^1$, and we do this here by defining an indexing set $\mathcal{A}^+$ of triples (i, j, l). Our strategy for approaching Theorem 6.1.1 is to perform an inductive calculation based on a Howe factorization of the character $\chi \colon \mathbb{T}_h^1(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^{\times}$. In this section, we will also define a filtration of $\mathcal{A}^+$ corresponding to the two sequences $\{m_i\}, \{h_i\}$ associated with χ.

The algebraic group $\mathbb{G}_h^1$ can be described very explicitly: it consists of matrices $(A_{i,j})_{1 \leq i,j \leq n}$ where

$$A_{i,j} = \begin{cases} [1, A_{(i,j,1)}, A_{(i,j,2)}, \ldots, A_{(i,j,h-1)}] \in \mathbb{W}_h^1 & \text{if } i = j, \\ [A_{(i,j,0)}, A_{(i,j,1)}, \ldots, A_{(i,j,h-2)}] \in \mathbb{W}_{h-1} & \text{if } [i]_{n_0} > [j]_{n_0}, \\ [0, A_{(i,j,1)}, A_{(i,j,2)}, \ldots, A_{(i,j,h-1)}] \in \mathbb{W}_h & \text{if } [i]_{n_0} \leq [j]_{n_0} \text{ and } i \neq j. \end{cases}$$

Here, we recall that for $x \in \mathbb{Z}$, we write $[x]_{n_0}$ to denote the unique representative of $x\mathbb{Z}/n_0\mathbb{Z}$ in the set of coset representatives $\{1, \ldots, n_0\}$. We have a well-defined determinant map

$$\det \colon \mathbb{G}_h^1 \to \mathbb{W}_h^1.$$

In the way described above, $\mathbb{G}_h^1$ can be coordinatized by the indexing set

$$\mathcal{A}^+ := \left\{ (i,j,l) \in \mathbb{Z}^{\oplus 3} : \begin{array}{c} 1 \leq i, j \leq n \\ 0 \leq l \leq h-2 \text{ if } [i]_{n_0} > [j]_{n_0} \\ 1 \leq l \leq h-1 \text{ if } [i]_{n_0} \leq [j]_{n_0} \end{array} \right\}.$$

We also define:

$$\mathcal{A} := \{(i,j,l) \in \mathcal{A}^+ : i \neq j\},$$
$$\mathcal{A}^- := \{(i,j,l) \in \mathcal{A} : j = 1\}.$$

The indexing set $\mathcal{A}$ corresponds to the elements of $\mathbb{G}_h^1$ with 1's along the diagonal, and $\mathcal{A}^-$ remembers only the first column of elements of $\mathbb{G}_h^1$ with $(1,1)$-entry 1.

Definition 6.3.1. Define a norm on $\mathcal{A}^+$:

$$\mathcal{A}^+ \to \mathbb{R}_{\geq 0},$$
$$(i,j,l) \mapsto |(i,j,l)| := i - j + nl.$$

Definition 6.3.2. For $\lambda = (i,j,l) \in \mathcal{A}^+$, define

$$\lambda^\vee := (j,i,h-1-l).$$

The following seemingly innocuous lemma is in some sense the key reason that the indexing sets above allow us to carry over the calculations in [Cha19, Section 5] from $n' = 1$ setting to the present general n' setting with very few modifications.

Lemma 6.3.3. *Following the conventions as set up above, write $A = (A_{i,j})_{1 \leq i,j \leq n} \in \mathbb{G}_h^1$, where*

$$A_{i,j} = \begin{cases} [1, A_{(i,j,1)}, \ldots, A_{(i,j,h-1)}] \in \mathbb{W}_h^1 & \text{if } i = j, \\ [A_{(i,j,0)}, \ldots, A_{(i,j,h-2)}] \in \mathbb{W}_{h-1} & \text{if } [i]_{n_0} > [j]_{n_0}, \\ [0, A_{(i,j,1)}, \ldots, A_{(i,j,h-1)}] \in \mathbb{W}_h & \text{if } [i]_{n_0} \leq [j]_{n_0} \text{ and } i \neq j. \end{cases}$$

Assume that for $\lambda_1, \lambda_2 \in \mathcal{A}^+$, the variables A_{λ_1} and A_{λ_2} appear in the same monomial in $\det(A) \in \mathbb{W}_{h'}$ for some $h' \leq h$.

(a) Then $|\lambda_1| + |\lambda_2| \leq n(h'-1)$.
(b) If $|\lambda_1| + |\lambda_2| = n(h'-1)$, then $\lambda_2 = \lambda_1^\vee$, where $^\vee$ is taken relative to h'.

Proof. By definition,

$$\det(A) = \sum_{\gamma \in S_n} \prod_{1 \leq i \leq n} A_{i,\gamma(i)} \in \mathbb{W}_{h'}(\overline{\mathbb{F}}_q).$$

Let $l \leq h'-1$. If K has characteristic p, then the contributions to the ϖ^l-coefficient coming from $\gamma \in S_n$ are of the form

$$\prod_{i=1}^n A_{(i,\gamma(i),l_i)},$$

where $(l_1, \ldots, l_n)$ is a partition of l. Then

$$\sum_{i=1}^n |(i,\gamma(i),l_i)| = \sum_{i=1}^n i - \gamma(i) + nl_i = \sum_{i=1}^n nl_i = nl \leq n(h'-1). \tag{6.1}$$

If K has characteristic 0, then the major contributions to the ϖ^l-coefficient coming from γ are of the form

$$\prod_{i=1}^n A_{(i,\gamma(i),l_i)}^{e_i},$$

where the e_i are some nonnegative integers and where $(l_1, \ldots, l_n)$ is a partition of l. Hence

$$\sum_{i=1}^{n} |(i, \gamma(i), l_i)| = nl \leq n(h' - 1). \tag{6.2}$$

The minor contributions to the ϖ^l-coefficient coming from γ are polynomials in $\prod_{i=1}^{n} A_{(i,\gamma(i),l_i)}^{e_i'}$ where $l_1 + \cdots + l_n < l$ and the e_i' are some nonnegative integers. Hence $\sum_{i=1}^{n} |(i, \gamma(i), l_i)| < n(h' - 1)$.

Suppose now that $\lambda_1 = (i_1, j_1, l_1), \lambda_2 = (i_2, j_2, l_2) \in \mathcal{A}^+$ are such that A_{λ_1} and A_{λ_2} contribute to the same monomial in $\det(M) \in \mathbb{W}_{h'}^1$. Then there exists some $\gamma \in S_n$ such that $\gamma(i_1) = j_1$ and $\gamma(i_2) = j_2$, and by Equations (6.1) and (6.2),

$$|\lambda_1| + |\lambda_2| \leq n(h' - 1).$$

Observe that if K has characteristic 0 and λ_1 and λ_2 occur in a minor contribution, then $|\lambda_1| + |\lambda_2| < n(h')$. This proves (a), and furthermore, we see that if $|\lambda_1| + |\lambda_2| = n(h' - 1)$, then the simultaneous contribution of A_{λ_1} and A_{λ_2} comes from a major contribution. But now (b) follows: since the image of $\mathbb{G}_h^1$ under the determinant is $\mathbb{W}_h^1$, if $|\lambda_1| + |\lambda_2| = n(h' - 1)$, then necessarily the contribution of λ_1 and λ_2 to the $(h' - 1)$th coordinate of the determinant must come from a transposition. $\square$

Given two sequences of integers

$$1 =: m_0 \leq m_1 < m_2 < \cdots < m_{d'} \leq m_{d'+1} \leq m_{d+1} := n$$
$$h =: h_0 \geq h_1 > h_2 > \cdots > h_{d'} > h_{d'+1} = h_{d+1} := 1$$

satisfying $m_i \mid m_{i+1}$ for $0 \leq i \leq d$, we can define the following subsets of $\mathcal{A}$ for $0 \leq s, t \leq d$:

$$\mathcal{A}_{s,t} := \{(i, j, l) \in \mathcal{A} : i \equiv j \pmod{m_s}, i \not\equiv j \pmod{m_{s+1}}, l \leq h_t - 1\},$$
$$\mathcal{A}_{s,t}^- := \mathcal{A}_{s,t} \cap \mathcal{A}^-.$$

We will need to understand which $\lambda \in \mathcal{A}$ are such that x_λ contributes nontrivially to the determinant. We denote the set of all such λ by $\mathcal{A}^{\min}$. We may describe this explicitly:

$$\mathcal{A}^{\min} = \{\lambda \in \mathcal{A} : \lambda^\vee \in \mathcal{A}\} \tag{6.3}$$
$$= \left\{ (i, j, l) \in \mathcal{A} : \begin{array}{l} 0 \leq l \leq h - 2 \text{ if } [i]_{n_0} > [j]_{n_0} \\ 1 \leq l \leq h - 1 \text{ if } [i]_{n_0} < [j]_{n_0} \\ 1 \leq l \leq h - 2 \text{ if } [i]_{n_0} = [j]_{n_0} \end{array} \right\}.$$

For $0 \leq s, t \leq r$, by considering $^\vee$ relative to h_t, we may similarly define

$$\mathcal{A}_{s,t}^{\min} := \{\lambda \in \mathcal{A}_{s,t} : \lambda^\vee \in \mathcal{A}_{s,t}\}$$
$$= \left\{ (i, j, l) \in \mathcal{A}_{s,t} : \begin{array}{l} 0 \leq l \leq h_t - 2 \text{ if } [i]_{n_0} > [j]_{n_0} \\ 1 \leq l \leq h_t - 1 \text{ if } [i]_{n_0} < [j]_{n_0} \\ 1 \leq l \leq h_t - 2 \text{ if } [i]_{n_0} = [j]_{n_0} \end{array} \right\}.$$

Define $\mathcal{A}_{s,t}^{-;\min} := \mathcal{A}^- \cap \mathcal{A}_{s,t}^{\min} = \mathcal{A}_{s,t}^- \cap \mathcal{A}_{s,t}^{\min}$. Define the following decomposition of $\mathcal{A}_{s,t}^{-;\min}$:

$$\mathcal{I}_{s,t} := \{(i, 1, l) \in \mathcal{A}_{s,t}^{-;\min} : |(i, 1, l)| > n(h_t - 1)/2\},$$
$$\mathcal{J}_{s,t} := \{(i, 1, l) \in \mathcal{A}_{s,t}^{-;\min} : |(i, 1, l)| \leq n(h_t - 1)/2\}.$$

For any real number ν, define

$$\mathcal{A}^{\min}_{\geq \nu, t} := \bigsqcup_{s=\lceil \nu \rceil}^{r} \mathcal{A}^{\min}_{s,t}, \qquad \mathcal{A}^{-,\min}_{\geq \nu, t} = \mathcal{A}^{-} \cap \mathcal{A}^{\min}_{\geq \nu, t},$$

and observe that for $0 \leq s \leq r$ an integer,

$$\mathcal{A}^{\min}_{\geq s,t} = \left\{ (i,j,l) \in \mathcal{A} : \begin{array}{c} j \equiv i \pmod{m_s} \\ 0 \leq l \leq h_t - 2 \text{ if } [i]_{n_0} > [j]_{n_0} \\ 1 \leq l \leq h_t - 1 \text{ if } [i]_{n_0} < [j]_{n_0} \\ 1 \leq l \leq h_t - 2 \text{ if } [i]_{n_0} = [j]_{n_0} \end{array} \right\}.$$

Lemma 6.3.4. *There is an order-reversing injection $\mathcal{I}_{s,t} \to \mathcal{J}_{s,t}$ that is a bijection if and only if $\mathcal{A}^{-,\min}_{s,t}$ is even. Explicitly, it is given by*

$$\mathcal{I}_{s,t} \hookrightarrow \mathcal{J}_{s,t}, \qquad (i,1,l) \mapsto ([n-i+2]_n, 1, h_t - 2 - l).$$

Note that $\#\mathcal{A}^{-,\min}_{s,t}$ is even unless n and h_t are both even.

Proof. If $(i,1,l) \in \mathcal{A}^{-,\min}_{s,t}$, then by definition $i \equiv 1$ modulo m_s and $i \not\equiv 1$ modulo m_{s+1}. Thus $[n-i+2]_n \equiv 1$ modulo m_s and $[n-i+2]_n \not\equiv 1$ modulo m_{s+1}, which shows that $(i,1,l) \in \mathcal{A}^{-,\min}_{s,t}$ implies $([n-i+2]_n, 1, l) \in \mathcal{A}^{-,\min}_{s,t}$. Since $i \geq 2$ by assumption, we have $i + [n-i+2]_n = n+2$ and

$$|(i,1,l)| + |([n-i+2]_n, 1, h_t - 2 - l)| = n(h_t - 1).$$

Hence if $(i,1,l) \in \mathcal{I}_{s,t}$, then $([n-i+2]_n, 1, l) \in \mathcal{J}_{s,t}$. It is clear that the map is a bijection if and only if $\mathcal{J}_{s,t}$ does not contain an element of norm $n(h_t - 1)/2$. Such an element must necessarily be of the form $((n+2/2), 1, (h_t - 2)/2)$, which is integral if and only if n and h_t are both even. $\square$

6.4. **The cohomology of X_h^1.** The purpose of this section is to establish the following result:

Theorem 6.4.1. *For any character $\chi \colon \mathbb{T}_h^1(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$,*

$$\mathrm{Hom}_{\mathbb{G}_h^1(\mathbb{F}_q)}\left(\mathrm{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi), H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell) \right) = \begin{cases} \overline{\mathbb{Q}}_\ell^{\oplus q^{n f_\chi/2}} \otimes ((-q^{n/2})^{r_\chi})^{\deg} & \text{if } i = r_\chi, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, Fr_{q^n} acts on $H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)$ by multiplication by the scalar $(-1)^i q^{ni/2}$.

This is a technical calculation which follows the strategy developed in [Cha19] (in particular, see Sections 4 and 5 of *op. cit.*). We first rephrase space of homomorphisms in the statement of Theorem 6.4.1 in terms of the cohomology of a related variety. Every coset of $\mathbb{G}_h^1/\mathbb{T}_h^1$ has a unique coset representative g whose diagonal entries are identically 1. Over $\mathbb{F}_q$, we may identify $\mathbb{G}_h^1/\mathbb{T}_h^1$ with the affine space $\mathbb{A}[\mathcal{A}]$ (the affine space of dimension $\#\mathcal{A}$ with coordinates indexed by the set $\mathcal{A}$ of Section 6.3). Then the quotient morphism $\mathbb{G}_h^1 \to \mathbb{G}_h^1/\mathbb{T}_h^1$ has a section given by

$$s \colon \mathbb{G}_h^1/\mathbb{T}_h^1 \to \mathbb{G}_h^1, \qquad (x_{(i,j,l)})_{(i,j,l)\in\mathcal{A}} \mapsto (x_{i,j})_{i,j=1,\dots,n},$$

where

$$x_{i,j} = \begin{cases} 1 \in \mathbb{W}_h^1 & \text{if } i = j, \\ [x_{(i,j,0)}, x_{(i,j,1)}, \dots, x_{(i,j,h-2)}] \in \mathbb{W}_{h-1} & \text{if } [i]_{n_0} > [j]_{n_0}, \\ [0, x_{(i,j,1)}, x_{(i,j,2)}, \dots, x_{(i,j,h-1)}] \in \mathbb{W}_h & \text{if } [i]_{n_0} \leq [j]_{n_0} \text{ and } i \neq j. \end{cases}$$

As in [Cha19, Section 5.1.1], there exists a closed $\mathbb{F}_{q^n}$-subscheme Y_h^1 of $\mathbb{G}_h^1$ such that $X_h = L_q^{-1}(Y_h^1)$ which satisfies the condition that $\mathrm{Fr}_q^i(Y_h^1) \cap \mathrm{Fr}_q^j(Y_h^1) = \{1\}$ for all $i \neq j$. We are therefore in a setting where we can invoke [Cha19, Proposition 4.1.1].

Define
$$\beta : (\mathbb{G}_h^1/\mathbb{T}_h^1) \times \mathbb{T}_h^1 \to \mathbb{G}_h^1, \qquad (x, g) \mapsto s(\mathrm{Fr}_q(x)) \cdot g \cdot s(x).$$

The affine $\mathbb{F}_{q^n}$-scheme $\beta^{-1}(Y_h^1) \subset (\mathbb{G}_h^1/\mathbb{T}_h^1) \times \mathbb{T}_h^1$ comes with two maps:
$$\mathrm{pr}_1 : \beta^{-1}(Y_h^1) \to \mathbb{G}_h^1/\mathbb{T}_h^1 = \mathbb{A}[\mathcal{A}], \qquad \mathrm{pr}_2 : \beta^{-1}(Y_h^1) \to \mathbb{T}_h^1.$$

Recall from [Cha19, Lemma 4.1.2] that since the Lang morphism L_q is surjective,
$$(x, g) \in \beta^{-1}(Y_h^1) \qquad \Longleftrightarrow \qquad s(x) \cdot y \in X_h, \tag{6.4}$$
where $y \in \mathbb{T}_h^1$ is any element such that $L_q(y) = g$.

Proposition 6.4.2. *For any character* $\chi : \mathbb{T}_h^1(\mathbb{F}_q) \cong \mathbb{W}_h^1(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$, *let* $\mathcal{L}_\chi$ *denote the corresponding* $\overline{\mathbb{Q}}_\ell$-*local system on* $\mathbb{W}_h^1$. *For* $i \geq 0$, *we have* Fr_{q^n}-*compatible isomorphisms*
$$\mathrm{Hom}_{\mathbb{G}_h^1(\mathbb{F}_q)} \left(\mathrm{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell) \right) \cong H_c^i(\mathbb{A}[\mathcal{A}^-], P^*\mathcal{L}_\chi),$$
where $P : \mathbb{A}[\mathcal{A}^-] \to \mathbb{W}_h^1$ *is the morphism* $(x_{(i,1,l)})_{(i,1,l) \in \mathcal{A}^-} \mapsto L_q(\det(g_b^{\mathrm{red}}(1, x_2, \ldots, x_n)))^{-1}$ *for* $x_i := [x_{(i,1,0)}, x_{(i,1,1)}, \ldots, x_{(i,1,h-1)}]$.

Proof. By [Cha19, Proposition 4.1.1],
$$\mathrm{Hom}_{\mathbb{G}_h^1(\mathbb{F}_q)} \left(\mathrm{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell) \right) \cong H_c^i(\beta^{-1}(Y_h^1), \mathrm{pr}_2^* \mathcal{F}_\chi),$$
where $\mathcal{F}_\chi$ is the rank-1 local system on $\mathbb{T}_h^1$ corresponding to χ. By the same proof as [Cha19, Lemma 5.1.1], $\beta^{-1}(Y_h^1)$ is the graph of the morphism $P_0 : \mathbb{A}[\mathcal{A}] \to \mathbb{W}_h^1$ given by $x \mapsto L_q(\det(s(x)))^{-1}$. Furthermore, as morphisms on $\beta^{-1}(Y_h^1)$, we have $\mathrm{pr}_2 = i \circ P_0 \circ \mathrm{pr}_1$, where $i : \mathbb{W}_h^1 \to \mathbb{T}_h^1, x \mapsto \mathrm{diag}(x, 1, \ldots, 1)$. Therefore, as sheaves on $\mathrm{pr}_1(\beta^{-1}(Y_h^1))$, we have $\mathrm{pr}_2^* \mathcal{F}_\chi = P_0^* i^* \mathcal{F}_\chi = P_0^* \mathcal{L}_\chi$, so
$$H_c^i(\beta^{-1}(Y_h^1), \mathrm{pr}_2^* \mathcal{F}_\chi) = H_c^i(\mathrm{pr}_1(\beta^{-1}(Y_h^1)), P_0^* \mathcal{L}_\chi).$$

Next we claim that the projection $\mathbb{A}[\mathcal{A}] \to \mathbb{A}[\mathcal{A}^-]$ induces an isomorphism $\mathrm{pr}_1(\beta^{-1}(Y_h^1)) \to \mathbb{A}[\mathcal{A}^-]$. Injectivity is clear: using (6.4), we know that $x \in \mathrm{pr}_1(\beta^{-1}(Y_h^1))$ if $s(x) \cdot y \in X_h^1$ for some $y \in \mathbb{T}_h^1$. Since $s(x) \cdot y$ is uniquely determined by its first column, then $s(x)$ is uniquely determined by its first column, which is precisely the projection of x to $\mathbb{A}[\mathcal{A}^-]$. To see surjectivity, we need to show that for any $x \in \mathbb{A}[\mathcal{A}^-](\overline{\mathbb{F}}_q)$, there exists a $y \in \mathbb{T}_h^1(\overline{\mathbb{F}}_q)$ such that $g_b^{\mathrm{red}}(x) \cdot y \in X_h^1$. Pick any $y = \mathrm{diag}(y_1, \sigma(y_1), \ldots, \sigma(y_1)) \in \mathbb{T}_h^1(\overline{\mathbb{F}}_q)$ such that $\det(y) = \det(g_b^{\mathrm{red}}(x))^{-1}$. Then $g_b^{\mathrm{red}}(x) \cdot y \in X_h$ since $g_b^{\mathrm{red}}(x) \cdot y = g_b^{\mathrm{red}}(xy_1)$ and $\det(g_b^{\mathrm{red}}(x) \cdot y) = 1 \in \mathbb{W}_h^1(\mathbb{F}_q)$. Under the isomorphism $\mathrm{pr}_1(\beta^{-1}(Y_h^1)) \cong \mathbb{A}[\mathcal{A}^-]$, the sheaf $P_0^* \mathcal{L}_\chi$ is identified with $P^* \mathcal{L}_\chi$, and the proposition now follows. $\square$

Note that the last paragraph of the above proof is a simpler and more conceptual proof of [Cha19, Lemma 5.1.6]. To calculate $H_c^i(\mathbb{A}[\mathcal{A}^-], P^*\mathcal{L}_\chi)$, we will use an inductive argument on affine fibrations that relies on iteratively applying the next two propositions:

Proposition 6.4.3. *For* $0 \leq t \leq d'$, *we have* Fr_{q^n}-*compatible isomorphisms*
$$H_c^i(\mathbb{A}[\mathcal{A}_{\geq t,t}^{-,\min}], P^*\mathcal{L}_{\chi_{\geq t}}) \cong H_c^i(\mathbb{A}[\mathcal{A}_{\geq t,t+1}^{-,\min}], P^*\mathcal{L}_{\chi_{\geq t+1}})[2e_t] \otimes ((-q^{n/2})^{2e_t})^{\deg},$$
where $e_t = \#(\mathcal{A}_{\geq t,t}^{-,\min} \smallsetminus \mathcal{A}_{\geq t,t+1}^{-,\min})$.

Proof. The proof is the same as the proof of [Cha19, Proposition 5.3.1]. We give a sketch here. By definition, $\chi_{\geq t} = \chi_t \cdot \chi_{\geq t+1}$ and χ_t factors through the norm map $\mathbb{W}_{h_t}^1(\mathbb{F}_{q^n}) \to \mathbb{W}_{h_t}^1(\mathbb{F}_{q^{m_t}})$.

Let $\mathrm{pr}\colon \mathbb{W}_{h_t}^1 \to \mathbb{W}_{h_{t+1}}^1$. Since $P\colon \mathbb{A}[\mathcal{A}_{\geq t,t}^{-,\min}] \to \mathbb{W}_{h_t}^1$ factors through $L_{q^{m_t}}$, this implies that

$$P^*\mathcal{L}_{\chi_{\geq t}} = P^*\mathcal{L}_{\chi_t} \otimes P^*\mathrm{pr}^*\mathcal{L}_{\chi_{\geq t+1}} = \overline{\mathbb{Q}}_\ell \boxtimes P^*\mathcal{L}_{\chi_{\geq t+1}},$$

where $\overline{\mathbb{Q}}_\ell$ is the constant sheaf on $\mathbb{A}[\mathcal{A}_{\geq t,t}^{-,\min} \smallsetminus \mathcal{A}_{\geq t,t+1}^{-,\min}]$ and $P^*\mathcal{L}_{\chi_{\geq t+1}}$ is the pullback along $P\colon \mathbb{A}[\mathcal{A}_{\geq t,t+1}^{-,\min}] \to \mathbb{W}_{h_{t+1}}^1$. The conclusion then follows from the Künneth formula. $\square$

Proposition 6.4.4. *For $0 \leq t \leq d' - 1$, we have Fr_{q^n}-compatible isomorphisms*

$$H_c^i(\mathbb{A}[\mathcal{A}_{\geq t,t+1}^{-,\min}], P^*\mathcal{L}_{\chi_{\geq t+1}}) \cong H_c^i(\mathbb{A}[\mathcal{A}_{\geq t+1,t+1}^{-,\min}], P^*\mathcal{L}_{\chi_{\geq t+1}})^{\oplus q^{nf_t/2}}[f_t] \otimes ((-q^{n/2})^{f_t})^{\deg},$$

where $f_t = \#(\mathcal{A}_{\geq t,t+1}^{-,\min} \smallsetminus \mathcal{A}_{\geq t+1,t+1}^{-,\min}) = \#\mathcal{A}_{t,t+1}^{-,\min}$.

Proof. By replacing [Cha19, Lemmas 3.2.3, 3.2.6] with Lemmas 6.3.3, 6.3.4, the proof of [Cha19, Proposition 5.3.2] applies. (The proof is quite technical; simpler incarnations of this idea have appeared in [Boy12], [Cha16], [Cha18].) $\square$

Proof of Theorem 6.4.1. By Proposition 6.4.2, we need to calculate $H_c^i(\mathbb{A}[\mathcal{A}^-], P^*\mathcal{L}_\chi)$. Since $P(\mathbb{A}[\mathcal{A}^- \smallsetminus \mathcal{A}^{-,\min}]) = \{1\} \in \mathbb{W}_h^1$ and $\#(\mathcal{A}^- \smallsetminus \mathcal{A}^{-,\min}) = n' - 1$, we see that

$$H_c^i(\mathbb{A}[\mathcal{A}^-], P^*\mathcal{L}_\chi) = H_c^i(\mathbb{A}[\mathcal{A}^{-,\min}], P^*\mathcal{L}_\chi)[2(n'-1)] \otimes ((-q^{n/2})^{2(n'-1)})^{\deg}.$$

Using Propositions 6.4.3 and 6.4.4 iteratively, we have

$H_c^i(\mathbb{A}[\mathcal{A}^{-,\min}], P^*\mathcal{L}_\chi)$

$\quad = H_c^i(\mathbb{A}[\mathcal{A}_{\geq 0,0}^{-,\min}], P^*\mathcal{L}_{\chi_{\geq 0}})$ (by def)

$\quad \cong H_c^i(\mathbb{A}[\mathcal{A}_{\geq 0,1}^{-,\min}], P^*\mathcal{L}_{\geq 1})[2e_0] \otimes \left((-q^{n/2})^{2e_0}\right)^{\deg}$ (Prop 6.4.3)

$\quad \cong H_c^i(\mathbb{A}[\mathcal{A}_{\geq 1,1}^{-,\min}], P^*\mathcal{L}_{\geq 1})^{\oplus q^{nf_0/2}}[f_0 + 2e_0] \otimes \left((-q^{n/2})^{f_0+2e_0}\right)^{\deg}$ (Prop 6.4.4)

$\quad \cong H_c^i(\mathbb{A}[\mathcal{A}_{\geq 1,2}^{-,\min}], P^*\mathcal{L}_{\geq 2})^{\oplus q^{nf_0/2}}[f_0 + 2(e_0 + e_1)] \otimes \left((-q^{n/2})^{f_0+2(e_0+e_1)}\right)^{\deg}$ (Prop 6.4.3)

and so forth until

$$\cong H_c^i(\mathbb{A}[\mathcal{A}_{\geq d',d'+1}^{-,\min}], P^*\mathcal{L}_{\chi_{\geq d'+1}})^{\oplus q^{nf_\chi/2}}[f_\chi + 2e_\chi] \otimes \left((-q^{n/2})^{f_\chi+2e_\chi}\right)^{\deg},$$

where

$$f_\chi := f_0 + f_1 + \cdots + f_{d'-1}, \qquad e_\chi := e_0 + e_1 + \cdots + e_{d'}.$$

Since $\mathcal{A}_{\geq d',d'+1} = \varnothing$, now we have shown

$$H_c^i(\mathbb{A}[\mathcal{A}^{-,\min}], P^*\mathcal{L}_\chi) \cong H_c^i(*, \overline{\mathbb{Q}}_\ell)^{\oplus q^{nf_\chi/2}}[f_\chi + 2e_\chi] \otimes \left((-q^{n/2})^{f_\chi+2e_\chi}\right)^{\deg}. \tag{6.5}$$

Set $r_\chi := 2(n'-1) + f_\chi + 2e_\chi$. By Proposition 6.4.2, we now have

$$\mathrm{Hom}_{\mathbb{G}_h^1(\mathbb{F}_q)}\left(\mathrm{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\right) \cong \begin{cases} \overline{\mathbb{Q}}_\ell^{q^{nf_\chi/2}} & \text{if } i = r_\chi, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, since Fr_{q^n} acts trivially on $H_c^i(\mathbb{A}[\mathcal{A}_{\geq d',d'+1}^{-,\min}], P^*\mathcal{L}_{\chi_{\geq d'+1}}) = H^0(*, \overline{\mathbb{Q}}_\ell)$, then Fr_{q^n} acts by multiplication by $(-1)^{r_\chi}q^{nr_\chi/2}$ on the above space of homomorphisms.

To finish the proof of Theorem 6.4.1, we need only calculate e_χ, f_χ, r_χ. Unwinding the definitions of indexing sets given in Section 6.3, we have, for $0 \leq t \leq d'$,

$$\mathcal{A}_{\geq t,t}^{-,\min} = \left\{ (i,1,l) \in \mathbb{Z}^{\oplus 3} : \begin{array}{c} 2 \leq i \leq n,\ i \equiv 1 \pmod{m_t} \\ 0 \leq l \leq h_t - 2 \text{ if } [i]_{n_0} \neq 1 \\ 1 \leq l \leq h_t - 2 \text{ if } [i]_{n_0} = 1 \end{array} \right\},$$

$$\mathcal{A}_{\geq t,t+1}^{-,\min} = \left\{ (i,1,l) \in \mathbb{Z}^{\oplus 3} : \begin{array}{c} 2 \leq i \leq n,\ i \equiv 1 \pmod{m_t} \\ 0 \leq l \leq h_{t+1} - 2 \text{ if } [i]_{n_0} \neq 1 \\ 1 \leq l \leq h_{t+1} - 2 \text{ if } [i]_{n_0} = 1 \end{array} \right\}.$$

Therefore, we have

$$e_t = \left(\frac{n}{m_t} - 1 \right)(h_t - h_{t+1}) \qquad \text{if } 0 \leq t \leq d' - 1,$$

$$e_{d'} = \left(\frac{n}{m_{d'}} - 1 \right)(h_{d'} - 1) - \left(\frac{n}{\mathrm{lcm}(m_{d'}, n_0)} - 1 \right).$$

For $0 \leq t \leq d' - 1$, we have

$$\mathcal{A}_{t,t+1}^{-,\min} = \left\{ (i,1,l) \in \mathbb{Z}^{\oplus 3} : \begin{array}{c} 2 \leq i \leq n,\ i \equiv 1 \pmod{m_t},\ i \not\equiv 1 \pmod{m_{t+1}} \\ 0 \leq l \leq h_{t+1} - 2 \text{ if } [i]_{n_0} \neq 1 \\ 1 \leq l \leq h_{t+1} - 2 \text{ if } [i]_{n_0} = 1 \end{array} \right\}$$

so that

$$f_t = \left(\frac{n}{m_t} - \frac{n}{m_{t+1}} \right)(h_{t+1} - 1) - \left(\frac{n}{\mathrm{lcm}(m_t, n_0)} - \frac{n}{\mathrm{lcm}(m_{t+1}, n_0)} \right). \qquad \square$$

6.5. **The nonvanishing cohomological degree.** In this section, we use the results of the preceding sections to finish the proof of Theorem 6.1.1. Observe that from Theorem 6.4.1 together with Corollary 5.2.2, we have the following:

Corollary 6.5.1. *Let π be an irreducible constituent of $H_c^r(Z_h^1, \overline{\mathbb{Q}}_\ell)$ for some r. Then*

$$\mathrm{Hom}_{\mathbb{G}_h^1(\mathbb{F}_q)}\left(\pi, H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell) \right) = 0 \qquad \text{for all } i \neq r.$$

In particular, for any $\chi \colon \mathbb{T}_h^1(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$, there exists a positive integer s_χ such that

$$H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi] = \begin{cases} \text{irreducible} & \text{if } i = s_\chi, \\ 0 & \text{if } i \neq s_\chi. \end{cases}$$

Proof. This is the same as the proof of [Cha19, Corollary 5.1.3]. The irreducible $\mathbb{G}_h^1(\mathbb{F}_q)$-representation $\pi \subset H_c^r(X_h^1, \overline{\mathbb{Q}}_\ell)$ is a summand of $\mathrm{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi')$ for some χ'. Hence

$$\mathrm{Hom}_{\mathbb{G}_h^1(\mathbb{F}_q)}\left(\mathrm{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi'), H_c^r(X_h^1, \overline{\mathbb{Q}}_\ell) \right) \neq 0.$$

Theorem 6.4.1 implies that $r = r_{\chi'}$ and that there are no $\mathbb{G}_h^1(\mathbb{F}_q)$-equivariant homomorphisms from π to $H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)$ for $i \neq r_{\chi'}$. This proves the first assertion.

To see the second assertion, first recall from Corollary 5.2.2 that $H_c^*(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi]$ is (up to sign) an irreducible $\mathbb{G}_h^1(\mathbb{F}_q)$-representation. Therefore, we may apply the above argument to $H_c^*(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi]$ and we see that if $H_c^*(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi]$ is a summand of $\mathrm{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi')$, then

$$H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi] = \begin{cases} \text{irreducible} & \text{if } i = r_{\chi'}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the number $r_{\chi'}$ only depends on χ, we final assertion of the corollary holds taking $s_\chi = r_{\chi'}$. $\square$

We see now that the upshot of Theorem 6.4.1 is that we already know that $H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi]$ is concentrated in a single degree s_χ. However, it would be much more satisfying—for many reasons, computational, conceptual, idealogical—if we could pinpoint this nonvanishing cohomological degree. Taking a hint from the proof of Corollary 6.5.1, one strategy to prove that $s_\chi = r_\chi$ is to prove that $H_c^{s_\chi}(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi]$ is a summand of $\mathrm{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi)$. This is our next result.

Theorem 6.5.2. *For any $\chi \colon \mathbb{T}_h^1(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$,*

$$\mathrm{Hom}_{\mathbb{G}_h^1(\mathbb{F}_q)} \left(\mathrm{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi), H_c^{s_\chi}(Z_h^1, \overline{\mathbb{Q}}_\ell)[\chi] \right) \neq 0.$$

In particular, $s_\chi = r_\chi$.

The proof of Theorem 6.5.2 is essentially the same proof as [Cha19, Theorem 6.2.4]. By Frobenius reciprocity, it is enough to show

$$\mathrm{Hom}_{\mathbb{T}_h^1(\mathbb{F}_q)} \left(\chi, H_c^{s_\chi}(X_h^1, \overline{\mathbb{Q}}_\ell)[\theta] \right) \neq 0. \tag{6.6}$$

We will sometimes write $\mathbb{T}_h^1 = \mathbb{T}_{h,n,q}^1$ and $\mathbb{G}_h^1 = \mathbb{G}_{h,n,q}^1$, $X_h^1 = X_{h,n,q}^1$, $g_b^{n,q}$, and $s_\chi = s_\chi^{n,q}$ to emphasize the dependence on n, q. It is clear that once (6.6) is established, then by Theorem 6.4.1, it follows that $s_\chi = r_\chi$. For notational convenience, we write $H_c^i(X)$ to mean $H_c^i(X, \overline{\mathbb{Q}}_\ell)$. We first establish a few lemmas.

Lemma 6.5.3. *For any $\zeta \in \mathbb{F}_{q^n}^\times$ with trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer and any $g \in \mathbb{T}_h^1(\mathbb{F}_q)$,*

$$\mathrm{Tr} \left((\zeta, 1, g); H_c^{s_\chi}(X_h^1)[\chi] \right) = (-1)^{s_\chi} \chi(g).$$

Proof. Recall that the action of $(\zeta, 1, 1) \in \Gamma_h$ is given by conjugation. Observe that if $x \in (X_h^1)^{(\zeta,1,1)}$, then $x = g_b(v_1, 0, \ldots, 0)$. Furthermore, this forces $v_1 \in \mathbb{W}_h^1(\mathbb{F}_{q^n})$. Therefore $(X_h^1)^{(\zeta,1,1)} = \mathbb{T}_h^1(\mathbb{F}_q)$. By the Deligne–Lusztig fixed point formula,

$$\mathrm{Tr} \left((\zeta, g, 1)^*; H_c^*(X_h^1)[\chi] \right) = \frac{1}{\#\mathbb{T}_h^1(\mathbb{F}_q)} \sum_{t \in \mathbb{T}_h^1(\mathbb{F}_q)} \chi(t)^{-1} \mathrm{Tr} \left((\zeta, g, t)^*; H_c^*(X_h^1) \right)$$

$$= \frac{1}{\#\mathbb{T}_h^1(\mathbb{F}_q)} \sum_{t \in \mathbb{T}_h^1(\mathbb{F}_q)} \chi(t)^{-1} \mathrm{Tr} \left((1, g, t)^*; H_c^*((X_h^1)^{(\zeta,1,1)}) \right)$$

$$= \frac{1}{\#\mathbb{T}_h^1(\mathbb{F}_q)} \sum_{t \in \mathbb{T}_h^1(\mathbb{F}_q)} \chi(t)^{-1} \mathrm{Tr} \left((1, g, t)^*; H_c^*(\mathbb{T}_h^1(\mathbb{F}_q)) \right)$$

$$= \frac{1}{\#\mathbb{T}_h^1(\mathbb{F}_q)} \sum_{t \in \mathbb{T}_h^1(\mathbb{F}_q)} \chi(t)^{-1} \sum_{\chi' \colon \mathbb{T}_h^1(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times} \chi'(g)\chi'(t) = \chi(g).$$

The conclusion of the lemma now follows from Corollary 6.5.1. $\square$

Lemma 6.5.4. *Let p_0 be a prime dividing n. For any $\zeta \in \mathbb{F}_{q^{p_0}}^\times \smallsetminus \mathbb{F}_q^\times$ and any $g \in \mathbb{T}_h^1(\mathbb{F}_q)$,*

$$(-1)^{s_\chi^{n,q}} \mathrm{Tr} \left((\zeta, 1, g); H_c^{s_\chi^{n,q}}(X_{h,n,q}^1)[\chi] \right) = (-1)^{s_\chi^{n/p_0, q^{p_0}}} \mathrm{Tr} \left((1, 1, g); H_c^{s_\chi^{n/p_0, q^{p_0}}}(X_{h,n/p_0,q^{p_0}}^1)[\chi] \right).$$

Proof. Recall that the action of $(\zeta, 1, 1) \in \Gamma_h$ is given by conjugation. Observe that if $x \in (X_h^1)^{(\zeta,1,1)}$, then $x = g_b(v_1, \ldots, v_n)$ where $v_i = 0$ for all $i \not\equiv 1$ modulo p_0. The map

$$f\colon (X_{h,n,q}^1)^{(\zeta,1,1)} \to X_{h,n/p_0,q^{p_0}}^1$$

$$g_b^{n,q}(v_1, v_2, \ldots, v_n) \mapsto g_b^{n/p_0,q^{p_0}}(v_1, v_{p_0+1}, v_{2p_0+1}, \ldots, v_{n-p_0+1})$$

defines an isomorphism equivariant under the action of $\mathbb{T}_{h,n,q}^1(\mathbb{F}_q) \times \mathbb{T}_{h,n,q}^1(\mathbb{F}_q) \cong \mathbb{T}_{h,n/p_0,q^{p_0}}^1(\mathbb{F}_q) \times \mathbb{T}_{h,n/p_0,q^{p_0}}^1(\mathbb{F}_q)$. (Note that the determinant condition on the image can be seen by observing that the rows and columns of $x := g_b^{n,q}(v_1, \ldots, v_n)$ can be rearranged so that the matrix becomes block-diagonal of the form $\mathrm{diag}(f(x), \sigma^l(f(x)), \ldots, \sigma^{[l(p_0-1)]_n}(f(x)))$. Hence the determinant of x is fixed by σ if and only if the determinant of $f(x)$ is fixed by σ^{p_0}.)

By the Deligne–Lusztig fixed-point formula,

$$\mathrm{Tr}\left((\zeta, g, t)^*; H_c^*(X_{h,n,q}^1)\right) = \mathrm{Tr}\left((1, g, t)^*; H_c^*(X_{h,n,q}^1)^{(\zeta,1,1)}\right),$$

so that

$$\mathrm{Tr}\left((\zeta, g, 1)^*; H_c^*(X_{h,n,q}^1)[\chi]\right) = \frac{1}{\#\mathbb{T}_h^1(\mathbb{F}_q)} \sum_{t \in \mathbb{T}_h^1(\mathbb{F}_q)} \chi(t)^{-1} \mathrm{Tr}\left((\zeta, g, t)^*; H_c^*(X_{h,n,q}^1)\right)$$

$$= \frac{1}{\#\mathbb{T}_h^1(\mathbb{F}_q)} \sum_{t \in \mathbb{T}_h^1(\mathbb{F}_q)} \chi(t)^{-1} \mathrm{Tr}\left((1, g, t)^*; H_c^*((X_{h,n,q}^1)^{(\zeta,1,1)})[\chi]\right)$$

$$= \frac{1}{\#\mathbb{T}_h^1(\mathbb{F}_q)} \sum_{t \in \mathbb{T}_h^1(\mathbb{F}_q)} \chi(t)^{-1} \mathrm{Tr}\left((1, g, t)^*; H_c^*(X_{h,n/p_0,q^{p_0}}^1)\right)$$

$$= \mathrm{Tr}\left((1, g, 1)^*; H_c^*(X_{h,n/p_0,q^{p_0}}^1)[\chi]\right).$$

The conclusion of the lemma now holds by Corollary 6.5.1. $\qquad\square$

Lemma 6.5.5. *Let $\chi\colon \mathbb{T}_h^1(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$. Assume that we are in one of the following cases:*

(1) $n > 1$ is odd and p_0 is a prime divisor of n.
(2) $n > 1$ is even and $p_0 = 2$.

Fix a $\zeta \in \mathbb{F}_{q^{p_0}}^\times$ such that $\langle \zeta \rangle = \mathbb{F}_{q^{p_0}}^\times$ and consider the extension of χ defined by

$$\widetilde{\chi}\colon \mathbb{F}_{q^{p_0}}^\times \times \mathbb{T}_h^1(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times, \qquad (\zeta^i, g) \mapsto \begin{cases} \chi(g) & \text{if } q \text{ is even,} \\ ((-1)^{s_\chi^{n,q} + s_\chi^{n/p_0,q^{p_0}}})^i \cdot \chi(g) & \text{if } q \text{ is odd.} \end{cases}$$

Then

$$\sum_{x \in \mathbb{F}_{q^{p_0}}^\times \smallsetminus \mathbb{F}_q} \widetilde{\chi}(x, 1)^{-1} \neq 0.$$

Proof. This is the same proof as [Cha19, Lemma 6.2.6]. $\qquad\square$

Proof of Theorem 6.5.2. The proof is exactly as in [Cha19, Theorem 6.2.4]. We give a sketch here. Since $X_{h,1,q}^1 = \mathbb{T}_h^1(\mathbb{F}_q)$ and hence for any $\chi\colon \mathbb{T}_h^1(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$, we have

$$H_c^{s_\chi^{1,q}}(X_{h,1,q}^1)[\chi] = H_c^0(\mathbb{T}_h^1(\mathbb{F}_q))[\chi] = \chi,$$

so Equation (6.6) holds for $n = 1$ and q arbitrary. We induct on the number of prime divisors of n: assume that for a fixed integer $l \geq 0$, Equation (6.6) holds for any $\prod_{i=1}^l p_i$ and arbitrary q, where the p_i are (possibly non-distinct) primes. We will show that Equation (6.6) holds for any $\prod_{i=0}^l p_i$ and arbitrary q. If n is even, let $p_0 = 2$; otherwise, p_0 can be taken to be

anything. Let $\widetilde{\chi}$ be as in Lemma 6.5.5. Then

$$\sum_{(x,g)\in\mathbb{F}_{q^{p_0}}^\times\times\mathbb{T}_h^1(\mathbb{F}_q)} \widetilde{\chi}(x,g)^{-1}\operatorname{Tr}\left((x,1,g);H_c^{s_\chi^{n,q}}(X_{h,n,q}^1)[\chi]\right)$$

$$= \#(\mathbb{F}_q^\times\times\mathbb{T}_h^1(\mathbb{F}_q))\cdot\dim\operatorname{Hom}_{\mathbb{F}_q^\times\times\mathbb{T}_h^1(\mathbb{F}_q)}\left(\widetilde{\chi},H_c^{s_\chi^{n,q}}(X_{h,n,q}^1)[\chi]\right)$$

$$+ \sum_{\substack{(x,g)\in\mathbb{F}_{q^{p_0}}^\times\times\mathbb{T}_h^1(\mathbb{F}_q)\\ x\in\mathbb{F}_{q^{p_0}}^\times\smallsetminus\mathbb{F}_q^\times}} \widetilde{\chi}(x,g)^{-1}\cdot(-1)^{s_\chi^{n,q}+s_\chi^{n/p_0,q^{p_0}}}\cdot\operatorname{Tr}\left((1,1,g);H_c^{s_\chi^{n/p_0,q^{p_0}}}(X_{h,n/p_0,q^{p_0}}^1)[\chi]\right).$$

By the inductive hypothesis together with Lemma 6.5.5, the second summand is a nonzero number, and hence necessarily either the left-hand side is positive or the first summand is positive. In either case, Equation (6.6) must hold. □

For the reader's benefit, we summarize the discussion of this section to prove Theorem 6.1.1.

Proof of Theorem 6.1.1. By Corollary 5.2.2, we know that $H_c^*(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi]$ is (up to sign) an irreducible $\mathbb{G}_h^1(\mathbb{F}_q)$-representation. By Theorem 6.4.1, for any character χ',

$$\operatorname{Hom}_{\mathbb{G}_h^1(\mathbb{F}_q)}\left(\operatorname{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi'),H_c^i(X_h^1,\overline{\mathbb{Q}}_\ell)\right)\neq 0 \qquad\Longleftrightarrow\qquad i=r_{\chi'}.$$

As explained in Corollary 6.5.1, this implies that if $H_c^*(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi]$ is a summand of $\operatorname{Ind}_{\mathbb{T}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h^1(\mathbb{F}_q)}(\chi')$ for some χ', then

$$H_c^i(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi]\neq 0 \qquad\Longleftrightarrow\qquad i=r_{\chi'}=:s_\chi.$$

By Theorem 6.5.2, we see that in fact we can take $\chi'=\chi$, and therefore the nonvanishing cohomological degree of $H_c^i(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi]$ is in fact $i=r_\chi$. The final assertion about the action of Fr_{q^n} on $H_c^{r_\chi}(X_h^1,\overline{\mathbb{Q}}_\ell)[\theta]=(-1)^{r_\chi}H_c^*(X_h^1,\overline{\mathbb{Q}}_\ell)[\theta]$ now follows from Theorem 6.4.1. □

6.6. **Dimension formula.** We use Theorem 6.1.1 to give an explicit dimension formula for the $\mathbb{G}_h^1(\mathbb{F}_q)$-representation $H_c^*(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi]$.

Corollary 6.6.1. *If $\chi\colon\mathbb{T}_h^1(\mathbb{F}_q)\cong\mathbb{W}_h^1(\mathbb{F}_{q^n})\to\overline{\mathbb{Q}}_\ell^\times$ is any character, then*

$$\dim H_c^{r_\chi}(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi]=q^{(n^2-n)(h-1)-nr_\chi/2}.$$

In particular, if χ has trivial $\operatorname{Gal}(L/k)$-stabilizer, then

$$\log_q(\dim H_c^{r_\chi}(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi])=\frac{n}{2}\left(n(h_1-1)-(h_{d'}-1)-(n'-1)-\sum_{t=1}^{d'-1}\frac{n}{m_t}(h_t-h_{t+1})\right).$$

Proof. By applying [Boy12, Lemma 2.12] to calculate the character of $H_c^{r_\chi}(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi]$ at the identity, we have

$$\dim H_c^{r_\chi}(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi]=\frac{(-1)^{r_\chi}}{\lambda\cdot\#\mathbb{T}_h^1(\mathbb{F}_q)}\sum_{t\in\mathbb{T}_h^1(\mathbb{F}_q)}\chi(t)\cdot\#S_{1,t},$$

where $S_{1,t}=\{x\in X_h^1(\overline{\mathbb{F}}_q):\sigma(\operatorname{Fr}_{q^n}(x))=x\cdot t\}$ and λ is the scalar by which Fr_{q^n} acts on $H_c^{r_\chi}(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi]$. Suppose that $x\in S_{1,t}$. Then by the same argument as [CI18, Lemma 9.3], $\det(b\sigma(g_b(x)))=t\cdot\det(b)\det(g_b(x))$, which then forces $t=1$. By construction, $S_{1,1}=\mathbb{G}_h^1(\mathbb{F}_q)$, so therefore

$$\dim H_c^{r_\chi}(X_h^1,\overline{\mathbb{Q}}_\ell)[\chi]=\frac{\#\mathbb{G}_h^1(\mathbb{F}_q)}{q^{nr_\chi/2}\cdot\#\mathbb{T}_h^1(\mathbb{F}_q)}=q^{(n^2-n)(h-1)-nr_\chi/2},$$

where we also use the fact that $\lambda = (-1)^{r_\chi} q^{nr_\chi/2}$ from Theorem 6.1.1. The assertion in the case that χ has trivial $\mathrm{Gal}(L/k)$-stabilizer follows from Corollary 6.1.2. $\qquad\square$

7. Conjectures

7.1. **Concentration in a single degree.** Recall that from Corollary 5.2.2, we know that if $\theta\colon \mathbb{T}_h(\mathbb{F}_q) \cong \mathbb{W}_h^\times(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ is a character with trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0 r}})$-stabilizer, then the alternating sum $H_c^*(X_h \cap \mathbb{L}_h^{(r)}\mathbb{G}_h^1, \overline{\mathbb{Q}}_\ell)[\theta]$ is (up to sign) an irreducible $\mathbb{L}_h^{(r)}(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q)$-representation. We conjecture that in fact these cohomology groups should be concentrated in a single degree.

Conjecture 7.1.1. *Let $r \mid n'$ and let $\theta\colon \mathbb{T}_h(\mathbb{F}_q) \cong \mathbb{W}_h^\times(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ be a character with trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0 r}})$-stabilizer. Then there exists an integer $i_{\theta,r}$ such that*

$$H_c^i(X_h \cap \mathbb{L}_h^{(r)}\mathbb{G}_h^1, \overline{\mathbb{Q}}_\ell)[\theta] \neq 0 \qquad \Longleftrightarrow \qquad i = i_{\theta,r}.$$

In this paper, we proved this conjecture in the case $r = n'$ and in fact pinpointed the nonvanishing cohomological degree $i_{\theta,n'}$ (Theorem 6.1.1). We expect that a similar formula for $i_{\theta,r}$ should be obtainable, where the methods in this paper can be used to reduce the determination of $i_{\theta,r}$ to a "depth-zero" setting. The hypotheses of Conjecture 7.1.1 should be equivalent to saying that the consequent depth-zero input comes from the θ_0-isotypic part of the cohomology of a classical Deligne–Lusztig variety (of dimension $r - 1$) for the twisted Levi $\mathbb{L}_{1,r}$ in $\mathbb{G}_1$, where θ_0 is a character of $\mathbb{T}_1(\mathbb{F}_q) \cong \mathbb{F}_{q^n}^\times$ in general position.

7.2. **Relation to loop Deligne–Lusztig varieties.** The varieties X_h are closely related to a conjectural construction of Deligne–Lusztig varieties for p-adic groups initiated by Lusztig [Lus79]. We call these sets *loop Deligne–Lusztig varieties*, although the algebro-geometric structure is still unknown in general.

In [CI18], we studied this question for a certain class of these sets attached to inner forms of GL_n. We prove (see also [CI19b, Proposition 2.6]) that the fpqc-sheafification X of the presheaf on category $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ of perfect $\mathbb{F}_q$-schemes

$$X\colon R \mapsto \{x \in LG(R) : x^{-1}F(x) \in LU(R)\}/L(U \cap F^{-1}U)$$

is representable by a perfect $\overline{\mathbb{F}}_q$-scheme and that X is the perfection of

$$\bigsqcup_{g \in G(k)/G_{x,0}(\mathcal{O}_k)} g \cdot \varprojlim_h X_h.$$

We see that an intermediate step to understanding the cohomology of loop Deligne–Lusztig is to calculate the cohomology of X_h.

However, for various reasons, it is often easier to calculate the cohomology of the Drinfeld stratification. For example, in [CI19b], to prove cuspidality of $H_*(X, \overline{\mathbb{Q}}_\ell)[\theta]$ for a broad class of characters $\theta\colon T(k) \to \overline{\mathbb{Q}}_\ell^\times$, we calculate the formal degree of this representation, which we achieve by calculating the dimension of $H_c^*(X_h^{(n')}, \overline{\mathbb{Q}}_\ell)[\theta]$ from the Frobenius eigenvalues (see Corollary 6.6.1). In this setting, we can prove a comparison formula between the cohomology of $X_h^{(n')}$ and the cohomology of X_h (see Section 7.2.1.2).

We conjecture the following comparison theorem between the cohomology of X_h and its Drinfeld stratification. In Section 7.2.1, we present evidence supporting the truth of this conjecture.

Conjecture 7.2.1. *Let $r \mid n'$ and let $\theta\colon \mathbb{T}_h(\mathbb{F}_q) \cong \mathbb{W}_h^\times(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ be a character with trivial $\mathrm{Gal}(L/k)$-stabilizer. Let $\chi := \theta|_{\mathbb{W}_h^1(\mathbb{F}_{q^n})}$ and assume that the stabilizer of χ in $\mathrm{Gal}(L/k)$ is*

equal to the unique index-$n_0 r$ subgroup. Then we have an isomorphism of virtual $\mathbb{G}_h(\mathbb{F}_q)$-representations

$$H_c^*(X_h, \overline{\mathbb{Q}}_\ell)[\theta] \cong H_c^*(X_h^{(r)}, \overline{\mathbb{Q}}_\ell)[\theta].$$

Combining Conjectures 7.1.1 and 7.2.1 with Corollary 5.2.2, the above conjecture asserts that as elements of the Grothendieck group of $\mathbb{G}_h(\mathbb{F}_q)$,

$$H_c^*(X_h, \overline{\mathbb{Q}}_\ell)[\theta] = (-1)^{i_{\theta,r}} H_c^{i_{\theta,r}}(X_h^{(r)}, \overline{\mathbb{Q}}_\ell)[\theta]$$

$$= (-1)^{i_{\theta,r}} \operatorname{Ind}_{\mathbb{L}_h^{(r)}(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q)}^{\mathbb{G}_h(\mathbb{F}_q)} \left(H_c^{i_{\theta,r}}(X_h \cap \mathbb{L}_h^{(r)}\mathbb{G}_h^1, \overline{\mathbb{Q}}_\ell)[\theta] \right).$$

7.2.1. *Evidence.* At present, we can prove Conjecture 7.2.1 in some special cases. We discuss these various cases, their context, and the ideas involved in the proof.

7.2.1.1. The most degenerate setting of Conjecture 7.2.1 is when G is a division algebra over k. Then $n' = 1$ and so the closed Drinfeld stratum $X_h^{(n')} = X_h^{(1)}$ is the only Drinfeld stratum. Additionally, we have that $X_h^{(n')}$ is a disjoint union of $\#\mathbb{G}_h(\mathbb{F}_q)$ copies of $X_h^1 :=$ $X_h \cap \mathbb{G}_h^1$. In [Cha19], all the technical calculations happen at the level of X_h^1 (though in different notation in *op. cit.*), and using the new methods developed there, one knows nearly everything about the representations $H_c^i(X_h^1, \overline{\mathbb{Q}}_\ell)[\chi]$ for arbitrary characters $\chi \colon \mathbb{T}_h^1(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$. However, the expected generalization of these techniques extend not to $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$, but to $H_c^i(X_h^{(r)}, \overline{\mathbb{Q}}_\ell)[\chi]$—hence one is really forced to work on the stratum in order to approach X_h (at least with the current state of technology).

7.2.1.2. Now let G be any inner form of GL_n (as it has been this entire paper, outside Section 3). We are close to establishing Conjecture 7.2.1 when $\chi = \theta|_{\mathbb{W}_h^1(\mathbb{F}_{q^n})}$ has trivial $\operatorname{Gal}(L/k)$-stabilizer. In this case, Conjecture 7.2.1 says that $H_c^*(X_h, \overline{\mathbb{Q}}_\ell)[\theta] \cong H_c^*(X_h^{(n')}, \overline{\mathbb{Q}}_\ell)[\theta]$ as virtual $\mathbb{G}_h(\mathbb{F}_q)$-representations. In [CI19b, Theorem 4.1], we prove this isomorphism holds under the additional assumption that $p > n$. The idea here is to use a highly nontrivial generalization of a method of Lusztig to calculate the inner product $\langle H_c^*(X_h, \overline{\mathbb{Q}}_\ell)[\theta], H_c^*(X_h^{(n')}, \overline{\mathbb{Q}}_\ell)[\theta] \rangle$ in the space of conjugation-invariant functions on $\mathbb{G}_h(\mathbb{F}_q)$.

7.2.1.3. In Appendix A, we present a possible geometric approach to Conjecture 7.2.1 which has its roots in the GL_2 setting of the proof of [Iva16, Theorem 3.5]. The idea is to study the fibers of the natural projection[2] $\pi \colon X_h \to X_{h-1}$. We can show that the behavior of $\pi^{-1}(x)$ depends *only* on the location of x relative to the Drinfeld stratification of X_h: If r is the smallest divisor of n' such that $x \in X_h^{(r)}$ (i.e. x is in the rth Drinfeld stratum $X_{h,r}$ of X_h), then there exists a morphism

$$\pi^{-1}(x) \to \bigsqcup_{\mathbb{W}_h^{h-1}(\mathbb{F}_{q^{n_0 r}})} \mathbb{A}^{n-1}$$

which is a composition of isomorphisms and purely inseparable morphisms. Moreover, the action of $\ker(\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) \to \mathbb{W}_h^{h-1}(\mathbb{F}_{q^{n_0 r}}))$ on $\pi^{-1}(x)$ fixes the set of connected components. The crucial point here is that the fibers of the natural map

$$X_{h,r}/\ker(\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) \to \mathbb{W}_h^{h-1}(\mathbb{F}_{q^{n/(n_0 r)}})) \to X_{h-1,r}$$

---

[2]When $G = \operatorname{GL}_n$, then this is literally what we do in Appendix A. When G is a nonsplit inner form of GL_n, in order to get a shape analogous to the split case, we work with an auxiliary scheme which is an affine fibration over X_h.

are again isomorphic to $\bigsqcup_{\mathbb{W}_h^{h-1}(\mathbb{F}_{q^{n_0 r}})} \mathbb{A}^{n-1}$ and therefore $\ker(\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) \to \mathbb{W}_h^{h-1}(\mathbb{F}_{q^{n/(n_0 r)}}))$ acts trivially on the cohomology of $X_{h,r}$:

$$H_c^*(X_{h,r}, \overline{\mathbb{Q}}_\ell) \cong H_c^*(X_{h,r}, \overline{\mathbb{Q}}_\ell)^{\ker(\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) \to \mathbb{W}_h^{h-1}(\mathbb{F}_{q^{n/(n_0 r)}}))}.$$

Using open/closed decompositions of X_h via Drinfeld strata, we have that if θ is trivial on $\ker(\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) \to \mathbb{W}_h^{h-1}(\mathbb{F}_{q^{n/(n_0 r)}}))$, then

$$H_c^*(X_h, \overline{\mathbb{Q}}_\ell)[\theta] \cong H_c^*(X_h^{(r)}, \overline{\mathbb{Q}}_\ell)[\theta]$$

as virtual $\mathbb{G}_h(\mathbb{F}_q)$-representations. It seems reasonable to guess that if one can generalize Appendix A to study the fibers of $X_h \to X_1$, then one could establish Conjecture 7.2.1 using a similar reasoning as above.

Appendix A. The geometry of the fibers of projection maps

In this section, we study the fibers of the projection maps $X_h \to X_{h-1}$. This is a technical computation which we perform by first using the isomorphism $X_h \cong X_h(b, b_{\mathrm{cox}})$ for a particular choice of b which we call the *special representative*. This is the first time in this paper that we see the convenience of having the alternative presentations of X_h discussed in Sections 3.2 and 4.5.

A.1. **The special representative.** We first recall the content of Section 4.5 in the context of a particular representative of the σ-conjugacy class corresponding to the fixed integer κ.

Definition A.1.1. The *special representative* b_{sp} attached to κ is the block-diagonal matrix of size $n \times n$ with $(n_0 \times n_0)$-blocks of the form $\begin{pmatrix} 0 & \varpi \\ 1_{n_0-1} & 0 \end{pmatrix}^\kappa$.

By [CI18, Lemma 5.6], there exists a $g_0 \in G_{x,0}(\mathcal{O}_{\check{k}})$ such that $b_{\mathrm{sp}} = g_0 b_{\mathrm{cox}} \sigma(g_0)^{-1}$. Observe further that since $b_{\mathrm{sp}}, b_{\mathrm{cox}}$ are σ-fixed and $b_{\mathrm{sp}}^n = b_{\mathrm{cox}}^n = \varpi^{kn}$,

$$\sigma^n(g_0) = g_0.$$

Therefore b_{sp} satisfies the conditions of Lemma 4.5.3. Recall from Section 4.5 that we have

$$X_h \cong X_h(b_{\mathrm{sp}}, b_{\mathrm{cox}}) \cong \{v \in \mathscr{L}_h : \sigma(\det g_{b_{\mathrm{sp}}}(v)) = (-1)^{n-1} \det g_{b_{\mathrm{sp}}}(v) \in \mathbb{W}_h^\times\}, \qquad (A.1)$$

where

$$\mathscr{L}_h = (\mathbb{W}_h \oplus (V\mathbb{W}_{h-1})^{\oplus n_0-1})^{\oplus n'} \subset \mathbb{W}_h^{\oplus n}$$

$$g_{b_{\mathrm{sp}}}(v) = (v_1 \mid v_2 \mid v_3 \mid \cdots \mid v_n)$$

$$\text{where } v_i = \varpi^{\lfloor (i-1)k_0/n_0 \rfloor} \cdot (b_{\mathrm{sp}}\sigma)^{i-1}(v) \text{ for } 1 \le i \le n-1.$$

In this section, we will work with

$$X_h^+ := \{v \in \mathscr{L}_h^+ : \sigma(\det g_{b_{\mathrm{sp}}(v)}) = \det g_{b_{\mathrm{sp}}}(v) \in \mathbb{W}_h^\times\} \qquad (A.2)$$

where $\mathscr{L}_h^+$ is now the subquotient of $\mathbb{W}_{h+1}^{\oplus n}$

$$\mathscr{L}_h^+ := (\mathbb{W}_h \oplus (V\mathbb{W}_h)^{\oplus n_0-1})^{\oplus n'},$$

and $g_{b_{\mathrm{sp}}}(v)$ is defined as before. Note that (A.1) differs from (A.2) in that the former takes place in $G_{x,0}/G_{x,(h-1)+}$ and the latter takes place in $G_{x,0}/G_{x,h}$. A straightforward computation shows that the defining equation of X_h^+ does not depend on the quotient $\mathscr{L}_h^+/\mathscr{L}_h = \mathbb{A}^{n-n'}$.

Observe that $\det g_{b\,\mathrm{sp}}(\zeta v) = \mathrm{Nm}(\zeta) \cdot \det g_{b\,\mathrm{sp}}(\zeta v)$ where $\mathrm{Nm}(\zeta) = \zeta \cdot \sigma(\zeta) \cdot \sigma^2(\zeta) \cdots \sigma^{n-1}(\zeta)$. Picking any ζ such that $\sigma(\mathrm{Nm}(\zeta)) = (-1)^{n-1}\,\mathrm{Nm}(\zeta)$ allows us to undo the $(-1)^{n-1}$ factor in the defining equation in (A.1). In particular, this means

$$H_c^i(X_h^+, \overline{\mathbb{Q}}_\ell) = H_c^{i+2(n-n')}(X_h, \overline{\mathbb{Q}}_\ell), \qquad \text{for all } i \geq 0.$$

For each divisor $r \mid n'$, we define the rth Drinfeld stratum $X_{h,r}^+$ of X_h^+ to be the preimage of $X_{h,r}$ under the natural surjection $X_h^+ \to X_h$.

A.2. **Fibers of $X_{h,r}^+ \to X_{h-1,r}^+$.** For notational convenience, we write $b = b_{\mathrm{sp}}$. We may identify $\mathscr{L}_h^+ = \mathbb{A}^{n(h-1)}$ with coordinates $x = \{x_{i,j}\}_{1 \leq i \leq n, 0 \leq j \leq h-1}$ which we typically write as $x = (\widetilde{x}, x_{1,h-1}, x_{2,h-1}, \ldots, x_{n,h-1}) \in \mathscr{L}_{h-1}^+ \times \mathbb{A}^n$; here, an element $v = (v_1, \ldots, v_n) \in \mathscr{L}_h^+$ is such that $v_i = [x_{i,0}, x_{i,1}, \ldots, x_{i,n}]$ if $i \equiv 1 \pmod{n_0}$ and $v_i = [0, x_{i,0}, x_{i,1}, \ldots, x_{i,n}]$ if $i \not\equiv 1 \pmod{n_0}$.

In this section, fix a divisor $r \mid n'$. From the definitions, $X_{h,r}^+$ can be viewed as the subvariety of $X_{h-1,r}^+ \times \mathbb{A}^n$ cut out by the equation

$$0 = P_0(x)^q - P_0(x),$$

where P_0 is the coefficient of ϖ^{h-1} in the expression $\det g_b^{\mathrm{red}}(v)$. Let c denote the polynomial consisting of the terms of $P_0(x)$ which only depend on $\widetilde{x}$. An explicit calculation shows that there exists a polynomial P_1 in x such that

$$P_0(x) = c(\widetilde{x}) + \sum_{i=0}^{n_0-1} P_1(x)^{q^i}. \tag{A.3}$$

Therefore $X_{h,r}^+$ is the subvariety of $X_{h-1,r}^+ \times \mathbb{A}^n$ cut out by

$$P_1(x)^{q^{n_0}} - P_1(x) = c(\widetilde{x}) - c(\widetilde{x})^q.$$

One can calculate P_1 explicitly (see [CI18, Proposition 7.5]):

Lemma A.2.1. *Explicitly, the polynomial P_1 is*

$$P_1(x) = \sum_{1 \leq i,j \leq n'} m_{ji} x_{1+n_0(i-1),h-1}^{q^{(j-1)n_0}},$$

where $m := (m_{ji})_{j,i}$ is the adjoint matrix of $\overline{g_b}(\bar{x})$ and $\bar{x}$ denotes the image of x in $\overline{V} = \mathscr{L}_0/\mathscr{L}_0^{(1)}$. Explicitly, $m \cdot \overline{g_b}(\bar{x}) = \det \overline{g_b}(\bar{x})$ and the (j,i)th entry of m is $(-1)^{i+j}$ times the determinant of the $(n'-1) \times (n'-1)$ matrix obtained from $\overline{g_b}(\bar{x})$ by deleting the ith row and jth column.

The main result of this section is:

Proposition A.2.2. *There exists an $X_{h-1,r}^+$-morphism*

$$M_r \colon X_{h-1,r}^+ \times \mathbb{A}^n \to X_{h-1,r}^+ \times \mathbb{A}^n$$

(the left $\mathbb{A}^n$ in terms of the coordinates $\{x_{i,h-1}\}_{i=1}^n$ and the right $\mathbb{A}^n$ in terms of new coordinates $\{z_i\}_{i=1}^n$) satisfying the following properties:

(i) M_r is a composition of $X_{h-1,r}^+$-isomorphisms and purely inseparable $X_{h-1,r}^+$-morphisms.
(ii) $M_r(X_{h,r}^+)$ is the closed subscheme defined by the equation

$$z_1^{q^{n_0 r}} - z_1 = c(\widetilde{x}) - c(\widetilde{x})^q,$$

where c is as in (A.3).

(iii) M_r is $\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$-equivariant after equipping the left $X_{h-1,r}^+ \times \mathbb{A}^n$ with the $\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$-action

$$1 + \varpi^{h-1}a \colon x_{i,h-1} \mapsto x_{i,h-1} + x_{i,0}a, \qquad \text{for all } 1 \leq i \leq n,$$

and the right $X_{h-1,r}^+ \times \mathbb{A}^n$ with the $\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$-action

$$1 + \varpi^{h-1}a \colon z_i \mapsto \begin{cases} z_1 + \mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0 r}}}(a) & \text{if } i = 1, \\ z_2 + a & \text{if } r \neq n' \text{ and } i = 2, \\ z_i & \text{otherwise.} \end{cases}$$

In the rest of this section we prove Proposition A.2.2. To simplify the notation we will first establish the proposition in the case $\kappa = 0$ (i.e. $G = \mathrm{GL}_n$), and at the end generalize it to all κ. The first part of the proof of Proposition A.2.2 is given by the lemma below. Before stating it, we establish some notation. For an ordered basis $\mathscr{B}$ of V and $v \in V$, let $v_{\mathscr{B}}$ denote the coordinate vector of v in the basis $\mathscr{B}$. For two ordered bases $\mathscr{B}, \mathscr{C} = \{c_i\}_{i=1}^n$ of V, let $M_{\mathscr{B},\mathscr{C}}$ denote the base change matrix between them, that is, the ith column vector of $M_{\mathscr{B},\mathscr{C}}$ is $c_{i,\mathscr{B}}$. It is clear that

- $M_{\mathscr{C},\mathscr{B}} = M_{\mathscr{B},\mathscr{C}}^{-1}$,
- for any $v \in V$, $M_{\mathscr{B},\mathscr{C}} v_{\mathscr{C}} = v_{\mathscr{B}}$,
- for a third ordered basis $\mathscr{D}$ of V, one has $M_{\mathscr{B},\mathscr{C}} M_{\mathscr{C},\mathscr{D}} = M_{\mathscr{B},\mathscr{D}}$.

For a linear map $f \colon V \to V$, let $M_{\mathscr{B},\mathscr{C}}(f)$ denote the matrix representation of f; that is, $M_{\mathscr{B},\mathscr{C}}(f) \cdot v_{\mathscr{C}} = f(v)_{\mathscr{B}}$. In V we have the two ordered bases:

$$\mathscr{E} := \text{ the standard basis of } V, \text{ arising from the basis } \{e_i\} \text{ of the lattice } \mathscr{L}_0,$$

$$\mathscr{B}_x := \{\sigma_b^{i-1}(x)\}_{i=1}^n, \text{attached to the given } x \in X_0^+.$$

We identify V with $\overline{\mathbb{F}}_q^n$ via the standard basis $\mathscr{E}$ and write $v = v_{\mathscr{E}}$ for all $v \in V$.

Lemma A.2.3. *Assume $\kappa = 0$. There exists an $X_{h-1,r}^+$-isomorphism $X_{h-1,r}^+ \times \mathbb{A}^n \xrightarrow{\sim} X_{h-1,r}^+ \times \mathbb{A}^n$ given by a linear change of variables $x_{i,h-1} \rightsquigarrow x_{i,h-1}'$, such that P_1 in the new coordinates $x_{i,h-1}'$ takes the form*

$$P_1 = x_{1,h-1}' + x_{1,h-1}'^{,q} + \cdots + x_{1,h-1}'^{,q^{n-1}} + \sum_{j=0}^{s} \sum_{\lambda=i_j+1}^{i_{j+1}} x_{s+2-j,h-1}'^{,q^\lambda},$$

and the action of $1 + \varpi^{h-1}a \in W_h^{h-1}(\mathbb{F}_{q^n})$ on the coordinates $x_{i,h-1}'$ is given by

$$x_{i,h-1}' \mapsto \begin{cases} x_{1,h-1}' + a & \text{if } i = 1, \\ x_{i,h-1} & \text{if } i \geq 2. \end{cases} \tag{A.4}$$

Proof of Lemma A.2.3. We have to find a morphism $C := (c_{ij}) \colon X_{h-1,r}^+ \to \mathrm{GL}(V) = \mathrm{GL}_{n,\mathbb{F}_q}$ (this identification uses the standard basis $\mathscr{E}$ of V) such that the corresponding linear change of coordinates

$$x_{i,h-1} = c_{i,1}x_{1,h-1}' + c_{i,2}x_{2,h-1}' + \cdots + c_{i,n}x_{n,h-1}', \text{ for all } 1 \leq i \leq n. \tag{A.5}$$

brings P_1 to the requested form. Moreover, it suffices to do this fiber-wise by first determining $C(\tilde{x})$ for any point $\tilde{x} \in X_{h-1,r}^+$ and then seeing that $\tilde{x} \mapsto C(\tilde{x})$ is in fact an algebraic morphism.

Fix $\tilde{x} \in X_{h-1,r}^+$ with image $x \in X_1^+$, and write C instead of $C(\tilde{x})$ to simplify notation. Let C_i denote the ith column of C. Our coordinate change replaces P_1 by the polynomial (after

dividing by the irrelevant non-zero constant $\det g_b(x) \in \mathbb{F}_q^\times$)

$$
\begin{aligned}
P_1 = {}& x'_{1,h-1}(m_1 \cdot C_1) + x'^{,q}_{1,h-1}(m_2 \cdot \sigma_b(C_1)) + x'^{,q^2}_{1,h-1}(m_3 \cdot \sigma_b^2(C_1)) + \cdots + x'^{,q^{n-1}}_{1,h-1}(m_n \cdot \sigma_b^{n-1}(C_1)) \\
&+ x'_{2,h-1}(m_1 \cdot C_2) + x'^{,q}_{2,h-1}(m_2 \cdot \sigma_b(C_2)) + x'^{,q^2}_{2,h-1}(m_3 \cdot \sigma_b^2(C_2)) + \cdots + x'^{,q^{n-1}}_{2,h-1}(m_n \cdot \sigma_b^{n-1}(C_2)) \\
&+ \cdots + \\
&+ x'_{n,h-1}(m_1 \cdot C_n) + x'^{,q}_{n,h-1}(m_2 \cdot \sigma_b(C_n)) + x'^{,q^2}_{n,h-1}(m_3 \cdot \sigma_b^2(C_n)) + \cdots + x'^{,q^{n-1}}_{n,h-1}(m_n \cdot \sigma_b^{n-1}(C_n))
\end{aligned}
\tag{A.6}
$$

in the indeterminates $\{x'_{i,h-1}\}_{i=1}^n$. Here, we write m_i to mean the ith row of the matrix m (adjoint to $g_b(x)$) from Lemma A.2.1. For $z \in V$, we put

$$
m * z = \sum_{i=1}^n (m_i \cdot (b\sigma)^{i-1}(z)) e_i. \tag{A.7}
$$

The intermediate goal is to describe the map $m*: V \to V$ in terms of a coordinate matrix. Of course, $m*$ is not linear, but its composition with the projection on the ith component (corresponding to the ith standard basis vector) is σ^{i-1}-linear. Thus we instead will describe the linear map $(m*)': V \to V$, which is the composition of $m*$ and the map $\sum_i v_i e_i \mapsto \sum_i \sigma^{-(i-1)}(v_i) e_i$. This is done by the following lemma.

Lemma A.2.4. *Assume $\kappa = 0$. We have*

$$
M_{\mathcal{E},\mathscr{B}_x}((m*)') =
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & \sigma^{-1}(y_1) \\
\vdots & \vdots & & \vdots & \sigma^{-2}(y_2) & * \\
1 & 0 & 0 & & * & \vdots \\
1 & 0 & \sigma^{-(n-2)}(y_{n-2}) & & \vdots & * \\
1 & \sigma^{-(n-1)}(y_{n-1}) & * & \cdots & * & *
\end{pmatrix}
$$

where the y_i's are defined by the equation

$$
(b\sigma)^n(\mathfrak{v}) = v + \sum_{i=1}^{n-1} y_i (b\sigma)^i(\mathfrak{v}).
$$

More precisely, if $\mu_{i,j}$ denotes the (i,j)th entry of $\det(g_b(\bar{x}))^{-1} M_{\mathcal{E},\mathscr{B}_x}((m*)')$, then for $1 \le i, j \le n$ we have

$$
\mu_{i,j} =
\begin{cases}
1 & \text{if } j = 1 \\
0 & \text{if } i + j \le n + 1 \text{ and } j > 1 \\
\sigma^{-(i-1)}(y_{i-1}) & \text{if } i + j = n + 2 \\
\mu_{i-1,j} + \sigma^{-(i-1)}(y_{i-1})\sigma^{n-(i-1)}(\mu_{n,j+i-(n+1)}) & \text{if } i + j \ge n + 3 \text{ and } i \ge 3.
\end{cases}
$$

In particular, if $i + j \ge n + 3$ and $y_{i-1} = 0$, then $\mu_{i,j} = \mu_{i-1,j}$.

Proof of Lemma A.2.4. Let $z = \sum_{i=1}^n z_i (b\sigma)^{i-1}(x)$ be a generic element of V, written in $\mathscr{B}_x$-coordinates, that is $z_{\mathscr{B}_x}$ is the n-tuple $(z_i)_{i=1}^n$. The (i,j)th entry of $M_{\mathcal{E},\mathscr{B}_x}((m*)')$ is equal to $\sigma^{-(i-1)}$ applied to the coefficient of $\sigma^{i-1}(z_j)$ in the ith entry of $(b\sigma)^{i-1}(z)_{\mathscr{B}_x}$ (= the ith entry of $m * z$).

The coordinate matrix of the σ-linear operator $b\sigma \colon V \to V$ in the basis $\mathscr{B}_x$,

$$M_{\mathscr{B}_x, \mathscr{B}_x}(b\sigma) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & y_1 \\ 0 & 1 & \ddots & \vdots & y_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & y_{n-1} \end{pmatrix}.$$

That is, for any $z \in V$,

$$b\sigma(z)_{\mathscr{B}_x} = M_{\mathscr{B}_x, \mathscr{B}_x}(b\sigma) \cdot \sigma(z_{\mathscr{B}_x}), \tag{A.8}$$

where the last σ is applied entry-wise. Explicitly, the first entry of $b\sigma(z)_{\mathscr{B}_x}$ is $\sigma(z_n)$, and for $2 \le i \le n$ the ith entry of $b\sigma(z)_{\mathscr{B}_x}$ is $\sigma(z_{i-1}) + y_{i-1}\sigma(z_n)$. This allows to iteratively compute $b\sigma^i(z)$ for all i, which we do to finish the proof.

First, we see that z_1 can occur in the nth (i.e. last) entry of $(b\sigma)^{\lambda-1}(z)_{\mathscr{B}_x}$ only if $\lambda \ge n$; hence its contribution to the ith entry of $(b\sigma)^{i-1}(z)_{\mathscr{B}_x}$ for $i \le n$ is simply $\sigma^{i-1}(z_1)$. This shows that the first column of $M_{\mathscr{E}, \mathscr{B}_x}((m*)')$ consists of 1's. Assume now $j \ge 2$. Then there is a smallest (if any) i_0, such that z_j occurs in the i_0th entry of $(b\sigma)^{i_0-1}(z)_{\mathscr{B}_x}$. Note that as $j \ge 2$, one has $i_0 \ge 2$. Then z_j must have been occurred in the nth entry of $(b\sigma)^{i_0-2}(z)_{\mathscr{B}_x}$. As z_j occurs in $z_{\mathscr{B}_x}$ in exactly the jth entry, and it needs $(n-j)$ times to apply $b\sigma$ to get it to the nth entry, we must have $i_0 - 2 \ge n - j$. This shows that the (i,j)th entry of $M_{\mathscr{E}, \mathscr{B}_x}((m*)')$ is 0, unless $i \ge n + 2 - j$. The same consideration shows that if $i = n + 2 - j$, then $\sigma^{i-1}(z_j)$ has the coefficient y_{i-1} in $\sigma_b^{i-1}(z)_{\mathscr{B}_x}$. This gives the entries of $M_{\mathscr{E}, \mathscr{B}_x}((m*)')$ on the diagonal $i = n + 2 - j$. It remains to compute the entries below it, so assume $i > n + 2 - j$. Again, by the characterization of the entries of $M_{\mathscr{E}, \mathscr{B}_x}((m*)')$ in the beginning of the proof and by the explicit description of how σ_b acts (in the $\mathscr{B}_x$-coordinates), it is clear that the (i,j)th entry of $M_{\mathscr{E}, \mathscr{B}_x}((m*)')$ is just the sum of the $(i-1,j)$th entry and $\sigma^{-(i-1)}(y_{i-1})\sigma^{n-(i-1)}((n, j-1)$th entry$)$. This finishes the proof of Lemma A.2.4. $\qquad\square$

Now we continue the proof of Lemma A.2.3. Let $\mathscr{C}$ denote the ordered basis of V consisting of columns $C_1, C_2, \ldots, C_n$ of C. We have $M_{\mathscr{B}_x, \mathscr{C}} = (\det g_b(x))^{-1} m \cdot C$. In particular, to give the invertible matrix C it is equivalent to give the invertible matrix $M_{\mathscr{B}_x, \mathscr{C}}$. But the ith column of $M_{\mathscr{B}_x, \mathscr{C}}$ is the coordinate vector of C_i in the basis $\mathscr{B}_x$, i.e., what we denoted $C_{i, \mathscr{B}_x}$. We now show that one can find an invertible $M_{\mathscr{B}_x, \mathscr{C}}$, such that for its columns $C_{i, \mathscr{B}_x}$ we have

$$m * C_{1, \mathscr{B}_x} = \sum_{\lambda=1}^{n} e_\lambda$$

$$m * C_{s+2-j, \mathscr{B}_x} = \sum_{\lambda=i_j+1}^{i_j} e_\lambda \qquad \text{for } s \ge j \ge 0, \tag{A.9}$$

$$m * C_{j', \mathscr{B}_x} = 0 \qquad \text{if } j' > s + 2.$$

Taking into account equation (A.6) and the definition of $m*$ in (A.7), this (plus the fact that $x \mapsto M_{\mathscr{B}_x, \mathscr{C}}$ will in fact an algebraic morphism) finishes the proof of Lemma A.2.3, except for the claim regarding the $W_h^{h-1}(\mathbb{F}_{q^n})$-action.

To find $M_{\mathscr{B}_x, \mathscr{C}}$ satisfying (A.9), first observe that by Lemma A.2.4, there is some invertible matrix S depending on $\tilde{x} \in X_{h-1,r}^+$ (in fact, only on its image $x \in X_1^+$), such that $M_{\mathscr{E}, \mathscr{B}_x}((m*)') \cdot S$ has the following form: its first column consists of 1's; its ith column is 0, unless $i = n + 1 - i_j$ for some $s \ge j \ge 0$; for $s \ge j \ge 0$, the λth entry of its $(n + 1 - i_j)$th column is 1 if $i_j + 1 \le \lambda \le i_{j+1}$ (we put $i_{s+1} := n$ here) and zero otherwise. (To show this, use the general shape of $M_{\mathscr{E}, \mathscr{B}_x}((m*)')$ provided by Lemma A.2.4, and then consecutively

apply row operations to it and use the last statement of Lemma A.2.4). Moreover, it is also clear from Lemma A.2.4 that S will be upper triangular with the upper left entry $= 1$.

Secondly, let T be a matrix such that: the first row has 1 in the first position and zeros otherwise; all except for the first entry of the first column are 0; for $s \geq j \geq 0$, the $(n+1-i_j)$th row has 1 in the $(s+2-j)$th position and 0's otherwise; the remaining rows can be chosen arbitrarily. Obviously, T can be chosen to be a permutation matrix with entries only 0 or 1, and in particular invertible and independent of x. Finally, put $M_{\mathscr{B}_x, \mathscr{C}} := S \cdot T$. Explicitly the columns of the matrix

$$M_{\mathscr{E}, \mathscr{B}_x}((m*)') \cdot M_{\mathscr{B}_x, \mathscr{C}} = (M_{\mathscr{E}, \mathscr{B}_x}((m*)') \cdot S) \cdot T \tag{A.10}$$

are as follows: the first column consist of 1's; for $s \geq j \geq 0$, the the λth entry of the $(s+2-j)$th column is 1 if $i_j + 1 \leq \lambda \leq i_{j+1}$, and zero otherwise; all other columns consist of 0's. On the other side, the jth column of of $M_{\mathscr{E}, \mathscr{B}_x}((m*)') \cdot M_{\mathscr{B}_x, \mathscr{C}}$ is precisely $m * C_{j, \mathscr{B}_x}$ (up to the unessential σ^{-*}-twist in each entry). This justifies (A.9).

The action of $1 + \varpi^h a \in W_h^{h-1}(\mathbb{F}_{q^n})$ on the coordinates $x_{i,h}$ is given by $(x_{i,h})_{i=1}^n \mapsto (x_{i,h} + ax_{i,0})_{i=1}^n$. We determine the action $1 + \varpi^h a$ in the coordinates $x'_{i,h}$. Indeed, let $C^{-1} = (d_{i,j})_{1 \leq i,j \leq n}$. Then $1 + \varpi^h a$ acts on $x'_{i,h}$ by

$$x'_{i,h} = \sum_{j=1}^n d_{i,j} x_{j,h} \mapsto \sum_{j=1}^n d_{i,j}(x_{j,h} + ax_{j,0}) = x'_{i,h} + a \sum_{j=1}^n d_{i,j} x_{j,0}.$$

Organizing the $x_{i,h}$ for $1 \leq i \leq n$ in one (column) vector, we can rewrite this as

$$1 + \varpi^h a \colon (x'_{i,h})_{i=1}^n \mapsto (x'_{i,h})_{i=1}^n + aC^{-1} \cdot x.$$

We determine $C^{-1} \cdot x$. As $M_{\mathscr{B}_x, \mathscr{C}} = \det(g_b(x))^{-1} mC = g_b(x)^{-1} C$ (as $\det(g_b(x))^{-1} m = g_b(x)^{-1}$), we have $C^{-1} = M_{\mathscr{B}_x, \mathscr{C}}^{-1} g_b(x)^{-1}$. But x is the first column of $g_b(x)$, thus

$$C^{-1} \cdot x = M_{\mathscr{B}_x, \mathscr{C}}^{-1} g_b(x)^{-1} \cdot x = M_{\mathscr{B}_x, \mathscr{C}}^{-1} \cdot (1, 0, \ldots, 0)^{\mathsf{T}},$$

so $C^{-1} \cdot x$ is the first column of $M_{\mathscr{B}_x, \mathscr{C}}^{-1} = (ST)^{-1} = T^{-1} S^{-1}$. But S is upper triangular with upper left entry $= 1$, so the first column of $M_{\mathscr{B}_x, \mathscr{C}}^{-1}$ is the first column of T^{-1}, which is $(1, 0, \ldots, 0)^{\mathsf{T}}$. This finishes the proof of the lemma. $\square$

The second part of the proof is given by the following lemma.

Lemma A.2.5. *Assume $\kappa = 0$. There exists a $X_{h-1,r}^+$-morphism $X_{h-1,r}^+ \times \mathbb{A}^n \to X_{h-1,r}^+ \times \mathbb{A}^n$ such that if $\{z_i\}$ denotes the coordinates on $\mathbb{A}^n$ on the target $\mathbb{A}^n$, then the image of $X_{h,r}^+$ in $X_{h-1,r}^+ \times \mathbb{A}^n$ and the action of $\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$ on z_i are given by Proposition A.2.2(ii),(iii). Moreover, such a morphism is given by the composition of the change-of-variables $x'_{i,h}$ and purely inseparable morphisms of the form $x'_{i,h-1} \mapsto x_{h-1}^{',q^{-j}}$ for appropriate i, j.*

Proof. If $r = n$, this is literally Lemma A.2.3. Assume $r < n$. First, for $s \geq j \geq 0$, replace x'_{s+2-j} by $x_{s+2-j}^{',q^{i_j+1}}$. Then, by applying a series of iterated changes of variables of the form $x'_c =: x'_c + x_d^{',q^\lambda}$ for appropriate $2 \leq c, d \leq s+2$ and λ (essentially following the Euclidean algorithm to find the gcd of the integers $(i_{j+1} - i_j)$ (this gcd is equal to r)), we transform P_1 from Lemma A.2.3 to the form

$$P_1 = \sum_{i=0}^{n-1} x_{1,h}^{',q^i} + \sum_{i=0}^{r-1} x_{2,h}^{',q^i}.$$

As these operations does not involve $x'_{1,h}$, the formulas (A.4) remain true. Now make the change of variables given by $z_1 := x'_{2,h} + \sum_{j=0}^{\frac{n}{r}-1} x'^{,q^{rj}}_{1,h}$ and $z_2 := x'_{1,h-1}$. In this coordinates, $P_1 = \sum_{i=0}^{r-1} z_1^{q^i}$ and the action is as claimed. $\qquad\square$

We are now ready to complete the proof of Proposition A.2.2.

Proof of Proposition A.2.2. Combining Lemmas A.2.3 and A.2.5 we obtain Proposition A.2.2 in the case $\kappa = 0$. Now let κ be arbitrary. It is clear that the proof of Lemma A.2.3 can be applied to this more general situation. One then obtains the same statement, with the only difference being that now our change of variables does not affect the variables $x_{i,h-1}$ for $i \not\equiv 1$ mod n_0 (these are exactly the variables which do not show up in P_1). That is, the right-hand side $X^+_{h-1,r} \times \mathbb{A}^n$ will have the coordinates $\{x'_{i,h-1} : i \equiv 1 \mod n_0, 1 \leq i \leq n\} \cup \{x_{i,h-1} : i \not\equiv 1 \mod n_0, 1 \leq i \leq n\}$ and the polynomial defining $X^+_{h,r}$ as a relative $X^+_{h-1,r}$ hypersurface in $X^+_{h-1,r} \times \mathbb{A}^n$ is

$$P_1 = x'_{1,h-1} + x'^{,q^{n_0}}_{1,h-1} + \cdots + x'^{,q^{n_0(n'-1)}}_{1,h-1} + \sum_{j=0}^{s} \sum_{\lambda=i_j+1}^{i_{j+1}} x'^{,q^{n_0\lambda}}_{s+2-j,h-1},$$

and the $\mathbb{W}^{h-1}_h(\mathbb{F}_{q^n})$-action is given by

$$1 + \varpi^{h-1}a : x'_{i,h-1} \mapsto \begin{cases} x'_{1,h-1} + a & \text{if } i = 1 \\ x'_{i,h-1} & \text{if } i \equiv 1 \mod n_0 \text{ and } i > 1 \\ x_{i,h-1} + x_{i,0}a & \text{if } i \not\equiv 1 \mod n_0. \end{cases}$$

We now apply the change of variables replacing $x_{i,h-1}$ by $x'_{i,h-1} := x_{h-1} - x_{i,0}x'_{1,h-1}$ for all $i \not\equiv 1 \mod n_0$. This exactly gives us Lemma A.2.3 for arbitrary κ (the only difference being the q^{n_0}-powers occurring in P_1). Now Lemma A.2.5 can be applied as in the case $\kappa = 0$, and this finishes the proof of Proposition A.2.2. $\qquad\square$

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DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MA 02139, USA
E-mail address: `charchan@mit.edu`

MATHEMATISCHES INSTITUT, UNIVERSITÄT BONN, ENDENICHER ALLEE 60, 53115 BONN, GERMANY
E-mail address: `ivanov@math.uni-bonn.de`