AFFINE DELIGNE-LUSZTIG VARIETIES OF HIGHER LEVEL AND THE LOCAL LANGLANDS CORRESPONDENCE FOR GL_2

A. B. IVANOV

Abstract. In the present article we define coverings of affine Deligne-Lusztig varieties attached to a connected reductive group over a non-archimedean local field. In the case of GL_2 and positive characteristic, the unramified cuspidal part of the local Langlands correspondence is realized in the ℓ-adic cohomology of these varieties. We show this by giving a detailed comparison with the realization of local Langlands via cuspidal types by Bushnell-Henniart. All proofs are purely local.

1. Introduction

The classical Deligne-Lusztig theory aims for a geometric construction of representations of finite groups of Lie-type. In [9], Deligne and Lusztig constructed the so-called Deligne-Lusztig varieties attached to a connected reductive group over a finite field and could show that any irreducible representation of the group of $\mathbb{F}_q$-valued points occurs in the ℓ-adic cohomology of these varieties. Since then one was trying to find similar constructions in the affine setting, aiming for a geometric realization of the local Langlands correspondence. However, usual geometric realizations of local Langlands make use of p-adic methods, formal schemes and adic spaces, also using the global theory. In the present article we introduce a very natural affine analog of Deligne-Lusztig varieties of arbitrary level attached to a connected reductive group over a local field F of positive characteristic. Using these varieties we realize the unramified part of the local Langlands correspondence for GL_2 over F using only schemes over $\mathbb{F}_q$ and purely local methods. Moreover, we will give a detailed comparison of our construction with the theory of cuspidal types of Bushnell-Kutzko [3] (we use the language of Bushnell-Henniart [2]) and on the 'algebraic' side we will show an improvement of the Intertwining theorem [2] 15.1.

To begin with, let F be a non-archimedean local field and let L denote the completion of the maximal unramified extension of F. Let $\mathcal{O}_F$ resp. $\mathcal{O}_L$ be the ring of integers of F resp. L. We denote by k resp. $\bar{k}$ the residue field of F resp. L, and by q the cardinality of k. Let $\sigma\colon \bar{k} \to \bar{k}$ denote the k-automorphism given by $x \mapsto x^q$. We also denote by σ the unique F-automorphism of L, lifting $\sigma\colon \bar{k} \to \bar{k}$.

Let G be a connected reductive group over F and let $\mathbb{G}$ be a smooth model of G over $\mathcal{O}_F$. It is a central problem to realize smooth representations of the locally compact group $G(F)$ in the ℓ-adic cohomology of certain schemes (or formal schemes, or adic spaces, ...) over k (where ℓ is prime to $\mathrm{char}(k)$). Usually such schemes come up with an action of $G(F) \times T(F)$, where T is some maximal torus of G and as a consequence the representations of $G(F)$ occurring in their ℓ-adic cohomology are parametrized by characters of $T(F)$, lying in sufficiently general position. After the fundamental work of Deligne and Lusztig [9], which followed the pioneering example of Drinfeld concerning $\mathrm{SL}_2(k)$, and deals with representations of the finite group $\mathbb{G}(k)$, many generalizations of their ideas aiming a construction of representations of $\mathbb{G}(\mathcal{O}_F/t^r)$ for $r \geqslant 2$ resp. of $G(F)$ were made. We give some examples. In [12] Lusztig suggested such construction (without proofs) and more recently he gave proofs in [13]. (A minor variation of) this construction was worked out for division algebras by Boyarchenko [1] and Chan [7] (see also [6]). A further closely related approach, was given by Stasinski in [17], who suggested a method to construct the so called extended Deligne-Lusztig varieties attached to $\mathbb{G}(\mathcal{O}_F/t^r)$. The advantages of our construction are that it (i) has a quite simple definition in terms of the Bruhat-Tits building of G, (ii) establishes a direct link with affine Deligne-Lusztig varieties, which are well-studied in various contexts. In particular, this allows to use the whole combinatoric machinery developed for their study.

A starting point for our construction is Rapoport's definition of affine Deligne-Lusztig varieties in [16] Definition 4.1. We recall this definition (in the Iwahori case). Let $\mathscr{B}_L$ be the Bruhat-Tits building of the adjoint group $G_{L,ad}$. The Bruhat-Tits building of G_{ad} over F can be identified with the σ-invariant subset of $\mathscr{B}_L$. Let S be a maximal L-split torus in G, which is defined over F (such a torus exists due to [5] 5.1.12). Let $I \subseteq G(L)$ be the Iwahori subgroup attached to a σ-stable alcove in the apartment corresponding to S. Let $\mathscr{F}$ be the affine flag manifold of G, seen as an ind-scheme over k if F has positive characteristic, and seen as a perfect algebraic space in the sense of [19] otherwise. Its $\bar{k}$-points can be identified with $G(L)/I$. Let $\tilde{W}$ denote the extended affine Weyl group of G attached to S. The Bruhat decomposition of $G(L)$ induces the invariant position map

$$\mathrm{inv}\colon \mathscr{F}(\bar{k}) \times \mathscr{F}(\bar{k}) \to \tilde{W}.$$

For $w \in \tilde{W}$ and $b \in G(L)$ the affine Deligne-Lusztig variety attached to w and b is the locally closed subset

$$X_w(b) = \{gI \in \mathscr{F} \colon \mathrm{inv}(gI, b\sigma(g)I) = w\}$$

of $\mathscr{F}$, which is given the structure of the reduced induced sub-Ind-scheme resp. perfect algebraic subspace. Let J_b be the σ-stabilizer of b, i.e., the algebraic group over F defined by

$$J_b(R) = \{g \in G(R \otimes_F L) \colon g^{-1} b\sigma(g) = b\}$$

for any F-algebra R. Then $J_b(F)$ acts on $X_w(b)$.

We sketch now the construction of natural covers of these varieties, which still admit an action by $J_b(F)$. The details are given in Section 2. Let $\Phi = \Phi(G, S)$ be the relative root system. We see 0 as the 'root' corresponding to the centralizer T of S in G (as G is quasi-split, this is a maximal torus). After choosing a σ-stable special base point x in $\mathscr{B}_L$, with a concave function f on $\Phi \cup \{0\}$ (for a definition cf. Section 2.1) one can associate a subgroup $G(L)_f \subseteq G(L)$. In [18], Yu defined a smooth model $\underline{G}_f$ of G_L over $\mathcal{O}_L$, such that $\underline{G}_f(\mathcal{O}_L) = G(L)_f$. Assume that $G(L)_f \subseteq I$ and that $G(L)_f$ is σ-stable. Then $\underline{G}_f$ descends to a smooth group scheme over $\mathcal{O}_F$. Further, $G(L)/G(L)_f$ is the set of $\bar{k}$-points of an Ind-scheme resp. an Ind perfect algebraic space $\mathscr{F}^f$, which defines a natural cover of $\mathscr{F}$, as follows from the work of Pappas and Rapoport [15] Theorem 1.4 resp. [19] Theorem 1.5. Moreover, if $G(L)_f$ is normal in I, then $\mathscr{F}^f \to \mathscr{F}$ is a (right) principal homogeneous space under $I/G(L)_f$. There is a map

$$\mathrm{inv}^f \colon \mathscr{F}^f(\bar{k}) \times \mathscr{F}^f(\bar{k}) \to D_{G,f},$$

which covers the map inv. Here $D_{G,f}$ is a set of representatives of double cosets of $G(L)_f$ in $G(L)$. For $w \in D_{G,f}$, $b \in G(L)$, we define the *affine Deligne-Lusztig variety of level f* attached to w and b as the locally closed subset

$$X_w^f(b) = \{\bar{g} = gG(L)_f \in \mathscr{F}^f(\bar{k}) \colon \mathrm{inv}^f(\bar{g}, b\sigma(\bar{g})) = w\},$$

endowed with its induced structure of a reduced sub-Ind-scheme resp. sub-(Ind) perfect algebraic space (for the mixed characteristic case, compare [19] Section 0.3). In fact, in cases of interest this is a scheme resp. perfect scheme locally of finite type over k. Assume $G(L)_f$ is normal in I. Then I acts on $D_{G,f}$ by σ-conjugation $w \mapsto i^{-1}w\sigma(i)$, hence we can consider the stabilizer $I_{f,w} \subseteq I$ of w under this action. It acts on $X_w^f(b)$ on the right and this action commutes with the left action of $J_b(F)$. Moreover this $I_{f,w}$-action can be extended to an action of $Z(F)I_{f,w}$, where Z is the center of G. Thus we obtain the desired variety resp. perfect scheme $X_w^f(b)$ with an action of $G(F) \times Z(F)I_{f,w}$. In the mixed characteristic case, note that perfect schemes has enough structure such that étale cohomology groups can be attached to them.

We study further properties of $I_{f,w}$ and $X_w^f(b)$ for general G elsewhere. The rest of the paper is devoted to the detailed study of $G = \mathrm{GL}_2$ in the equal characteristic case. Now we explain our results in this case. As the levels indexed by concave functions are cofinal, we restrict attention to special functions f_m and elements $w \in D_{G,f_m}$ (cf. Sections 2.1,3.1) for integers $m \geqslant 0$ and write I^m instead of $G(L)_{f_m}$, $X_w^m(1)$ instead of $X_w^{f_m}(b)$, etc. We determine the varieties $X_w^m(1)$ and the $G(F)$-representations in the cohomology of these varieties with $\overline{\mathbb{Q}_\ell}$-coefficients. Further we compare our results with the algebraic construction of the same representations in [2] using the theory of cuspidal types. We sketch our results here; for a precise treatment cf. Section 4.1. Let E/F be the unramified extension degree 2. If the image of w in the finite Weyl group is non-trivial, then $Z(F)I_{m,w}$ has a natural quotient isomorphic to E^*, and the $Z(F)I_{m,w}$-action in the ℓ-adic cohomology of $X_w^m(1)$ factors through an E^*-action. In this way we obtain a $G(F)$-representation in the spaces $\mathrm{H}_c^i(X_w^m(1), \overline{\mathbb{Q}}_\ell)[\chi]$, where χ goes through smooth $\overline{\mathbb{Q}}_\ell^*$-valued characters of E^*. It turns out that if χ is minimal of level m, lies in sufficiently general position, then there is an integer i_0, such that $\mathrm{H}_c^i(X_w^m(1), \overline{\mathbb{Q}}_\ell)[\chi] = 0$ for all $i \neq i_0$ and

$$R_\chi = \mathrm{H}_c^{i_0}(X_w^m(1), \overline{\mathbb{Q}}_\ell)[\chi]$$

is an unramified irreducible cuspidal representation of $G(F)$, of level m (we also define R_χ for χ non-minimal). Here for an irreducible cuspidal representation π of $G(F)$ to be *unramified* means essentially that π arises by an automorphic induction process from a character of an (anisotropic modulo center) torus of $G(F)$, which is split after an *unramified* extension of F. Alternatively, unramified cuspidal representations are those which contain unramified fundamental strata in terms of [2]. Cuspidal representations attached to (at least tamely) ramified tori can also be studied via Deligne-Lusztig type constructions, see for example the work of Stasinski [17] or a future work of the author [11].

Let $\mathbb{P}_2^{\mathrm{nr}}(F)$ be the set of all isomorphism classes of admissible pairs over F attached to E/F (cf. [2] 18.2). Let $\mathscr{A}_2^{\mathrm{nr}}(F)$ be the set of all isomorphism classes of unramified irreducible cuspidal representations of $G(F)$. We defined a map

$$R \colon \mathbb{P}_2^{\mathrm{nr}}(F) \to \mathscr{A}_2^{\mathrm{nr}}(F), \quad (E/F, \chi) \mapsto R_\chi. \tag{1.1}$$

As a consequence of our trace computations in Sections 4.2-4.4, we see that this map is injective (cf. Proposition 4.26). Using the theory of cuspidal types and strata, Bushnell-Henniart attached to an admissible pair $(E/F, \chi)$ an irreducible cuspidal $G(F)$-representation π_χ ([2] §19; we recall the construction briefly in Section 4.6). The tame parametrization theorem ([2] 20.2 Theorem) then shows that the map

$$\mathbb{P}_2^{\mathrm{nr}}(F) \xrightarrow{\sim} \mathscr{A}_2^{\mathrm{nr}}(F), \quad (E/F,\chi) \mapsto \pi_\chi$$

is a bijection (also for even q). Here is our main result (which also works for even q).

Theorem 4.3. *Let $(E/F,\chi)$ be an admissible pair. The representation R_χ is irreducible cuspidal, unramified, has level $\ell(\chi)$ and central character $\chi|_{F^*}$. Moreover, R_χ is isomorphic to π_χ. In particular, the map (1.1) is a bijection.*

The proof is purely local. Two ideas in the proof follow [1],[6]: it is Boyarchenko's trace formula (cf. Lemma 4.7) and maximality of certain closed subvarieties of $X_w^m(1)$ (note that $X_w^m(1)$ itself is not maximal due to the presence of a 'level 0 part'). The rest of the proof is independent of [1],[6].

Finally, we remark that for $G = \mathrm{GL}_2$ and b superbasic, $J_b(F) = D^*$ for D a quaternion algebra over F and the varieties $X_{x_m}^m(b)$ seem to be very close (but unequal) to the varieties studied by Chan in [7] (cf. Section 3.6).

Outline of the paper. In Section 2 we define affine Deligne-Lusztig varieties for a connected reductive group G of level attached to a concave function on the roots. In Section 3 we compute these varieties for $G = \mathrm{GL}_2$, $b = 1$ and determine their ℓ-adic cohomology. In Section 4.1 we recall the setup and state our main result for GL_2, Theorem 4.3. After performing necessary trace calculations in Sections 4.2-4.4, we compare our construction with that in [2] in Sections 4.5-4.6, and finish the proof of Theorem 4.3.

Acknowledgments. The author is especially grateful to Eva Viehmann, Christian Liedtke, Stephan Neupert and to a referee for very helpful comments concerning this work. Also he is grateful to Paul Hamacher, Bernhard Werner and other people for interesting discussions concerning this work. Further, the author wants to thank Michael Rapoport and Ulrich Görtz for patiently introducing him to the field of affine Deligne-Lusztig varieties some years ago. The author was partially supported by ERC starting grant 277889 "Moduli spaces of local G-shtukas".

2. Coverings of affine Deligne-Lusztig varieties

The goal of this section is to define coverings of affine Deligne-Lusztig varieties.

2.1. **Concave functions and smooth models.** Let G be a connected reductive group over F. As $\bar{k}$ is algebraically closed, G_L is quasi-split over L. Let $S \subseteq G$ be a maximal L-split torus, which is defined over F. Let $T = \mathscr{Z}_G(S)$ be the centralizer of S. As G_L is quasi-split, T is a maximal torus. Let $\Phi = \Phi(G_L, S_L)$ denote the relative root system. For $a \in \Phi$, write U_a for the corresponding root subgroup and let $U_0 = T$. Let $\mathscr{B}_L$ be the Bruhat-Tits building of G_L and let $\mathscr{A}_S$ be the apartment corresponding to S_L. We fix a σ-stable base alcove $\underline{a}$ contained in $\mathscr{A}_S$ and let x be one of its special vertices. Let $\tilde{\mathbb{R}} = \mathbb{R} \cup \{r+ : r \in \mathbb{R}\} \cup \{\infty\}$ be the monoid as in [4] 6.4.1. Then x defines a filtration of $U_a(L)$ by subgroups $U_a(L)_{x,r}$ ($r \in \tilde{\mathbb{R}}$) for $a \in \Phi$ (cf. [4] §6.2 and §6.4).

Moreover, choose an admissible schematic filtration on tori in the sense of Yu [18] §4. This gives a filtration $U_0(L)_{x,r} = T(L)_r$ on T. If G satisfies condition (T) from [18] 4.7.1, then this filtration is independent of the choice of the admissible filtration and coincide with the Moy-Prasad filtration on $T(L)$, cf. [18] Lemma 4.7.4. Moreover, G satisfies (T) if it is either simply connected or adjoint or split over a tamely ramified extension [18] 8.1. We do not use this in the following.

A function $f\colon \Phi \cup \{0\} \to \tilde{\mathbb{R}}$ is called *concave* ([4] 6.4), if

$$\sum_{i=1}^{s} f(a_i) \geqslant f(\sum_{i=1}^{s} a_i).$$

Fix a concave function $f\colon \Phi \cup \{0\} \to \tilde{\mathbb{R}}_{\geqslant 0} \smallsetminus \{\infty\}$. Let $G(L)_{x,f}$ be the subgroup of $G(L)$ generated by $U_a(L)_{x,f(a)}$, $a \in \Phi \cup \{0\}$. By [18] Theorem 8.3, there is a unique smooth model $\underline{G}_{x,f}$ of G_L over $\mathcal{O}_L$ such that $\underline{G}_{x,f}(\mathcal{O}_L) = G(L)_{x,f}$. Moreover, if $G(L)_{x,f}$ is σ-stable, then $\underline{G}_{x,f}$ descends to a group scheme defined over $\mathcal{O}_F$ ([18] 9.1). We denote it again by $\underline{G}_{x,f}$.

Let $I \subseteq G(L)$ be the Iwahori subgroup associated with $\underline{a}$ and let $\Phi^+ \subseteq \Phi$ denote the set of positive roots determined by $\underline{a}$. Let f_I be the concave function on $\Phi \cup \{0\}$ defined by

$$f_I(a) = \begin{cases} 0 & \text{for } a \in \Phi^+ \cup \{0\} \\ f_I(a) = 0+ & \text{for } a \in \Phi^-. \end{cases}$$

Then $G(L)_{x,f_I} = I$ (cf. [18] 7.3). For $m \geqslant 0$ let $f_m\colon \Phi \cup \{0\} \to \tilde{\mathbb{R}}_{\geqslant 0} \smallsetminus \{\infty\}$ be the concave function defined by

$$f_m(a) = \begin{cases} m & \text{if } a \in \Phi^+ \\ m^+ & \text{if } a \in \Phi^- \cup \{0\}. \end{cases}$$

Write $I^m = G(L)_{x,f_m}$.

Lemma 2.1. *For $m \geqslant 0$, I^m is normal in I and I^m is σ-stable. In particular, I^m admits a unique smooth model $\underline{G}_{x,f_m}$. This model is already defined over $\mathcal{O}_F$.*

Proof. I (resp. I^m) is generated by $U_a(L)_{x,f_I(a)}$ (resp. $U_a(L)_{x,f_m(a)}$) for $a \in \Phi \cup \{0\}$. To show normality, it is enough to show that for any roots $a, b \in \Phi \cup \{0\}$, the commutator $(U_a(L)_{x,f_I(a)}, U_b(L)_{x,f_m(b)})$ is contained in I^m. By [4] (6.2.1) V3 (we can treat 0 as a root), $(U_a(L)_{x,f_I(a)}, U_b(L)_{x,f_m(b)})$ is contained in the subgroup generated by $U_{pa+qb}(L)_{pf_I(a)+qf_m(b)}$ for $p, q > 0$ such that $pa + qb \in \Phi \cup \{0\}$. Now $qf_m(b) \geqslant m$, hence $U_{pa+qb}(L)_{pf_I(a)+qf_m(b)} \nsubseteq I^m$ can only happen if $f_I(a) = 0$, $f_m(b) = m$, $f_m(pa + qb) = m^+$. This is equivalent to $a \in \Phi^- \cup \{0\}$, $b \in \Phi^+$, $a + b \in \Phi^- \cup \{0\}$. This is impossible, hence $U_{pa+qb}(L)_{pf_I(a)+qf_m(b)} \subseteq I^m$ and the normality is shown. We show now the σ-stability of I^m. On the one side, I is generated by the subgroups $U_{\sigma(a)}(L)_{x,f_I(\sigma(a))}$ (for varying a), and on the other side $I = \sigma(I)$ is generated by $\sigma(U_a(L)_{x,f_I(a)}) = U_{\sigma(a)}(L)_{\sigma(x),f_I(a)}$. Using parts (i), (ii) of [18] Theorem 8.3 we deduce that $U_{\sigma(a)}(L)_{x,f_I(\sigma(a))} = U_{\sigma(a)}(L)_{\sigma(x),f_I(a)}$. But then the same is true also for f_m instead of f_I. From this the σ-stability of I^m follows. □

To have common notation for mixed and equal characteristic cases, for a k-algebra R set

$$\mathbb{W}(R) := \begin{cases} R \mathbin{\widehat{\otimes}_k} \mathcal{O}_F & \text{if } \mathrm{char}(F) > 0 \\ W(R) \otimes_{W(k)} \mathcal{O}_F & \text{if } \mathrm{char}(F) = 0, \end{cases}$$

where $W(R)$ denotes the p-typical Witt ring of R. In the mixed characteristic case, $\mathbb{W}$ behaves only well, if R is a perfect k-algebra. In any case, let ϖ denote a uniformizer of F. Consider the loop group LG, which is the functor on the category of k-algebras,

$$LG\colon R \mapsto G(\mathbb{W}(R)[\varpi^{-1}]).$$

Assume the concave function f is such that $G(L)_{x,f}$ is σ-invariant. Let $L^+\underline{G}_{x,f}$ be the functor on the category of k-algebras defined by

$$L^+\underline{G}_{x,f}\colon R \mapsto \underline{G}_{x,f}(\mathbb{W}(R)),$$

and taking perfection in the mixed characteristic case (as in [19] Section 1.1). Consider the quotient of fpqc-sheaves

$$\mathscr{F}^f = LG/L^+\underline{G}_{x,f}$$

on the category of k-algebras in the equal characteristic case resp. on the category of perfect k-algebras in the mixed characteristic case. By [15] Theorem 1.4 resp. [19] Theorem 1.5 it is represented by an Ind-k-scheme of ind-finite type over $\bar{k}$ resp. by a ind perfectly proper perfect algebraic space over k and its $\bar{k}$-points are $\mathscr{F}^f(\bar{k}) = G(L)/G(L)_{x,f}$. Moreover, if $g \leqslant f$ are two concave functions as above, then we have a natural projection $\mathscr{F}^f \twoheadrightarrow \mathscr{F}^g$. We write $\mathscr{F} = \mathscr{F}^{f_I}$ for the affine flag manifold associated with $\underline{G}_{x,f_I}$, the smooth model of I and $\mathscr{F}^m = \mathscr{F}^{f_m}$ for $m \geqslant 0$.

2.2. **Affine Deligne-Lusztig varieties and covers.** We keep the notations from Section 2.1. We fix a concave function $f\colon \Phi \cup \{0\} \to \tilde{\mathbb{R}}_{\geqslant 0} \smallsetminus \{\infty\}$, such that $f \geqslant f_I$, i.e., $G(L)_{x,f} \subseteq I$ and s.t. $G(L)_{x,f}$ is σ-invariant, i.e., $\underline{G}_{x,f}$ is defined over $\mathcal{O}_F$. We write $I^f = G(L)_{x,f}$. There are natural σ-actions on $\mathscr{F}(\bar{k}), \mathscr{F}^f(\bar{k})$, which are compatible with natural projections.

Let N_T be the normalizer of T in G. Let $W = \mathrm{N}_T(L)/T(L)$ be the finite Weyl group associated with S and $\tilde{W}$ the extended affine Weyl group. If Γ denotes the absolute Galois group of L, then $\tilde{W}$ sits in the short exact sequence

$$0 \to X_*(T)_\Gamma \to \tilde{W} \to W \to 0.$$

Then the Iwahori-Bruhat decomposition states that

$$G(L) = \coprod_{w \in \tilde{W}} I\dot{w}I,$$

where $\dot{w}$ is any lift of w to $N(L)$. Consider now the set of double cosets

$$D_{G,f} = G(L)_{x,f}\backslash G(L)/G(L)_{x,f},$$

equipped with the natural projection map $D_{G,f} \twoheadrightarrow I\backslash G(L)/I \cong \tilde{W}$. If $m \geqslant 0$, we also write $D_{G,m}$ instead of D_{G,f_m}. At least for w 'big' enough, the fiber $D_{G,f}(w)$ over a fixed $w \in \tilde{W}$ can be given the structure of a finite-dimensional affine variety over $\bar{k}$, by parametrizing it using subquotients of (finite) root subgroups. As this seems quite technical and as in this article we only need the case $G = GL_2$ (cf. (3.3)), we omit the corresponding result in this article. We obtain a map, which covers the classical relative position map.

Definition 2.2. Define the map

$$\mathrm{inv}^f\colon \mathscr{F}^f(\bar{k}) \times \mathscr{F}^f(\bar{k}) \to D_{G,f}$$

on $\bar{k}$-points by $\mathrm{inv}^f(xG(L)_{x,f}, yG(L)_{x,f}) = w$, where w is the double $G(L)_{x,f}$-coset containing $x^{-1}y$.

We come to our main definition.

Definition 2.3. For $f \geqslant f_I$ concave, such that $G(L)_{x,f}$ is σ-invariant, $b \in G(L)$, and $w \in D_{G,f}$ we define the *affine Deligne-Lusztig variety of level* f associated with b, w as

$$X^f_w(b) = \{\bar g = gG(L)_{x,f} \in \mathscr{F}^f(\bar k) \colon \mathrm{inv}^f(\bar g, b\sigma(\bar g)) = w\},$$

with its induced reduced sub-Ind-scheme resp. sub-Ind perfect algebraic space structure.

We write $X^m_w(b)$ instead of $X^{f_m}_w(b)$. As usual, $X^f_w(b)$ is equipped with two group actions. For $b \in G(L)$, let J_b be the σ-stabilizer of b, i.e., the algebraic group over F defined by

$$J_b(R) = \{g \in G(R \otimes_F L) \colon g^{-1}b\sigma(g) = b\}$$

for any F-algebra R. Then $J_b(F)$ acts on $X^f_w(b)$ for any f and $w \in D_{G,f}$. If $f \geqslant f'$ and $w \in D_{G,f}$ lies over $w' \in D_{G,f'}$, then $X^f_w(b)$ lies over $X^{f'}_{w'}(b)$ and the $J_b(F)$-actions are compatible.

To describe the second group action, assume additionally that $G(L)_{x,f}$ is normal in I. For $\bar w \in \tilde W$, we have a left and a right I/I^f-action on $D_{G,f}(\bar w)$ by multiplication. We obtain the (right) I/I^f-action on $D_{G,f}(\bar w)$ by $(i, w) \mapsto i^{-1}w\sigma(i)$.

Lemma 2.4. *Assume I^f is normal in I. Let $b \in G(L), \bar w \in \tilde W$ and $w \in D_{G,f}(\bar w)$.*

(i) *$X^f_w(b)$ is locally of finite type over k.*
(ii) *For every $g \in G(L)$, the map $(h, xI^f) \mapsto (g^{-1}hg, g^{-1}xI^f)$ defines an isomorphism of pairs $(J_b(F), X^f_{w_f}(b)) \xrightarrow{\sim} (J_{g^{-1}b\sigma(g)}(F), X^f_{w_f}(g^{-1}b\sigma(g)))$.*
(iii) *For $i \in I$, the map $xI^f \mapsto xiI^f$ defines an isomorphism $X^f_w(b) \xrightarrow{\sim} X^f_{i^{-1}w\sigma(i)}(b)$.*

Proof. (ii) and (iii) are trivial computations. (i): The affine Deligne-Lusztig varieties $X_w(b)$ are locally of finite type, $\mathscr{F}^f \twoheadrightarrow \mathscr{F}$ is a I/I^f-bundle and I/I^f is of finite dimension over $\bar k$. □

By Lemma 2.4 (iii), the σ-stabilizer

$$I_{f,w} = \{i \in I \colon i^{-1}w\sigma(i) = w\}$$

of $w \in D_{G,f}(\bar w)$ in I acts on $X^f_w(b)$ by right multiplication, and this action factors through an action of $I_{f,w}/I^f$. Let Z denote the center of G. Note that $Z(F) \subseteq J_b(F)$, and that $J_b(F)$-action restricted to $Z(F)$ can also be seen as a right action, thus extending the right $I_{f,w}$-action on $X^f_w(b)$ to a right $Z(F)I_{f,w_f}$-action. If $m \geqslant 0$, we also write $I_{m,w}$ instead of $I_{f_m,w}$.

3. Computations for GL_2

From now on and until the end of the paper we set $G = \mathrm{GL}_2$ and restrict ourselves to the case of positive characteristic, i.e., $F = k((t))$ is the field of Laurent series with uniformizer t, $L = \bar k((t))$ and $\sigma \colon L \to L$ is given by $\sigma(\sum_i a_i t^i) = \sum_i a_i^q t^i$. In this section, we compute the associated varieties $X^m_w(1)$ and their ℓ-adic cohomology.

3.1. **Some notations and preliminaries.** We fix the diagonal torus T and the upper triangular Borel subgroup B of G. We set $K = G(\mathcal{O}_F)$ and fix the Iwahori subgroup I and its subgroups I^m for $m \geqslant 0$:

$$I^m = \begin{pmatrix} 1 + \mathfrak{p}_L^{m+1} & \mathfrak{p}_L^m \\ \mathfrak{p}_L^{m+1} & 1 + \mathfrak{p}_L^{m+1} \end{pmatrix} \subsetneq I = \begin{pmatrix} \mathcal{O}_L^* & \mathcal{O}_L \\ \mathfrak{p}_L & \mathcal{O}_L^* \end{pmatrix} \subseteq G(\mathcal{O}_L).$$

Note that the groups I^m coincide with those defined in Section 2.1 with respect to the valuation on the root datum, which corresponds to the vertex of the Bruhat-Tits building of G associated with the maximal compact subgroup $G(\mathcal{O}_L)$. The maximal torus T is split over F and hence the filtration on it does not depend on the choice of an admissible schematic filtration. It is given by $T(L)_r = \begin{pmatrix} 1+\mathfrak{p}^r & 0 \\ 0 & 1+\mathfrak{p}^r \end{pmatrix}$. Let $W_a \subseteq \tilde{W}$ be the affine and the extended affine Weyl group of G.

The variety $X_w(1)$ is empty, unless $w = 1$ or $w \in W_a$ with odd length (cf. e.g. [10] Lemma 2.4). The case $w = 1$ is not very interesting: $X_1(1)$ is a disjoint union of points and the cohomology of coverings of $X_1(1)$ contains the principal series representations of $G(F)$, as for classical Deligne-Lusztig varieties and as in [10] in case of level 0. Thus we restrict attention to elements of odd length in W_a. To simplify some computations, we fix once for all time an even integer $n > 0$ and the elements

$$\dot{w} = \begin{pmatrix} 0 & t^{-n} \\ -t^n & 0 \end{pmatrix}, \quad \dot{v} = \begin{pmatrix} t^{\frac{n}{2}} & \\ & t^{-\frac{n}{2}} \end{pmatrix} \in \mathrm{N}_T(L) \subseteq G(L) \tag{3.1}$$

and denote by w (resp. v) the image of $\dot{w}$ (resp. $\dot{v}$) in W_a (the elements with $n < 0$ can be obtained by conjugation; the elements with n odd lead to similar results). We denote the image of $\dot{w}$ in $D_{G,m}$ again by $\dot{w}$ (from the context it is always clear, in which set $\dot{w}$ lies). Let $pr_m \colon \mathscr{F}^m \to \mathscr{F}$ be the natural projection. Let $C_v \subseteq \mathscr{F}$ denote the (open) Schubert cell attached to v. We have the following parametrizations of $C_v^m = pr_m^{-1}(C_v)$ and $D_{G,m}(w)$. For $m \geqslant 0$, let R_m denote the Weil restriction functor $\mathrm{Res}_{(k[t]/t^{m+1})/k}$ from $k[t]/t^{m+1}$-schemes to k-schemes. C_v is parametrized by $R_{n-1}\mathbb{G}_a \to C_v$, $a \mapsto \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} v$, where $a = \sum_{i=0}^{n-1} a_i t^i$. Then for $m \geqslant 0$, C_v^m is parametrized by

$$\begin{aligned} \psi_v^m \colon \mathrm{R}_{n-1}\mathbb{G}_a \times \mathrm{R}_m\mathbb{G}_m^2 \times \mathrm{R}_m\mathbb{G}_a^2 \quad &\stackrel{\sim}{\longrightarrow} \quad C_v^m = IvI/I^m \\ a, C, D, A, B \quad &\mapsto \quad \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \dot{v} \begin{pmatrix} C & \\ & D \end{pmatrix} \begin{pmatrix} 1 & A \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ tB & 1 \end{pmatrix} I^m \end{aligned} \tag{3.2}$$

We write $a = \sum_{i=0}^{n-1} a_i t^i$, $A = \sum_{i=0}^{m} A_i t^i$ and $C = c_0(1 + \sum_{i=1}^{m} c_i t^i)$. Moreover, for $m \leqslant n$, $D_{G,m}(w)$ is parametrized by

$$\begin{aligned} \phi_w^m \colon \mathrm{R}_m\mathbb{G}_m^2(\bar{k}) \times \mathrm{R}_{m-1}\mathbb{G}_a^2(\bar{k}) \quad &\stackrel{\sim}{\longrightarrow} \quad D_{G,m}(w) = I^m \backslash IwI/I^m \\ (C, D), (E, B) \quad &\mapsto \quad I^m \begin{pmatrix} 1 & \\ tE & 1 \end{pmatrix} \dot{w} \begin{pmatrix} C & \\ & D \end{pmatrix} \begin{pmatrix} 1 & \\ tB & 1 \end{pmatrix} I^m. \end{aligned} \tag{3.3}$$

The proof that ψ_v^m resp. ϕ_w^m is an isomorphism of varieties resp. sets amounts to a simple computation. We omit the details.

Finally, we remark the existence of the following determinant maps. Let $x \in W_a$. There is a natural k-morphism of k-varieties:

$$\det{}^m \colon C_x^m = IxI/I^m \to \mathrm{R}_m\mathbb{G}_m, \quad yI^m \mapsto \det(y) \mod t^{m+1}.$$

In the same way we have the k-morphism

$$\det{}^m \colon D_{G,m}(w) \to \mathrm{R}_m\mathbb{G}_m, \quad I^m y I^m \mapsto \det(y) \mod t^{m+1}.$$

3.2. The structure of $X_{\dot{w}}^m(1)$.

Lemma 3.1. *Let $m \geqslant 0$. There is a natural isomorphism*

$$I_{m,\dot{w}}/I^m \xrightarrow{\sim} \left\{ \begin{pmatrix} C & A \\ 0 & D \end{pmatrix} \in G(\bar{k}[t]/t^{m+1}) \colon \sigma^2(C) = C, D = \sigma(C) \right\}.$$

Proof. An easy computation using (3.3) shows the lemma. □

For $r > m$, let $\tau_m \colon \bar{k}[t]/t^r \twoheadrightarrow \bar{k}[t]/t^{m+1}$ denote the reduction modulo t^{m+1}. Using coordinates from (3.2), let $S = \tau_m(\sigma(a) - a)$ and let $Y_v^m \subseteq C_v^m$ be the locally closed subset defined by

$$\begin{aligned} a_0 &\notin k \\ B &= 0 \\ \sigma(C)D^{-1}S^{-1} &= 1 \\ \sigma(D)C^{-1}S &= 1 \end{aligned} \tag{3.4}$$

Let $D_v \subseteq C_v$ be the open subset defined by the condition $a_0 \notin k$. The composition $Y_v^m \to C_v^m \to C_v$ factors through $Y_v^m \to D_v$. The natural K-action on C_v^m by left multiplication restricts to an action on Y_v^m (this will follow implicitly from the proof of Theorem 3.2). Moreover, Lemma 3.1 implies that the natural right I/I^m-action on C_v^m restricts to a right action of $I_{m,\dot{w}}/I^m$ on Y_v^m.

Theorem 3.2. *Let $0 \leqslant m < n$. Let $w' = \phi_w^m(C, D, E, B) \in D_{G,m}(w)$. Then $X_{w'}^m(1)$ is non-empty if and only if one has $B = -\sigma(E)$. If this holds true, then w' is I-σ-conjugate to $\dot{w} = \phi_w^m(1, 1, 0, 0)$ in $D_{G,m}(w)$ (that is $w' = i^{-1}\dot{w}\sigma(i) \in D_{G,m}$ for some $i \in I$). In particular, $X_{w'}^m(1) \cong X_{\dot{w}}^m(1)$, compatible with appropriate group actions. Further, there is an isomorphism equivariant for the left $G(F)$- and right $(I/I^m)_{\dot{w}}$-actions:*

$$X_{\dot{w}}^m(1) \cong \coprod_{G(F)/K} Y_v^m.$$

Proof. In [10] it was shown that $X_w(1) = \coprod_{g \in G(F)/K} gD_v$ is the decomposition of $X_w(1)$ in connected components. As the natural projection $\mathscr{F}^m \to \mathscr{F}$ restricts to a map $\mathrm{pr}_m \colon X_{w'}^m(1) \to X_w(1)$, we have

$$X_{w'}^m(1) \cong \coprod_{G(F)/K} \mathrm{pr}_m^{-1}(gD_v) = \coprod_{G(F)/K} g\,\mathrm{pr}_m^{-1}(D_v).$$

Thus it is enough to determine $pr_m^{-1}(D_v)$. One sees from Lemma 3.3, that if $w' = \phi_w^m(C, D, E, B)$ does not satisfy $B = -\sigma(E)$, then $pr_m^{-1}(D_v) = \varnothing$. On the other hand, if w' satisfies this, then σ-conjugating w' first by $\begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} \in I$ and then by a diagonal $i = \begin{pmatrix} i_1 & \\ & i_2 \end{pmatrix} \in I$ such that $i_1^{-1}C\sigma(D)\sigma^2(i_1) = 1$ (such i_1 exists by Lang's theorem) and $i_2 = C\sigma(i_1)$, we deduce that w' is I-σ-conjugate to $\dot{w}$ in $D_{G,m}$. Thus by Lemma 2.4(iii) we may assume $w' = \dot{w}$. In this case Lemma 3.3 shows $pr_m^{-1}(D_v) = Y_v^m$, which finishes the proof. □

Lemma 3.3 (Key computation)**.** *Let $0 \leqslant m < n$. Let $\dot{x}I^m = \psi_v^m(a, C, D, A, B) \in C_v^m$ such that $a_0 \notin k$. Write $S = \tau_m(\sigma(a) - a)$. Then*

$$\mathrm{inv}^m(\dot{x}I^m, \sigma(\dot{x})I^m) = \phi_w^m(\sigma(C)D^{-1}S^{-1}, \sigma(D)C^{-1}S, -B, \sigma(B)).$$

Proof. Let

$$\dot{x} = \begin{pmatrix} t^{\frac{n}{2}} & t^{-\frac{n}{2}}a \\ & t^{-\frac{n}{2}} \end{pmatrix} \begin{pmatrix} C & \\ & D \end{pmatrix} \begin{pmatrix} 1 & A \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ tB & 1 \end{pmatrix} \in G(L).$$

We have to compute the (I^m, I^m)-double coset of $\dot{x}^{-1}\sigma(\dot{x})$. By assumption S is a unit and one computes (using $m < n$)

$$\begin{pmatrix} t^{\frac{n}{2}} & t^{-\frac{n}{2}}a \\ & t^{-\frac{n}{2}} \end{pmatrix}^{-1} \sigma \begin{pmatrix} t^{\frac{n}{2}} & t^{-\frac{n}{2}}a \\ & t^{-\frac{n}{2}} \end{pmatrix} = \begin{pmatrix} 1 & t^{-n}(\sigma(a) - a) \\ & 1 \end{pmatrix} \in I^m \begin{pmatrix} S & \\ & S^{-1} \end{pmatrix} \dot{w} I^m,$$

in $G(L)$. Thus by normality of I^m in I, we obtain:

$$\dot{x}^{-1}\sigma(\dot{x}) \in I^m \begin{pmatrix} 1 & \\ -tB & 1 \end{pmatrix} \begin{pmatrix} 1 & -A \\ & 1 \end{pmatrix} \begin{pmatrix} C^{-1} & \\ & D^{-1} \end{pmatrix} \cdot \begin{pmatrix} S & \\ & S^{-1} \end{pmatrix} \dot{w} \dots$$

$$\dots \begin{pmatrix} \sigma(C) & \\ & \sigma(D) \end{pmatrix} \begin{pmatrix} 1 & \sigma(A) \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ t\sigma(B) & 1 \end{pmatrix} I^m.$$

Then we can pull the term containing $-A$ to the right side of $\dot{w}$, without changing the other terms. The corresponding term, which then appear on the right side of $\dot{w}$ will lie in I^m, i.e., we can cancel it by normality of I^m in I. The same can be done then with the term containing $\sigma(A)$, by pulling it to the left side of $\dot{w}$ and canceling it. Computing the remaining matrices together, we obtain:

$$\dot{x}^{-1}\sigma(\dot{x}) \in I^m \begin{pmatrix} 1 & \\ -tB & 1 \end{pmatrix} \dot{w} \begin{pmatrix} \sigma(C)D^{-1}S^{-1} & \\ & C^{-1}\sigma(D)S \end{pmatrix} \begin{pmatrix} 1 & \\ t\sigma(B) & 1 \end{pmatrix} I^m.$$

This finishes the proof. □

3.3. **The structure of Y_v^m.** We keep notations from Sections 3.1 and 3.2. Let k_2/k denote the subextension of $\bar{k}/k$ of degree two. There is a natural surjection

$$I_{m,\dot{w}} \twoheadrightarrow T_{w,m} = \left\{ \begin{pmatrix} C & \\ & \sigma(C) \end{pmatrix} : C \in (k_2[t]/t^{m+1})^* \right\}.$$

Let $T_{w,m,0} = T_{w,m} \cap \mathrm{SL}_2(k_2[t]/t^{m+1})$ be the subgroup defined by the condition $C^{-1} = \sigma(C)$. Let $f\colon \bar{k} \to \bar{k}$ denote the map $f(x) = x^q - x$. For $X \in \bar{k}[t]/t^r$ we write $X = \sum_{i=0}^{r-1} X_i t^i$. We denote the affine space (over $\bar{k}$) spanned by coordinates $X_0, \dots, X_{r-1}$ by $\mathbb{A}^r(X_0, \dots, X_{r-1})$ resp. by $\mathbb{A}^r(X)$. Recall that $S = \tau_m(\sigma(a) - a)$ is a function on D_v with values in $(\bar{k}[t]/t^{m+1})^* = (\mathrm{R}_m\mathbb{G}_m)(\bar{k})$.

Proposition 3.4. *Let $0 \leqslant m \leqslant n$.*

(i) *The variety Y_v^m is isomorphic to the finite covering of $D_v \times \mathbb{A}^m(A)$ which is the closed subset of $D_v \times \mathbb{A}^m(A) \times \mathrm{R}_m\mathbb{G}_m$ cut out by the equation*

$$\sigma^2(C)C^{-1} = \sigma(S)S^{-1}. \tag{3.5}$$

in $(\mathrm{R}_m\mathbb{G}_m)(\bar{k})$. It is a finite étale Galois covering with Galois group $T_{w,m}$.

(ii) *The (set-theoretic) image of $\det^m\colon Y_v^m \to \mathrm{R}_m\mathbb{G}_m$ is the disjoint union of the k-rational points, which is as a set equal to $(k[t]/t^{m+1})^*$. Moreover, $\pi_0(Y_v^m) = (k[t]/t^{m+1})^*$. If $m > 0$, the map $\pi_0(Y_v^m) \twoheadrightarrow \pi_0(Y_v^{m-1})$ induced by the projection corresponds to the reduction modulo t^m map.*

(iii) *Let $Y_{v,0}^m$ be the connected component of Y_v^m corresponding to $1 \in (k[t]/t^{m+1})^*$. Then $Y_{v,0}^m$ is (isomorphic to) a finite covering of $D_v \times \mathbb{A}^m$ given by*

$$C\sigma(C) = S. \tag{3.6}$$

It is a connected finite étale Galois covering with Galois group $T_{w,m,0}$*. Moreover,* $Y^m_{v,0} \twoheadrightarrow Y^{m-1}_{v,0} \times \mathbb{A}^1(A_{m-1})$ *is given by*

$$c_m^q + c_m = \frac{f(a_m)}{f(a_0)} - \sum_{i=1}^{m-1} c_i^q c_{m-i}. \tag{3.7}$$

Proof. Note that for a point $\psi^m_v(a, C, D, A, 0) \in Y^m_v$, $S = \tau_m(a), C, D$ are units in $\bar{k}[t]/t^{m+1}$. From the last two equations in (3.4), we see that on Y^m_v, $D = \sigma(C)S^{-1}$ is uniquely determined by C and S and that Y^m_v is indeed given by the equation (3.5). Let us from now on proceed by induction on m. We see that $Y^0_v \to D_v$ is defined by $c_0^{q^2-1} = f(a_0)^{q-1}$, i.e., it is finite etale with Galois group isomorphic to k_2^*. Clearly, Y^m_v lies over $Y^{m-1}_v \times \mathbb{A}^1(A_{m-1})$. Bring equation (3.5) to the form $\sigma^2(C)S = C\sigma(S)$. Expanding this expression with respect to $C = c_0(1 + \sum_{i=1}^m c_i t^i)$, $S = \sum_i f(a_i)t^i$, shows that $Y^m_v \to Y^{m-1}_v \times \mathbb{A}^1(A_{m-1})$ is defined by an equation of the form

$$c_m^{q^2} - c_m = p(a_0, \ldots, a_m, c_0, \ldots, c_{m-1})$$

with p some regular function on $Y^{m-1}_v \times \mathbb{A}^1(A_{m-1})$. This is clearly a finite étale covering. Moreover, it is Galois and the Galois group is isomorphic to k_2, where $\lambda \in k_2$ acts by $c_m \mapsto c_m + \lambda$. By induction, $Y^m_v \to D_v \times \mathbb{A}^m(A)$ is also finite étale and has degree $(q^2-1)q^{2m}$. Equation (3.5) shows that the automorphism group of this covering contains $T_{w,m}$. Comparing the degrees we see that $Y^m_v \to D_v \times \mathbb{A}^m$ is Galois with Galois group $T_{w,m}$. This shows part (i). Let $\dot{x}I^m = \psi^m_v(a, C, D, A, 0) \in Y^m_v$. As $Y^m_v \subseteq X^m_{\dot{w}}(1)$, we obtain

$$1 = \det{}^m(\phi^m_w(1,1,0,0)) = \det{}^m(I^m\dot{x}^{-1}\sigma(\dot{x})I^m) = (CD)^{-1}\sigma(CD).$$

It follows that $\det^m(\dot{x}I^m) = CD \in (k[t]/t^{m+1})^*$. This shows the claim about the image of $\det^m$. In particular, we obtain a map $\pi_0(Y^m_v) \to (k[t]/t^{m+1})^*$. Its surjectivity follows using the action of $T_{w,m}$ on Y^m_v and the fact that $\det\colon T_{w,m} \to (k[t]/t^{m+1})^*$ is surjective. Let $Y^m_{v,0}$ be the preimage of 1 under $\det^m\colon Y^m_v \to \mathrm{R}_m\mathbb{G}_m$. Then $Y^m_{v,0}$ is connected: this is a byproduct of Lemma 3.9 (i) below. The compatibility of $\det^m$ with changing the level is immediate. Thus it remains to prove part (iii) of the proposition. Equation $CD = 1$, holding on $Y^m_{v,0}$, inserted into (3.4) shows the first claim of (iii). The second statement of (iii) is clear from parts (i),(ii). Inserting $C = c_0(1 + \sum_{i=1}^m c_i t^i)$, $S = \sum_i f(a_i)t^i$ into (3.6) shows (3.7). $\square$

3.4. **Cohomology of Y^m_v.** We keep the notations from Sections 3.1-3.3. Fix a prime $\ell \neq \mathrm{char}(k)$. We are interested in the ℓ-adic cohomology with compact support of the base change of Y^m_v to $\bar{k}$. To simplify notation, for a scheme X over k, we write $\mathrm{H}^i_c(X)$ for the space $\mathrm{H}^i_c(X \times_k \operatorname{Spec} \bar{k}, \overline{\mathbb{Q}_\ell})$. This space comes with a natural action of the Frobenius σ. Set $h^i_c(X) = \dim_{\overline{\mathbb{Q}}_\ell} \mathrm{H}^i_c(X)$. Further, $\mathrm{H}^i_c(X)(r)$ denotes the r-th Tate twist. Set:

$$N_- \;=\; \{a_0, c_0 \in \bar{k} \colon a_0 \in k_2 \smallsetminus k, c_0^{q+1} = f(a_0)\} \subseteq \bar{k} \times \bar{k},$$

and let C_+, C_- be affine curves over $\mathbb{F}_q$ defined by

$$C_\pm \colon x^q \pm x = y^{q+1},$$

where for C_- we additionally require $x \notin k$. We write $V_\pm = \mathrm{H}^1_c(C_\pm)$. One has $\dim_{\overline{\mathbb{Q}_\ell}} V_+ = q(q-1)$.

Theorem 3.5. *Let $0 \leqslant m < n$. Then $\mathrm{H}^i_c(Y_v^m) = \bigoplus_{(k[t]/t^{m+1})^*} \mathrm{H}^i_c(Y_{v,0}^m)$. Let $d_0 = d_0(n,m) = 2(n-1) + 2m + 1$. Then $\mathrm{H}^i_c(Y_{v,0}^m) = 0$ if $i > d_0 + 1$ or $i < d_0 - m$ and*

$$\begin{aligned}
\mathrm{H}_c^{d_0+1}(Y_{v,0}^m) &\cong \mathbb{Q}_\ell(-(n+m)) \\
\mathrm{H}_c^{d_0}(Y_{v,0}^m) &\cong V_-(-(n+m-1)) \\
\mathrm{H}_c^{d_0-j}(Y_{v,0}^m) &\cong \bigoplus_{N_-} \overline{\mathbb{Q}}_\ell^{q^{2(j-1)}(q-1)} \qquad \text{for any } 1 \leqslant j \leqslant m.
\end{aligned}$$

For $1 \leqslant j \leqslant m$ the action of Frob_{q^2} on $\mathrm{H}_c^{d_0-j}(Y_{v,0}^m)$ is given as follows: it acts by permuting the blocks corresponding to elements of N_- (by $(a_0, c_0) \mapsto (a_0, -c_0)$) and acts as multiplication with the scalar $(-1)^{d_0-j} q^{d_0-j}$ in each of these blocks.

Remark 3.6. We have chosen d_0 such that $\mathrm{H}_c^{d_0-j}(Y_v^m)$ corresponds to a $G(F)$-representation of level j (cf. Definition 4.2).

Proof. The first statement of the theorem follows from Proposition 3.4. We need some further notation:

$$\begin{aligned}
k_- = k_-(x) &= \{x \in k_2 \colon x^q + x = 0\} \subseteq k_2 \\
N_+ = N_+(x,y) &= \{x, y \in \bar{k} \colon x \in k_2, y^{q+1} = x^q + x\} \subseteq \bar{k} \times \bar{k}\,.
\end{aligned}$$

Let $Y_{v,0}^{m,\prime}$ be the finite étale covering of the open subset $\{a_0 \notin k\}$ of the $m+1$-dimensional affine space $\mathbb{A}^{m+1}(a_0, \dots, a_m)$, which is defined by the same equations defining $Y_{v,0}^m$ (cf. (3.7)). There is a projection $Y_{v,0}^m \twoheadrightarrow Y_{v,0}^{m,\prime}$ and the $I_{m,\dot{w}}/I^m$-action on $Y_{v,0}^m$ induces a $T_{w,m}$-action on $Y_{v,0}^{m,\prime}$. We have $Y_{v,0}^m \cong Y_{v,0}^{m,\prime} \times \mathbb{A}^{n-1}(a_{m+1}, \dots, a_{n-1}, A_0, \dots, A_{m-1})$ and hence $\mathrm{H}^i_c(Y_{v,0}^m) = \mathrm{H}_c^{i-2(n-1)}(Y_{v,0}^{m,\prime})(-(n-1))$. For $m \geqslant 0$, let $Z^m \subseteq Y_{v,0}^{m,\prime}$ be the closed subscheme defined by the equation $a_0^{q^2} - a_0 = 0$. Note that K- and $T_{w,m}$-actions on $Y_{v,0}^{m,\prime}$ restrict to actions on Z^m and that equation (3.7) defines $Z^m \subseteq Z^{m-1} \times \mathbb{A}^1(c_m)$ as a covering of Z^{m-1}. The equation $a_0^{q^2} - a_0 = 0$ divides Z^m into a disjoint union of $q^2 - q$ components, which are given by the same equations as $Y_{v,0}^{m,\prime}$ and on which a_0 is a fixed constant in $k_2 \smallsetminus k$. Thus on Z_m (for each $m \geqslant 1$) we may change our equations replacing a_m by $a_m' = f(a_0)^{-\frac{1}{q}} a_m - c_m$. For $a_0 \in k_2 \smallsetminus k$, one computes $f(a_0)^q = -f(a_0)$ and equation (3.7) simplifies over the locus $a_0 \in k_2 \smallsetminus k$ to

$$a_m'^{,q} + a_m' = \sum_{i=1}^{m-1} c_i^q c_{m-i}. \tag{3.8}$$

Now we make a coordinate change: for all $m \geqslant 1$ replace a_m' by $\alpha_m = a_m' - \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} c_i^q c_{m-i}$. This coordinate change turns equation (3.8) defining Z^m over Z^{m-1} into

$$\alpha_m^q + \alpha_m = \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} (c_i - c_i^{q^2}) c_{m-i}^q + \delta_m c_{m/2}^{q+1}, \tag{3.9}$$

where $\delta_m = 0$ if m is odd and $\delta_m = 1$ if m is even. All together, Z^m is isomorphic to the locally closed subset of $\mathbb{A}^{2m+2}(a_0, \alpha_1 \dots, \alpha_m, c_0, c_1, \dots, c_m)$ defined by $a_0^{q^2} - a_0 = 0$, $a_0^q - a_0 \neq 0$, $c_0^{q+1} = a_0^q - a_0$ and the m equations (3.9) for $m' = 1, 2, \dots, m$. The first three of these equations and the equation (3.9) for $m' = 1$ obviously divide Z^m into $N_- \times k_-(\alpha_1)$ components, which are all isomorphic, as one sees using K- and $T_{w,m}$-actions on Z^m. Thus $Z^m \cong \coprod_{N_- \times k_-(\alpha_1)} Z_0^m$, where Z_0^m is the closed subvariety of $\mathbb{A}^{2m-2}(\alpha_2, \dots, \alpha_m, c_1, \dots, c_m)$ defined by equations (3.9) for $m' = 2, \dots, m$.

Lemma 3.7. *Let $m \geqslant 1$. Then Z_0^m is connected, i.e., $\pi_0(Z^m) = N_- \times k_-(\alpha_1)$.*

Proof. We proceed by induction: for $m = 0$, Z_0^0 is a point, thus connected. Let $m \geqslant 1$ and assume that Z_0^{m-1} is connected. By Lemma 3.9(ii) below (this lemma is formulated for $\tilde{Z}_m$ instead of $Z^m \cong \tilde{Z}^{m-1} \times \mathbb{A}^1(c_m)$ – see below in the proof of the theorem), the fibers of $Z^m \to Z^{m-1}$ (and hence also of $Z_0^m \to Z_0^{m-1}$) over the open subset defined by $c_1^{q^2} - c_1 \neq 0$ are connected. Hence Z_0^m is connected. □

We see that Z_0^m is a connected étale covering of $\mathbb{A}^m(c_1, \dots, c_m)$. Hence $\mathrm{H}_c^i(Z^m) = 0$ for $i > 2m$ and $\mathrm{H}_c^{2m}(Z^m) = \bigoplus_{N_- \times k_-(\alpha_1)} \mathbb{Q}_\ell(-m)$. Consider now the decomposition in an open and a closed subset:

$$Y_{v,0}^{m,\prime} \smallsetminus Z^m \hookrightarrow Y_{v,0}^{m,\prime} \hookleftarrow Z^m. \tag{3.10}$$

Lemma 3.9 shows that $\mathrm{H}_c^i(Y_{v,0}^{m,\prime} \smallsetminus Z^m) = \mathrm{H}_c^{i-2m}(Y_{v,0}^{0,\prime} \smallsetminus Z^0)(-m)$ and $Y_{v,0}^{0,\prime} \smallsetminus Z^0$ can be identified with the open subset $C_- \smallsetminus N_-$ of the curve C_- defined in the variables a_0, c_0.

Lemma 3.8. *In the long exact sequence for $\mathrm{H}_c^*(\cdot)$ attached to* (3.10) *(cf.* [14] *III §1 Remark 1.30), the map*

$$\delta_m \colon \mathrm{H}_c^{2m}(Z^m) \to \mathrm{H}_c^{2m+1}(Y_{v,0}^{m,\prime} \smallsetminus Z^m) = \bigoplus_{N_-} \overline{\mathbb{Q}}_\ell(-m) \oplus V_-(-m).$$

is surjective onto the first summand.

Proof. By comparing the Frobenius-weights (which is possible due to Lemma 3.9) we see that the image is contained in the first summand. On the other hand, the natural projection $Y_{v,0}^{m,\prime} \to Y_{v,0}^{m-1,\prime}$ induces a morphism between the corresponding long exact sequences for $\mathrm{H}_c^*(\cdot)$, which induces a commutative diagram relating δ_m with δ_{m-1}. Iterating this for all levels $\geqslant 1$, we obtain a commutative diagram:

$$\begin{array}{ccccccc}
\bigoplus_{N_- \times k_-} \overline{\mathbb{Q}}_\ell(-m) & = & \mathrm{H}_c^{2m}(Z^m) & \xrightarrow{\delta_m} & \mathrm{H}_c^{2m+1}(Y_{v,0}^{m,\prime} \smallsetminus Z^m) & & \\
\downarrow & & \| & & \downarrow & & \\
\bigoplus_{N_-} \overline{\mathbb{Q}}_\ell & = & \mathrm{H}_c^0(Z^0) & \xrightarrow{\delta_0} & \mathrm{H}_c^1(C_- \smallsetminus N_-) & = & \bigoplus_{N_-} \overline{\mathbb{Q}}_\ell \oplus V_-
\end{array}$$

This diagram shows the lemma. □

The long exact sequence for $\mathrm{H}_c^*(\cdot)$ and Lemma 3.8 implies:

$$\mathrm{H}_c^i(Y_{v,0}^{m,\prime}) = \begin{cases} \mathrm{H}_c^i(Z^m) & \text{if } i < 2m \\ \bigoplus_{N_-} [\bigoplus_{k_-(\alpha_1)} \overline{\mathbb{Q}_\ell}(-m)]^{\Sigma_{\alpha_1} = 0} & \text{if } i = 2m \\ V_-(-m) & \text{if } i = 2m+1 \\ \mathbb{Q}_\ell(-m-1) & \text{if } i = 2m+2 \\ 0 & \text{if } i > 2m+2, \end{cases} \tag{3.11}$$

where $\sum_{\alpha_1} = 0$ means that we take the subspace of sum zero elements. Let $\tilde{Z}^{m-1}$ be the closed subspace of $\mathbb{A}^{2m+1}(a_0, \alpha_1, \dots, \alpha_m, c_0, c_1, \dots, c_{m-1})$ given by the same equations as Z^m: $a_0^{q^2} - a_0 = 0$, $a_0^q - a_0 \neq 0$, $c_0^{q+1} = a_0^q - a_0$ and equations (3.9) for $m' = 1, \dots, m$. Let $H = \{c_1^{q^2} - c_1 = 0\}$; this is a finite union of hyperplanes in the same affine space. Then $Z^m \cong \tilde{Z}^{m-1} \times \mathbb{A}^1(c_m)$. For $m \geqslant 3$ Lemma 3.9(ii) shows (here and until (3.15) we ignore Tate twists):

$$\mathrm{H}_c^i(\tilde{Z}^{m-1} \smallsetminus H) = \mathrm{H}_c^{i-2(m-2)}(\tilde{Z}^1 \smallsetminus H) = \begin{cases} 0 & \text{if } i \leqslant 2m-4 \\ \bigoplus_{N_- \times k_-(\alpha_1)} [(\bigoplus_{N_+(\alpha_2, c_1)} \overline{\mathbb{Q}}_\ell) \oplus V_+] & \text{if } i = 2m-3 \\ \bigoplus_{N_- \times k_-(\alpha_1)} \overline{\mathbb{Q}}_\ell & \text{if } i = 2m-2, \end{cases}$$

because $\tilde{Z}^1 \smallsetminus H \cong \coprod_{N_- \times k_-(\alpha_1)} (C_+ \smallsetminus N_+)$. Further, Lemma 3.10 shows $\tilde{Z}^{m-1} \cap H = \coprod_{k_-(\alpha_1) \times N_+(\alpha_2, c_1)} Z_{(1)}^{m-2}$, where $Z_{(1)}^{m-2} \cong Z^{m-2}$ and the index (1) indicates the shift in variables given by $\alpha_i \mapsto \alpha_{i+2}, c_i \mapsto c_{i+1}$ (for $i \geqslant 1$) and hence $\mathrm{H}_c^i(\tilde{Z}^{m-1} \cap H) = \bigoplus_{k_-(\alpha_1) \times N_+(\alpha_2, c_1)} \mathrm{H}_c^i(Z_{(1)}^{m-2})$ and, in particular, the top cohomology group of $\tilde{Z}^{m-1} \cap H$ is in degree $2m-4$ and is equal to

$$\mathrm{H}_c^{2m-4}(\tilde{Z}^{m-1} \cap H) = \bigoplus_{k_-(\alpha_1) \times N_+(\alpha_2, c_1)} \mathrm{H}_c^{2m-4}(Z_{(1)}^{m-2}) = \bigoplus_{k_-(\alpha_1) \times N_+(\alpha_2, c_1)} \bigoplus_{N_- \times k_-(\alpha_3)} \overline{\mathbb{Q}}_\ell.$$

as follows from Lemma 3.7 (note the index shift $\alpha_1 \mapsto \alpha_3$). All these, the long exact sequence for $\mathrm{H}_c^*(\cdot)$ attached to

$$\tilde{Z}^{m-1} \smallsetminus H \hookrightarrow \tilde{Z}^{m-1} \hookleftarrow \tilde{Z}^{m-1} \cap H,$$

the analog of Lemma 3.8 for this sequence and Lemma 3.10 show that for $m \geqslant 3$ we have:

$$\mathrm{H}_c^i(Z^m) = \mathrm{H}_c^{i-2}(\tilde{Z}^{m-1}) = \begin{cases} \bigoplus_{k_-(\alpha_1) \times N_+(\alpha_2, c_1)} \mathrm{H}_c^{i-2}(Z_{(1)}^{m-2}) & \text{if } i < 2m-2 \\ \bigoplus_{N_- \times k_-(\alpha_1) \times N_+(\alpha_2, c_1)} [\bigoplus_{k_-(\alpha_3)} \overline{\mathbb{Q}}_\ell]^{\sum_{\alpha_3} = 0} & \text{if } i = 2m-2 \\ \bigoplus_{N_- \times k_-(\alpha_1)} V_+ & \text{if } i = 2m-1 \\ \bigoplus_{N_- \times k_-(\alpha_1)} \overline{\mathbb{Q}}_\ell & \text{if } i = 2m. \end{cases} \tag{3.12}$$

Note that $\sharp N_+ = q^3, \sharp k_- = q$. Hence for $m \geqslant 3$, we have $h_c^{2m}(Z^m) = (\sharp N_-)q$ and $h_c^{2m-j}(Z^m) = (\sharp N_-)q^{2j}(q-1)$ for $j \in \{1, 2\}$. For Z^1, Z^2 one computes: $\mathrm{H}_c^2(Z^1) = \bigoplus_{N_- \times k_-} \overline{\mathbb{Q}}_\ell$ and $\mathrm{H}_c^i(Z^1) = 0$ if $i \neq 2$ and

$$\mathrm{H}_c^i(Z^2) = \mathrm{H}_c^{i-2}(\tilde{Z}^1) = \begin{cases} 0 & \text{if } i \leqslant 2 \text{ or } i \geqslant 5 \\ \bigoplus_{N_- \times k_-(\alpha_1)} V_+ & \text{if } i = 3 \\ \bigoplus_{N_- \times k_-(\alpha_1)} \overline{\mathbb{Q}}_\ell & \text{if } i = 4. \end{cases} \tag{3.13}$$

Let now $m \geqslant 3$. For $j > 0$, write $j = 2\lfloor \frac{j-1}{2} \rfloor + j'$, where $j' = 1$ if j odd, $j' = 2$ otherwise. Iterating (3.12) $\lfloor \frac{j-1}{2} \rfloor$ times, we get for all $0 < j < m$:

$$h_c^{2m-j}(Z^m) = q^{4\lfloor \frac{j-1}{2} \rfloor} h_c^{2(m-2\lfloor \frac{j-1}{2} \rfloor) - j'}(Z_{(\lfloor \frac{j-1}{2} \rfloor)}^{m-2\lfloor \frac{j-1}{2} \rfloor}) = (\sharp N_-)q^{2j}(q-1). \tag{3.14}$$

where $Z_{(l)}^m \cong Z^m$ using the index shift as above l times. Thus for all $m \geqslant 1$, $j > 0$:

$$h_c^{2m+1-j}(Z^m) = \begin{cases} (\sharp N_-)q & \text{if } j = 1 \\ (\sharp N_-)q^{2(j-1)}(q-1) & \text{if } 1 < j \leqslant m \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Combined with (3.11), this implies the dimension formula in the theorem. It remains to compute the Frobenius action. The Tate twists in the two top cohomology groups of $Y_{v,0}^m$ can be deduced easily by relating $Y_{v,0}^m$ with $Y_{v,0}^{m,\prime}$. To prove the claim about Frob_{q^2}-action in degrees $i \leqslant d_0-1$, note that Frob_{q^2} acts on N_- by $(a_0, c_0) \mapsto (a_0, -c_0)$. Further, let Z_1^m be the subvariety of $\mathbb{A}^{2m}(c_1, \ldots, c_m, a_1', \ldots, a_m')$ defined by m equations (3.8) for $m' = 1, \ldots, m$, i.e., $Z^m = N_- \times Z_1^m$, where N_- is seen as a discrete variety. Lemma 3.11 shows that Z_1^m is a maximal variety over $\mathbb{F}_{q^2}$ (for a definition cf. the paragraph preceding Lemma 3.11), i.e., Frob_{q^2} acts on $\mathrm{H}_c^i(Z_1^m)$ by $(-1)^i(q^2)^{i/2} = (-q)^i$ for any $i \in \mathbb{Z}$. Further we have for all $2 \leqslant j \leqslant m$:

$$\mathrm{H}_c^{d_0-j}(Y_{v,0}^m) = \mathrm{H}_c^{2m+1-j}(Y_{v,0}^{m,\prime})(-(n-1)) = \mathrm{H}_c^{2m+1-j}(Z^m)(-(n-1))$$

(note that for $j = 1$, this remains true if one replaces the second equality by an inclusion, cf. (3.11)). This implies the last statement of the theorem. ☐

Lemma 3.9. *With notations as in the proof of Theorem 3.5, we have:*

(i) *Let $m \geqslant 1$. The fibers of the natural projection $\pi\colon Y_{v,0}^{m,\prime} \smallsetminus Z^m \to Y_{v,0}^{m-1,\prime} \smallsetminus Z^{m-1}$ are isomorphic to $\mathbb{A}^1$. We have:*

$$\mathrm{H}_c^i(Y_{v,0}^{m,\prime} \smallsetminus Z^m) = \mathrm{H}^{i-2}(Y_{v,0}^{m-1,\prime} \smallsetminus Z^{m-1})(-1).$$

(ii) *Let $m \geqslant 3$. The fibers of the natural projection $\tilde{Z}^{m-1} \smallsetminus H \to \tilde{Z}^{m-2} \smallsetminus H$ are isomorphic to $\mathbb{A}^1$. We have:*

$$\mathrm{H}_c^i(\tilde{Z}^{m-1} \smallsetminus H) = \mathrm{H}_c^{i-2}(\tilde{Z}^{m-2} \smallsetminus H)(-1).$$

Proof. Let us prove part (i). The scheme $Y_{v,0}^{m,\prime} \smallsetminus Z^m$ is the closed subspace of $(Y_{v,0}^{m-1,\prime} \smallsetminus Z^{m-1}) \times \mathbb{A}^2(a_m, c_m)$ defined by the equation (3.7). Letting x be a point of $Y_{v,0}^{m-1,\prime} \smallsetminus Z^{m-1}$, we see that the fiber of π over x is given by the equation

$$c_m^q + c_m = f(a_0(x))^{-1}(a_m^q - a_m) + \lambda(x),$$

with $a_0(x) \notin k_2$ the a_0-coordinate of x and $\lambda(x) \in \bar{k}$ depending on x. Using the substitution $a_m' = f(a_0(x))^{-\frac{1}{q}} a_m - c_m$, this equation can be rewritten as

$$a_m'^{;q} - f(a_0)^{\frac{1}{q}-1} a_m' = (1 + f(a_0(x))^{\frac{1}{q}-1}) c_m - \lambda(x).$$

As $a_0(x) \notin k_2$ we have $f(a_0(x))^{\frac{1}{q}-1} \neq 0, -1$ and hence the fiber of π over x is isomorphic (over $\bar{k}$) to the Artin-Schreier covering of $\mathbb{A}^1(c_m)$, hence is itself isomorphic to the affine line. This shows the first statement of the lemma.

For the second statement, note that as the fibers of π are $\cong \mathbb{A}^1$, we have $\mathrm{R}_c^2\pi_*\overline{\mathbb{Q}_\ell} \cong \overline{\mathbb{Q}_\ell}(-1)$ and $\mathrm{R}_c^j\pi_*\overline{\mathbb{Q}_\ell} = 0$ for $j \neq 2$. This together with the spectral sequence

$$\mathrm{H}_c^i(Y_{v,0}^{m-1,\prime} \smallsetminus Z^{m-1}, \mathrm{R}_c^j\pi_*\overline{\mathbb{Q}_\ell}) \Rightarrow \mathrm{H}_c^{i+j}(Y_{v,0}^{m,\prime} \smallsetminus Z^m)$$

implies the second statement of part (i). Part (ii) of the lemma has a similar proof, using (3.9) instead of (3.7). ☐

Lemma 3.10. *With notations as in the proof of Theorem 3.5, for $m \geqslant 3$, we have* $\tilde{Z}^{m-1} \cap H \cong \coprod_{k_-(\alpha_1)} \coprod_{N_+(\alpha_2, c_1)} Z^{m-2}$.

Proof. On the union of hyperplanes H, the term $(c_1 - c_1^{q^2})c_{m-1}^q$ in the equation (3.9) defining $\tilde{Z}^{m-1}$ over $\tilde{Z}^{m-2}$ cancels, leaving the free variable c_m and the equation arising from it (after renaming the variables by $\alpha_i \mapsto \alpha_{i-2}$ for $i \geqslant 3$, $c_i \mapsto c_{i-1}$ for $i \geqslant 2$) is simply the equation defining Z^{m-2} over Z^{m-3}. The lemma follows from this observation. $\square$

We recall the definition of maximal varieties from the introduction of [6], where it appears in a similar setup. Let X be a scheme of finite type over a finite field $\mathbb{F}_Q$ with Q elements. Let Frob_Q denote the Frobenius over $\mathbb{F}_Q$. By [8] Theorem 3.3.1, for each i and each eigenvalue α of Frob_Q in $\mathrm{H}_c^i(X)$, there exists an integer $i' \leqslant i$, such that all complex conjugates of α have absolute value $Q^{i'/2}$. Hence by Grothendieck-Lefschetz formula we get an upper bound on the number of points on X:

$$\sharp X(\mathbb{F}_Q) = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}(\mathrm{Frob}_Q; \mathrm{H}_c^i(X)) \leqslant \sum_{i \in \mathbb{Z}} Q^{i/2} h_c^i(X), \tag{3.16}$$

where equality holds if and only if Frob_Q acts on $\mathrm{H}_c^i(X)$ by the scalar $(-1)^i Q^{i/2}$ for each $i \in \mathbb{Z}$. If this is the case, then $X/\mathbb{F}_Q$ is called *maximal*.

Lemma 3.11. *Let Z_1^m be as in the proof of Theorem 3.5. For $m \geqslant 1$, Z_1^m is a maximal variety over $\mathbb{F}_{q^2}$.*

Proof. Frob_{q^2} acts on $\mathrm{H}_c^i(Z_1^m)$ as an endomorphism with eigenvalues being Weil numbers with absolute value $(q^2)^{i'/2} \leqslant (q^2)^{i/2} = q^i$. From (3.16) we obtain the upper bound $u(Z_1^m, q^2)$ for the number of $\mathbb{F}_{q^2}$-points on Z_1^m:

$$\sharp Z_1^m(\mathbb{F}_{q^2}) \leqslant u(Z_1^m, q^2) = \sum_{i=0}^{2m} q^i h_c^i(Z_1^m).$$

Using equation (3.15), we see that $u(Z_1^m, q^2) = q^{3m}$. On the other hand, let $p_i(\underline{c}) = \sum_{j=1}^{i-1} c_j c_{i-j}^q$ and let $c_j \in \mathbb{F}_{q^2}$ for $j = 1, \ldots, m$ be given. Then we have $p_i(\underline{c})^q = p_i(\underline{c})$, i.e., $p_i((c_j)_{j=1}^{i-1}) \in \mathbb{F}_q$ for all $1 \leqslant i \leqslant m$. But the equation $x^q + x = \lambda \in \mathbb{F}_q$ has precisely q solutions in $\mathbb{F}_{q^2}$. Thus for each given point $(c_1, \ldots, c_m) \in \mathbb{A}^m(\mathbb{F}_{q^2})$, there are exactly q^m points in Z_1^m lying over it (cf. equation (3.8)). Thus $\sharp Z_1^m(\mathbb{F}_{q^2}) = q^{3m}$, which finishes the proof. $\square$

3.5. **Character subspaces.** We keep notations from Sections 3.1-3.4 and deduce some corollaries from Theorem 3.5.

Lemma 3.12. *Let $m \geqslant 0$. The $I_{m,\dot{w}}/I^m$-action on $\mathrm{H}_c^i(Y_v^m)$ factors through a $T_{w,m}$-action.*

Proof. This is immediate as the action of $\ker(I_{m,\dot{w}}/I^m \twoheadrightarrow T_{w,m})$ on $Y_v^m = Y_v^{m,\prime} \times \mathbb{A}^{n-1}(a_{m+1}, \ldots, a_{n-1}, A_0, \ldots, A_{m-1})$ (where $Y_v^{m,\prime}$ is defined analogously to $Y_{v,0}^{m,\prime}$ in the proof of theorem 3.5) comes from an action on $\mathbb{A}^{n-1}$, which contributes to the cohomology of Y_v^m only via a dimension shift. $\square$

For an abelian (locally compact) group A, let $A^\vee$ denote the group of (smooth) $\overline{\mathbb{Q}}_\ell^*$-valued characters of A. By Lemma 3.12 we have a decomposition

$$\mathrm{H}^i_c(Y^m_v) = \bigoplus_{\chi\in T^\vee_{w,m}} \mathrm{H}^i_c(Y^m_v)[\chi] \tag{3.17}$$

into isotypical components with respect to the action of $T_{w,m}$.

Corollary 3.13. *Let $m \geqslant 1$ and $1 \leqslant j \leqslant m$. Let $\chi\colon T_{w,m} \to \overline{\mathbb{Q}}^*_\ell$ be a character. Then* Frob_{q^2} *acts on $\mathrm{H}^{d_0-j}_c(Y^m_v)[\chi]$ by multiplication with the scalar $\chi(-1)(-1)^{d_0-j}q^{d_0-j}$.*

Proof. We have $\mathrm{H}^i_c(Y^m_v) = \bigoplus_{(k[t]/t^{m+1})^*} \mathrm{H}^i_c(Y^m_{v,0})$ and Frob_{q^2} acts trivially on the index set of the direct sum, so it is enough to study its action on $\mathrm{H}^i_c(Y^m_{v,0})$. With notations as in the proof of Theorem 3.5, we have for $1 < j \leqslant m$:

$$\mathrm{H}^{d_0-j}_c(Y^m_{v,0}) = \mathrm{H}^{2m+1-j}_c(Y^{m,\prime}_{v,0})(-(n-1)) = \mathrm{H}^{2m+1-j}_c(Z^m)(-(n-1)) = (\bigoplus_{N_-}\overline{\mathbb{Q}}_\ell)\otimes\mathrm{H}^{2m+1-j}_c(Z^m_1)(-(n-1)),$$

as $Z^m = N_- \times Z^m_1$, where N_- is seen as a disjoint union of points (for $j = 1$ this remains true if we replace the second equality by an inclusion, cf. (3.11)). Now, Frob_{q^2} acts on N_- by $(a_0, c_0) \mapsto (a_0, -c_0)$ and in $\mathrm{H}^{2m+1-j}_c(Z^m_1)$ by the scalar $(-1)^{2m+1-j}q^{2m+1-j}$. Note that $-1 \in T_{w,m}$ acts on N_- in the same way as Frob_{q^2} and trivially in $\mathrm{H}^{2m+1-j}_c(Z^m_1)$. Thus the eigenspaces for -1 and Frob_{q^2} coincide. There are only two such eigenspaces U_1 and U_{-1}, and Frob_{q^2} acts on $U_{\pm1}$ by the scalar $(\pm1)(-1)^{2m+1-j}q^{2m+1-j}$. Now let $\overline{\chi}$ be the restriction of χ to $\mu_2 \subseteq T_{w,m}$. Then

$$\mathrm{H}^i_c(Y^m_v)[\chi] \subseteq \mathrm{H}^i_c(Y^m_v)[\overline{\chi}] = U_{\overline{\chi}(-1)},$$

which proves the corollary. ◻

Let $T^i_{w,m}$ denote the subgroup of $T_{w,m}$ of elements which are congruent 1 modulo t^i. Let $T^{\vee,gen}_{w,m}$ denote the set of all characters of $T_{w,m}$, which are non-trivial on $T_{w,m,0} \cap T^m_{w,m}$. We also need the following purity result.

Corollary 3.14. *Let $m \geqslant 1$. Let $d_0 = d_0(n,m)$. The finite étale morphism $Y^m_v \to Y^{m-1}_v \times \mathbb{A}^1(A_{m-1})$ induces an isomorphism*

$$\mathrm{H}^i_c(Y^m_{v,0}) \cong \mathrm{H}^i_c(Y^{m-1}_{v,0} \times \mathbb{A}^1(A_{m-1})) \cong \mathrm{H}^{i-2}_c(Y^{m-1}_{v,0})(-1)$$

for all $i \neq d_0 - m$. If $\chi \in T^{\vee,\mathrm{gen}}_{w,m}$, then

$$\mathrm{H}^i_c(Y^m_v)[\chi] = 0 \quad \textit{for all } i \neq d_0 - m.$$

Conversely, if $\chi \in T^\vee_{w,m} \smallsetminus T^{\vee,\mathrm{gen}}_{w,m}$, then $\mathrm{H}^{d_0-m}_c(Y^m_v)[\chi] = 0$.

Proof. The first statement follows directly from Theorem 3.5 by comparing dimensions. Let $N = \ker((k[t]/t^{m+1})^* \twoheadrightarrow (k[t]/t^m)^*)$. The finite étale covering $Y^m_v \to Y^{m-1}_v \times \mathbb{A}^1(A_{m-1})$ factors as $Y^m_v \to \coprod_N Y^{m-1}_v \times \mathbb{A}^1(A_{m-1}) \to Y^{m-1}_v \times \mathbb{A}^1(A_{m-1})$, where the first morphism has Galois group $T_{w,m,0} \cap T^m_{w,m}$. The first statement of the corollary implies that the first morphism in this factorization induces an isomorphism in the cohomology for all $i \neq d_0 - m$. The second statement of the corollary follows from it. If χ is trivial on $T_{w,m,0} \cap T^m_{w,m}$, then

$$\mathrm{H}^{d_0-m}_c(Y^m_v)[\chi] \subseteq \mathrm{H}^{d_0-m}_c(\coprod_N Y^{m-1}_v \times \mathbb{A}^1(A_{m-1})) = \bigoplus_N \mathrm{H}^{d_0-m-2}_c(Y^{m-1}_v) = \bigoplus_N \mathrm{H}^{d_0(n,m-1)-m}_c(Y^{m-1}_v)$$

and the last group is 0 by Theorem 3.5. Hence the third statement of the corollary. □

3.6. **Superbasic case.** Before going on, we make a digression and study the varieties $X^m_{x_m}(b)$ in the superbasic case $b = \begin{pmatrix} & 1 \\ t & \end{pmatrix}$. Let $\dot{x} = \begin{pmatrix} & t^{-n} \\ t^{n+1} & \end{pmatrix}$ with even $n > 0$ and let x resp. $\dot{x}$ be the image of $\dot{x}$ in $\tilde{W}$ resp. in $D_{G,m}(x)$. Let $\dot{v}$, v be as in (3.1). The group $J_b(F)$ is the group of units D^* of the quaternion algebra D over F. If $\mathcal{O}_D$ are the integers of D, then $U_D = \mathcal{O}_D^*$ is a maximal compact subgroup of D^* and $D^*/U_D \cong \mathbb{Z}$. Then [10] Theorem 3.3(i) shows

$$X_x(b) = \coprod_{D^*/U_D} C_v.$$

The same arguments as used in the proof of Theorem 3.2, show that for $m < 2k$ one has:

$$X^m_{\dot{x}}(b) = \coprod_{D^*/U_D} Y^m_v(b),$$

where $Y^m_v(b) \subseteq C^m_v$ is the closed subscheme given by equations

$$\begin{aligned} D^{-1}\sigma(C)(1 - ta\sigma(a))^{-1} &= 1 \\ C^{-1}\sigma(D)(1 - ta\sigma(a)) &= 1 \\ B &= 0. \end{aligned}$$

Again after eliminating D, it is defined in the coordinates a, C, A by

$$C\sigma(1 - ta\sigma(a)) = (1 - ta\sigma(a))\sigma^2(C). \tag{3.18}$$

An explicit comparison with results of Boyarchenko [1], who carried out the closely related construction of Lusztig for a division algebra over F of invariant $\frac{1}{n}$ (for levels $m = 1, 2$, with a suggestion of how one can continue for higher levels) and Chan [7] (who then extended Boyarchenko's results to all levels for the quaternion algebra) shows that the varieties X_h defined in the quoted papers are very similar to varieties $X^m_{\dot{x}}(b)$ defined by (3.18), but do not coincide completely, at least due to the presence of the additional coordinate A in our approach. Also note that level $h \geqslant 2$ in the quoted papers correspond to level $m = h - 1 \geqslant 1$ in the present article.

4. Representation theory of $GL_2(F)$

We continue to assume $G = GL_2$ throughout this section and keep the notations from Section 3.1 and the beginning of Section 3.2. Let us collect some further important notation here. We try to keep it consistent with the notation in [2]. The only major difference is that we write K (and not $U = U_{\mathfrak{M}}$) for the maximal compact subgroup $G(\mathcal{O}_F)$ of $G(F)$. For $\lambda \in X_*(T)$ we write $t^\lambda \in T(L)$ for the image of the uniformizer t under λ. For an element $x \in \bar{k}[t]/t^{m+1}$, we mean by its t-adic valuation $v_t(x)$ the largest integer $\mu \geqslant 0$, such that $x \in t^\mu \cdot \bar{k}[t]/t^{m+1}$. Moreover:

- Z is the center of $G(F)$
- $E = k_2((t)) \subset L$ is the unramified degree two extension of F
- U_M (resp. U^m_M for $m \geqslant 1$) denote the units (resp. the m-units) of a local field M
- $\mathfrak{M} = M_{22}(\mathcal{O}_F)$; it is an $\mathcal{O}_F$-algebra
- $K = G(\mathcal{O}_F) = \mathfrak{M}^*$; it is a maximal compact subgroup of $G(F)$

- $K^i = 1 + t^i\mathfrak{M}$ for $i \geqslant 0$, and $(K^i)_{i=0}^{\infty}$ defines a descending filtration of K by open normal subgroups
- $K_m = K/K^{m+1} \cong G(\mathcal{O}_F/t^{m+1})$
- $K_m^i = K^i/K^{m+1} = 1 + t^i\mathfrak{M}/1 + t^{m+1}\mathfrak{M}$ define a filtration on K_m
- $T_{w,m}, T_{w,m,0}$ are as in the beginning of Section 3.3 and $T_{w,m}^i, T_{w,m}^{\vee,gen}$ as in the paragraph preceding Corollary 3.14

For a *locally* compact abelian group A, a $\overline{\mathbb{Q}}_\ell$-vector space W with a right A-action and a $\overline{\mathbb{Q}}_\ell^*$-valued character χ of A, we let $W[\chi]$ be the maximal *quotient* of W, on which A acts by χ (if A is compact, W finite dimensional, $W[\chi]$ is canonically isomorphic to the maximal χ-isotypical subspace of W). A left $G(F)$-action on W, which commutes with the A-action, induces a left $G(F)$-action on $W[\chi]$.

4.1. **Definitions and results.** Let $\iota_E \colon E \hookrightarrow M_{22}(L)$ be the embedding of F-algebras given by $e \mapsto \mathrm{diag}(e, \sigma(e))$. We have $\iota_E(U_E)/\iota_E(U_E^{m+1}) = T_{w,m}$. Inflating the $T_{w,m}$-action to $\iota_E(U_E)$ and pulling back via ι_E, we obtain an U_E-action on $X_{\dot{w}}^m(1)$. The center Z of $G(F)$ is F^*, thus (as in the last lines of Section 2.2), the action of U_E on $X_{\dot{w}}^m(1)$ extends to an action of $E^* = F^*U_E$, which commutes with the left $G(F)$-action.

Let χ be a non-trivial character of E^*. The *level* $\ell(\chi)$ of χ is the least integer $m \geqslant 0$, such that $\chi|_{U_E^{m+1}}$ is trivial. Moreover, the pair $(E/F, \chi)$ is said to be *admissible* ([2] 18.2) if χ does not factor through the norm $\mathrm{N}_{E/F} \colon E^* \to F^*$. Two pairs $(E/F, \chi), (E/F, \chi')$ are F-isomorphic, if there is some $\gamma \in \mathrm{G}_{E/F}$ such that $\chi' = \chi \circ \gamma$. Let $\mathbb{P}_2^{\mathrm{nr}}(F)$ denote the set of all isomorphism classes of admissible pairs over F attached to E/F. An admissible pair $(E/F, \chi)$ is called *minimal* if $\chi|_{U_E^m}$ does not factor through $\mathrm{N}_{E/F}$, where m is the level of χ.

Definition 4.1. Let χ be a character of E^*, such that $(E/F, \chi)$ is admissible. The *essential level* $\ell_{\mathrm{ess}}(\chi)$ of χ is the smallest integer $m' \geqslant 0$, such that there is a character ϕ of F^* with $\ell(\phi_E\chi) = m'$, where $\phi_E = \phi \circ \mathrm{N}_{E/F}$.

Clearly, $\ell_{\mathrm{ess}}(\chi) \leqslant \ell(\chi)$. Moreover, an admissible pair $(E/F, \chi)$ is minimal if and only if $\ell(\chi) = \ell_{\mathrm{ess}}(\chi)$. Using the geometric constructions from last sections, we associate to any admissible pair $(E/F, \chi)$ a $G(F)$-representation. Recall the element $\dot{w} \in N_T(L)$ introduced in (3.1), which depends on an even integer $n > 0$. Recall from the beginning of Section 3.4, that we write $\mathrm{H}_c^i(X)$ instead of $\mathrm{H}_c^i(X \times_k \mathrm{Spec}\,\bar{k}, \overline{\mathbb{Q}_\ell})$ for a k-scheme X.

Definition 4.2. Let $(E/F, \chi)$ be admissible. Let $\ell(\chi) = m \geqslant \ell_{\mathrm{ess}}(\chi) = m'$. We take $n > 0$ even such that $0 \leqslant m < n$ and let $d_0(n,m)$ be as in Theorem 3.5. Define R_χ to be the $G(F)$-representation

$$R_\chi = \mathrm{H}_c^{d_0(n,m)-m'}(X_{\dot{w}}^m(1))[\chi].$$

One easily sees that this definition is independent of the choice of n (cf. Theorem 3.2 and the definition of Y_v^m). To state our main result, we need some terminology from [2], which we will freely use here. In particular, the *level* $\ell(\pi) \in \frac{1}{2}\mathbb{Z}$ of an irreducible $G(F)$-representation is defined in [2] 12.6. Moreover, in [2] 20.1, 20.3 Lemma it is explained when an irreducible cuspidal representation π of $G(F)$ is called *unramified*. We denote by $\mathscr{A}_2^{\mathrm{nr}}(F)$ the set of all isomorphism classes of irreducible cuspidal unramified representations of $G(F)$. This is a subset of the set $\mathscr{A}_2^0(F)$ of the isomorphism classes of all irreducible cuspidal representations of $G(F)$ ([2], §20). The (unramified part of the) tame parametrization theorem ([2] 20.2 Theorem) states the existence of a certain bijection (also for even q):

$$\mathbb{P}_2^{\mathrm{nr}}(F) \xrightarrow{\sim} \mathscr{A}_2^{\mathrm{nr}}(F), \quad (E/F,\chi) \mapsto \pi_\chi \tag{4.1}$$

where π_χ is a certain $G(F)$-representation constructed in [2] §19. Below, in Section 4.6 we briefly recall this construction. Here is our main result.

Theorem 4.3. *Let $(E/F,\chi)$ be an admissible pair. The representation R_χ is irreducible cuspidal, unramified, has level $\ell(\chi)$ and central character $\chi|_{F^*}$. Moreover, R_χ is isomorphic to π_χ, i.e., the map*

$$R\colon \mathbb{P}_2^{\mathrm{nr}}(F) \to \mathscr{A}_2^{\mathrm{nr}}(F), \quad (E/F,\chi) \mapsto R_\chi \tag{4.2}$$

is a bijection and coincides with the map from the tame parametrization theorem (4.1).

The theorem will be proven at the end of Section 4.6, after the necessary preparations in Sections 4.2-4.6 are done. We wish to point out here, that the injectivity of (4.2) follows from Proposition 4.26 and does not use the theory developed in [2], whereas to prove surjectivity of (4.2), we use the full machinery of [2].

In the rest of this section, we only deal with the central character, reduce to the minimal case and introduce some further notation. For a character ϕ of F^* and a representation π of $G(F)$, we write $\phi\pi$ for the $G(F)$-representation given by $g \mapsto \phi(\det(g))\pi(g)$ and we let $\phi_E = \phi \circ N_{E/F}$ be the corresponding character of E^*. If ϕ is a character of F^* and $(E/F,\chi)$ an admissible pair, then (4.1) satisfies: the central character of π_χ is $\chi|_{F^*}$ and $\phi\pi_\chi = \pi_{\chi\phi_E}$. We have an analogous statement for R_χ.

Lemma 4.4. *Let $(E/F,\chi)$ be admissible. Then the central character of R_χ is $\chi|_{F^*}$. If ϕ is a character of F^*, then $\phi R_\chi = R_{\chi\phi_E}$.*

Proof. The first statement follows from the definition of R_χ as the χ-isotypic component of some cohomology space, and the fact that the actions of $F^* \cong Z \subseteq G(F)$ and $F^* \subseteq E^*$ in this cohomology space coincide as they already coincide on the level of varieties. The second statement follows by unraveling the definition of R_χ, using the natural isomorphism $\mathrm{H}_c^{d_0(n,m)-m'}(Y_{v,0}^m) \cong \mathrm{H}_c^{d_0(n,\lambda)-m'}(Y_{v,0}^\lambda)$ for $m \geqslant \lambda \geqslant m'$ from Theorem 3.5 and $\chi|_{\ker(\mathrm{N}_{E/F})} = \phi_E\chi|_{\ker(\mathrm{N}_{E/F})}$. ☐

We fix some notations for the rest of Section 4. Let $(E/F,\chi)$ be a minimal pair, let m be the level of χ and write $i_0 = d_0(n,m) - m$. Let $\tilde{Y}_v^m$ be the (disjoint) union of all Z-translates of Y_v^m inside $X_w^m(1)$. Note that Y_v^m is fixed by K, hence $\tilde{Y}_v^m$ is fixed by ZK. Define the ZK-representation Ξ_χ by

$$\Xi_\chi = \mathrm{H}_c^{i_0}(\tilde{Y}_v^m)[\chi].$$

Then Theorem 3.2 shows

$$R_\chi = \mathrm{c-Ind}_{ZK}^{G(F)}\,\Xi_\chi.$$

Moreover, let ξ_χ be the restriction of Ξ_χ to K, i.e., Ξ_χ is the unique extension of ξ_χ to ZK such that $t^{(1,1)}$ acts as $\chi(t^{(1,1)})$. Let V_χ denote the space in which Ξ_χ (resp. ξ_χ) acts. Note that ξ_χ is inflated from a representation of the finite group $K_m = K/K^{m+1}$, as K^{m+1} acts trivially in the cohomology of Y_v^m.

Lemma 4.5. $\xi_\chi \cong \mathrm{H}_c^{i_0}(Y_v^m)[\chi|_{U_E}]$.

Proof. Let W be a $\overline{\mathbb{Q}}_\ell$-vector space in which K acts on the left, U_E on the right such that these actions commute. Let

$$W[t] = \left\{ \sum_{\substack{i\in\mathbb{Z} \\ -\infty \ll i \ll \infty}} w_i t^i : w_i \in W \right\}$$

with obvious K and U_E-actions. Extend them to ZK- resp. E^*-actions by letting t act as $t.(\sum_i w_i t^i) = \sum_i w_{i-1} t^i$. Then one checks that $(W[t])[\chi|_{U_E}] = (W[\chi|_{U_E}])[t]$ and that the composed map $W[\chi|_{U_E}] \hookrightarrow (W[\chi|_{U_E}])[t] \twoheadrightarrow (W[t])[\chi]$ is a bijection. Apply this to $W = \mathrm{H}_c^{i_0}(Y_v^m)$ and $W[t] \cong \mathrm{c}-\mathrm{Ind}_K^{ZK} W = \mathrm{H}_c^{i_0}(\tilde{Y}_v^m)$. ☐

4.2. **Trace computations I: preliminaries.** Let $(E/F,\chi)$ be a minimal pair of level $m \geqslant 0$. Via ι_E, $\chi|_{U_E}$ induces a character of $T_{w,m}$. We denote this character of $T_{w,m}$ also by χ. The context excludes any ambiguity.

Lemma 4.6. *We have $\chi \in T_{w,m}^{\vee,gen}$, i.e., χ is non-trivial on $T_{w,m,0} \cap T_{w,m}^m$.*

Proof. As $(E/F,\chi)$ is minimal, $\chi|_{\ker(\mathrm{N}_{E/F}\colon U_E^m \to U_F^m)}$ is non-trivial. The level of χ is m, i.e., χ is trivial on U_E^{m+1}. Now we have $U_E^m/U_E^{m+1} \cong T_{w,m}^m$ via ι_E, the norm map induces the map $\bar{\mathrm{N}}\colon U_E^m/U_E^{m+1} \to (k[t]/t^{m+1})^*$, and moreover, $\det \circ \iota_E = \bar{\mathrm{N}}$, where $\det\colon T_{w,m}^m \to (k[t]/t^{m+1})^*$ is the determinant. Now $\ker(\det\colon T_{w,m}^m \to (k[t]/t^{m+1})^*) = T_{w,m,0} \cap T_{w,m}^m$ and $\chi \in T_{w,m}^{\vee,gen}$ if and only it is non-trivial on $T_{w,m,0} \cap T_{w,m}^m$. This shows the lemma. ☐

In Section 4.1 we defined the K_m-representation ξ_χ. Our goal in Sections 4.2-4.4 will be to compute the trace of ξ_χ on some important subgroups of K_m. We will use the following trace formula from [1], which is similar to [9] Theorem 3.2 and is adapted to cover the situation with wild ramification.

Lemma 4.7 ([1] Lemma 2.12). *Let X be a separated scheme of finite type over a finite field $\mathbb{F}_Q$ with Q elements, on which a finite group A acts on the right. Let $g\colon X \to X$ be an automorphism of X, which commutes with the action of A. Let $\psi\colon A \to \overline{\mathbb{Q}}_\ell^*$ be a character of A. Assume that $\mathrm{H}_c^i(X)[\psi] = 0$ for $i \neq i_0$ and Frob_Q acts on $\mathrm{H}_c^{i_0}(X)[\psi]$ by a scalar $\lambda \in \overline{\mathbb{Q}}_\ell^*$. Then*

$$\mathrm{Tr}(g^*, \mathrm{H}_c^{i_0}(X)[\psi]) = \frac{(-1)^{i_0}}{\lambda \cdot \sharp A} \sum_{\tau \in A} \psi(\tau) \cdot \sharp S_{g,\tau},$$

where $S_{g,\tau} = \{x \in X(\overline{\mathbb{F}_q})\colon g(\mathrm{Frob}_Q(x)) = x \cdot \tau\}$.

We adapt this to our situation. Recall from (3.4) and Proposition 3.4(i) that Y_v^m was parametrized by coordinates $a \in \bar{k}[t]/t^{n-1}, C \in (\bar{k}[t]/t^{m+1})^*, A \in \bar{k}[t]/t^{m+1}$ with $a_0 = a \mod t \notin k$. We use Lemma 4.7 with $Q = q^2$.

Lemma 4.8. *Let $(E/F,\chi)$ be a minimal pair and let $g \in K$. Assume g acts on Y_v^m such that there is some rational expression $p(g,a) \in (\bar{k}[t]/t^{m+1})^*$ in a, such that $g.(a,C,A) = (g.a, p(g,a)\cdot C, g.A)$ on coordinates. Let $\tau \in T_{w,m}$. Then*

$$\mathrm{Tr}(\mathrm{g}^*; \mathrm{H}_\mathrm{c}^{\mathrm{i}_0}(\mathrm{Y}_\mathrm{v}^\mathrm{m})[\chi]) = \frac{1}{\mathrm{q}^{\mathrm{m}+1}} \sum_{\tau \in \mathrm{T}_{\mathrm{w},\mathrm{m}}} \chi(\tau) \sharp \mathrm{S}'_{\mathrm{g},\tau},$$

where $S'_{g,\tau}$ *is the set of solutions in the variable* $a = a \mod t^{m+1} \in \bar{k}[t]/t^{m+1}$ *(with* $a_0 \in \bar{k} \smallsetminus k$*) of the equations*

$$\begin{aligned} \sigma(\sigma(a)-a)(\sigma(a)-a)^{-1} &= -p(\sigma^2(a),g)^{-1}\tau \\ g.\sigma^2(a) &= a. \end{aligned} \tag{4.3}$$

Proof. The contribution to the cohomology of the affine space $\mathbb{A}^m(A)$ is just a shift in degree, the character χ is trivial on $\ker(I_{m,\dot{w}}/I^m \twoheadrightarrow T_{w,m})$ and a computation shows that g acts on A by $g.A = A + r(g,a,C)$ for some rational expression r depending only on g,a,C. If we write $Y_v^m = Y' \times \mathbb{A}^m(A)$, where Y' is given in coordinates a, C by the same equations as Y_v^m, we are reduced to apply Lemma 4.7 to Y', on which also the finite group $T_{w,m}$ acts. We claim that for this scheme $\sharp S_{g,\tau} = (q^2-1)q^{2(n-1)} \sharp S'_{g,-\tau}$. We observe that a point $(a,C) \in Y'(\bar{k})$ lies in $S_{g,\tau}$ if and only if it satisfies the following equations:

$$\begin{aligned} \sigma^2(C)C^{-1} &= \sigma(\sigma(a)-a)(\sigma(a)-a)^{-1} \mod t^{m+1} \\ g.\sigma^2(a) &= a \\ p(\sigma^2(a),g)\cdot\sigma^2(C) &= C\cdot\tau \end{aligned}$$

The coordinates $a_{m+1},\dots,a_{n-1}$ occur only in the second equation and hence contributes a factor q^2 to the number of solutions each. Finally, for a fixed a, the third equation has exactly $(q^2-1)q^{2m}$ different solutions in $C \in (\bar{k}[t]/t^{m+1})^*$, and we can eliminate C by putting the first and the third equations together. Thus $\sharp S_{g,\tau}$ is equal to $(q^2-1)q^{2(n-1)}$ times the number of solutions of equations (4.3) in $a \in \bar{k}[t]/t^{m+1}$ with τ replaced by $-\tau$. This shows the claim.

Now, Corollaries 3.13 and 3.14 show that the conditions of Lemma 4.7 are satisfied and the lemma follows from an easy computation involving the above claim. $\square$

Definition 4.9. For $x \in G(\mathcal{O}_L/t^{m+1}) \smallsetminus \{1\}$ the *level* of x is the maximal integer $\ell(x) \geqslant 0$, such that $x \equiv 1 \mod t^{\ell(x)}$. The level of 1 is $m+1$.

This definition is auxiliary and will be used in the next two sections. Note that the level is invariant under conjugation in $G(\mathcal{O}_L/t^{m+1})$.

4.3. **Trace computations II: N_m-action on V_χ.** We keep notations from Sections 4.1,4.2. Let further $N_m \subset K_m$ denote the subgroup

$$N_m = \left\{ \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} : x \in k[t]/t^{m+1} \right\}.$$

For $0 \leqslant i \leqslant m+1$, let N_m^i denote the subgroup of N_m consisting of elements congruent to 1 modulo t^i and let $N_m^{\vee,gen}$ denote the set of characters of N_m, which are non-trivial on N_m^m.

Proposition 4.10. *As N_m-representations one has*

$$\xi_\chi = \mathrm{Ind}_1^{N_m} 1 - \mathrm{Ind}_{N_m^m}^{N_m} 1 = \bigoplus_{\psi \in N_m^{\vee,gen}} \psi.$$

In particular, $\dim_{\overline{\mathbb{Q}}_\ell} V_\chi = (q-1)q^m$.

Proof. We claim that for $g \in N_m$ we have

$$\operatorname{Tr}(g^*, V_\chi) = \begin{cases} q^{m+1} - q^m & \text{if } g = 1 \\ -q^m & \text{if } g \in N_m^m \smallsetminus \{1\} \\ 0 & \text{if } g \notin N_m^m. \end{cases} \tag{4.4}$$

The proposition follows from this claim by comparing the traces of the N_m-representations on the left and the right sides. We need the following lemma. Let $S'_{g,\tau}$ be as in Lemma 4.8.

Lemma 4.11. *Let $g \in N_m$ of level $\ell(g) \leqslant m+1$. Then $S'_{g,\tau} = \varnothing$, unless $v_t(1-\tau) = \ell(g)$ and $\tau \cdot \sigma(\tau) = 1$ in $k_2[t]/t^{m+1}$. If both are satisfied, then*

$$\sharp S'_{g,\tau} = \begin{cases} (q-1)q^{2m+1} & \text{if } g = 1 \text{ (and hence } \tau = 1), \\ q^{m+1+\ell(g)} & \text{if } g \in N_m \smallsetminus \{1\}. \end{cases}$$

Proof. As both, the $T_{w,m}$- and the K_m-actions on Y_v^m have their origin in matrix multiplication, one sees easily that $S_{g,\tau} = \varnothing$, unless $\det(\tau) = \det(g)$. Thus we can assume this, i.e., $\tau\sigma(\tau) = 1$. Write $g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in N_m$ with $x \in k[t]/t^{m+1}$. Then $v_t(x) = \ell(g)$. The action of g can be describen by $g.(a, C) = (a + x, C)$. By Definition, $S'_{g,\tau}$ is the set of solutions of

$$\begin{aligned} \sigma(\sigma(a) - a)(\sigma(a) - a)^{-1} &= -\tau \\ \sigma^2(a) + x &= a \end{aligned} \tag{4.5}$$

in $a \in \bar{k}[t]/t^{m+1}$ with $a_0 \notin k$. Let $s = \sigma(a) - a$. For a fixed $s \in (\bar{k}[t]/t^{m+1})^*$, the equation $\sigma(a) - a = s$ in a has exactly q^{m+1} solutions. After adding $-\sigma(a)$ to both sides of the second equation in (4.5), this second equation gets $\sigma(s) + x = -s$. Thus we are reduced to solve the equations

$$\begin{aligned} \sigma(s) + x &= -s \\ \sigma(s)s^{-1} &= -\tau \end{aligned} \tag{4.6}$$

in the variable $s \in (\bar{k}[t]/t^{m+1})^*$. Putting $\sigma(s) = -(x+s)$ into the second equation, we obtain $1 - \tau = -xs^{-1}$ and one checks that equations (4.6) are equivalent to

$$\begin{aligned} \sigma(s) + x &= -s \\ 1 - \tau &= -xs^{-1} \end{aligned} \tag{4.7}$$

From the second equation of (4.7) and since s must be a unit, we see that either $S'_{g,\tau} = \varnothing$ or $v_t(1-\tau) = v_t(x)$. Assume the second holds and let $\mu = v_t(x)$. If $\mu = m+1$, then $g = 1$, $\tau = 1$ and the lemma follows. Assume now $0 < \mu < m+1$. Then $x = t^\mu \tilde{x}$ for some $\tilde{x} \in (\bar{k}[t]/t^{m+1-\mu})^*$ and $\tau = 1 + \tilde{\tau} t^\mu$ for some $\tilde{\tau} \in (k_2[t]/t^{m+1-\mu})^*$. The condition $\tau\sigma(\tau) = 1$ is equivalent to

$$\tilde{\tau} + \sigma(\tilde{\tau}) + \tilde{\tau}\sigma(\tilde{\tau})t^\mu \equiv 0 \mod t^{m+1-\mu}. \tag{4.8}$$

The second equation of (4.7) is equivalent to $s \equiv \tilde{\tau}^{-1}\tilde{x} \mod t^{m+1-\mu}$, i.e., s is uniquely determined modulo $t^{m+1-\mu}$. Moreover, if $s \equiv \tilde{\tau}^{-1}\tilde{x} \mod t^{m+1-\mu}$ we have

$$\sigma(s)+x+s\equiv\sigma(\tilde\tau^{-1}\tilde x)+x+\tilde\tau^{-1}\tilde x=\tilde x(t^\mu+\sigma(\tilde\tau)^{-1}+\tilde\tau^{-1})\equiv 0\mod t^{m+1-\mu},$$

where the last equation follows from (4.8). Thus $s\equiv\tilde\tau^{-1}\tilde x\mod t^{m+1-\mu}$ is the unique solution of equation (4.7) modulo $t^{m+1-\mu}$. One easily sees that over any solution of the first equation of (4.7) modulo t^λ lie precisely q solutions of it modulo $t^{\lambda+1}$. This shows that (4.7) has precisely q^μ solutions if $\mu>0$. It remains to handle the case $\mu=0$, but this is done similarly to the case $\mu>0$. ☐

For $g=1$, we have $S'_{1,\tau}=\varnothing$ unless $\tau=1$ and $\sharp S'_{1,1}=q^m(q-1)$. The claim (4.4) follows immediately from Lemma 4.8. Let now $g\in N_m\smallsetminus\{1\}$ of level $\ell=\ell(g)\leqslant m$. Then Lemmas 4.8 and 4.11 show

$$\begin{aligned}\operatorname{Tr}(g^*,V_\chi) &= \frac{1}{q^{m+1}}\sum_{\tau\in T_{w,m,0}\cap(T^\ell_{w,m}\smallsetminus T^{\ell+1}_{w,m})}\chi(\tau)\sharp S'_{g,\tau}\\ &= q^\ell\cdot\sum_{\tau\in T_{w,m,0}\cap(T^\ell_{w,m}\smallsetminus T^{\ell+1}_{w,m})}\chi(\tau)=-q^\ell\cdot\sum_{\tau\in T_{w,m,0}\cap T^{\ell+1}_{w,m}}\chi(\tau),\end{aligned}$$

the last equation being true as χ is a non-trivial character on $T_{w,m,0}\cap T^\ell_{w,m}$. Unless $\ell=m$, the sum in the last expression vanishes, as χ is still a non-trivial character on $T_{w,m,0}\cap T^{\ell+1}_{w,m}$. If $\ell=m$, we have $T^{\ell+1}_{w,m}=\{1\}$, and (4.4) follows. ☐

Corollary 4.12. *(ξ_χ,V_χ) is irreducible as $B(\mathcal{O}_F/t^{m+1})$-representation and hence also as K_m-representation.*

Proof. For $\psi\in N_m^{\vee,gen}$, let $V_\chi[\psi]$ denote the ψ-isotypic component of V_χ. By Proposition 4.10, $V_\chi[\psi]$ is one-dimensional. For $x\in T(\mathcal{O}_F/t^{m+1})$, let ψ^x be the character of N_m defined by $\psi^x(g)=\psi(x^{-1}gx)$. Then $x.V_\chi[\psi]\subseteq V_\chi[\psi^x]$. Let $0\neq W\subseteq V_\chi$ be a $B(\mathcal{O}_F/t^{m+1})$-invariant subspace. Then W decomposes as the sum of its N_m-isotypical components $W[\psi]$ ($\psi\in N_m^{\vee,gen}$) and $W[\psi]\subseteq V_\chi[\psi]$. As $V_\chi[\psi]$ is one-dimensional, $W[\psi]$ is either 0 or equal to $V_\chi[\psi]$. But as $W\neq 0$, there is a ψ, such that $W[\psi]=V_\chi[\psi]$. Note that the natural action of $T(\mathcal{O}_F/t^{m+1})$ on $N_m^\vee$ restricts to a transitive action on $N_m^{\vee,gen}$. This transitivity implies that $W[\psi]=V_\chi[\psi]$ for all $\psi\in N_m^{\vee,gen}$, i.e., $W=V_\chi$. ☐

4.4. **Trace computations III: H_m-action on V_χ.** We keep notations from Sections 4.1 and 4.2. Let $H_m\subset K_m$ be a non-split torus, i.e., a subgroup which is conjugate to $T_{w,m}$ inside $G(\mathcal{O}_L/t^{m+1})$. Let Z_m be the center of K_m. One has $Z_m\subseteq H_m$. We fix an isomorphism $c_s\colon T_{w,m}\xrightarrow{\sim}H_m$, given by conjugation with $s\in G(k_2[[t]])$, and let $H^i_m=c_s(T^i_{w,m})$. Let $\tilde\chi=\chi\circ c_s^{-1}$ and $\tilde\chi^\sigma=\chi^\sigma\circ c_s^{-1}$, where $\chi^\sigma=\chi\circ\sigma$.

Note that if $s'\in G(L)$ is another matrix conjugating $T_{w,m}$ into H_m, then $c'_sc_s^{-1}$ is either identity or σ. In particular, up to σ-action, $\tilde\chi$ does not depend on the choice of the element s.

For a character $\psi\in H_m^\vee$, let $i(\psi)\in\{0,\dots,m+1\}$ be the smallest integer such that ψ coincides with $\tilde\chi$ or $\tilde\chi^\sigma$ on the subgroup $H_m^{i(\psi)}$ (in particular, $i(\psi)=0$ if and only if $\psi=\tilde\chi$ or $\tilde\chi^\sigma$).

Theorem 4.13. *Let ψ be a character of H_m. Then $\langle\psi,\xi_\chi\rangle_{H_m}=0$ unless $\psi|_{Z_m}=\tilde\chi|_{Z_m}$. Assume $\psi|_{Z_m}=\tilde\chi|_{Z_m}$. Then*

$$\langle\psi,\xi_\chi\rangle_{H_m}=\begin{cases}1 & \textit{if } m-i(\psi)\ \textit{odd}\\ 0 & \textit{if } m-i(\psi)\ \textit{even.}\end{cases}$$

The proof of Theorem 4.13 is given at the end of this section. To prepare it, we have to compute the traces of elements $x \in H_m$ in V_χ. This is done in Proposition 4.20 below, for which Lemma 4.17 is the main technical tool. We have an immediate consequence of Theorem 4.13.

Corollary 4.14. *Let $m > 0$. The character $\tilde{\chi}$ (and hence also χ) is up to σ-conjugacy uniquely determined by ξ_χ among all characters of H_m as follows: it is the unique (up to σ-conjugacy) character $\psi \in H_m^\vee$, such that $\psi|_{Z_m}$ is the central character of ξ_χ and*

(i) *if m is odd: ψ occurs in ξ_χ, and all characters $\psi' \neq \psi$, which coincide with ψ on $Z_m H_m^1$ do not occur in ξ_χ.*

(ii) *if m is even: ψ does not occur in ξ_χ, and all characters $\psi' \neq \psi$, which coincide with ψ on $Z_m H_m^1$ occur in ξ_χ.*

Moreover, the map $(E/F, \chi) \mapsto \Xi_\chi$ from $\mathbb{P}_2^{\mathrm{nr}}(F)$ to the set of isomorphism classes of ZK-representations is injective.

Proof. The first statement is immediate from Theorem 4.13. We show injectivity of $\chi \mapsto \Xi_\chi$. From the first statement of the corollary, $\chi|_{U_E}$ is uniquely determined by the K_m-representation ξ_χ, hence also by its inflation to K, which is equal to $\Xi_\chi|_K$. Moreover, by Lemma 4.4, $\chi|_{F^*}$ is equal to the restriction of Ξ_χ to the center $Z \cong F^*$ of ZK. This finishes the proof, as $E^* = F^* U_E$. □

Definition 4.15. We say that x is *maximal* if $\ell(x) \geqslant \ell(zx)$ for all $z \in Z_m$.

Lemma 4.16. *Let $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in H_m \smallsetminus \{1\}$. Then x maximal if and only if $v_t(x_3) = \ell(x)$.*

Proof. Assume first $\ell = \ell(x) > 0$. Consider $\tau_x = c_s^{-1}(x) \in T_{w,m}$ and write $\tau_x = 1 + \tau_{x,\ell} t^\ell + \cdots + \tau_{x,m} t^m$. One sees immediately that maximality of an element is invariant under conjugation, hence x is maximal if and only if τ_x is, i.e., if and only if $\tau_{x,\ell} \notin k$. A computation (using the fact that all entries of s must be units) shows that $x_3 = t^\ell u(\tau_x - \sigma(\tau_x))$ with some unit $u \in (\bar{k}[t]/t^{m+1})^*$. The lemma follows in the case $\ell(x) > 0$. The case $\ell(x) = 0$ is similar. □

We introduce the following version of the characteristic polynomial of an element $x \in K_m$. Let $\ell = \ell(x)$ be the level of x. Let $\tilde{x}$ be some lift of x to K. Then the characteristic polynomial of $\tilde{x}$ can be seen as the function

$$p_{\tilde{x}} \colon \mathcal{O}_L \to \mathcal{O}_L \quad \lambda \mapsto p_{\tilde{x}}(\lambda) = \det(\lambda \cdot \mathrm{Id} - x).$$

Note that $p_{\tilde{x}}(U_L^\ell) \subseteq t^{2\ell}\mathcal{O}_L$. Let now $\lambda \in U_L^\ell$ and $\tilde{x}_1, \tilde{x}_2$ two lifts of x to K. Then $p_{\tilde{x}_1}(\lambda) - p_{\tilde{x}_2}(\lambda) \in t^{m+\ell+1}\mathcal{O}_L$, i.e., $p_{\tilde{x}}(\lambda)$ modulo $t^{m+\ell+1}$ depends only on x, not on the lift $\tilde{x}$. This gives a well-defined map $p_x' \colon U_L^\ell \to t^{2\ell}\mathcal{O}_L/t^{m+\ell+1}\mathcal{O}_L$. Moreover, one immediately computes that this induces the following map defined as the composition:

$$p_x \colon U_L^\ell/U_L^{m+1} \xrightarrow{p_x'} t^{2\ell}\mathcal{O}_L/t^{m+\ell+1}\mathcal{O}_L \to \mathcal{O}_L/t^{m-\ell+1}\mathcal{O}_L, \tag{4.9}$$

where the second arrow is multiplication by $t^{-2\ell}$. Explicitly, if $\ell > 0$ and $x = 1 + t^\ell \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$, then

$$p_x(1 + \tilde{\tau}t^\ell) = (\tilde{\tau} - y_1)(\tilde{\tau} - y_4) - y_2 y_3.$$

We identify $T_{w,m}$ with $(k_2[t]/t^{m+1})^* \subseteq (\bar{k}[t]/t^{m+1})^*$ by sending $\begin{pmatrix} \tau & \\ & \sigma(\tau) \end{pmatrix}$ to τ. In particular, for $x \in K_m^\ell$ and $\tau \in T_{w,m}^\ell$ we have the element $p_x(\tau) \in \bar{k}[t]/(t^{m-\ell+1})$, where ℓ is the level of x. Let $S'_{x,\tau}$ be as in Lemma 4.8.

Lemma 4.17. *Let $x \in H_m$ be maximal of level $\ell = \ell(x) \leqslant m$. Let $\tau \in T_{w,m}$. Then $S'_{x,\tau} = \varnothing$, unless $\tau \in T_{w,m}^\ell$ and $\det(\tau) = \det(x)$. For $\tau \in T_{w,m}^\ell$ with $\det(\tau) = \det(x)$ we have:*

$$\sharp S'_{x,\tau} = \begin{cases} q^{m+\ell} & \text{if } p_x(\tau) = 0 \text{ and } m-\ell \text{ even} \\ q^{m+\ell+1} & \text{if } p_x(\tau) = 0 \text{ and } m-\ell \text{ odd} \\ 0 & \text{if } v_t(p_x(\tau)) < \infty \text{ is odd} \\ (q+1)q^{m+\ell} & \text{if } v_t(p_x(\tau)) < \infty \text{ is even.} \end{cases}$$

Proof. Let $x \in H_m$ be maximal of level $\ell \leqslant m$ and let $\tau_x = c_s^{-1}(x)$. Let $\tau \in T_{w,m}$. From the definition of $S'_{x,\tau}$ one immediately deduces that $S'_{x,\tau} = \varnothing$, unless $\det(x) = \det(\tau)$, i.e., $\tau \in \tau_x T_{w,m,0} = T_x(0)$. Hence we can assume $\tau\sigma(\tau) = \det(x)$. Write $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. A point of Y_v^m is parametrized by the coordinates a, C and A as above. One computes:

$$x.(a,C) = (x.a, x.C) = (\frac{x_1 a + x_2}{x_3 a + x_4}, \frac{\det(x) C}{x_3 a + x_4}). \tag{4.10}$$

By Lemma 4.8, $\sharp S'_{x,\tau}$ is the number of solutions of equations (4.3) in the variable $a \in \bar{k}[t]/t^{m+1}$ (satisfying $a_0 \notin k$). Explicitly, these equations are:

$$\begin{aligned} x_1\sigma^2(a) + x_2 &= a(x_3 a\sigma^2(a) + x_4) \\ (\sigma^2(a) - \sigma(a))\sigma(\tau) &= -(x_3\sigma^2(a) + x_4)(\sigma(a) - a). \end{aligned}$$

Inserting the first equation into the second and applying σ^{-1} to the result, we see that the equations are equivalent to

$$x_3 a\sigma^2(a) - x_1\sigma^2(a) + x_4 a - x_2 = 0 \tag{4.11}$$

$$x_3 a\sigma(a) + (\tau - x_1)\sigma(a) - (\tau - x_4)a - x_2 = 0. \tag{4.12}$$

Sublemma 4.18. *For $i \geqslant 1$, there are precisely q^2 solutions of equation (4.11) modulo t^{i+1} lying over a given solution (satisfying $a_0 \notin k$) of (4.11) modulo t^i.*

Proof. Write $a = \sum_{j=0}^i a_j t^j$, $x_\lambda = \sum_{j=0}^i x_{\lambda j}$. The coefficient of t^i on the right side of (4.11) modulo t^{i+1} is

$$(x_{30}a_0 - x_{10})a_i^{q^2} + (x_{30}a_0^{q^2} + x_{40})a_i + R, \tag{4.13}$$

where $R \in \bar{k}$ depends only on $a_0, \ldots, a_{i-1}$ and x and not on a_i. As $a_0 \notin k$ and $x \in G(k)$, it is clear that $x_{30}a_0 - x_{10} \neq 0$ and $x_{30}a_0^{q^2} + x_{40} \neq 0$. Thus (4.13) is a separable polynomial in a_i of degree q^2, i.e., it has exactly q^2 different roots. □

Now we concentrate on the case $\ell > 0$, i.e., $x = 1 + t^\ell \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$. Equation (4.12) modulo t^ℓ shows $(\tau - 1)\sigma(a) = (\tau - 1)a \mod t^\ell$. If $\tau \not\equiv 1 \mod t^\ell$, then this forces $a_0 \in k$, which contradicts

$(a,c,A)\in Y_v^m$. Hence $S'_{x,\tau}=\varnothing$, unless $\tau\in T^\ell_{w,m}$. Assume $\tau\in T^\ell_{w,m}$. and $\tau=1+\tilde\tau t^\ell$, with some $\tilde\tau\in k_2[t]/t^{m+1-\ell}$. Note that the condition $\det(x)=\det(\tau)$ satisfied by x,τ is equivalent to

$$y_1+y_4+(y_1y_4-y_2y_3)t^\ell\equiv\tilde\tau+\sigma(\tilde\tau)+\tilde\tau\sigma(\tilde\tau)t^\ell \mod t^{m-\ell+1}. \tag{4.14}$$

Equations (4.11) and (4.12) transform to

$$t^\ell(y_3a\sigma^2(a)-y_1\sigma^2(a)+y_4a-y_2) = \sigma^2(a)-a \tag{4.15}$$

$$y_3a\sigma(a)+(\tilde\tau-y_1)\sigma(a)-(\tilde\tau-y_4)a-y_2 \equiv 0 \mod t^{m-\ell+1} \tag{4.16}$$

Sublemma 4.18 shows that the number of solutions of (4.15), (4.16) is equal to $q^{2\ell}$ times the number of solutions of (4.15) and (4.16) $\mod t^{m-\ell+1}$.

Let us write $Q=p_x(\tau)$ with p_x as in (4.9). A computation involving (4.14) implies

$$\tilde\tau+\sigma(\tilde\tau)-y_1-y_4\equiv t^\ell\tau^{-1}Q \mod t^{m-\ell+1}. \tag{4.17}$$

Sublemma 4.16 allows us to make the linear change of variables $a=b-\frac{\tilde\tau-y_1}{y_3}$ and equations (4.15), (4.16) modulo $t^{m-\ell+1}$ take the following form (using (4.17) and the fact that $\sigma^2(a)-a=\sigma^2(b)-b$):

$$t^\ell\left(y_3b\sigma^2(b)-\tilde\tau\sigma^2(b)-(t^\ell\tau^{-1}Q-\sigma(\tilde\tau))b+y_3^{-1}Q\right) \equiv \sigma^2(b)-b \mod t^{m-\ell+1} \tag{4.18}$$

$$y_3b\sigma(b)-t^\ell\tau^{-1}Qb+y_3^{-1}Q \equiv 0 \mod t^{m-\ell+1}. \tag{4.19}$$

Write $b=\sum_{i=0}^m b_it^i$. We have three cases: $v_t(Q)=\infty$, $v_t(Q)<\infty$ odd, $v_t(Q)<\infty$ even. Assume first $v_t(Q)=\infty$, i.e., $Q=0$. Then (4.19) is equivalent to $b_0=b_1=\cdots=b_{\lfloor\frac{m-\ell}{2}\rfloor}=0$. As $b=0$ is also a solution of (4.18) $\mod t^{\lfloor\frac{m-\ell}{2}\rfloor+1}$, it follows from Sublemma 4.16 that the number of solutions of (4.18) and (4.19) $\mod t^{m-\ell+1}$ is exactly $(q^2)^{m-\ell-\lfloor\frac{m-\ell}{2}\rfloor}$ and the lemma follows in this case, once we have shown that no of these solutions lies in the 'forbidden' subset, determined by $a_0\in k$. This is done in Sublemma 4.19 below.

Now assume $v_t(Q)<\infty$. Equation (4.19) shows that we must have $v_t(Q)=2v_t(b)$. In particular, $\sharp S'_{x,\tau}=\varnothing$ if $v_t(Q)$ is odd. Assume $v_t(Q)=2j<\infty$ is even and write $Q=t^{2j}Q'$. Then $b\in\bar k[t]/t^{m-\ell+1}$ solves (4.19) if and only if $b=t^jb'$ (i.e., $b_0=\cdots=b_{j-1}=0$) and $b'=\sum_{i=j}^{m-\ell+1}b_it^{i-j}$ solves

$$y_3b'\sigma(b')-t^{\ell+j}\tau^{-1}Q'b'+y_3^{-1}Q'\equiv 0 \mod t^{m-\ell-2j+1}. \tag{4.20}$$

Note that such a solution b' is necessarily a unit. Using this, we can express $\sigma(b')$ in terms of b', apply σ to it, and then insert again the expression of $\sigma(b')$ in (4.20). This shows:

$$\sigma^2(b')\equiv\frac{\sigma(Q')}{\frac{Q'}{b'}-y_3t^{\ell+j}\tau^{-1}Q'}+y_3^{-1}t^{\ell+j}\sigma(\tau^{-1}Q')\,(\mathrm{mod}\ t^{m-\ell-2j+1}),$$

which multiplied by t^j gives an expression of $\sigma^2(b)\mod t^{m-\ell-j+1}$ in terms of b'. Now a (very ugly, but straightforward) computation shows that if we put this expression for $\sigma^2(b)$ into equation (4.18) modulo $t^{m-\ell-j+1}$, we obtain the tautological equation $0=0$. This simply means that any solution b of (4.19) $\mod t^{m-\ell-j+1}$ is a solution of (4.18) $\mod t^{m-\ell-j+1}$. Similarly as in Sublemma 4.18, one checks that (4.19) modulo $t^{m-\ell-j+1}$ has precisely $(q+1)q^{m-\ell-2j}$ solutions ($q+1$ corresponds to the freedom of choosing b_j and $q^{m-\ell-2j}$ corresponds to the freedom of choosing $b_{j+1},\dots,b_{m-\ell-j}$).

Again by Sublemma 4.16, the lemma also follows in this case, once we have shown that no of these solutions lie in the 'forbidden' subset, determined by $a_0 \in k$. This is done in Sublemma 4.19.

In the case $\ell(x) = 0$, the lemma can be proven in the same way. □

Sublemma 4.19. *With notations as in the proof of Lemma 4.17, assume* $\tau \in T^{\ell}_{w,m}$ *and* $\det(\tau) = \det(x)$*. Let a be a solution of equations* (4.11), (4.12)*, then* $a_0 \notin k$*.*

Proof. For any $r \geqslant 1$ and an element $X \in \bar{k}[t]/(t^r)$, denote by $X_0 \in \bar{k}$ the reduction of X modulo t. Write $\tau_x = c_s^{-1}(x) \in T_{w,m}$. We handle the case $\ell > 0$ first. Write $\tau = 1 + \tilde{\tau}t^{\ell}$, $\tau_x = 1 + \tau_{x,\ell}t^{\ell} + \dots$. As a is a solution of (4.11), (4.12), $b = a + \frac{\tilde{\tau} - y_1}{y_3}$ is a solution of (4.18), (4.19). We have $a_0 = b_0 - \frac{\tilde{\tau}_0 - y_{10}}{y_{30}}$.

Assume first $v_t(Q) > 0$. Maximality of x (and hence of τ_x) implies $\tau_{x,\ell} \notin k$. Now, $v_t(Q) = v_t(p_x(\tau)) > 0$ is equivalent to $\tilde{\tau}_0 \equiv \tau_{x,\ell}$ or $\equiv \sigma(\tau_{x,\ell}) \mod t$. Hence $\tilde{\tau}_0 \notin k$. On the other hand, the solution b must satisfy $b_0 = 0$ and we have $y_{10}, y_{30} \in k$. As $\tilde{\tau}_0 \notin k$ we obtain $a_0 \notin k$.

Now assume $v_t(Q) = 0$ and suppose that $a_0 \in k$, i.e., $a_0^q = a_0$. Then for b_0 we must have:

$$b_0^q = b_0 + \frac{\tilde{\tau}_0^q - \tilde{\tau}_0}{y_{30}}. \tag{4.21}$$

Putting this into equation (4.19) $\mod t$, we deduce that b_0 must satisfy

$$b_0^2 + \frac{\tilde{\tau}_0^q - \tilde{\tau}_0}{y_{30}} b_0 + \frac{Q_0}{y_{30}^2} = 0, \tag{4.22}$$

where $Q_0 = (\tilde{\tau}_0 - \tau_{x,\ell})(\tilde{\tau}_0 - \sigma(\tau_{x,\ell}))$. By assumption we have $\det(\tau_x) = \det(x) = \det(\tau) \mod t^{\ell+1}$, hence

$$\tilde{\tau}_0 + \sigma(\tilde{\tau}_0) = \tau_{x,\ell} + \sigma(\tau_{x,\ell}). \tag{4.23}$$

Assume first $\operatorname{char}(k) > 2$. A computation shows that the discriminant of equation (4.22) is $D = y_{30}^{-2}(\sigma(\tau_{x,\ell}) - \tau_{x,\ell})^2$ and hence the solutions of it are

$$b_{0,\pm} = -\frac{\sigma(\tilde{\tau}_0) - \tilde{\tau}_0}{2y_{30}} \pm \frac{\sigma(\tau_{x,\ell}) - \tau_{x,\ell}}{2y_{30}}.$$

Putting any of this solutions into equation (4.21) shows $\tau_{x,\ell} = \sigma(\tau_{x,\ell})$, which is a contradiction to maximality of x. This finishes the proof in the case $\operatorname{char}(k) > 2$.

Assume now $\operatorname{char}(k) = 2$. Let $\mu = \frac{\tilde{\tau}_0^q + \tilde{\tau}_0}{y_{30}}$. Then $\mu \in k$. Further, (4.23) shows $\mu \neq 0$ (otherwise, $\tau_{x,\ell} \in k$, which is a contradiction to maximality of x). Set also $\delta = \frac{Q_0}{y_{30}^2 \mu^2}$. Note that by (4.23), $Q_0 \in k$ and hence also $\delta \in k$. Make the change of variables $b_0 = \mu s$, i.e., b_0 satisfies (4.22), (4.21) if and only if s satisfies

$$\begin{aligned} s^q + s + 1 &= 0 \\ s^2 + s + \delta &= 0. \end{aligned}$$

The second of these equations implies $s^q = s + \operatorname{Tr}_{k/\mathbb{F}_2}(\delta)$. This together with the first equation implies $\operatorname{Tr}_{k/\mathbb{F}_2}(\delta) = 1$. On the other hand, let $R = \frac{\tilde{\tau}_0 + \tau_{x,\ell}}{\tau_{x,\ell} + \tau_{x,\ell}^q}$. Using (4.23), we see that

$$R + R^2 = \frac{(\tilde{\tau}_0 + \tau_{x,\ell})(\tilde{\tau}_0 + \tau_{x,\ell}^q)}{(\tau_{x,\ell} + \tau_{x,\ell}^q)^2} = \delta.$$

This implies $\operatorname{Tr}_{k/\mathbb{F}_2}(\delta) = 0$, which is a contradiction. This proves the lemma in the case $\operatorname{char}(k) = 2$.

Let us now consider the case $\ell(x) = 0$. By maximality of x, we have $x_3 \in (k[t]/t^r)^*$. If $\mathrm{char}(k) > 2$, the variable change $a = b - x_3^{-1}(\tau - x_1)$ analogous to the one in the case $\ell(x) > 0$ leads to a very similar proof. Assume $\mathrm{char}(k) = 2$. We have to show that equations (4.12) $\mod t$ and $a_0^q = a_0$ do not have a common solution. Let

$$\lambda = x_{30}^{-1}(x_{10} + x_{40}) = x_{30}^{-1}(\tau_{x,0}^q + \tau_{x,0}),$$

Then $\lambda^q = \lambda$ and $\lambda \neq 0$ by maximality of x. Make the change of variables given by $a_0 = \lambda r + x_{30}^{-1}x_{10}$. Then $a_0^q = a_0$ transforms into $r^q = r$ and (4.12) gets (after using $r^q = r$ and canceling)

$$x_{30}\lambda^2 r^2 + x_{30}\lambda^2 r + x_{30}^{-1}\det(x) = 0,$$

or equivalently

$$r^2 + r + \frac{\tau_{x,0}^{q+1}}{(\tau_{x,0}^q + \tau_{x,0})^2} = 0.$$

We have to show that this equation has no solution in k. But observe that the two solutions of it are given by $\frac{\tau_{x,0}}{\tau_{x,0}^q+\tau_{x,0}}$ and $\frac{\tau_{x,0}^q}{\tau_{x,0}^q+\tau_{x,0}}$ lie in $k_2 \smallsetminus k$ (note that they are different by maximality of x). This finishes the proof also in this case. □

Proposition 4.20. *Let $x \in H_m$ be maximal of level $\ell(x) \leqslant m$. Then*

$$\mathrm{tr}(x; V_\chi) = (-1)^{m-\ell(x)+1} q^{\ell(x)} (\tilde{\chi}(x) + \tilde{\chi}^\sigma(x)).$$

Proof. Let $\tau_x = c_s^{-1}(x) \in T_{w,m}^\ell$. For $j' \in \{0, 1, \ldots, m-\ell, \infty\}$, let

$$\begin{aligned} T_x(j') &= \{\tau \in \tau_x T_{w,m,0}^\ell \cup \sigma(\tau_x) T_{w,m,0}^\ell \colon \tau \equiv \tau_x \text{ or } \sigma(\tau_x) \mod t^{\ell+j'}\} \\ &= \tau_x \ker(T_{w,m,0}^\ell \twoheadrightarrow T_{w,\ell+j'-1,0}^\ell) \cup \sigma(\tau_x)\ker(T_{w,m,0}^\ell \twoheadrightarrow T_{w,\ell+j'-1,0}^\ell) \subseteq T_{w,m}^\ell \end{aligned}$$

be the union of the two $\ker(T_{w,m,0}^\ell \twoheadrightarrow T_{w,\ell+j'-1,0}^\ell)$-cosets inside $T_{w,m}^\ell$ in which τ_x and $\sigma(\tau_x)$ lie (note that these cosets are disjoint if $j' > 0$ and equal if $j' = 0$). Note that $\tau \in T_x(0)$ if and only if $\det(\tau) = \det(x)$ and $\tau \in T_{w,m}^\ell$.

Sublemma 4.21. *For $\tau \in T_{w,m}^\ell$ with $\det(\tau) = \det(x)$ we have: $\tau \in T_x(j') \Leftrightarrow v_t(p_x(\tau)) \geqslant j'$.*

Proof of Sublemma 4.21. Write $\tau = 1 + t^\ell \tilde{\tau}$ and $\tau_x = c_s^{-1}(x) = 1 + t^\ell \tilde{\tau}_x$. The characteristic polynomial is invariant under conjugation, hence $v_t(p_x(\tau)) = v_t(p_{\tau_x}(\tau))$. Write $\tau_x = 1 + t^\ell \tilde{\tau}_x$. As x (and hence also τ_x) is maximal, $\tilde{\tau}_x - \sigma(\tilde{\tau}_x)$ is a unit. We have $p_{\tau_x}(\tau) = (\tilde{\tau} - \tilde{\tau}_x)(\tilde{\tau} - \sigma(\tilde{\tau}_x))$. Thus $v_t(p_x(\tau)) \geqslant j' \Leftrightarrow \tilde{\tau} \equiv \tilde{\tau}_x \mod t^{j'}$ or $\tilde{\tau} \equiv \sigma(\tilde{\tau}_x) \mod t^{j'}$. The sublemma follows. □

By Lemma 4.8 and the first statement of Lemma 4.17 we have:

$$\begin{aligned} \mathrm{tr}(x; V_\chi) &= \frac{1}{q^{m+1}} \sum_{\substack{\tau \in T_{w,m}^\ell \\ \det(\tau) = \det(x)}} \chi(\tau) \sharp S'_{x,\tau} \qquad (4.24) \\ &= \frac{1}{q^{m+1}} \Big(\sum_{\tau \in T_x(\infty)} \chi(\tau) \sharp S'_{x,\tau} + \sum_{j'=0}^{m-\ell} \sum_{\substack{\tau \in T_x(j') \\ \tau \notin T_x(j'+1)}} \chi(\tau) \sharp S'_{x,\tau} \Big). \end{aligned}$$

Write $T^\ell_{w,m,0} = T_{w,m,0} \cap T^\ell_{w,m}$. Lemma 4.17 implies for $0 < j' < m - \ell$:

$$\begin{aligned}\sum_{\substack{\tau\in T_x(j')\\ \tau\notin T_x(j'+1)}} \chi(\tau)\sharp S'_{x,\tau} &= (\tilde\chi(x) + \tilde\chi^\sigma(x)) \cdot (const) \cdot \sum_{\substack{\tau\in\ker(T^\ell_{w,m,0}\twoheadrightarrow T^\ell_{w,l+j'-1,0})\\ \tau\notin\ker(T^\ell_{w,m,0}\twoheadrightarrow T^\ell_{w,l+j',0})}} \chi(\tau)\\ &= (\tilde\chi(x) + \tilde\chi^\sigma(x)) \cdot (const) \cdot \sum_{\tau\in\ker(T^\ell_{w,m,0}\twoheadrightarrow T^\ell_{w,l+j',0})} \chi(\tau) = 0\end{aligned}$$

and similarly $\sum_{\substack{\tau\in T_x(0)\\ \tau\notin T_x(1)}} \chi(\tau)\sharp S'_{x,\tau} = 0$ as χ is non-trivial on $\ker(T^\ell_{w,m,0} \twoheadrightarrow T_{w,l+j',0})$ (one has to apply this twice). Further, if $m-\ell$ is odd, then $S'_{x,\tau}$ is empty for $\tau \in T_x(m-\ell) \smallsetminus T_x(\infty)$, hence in this case Lemma 4.17 implies:

$$\operatorname{tr}(x; V_\chi) = \frac{1}{q^{m+1}} \sum_{\tau\in T_x(\infty)} \chi(\tau)\sharp S'_{x,\tau} = q^\ell(\chi(\tau_x) + \chi^\sigma(\tau_x)).$$

If $m - \ell$ is even, then

$$\begin{aligned}\operatorname{tr}(x; V_\chi) &= \frac{1}{q^{m+1}} \left(\sum_{\tau\in T_x(\infty)} \chi(\tau) q^{m+\ell} + (\chi(\tau_x) + \chi^\sigma(\tau_x)) \sum_{\tau\in T^m_{w,m,0}\smallsetminus\{1\}} \chi(\tau)(q+1)q^{m+\ell} \right)\\ &= -q^\ell(\chi(\tau_x) + \chi^\sigma(\tau_x)).\end{aligned}$$

This finishes the proof of Proposition 4.20. □

Proof of Theorem 4.13. If $\psi|_{Z_m} \neq \tilde\chi|_{Z_m}$, then $\langle \psi, \xi_\chi\rangle_{H_m} = 0$ by Lemma 4.4. Note that for $x \in H_m, z \in Z_m$ we have $\operatorname{tr}(zx; V_\chi) = \chi(z)\operatorname{tr}(x; V_\chi)$. Let ψ be a character of H_m with $\psi|_{Z_m} = \chi|_{Z_m}$. Note that

$$\{x \in H_m \colon \max_{z\in Z_m} \ell(zx) = \ell\} = \begin{cases} Z_m H^\ell_m \smallsetminus Z_m H^{\ell+1}_m & \text{if } \ell \leqslant m\\ Z_m & \text{if } \ell = m+1.\end{cases}$$

As $\psi|_{Z_m} = \chi|_{Z_m}$ and $\operatorname{tr}(z; V_\chi) = (q-1)q^m\chi(z)$ for $z \in Z_m$ by Lemma 4.11, we have

$$\langle \psi, \xi_\chi\rangle_{H_m} = \frac{1}{(q^2-1)q^{2m}} \sum_{x\in H_m} \psi(x)\operatorname{tr}(x; V_\chi) = \frac{1}{(q^2-1)q^{2m}}((q-1)^2q^{2m} + \sum_{\ell=0}^m S_\ell), \tag{4.25}$$

where

$$\begin{aligned}S_\ell &= \sum_{x\in Z_m H^\ell_m\smallsetminus Z_m H^{\ell+1}_m} \psi(x)\operatorname{tr}(x; V_\chi) = (-1)^{m-\ell+1}q^\ell \sum_{x\in Z_m H^\ell_m\smallsetminus Z_m H^{\ell+1}_m} \psi(x)(\tilde\chi(x) + \tilde\chi^\sigma(x))\\ &= \sharp(Z_m H^\ell_m)\langle \psi, \tilde\chi + \tilde\chi^\sigma\rangle_{Z_m H^\ell_m} - \sharp(Z_m H^{\ell+1}_m)\langle \psi, \tilde\chi + \tilde\chi^\sigma\rangle_{Z_m H^{\ell+1}_m}\end{aligned}$$

for $0 \leqslant \ell \leqslant m$, by Proposition 4.20. For $\ell \geqslant 1$ we have $\sharp(Z_m H^\ell_m) = (q-1)q^{2m-\ell+1}$, and we compute:

$$S_0 = \begin{cases} (-1)^{m+1}q^{2m+1}(q-1) & \text{if } i(\psi) = 0 \\ (-1)^m (q-1)q^{2m} & \text{if } i(\psi) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and for $0 < \ell < m$:

$$S_\ell = \begin{cases} (-1)^{m-\ell+1}(q-1)^2 q^{2m} & \text{if } i(\psi) \leqslant \ell \\ (-1)^{m-\ell}(q-1)q^{2m} & \text{if } i(\psi) = \ell+1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S_m = \begin{cases} (-1)q^{2m}(q-1)(q-2) & \text{if } i(\psi) \leqslant m \\ 2(q-1)q^{2m} & \text{if } i(\psi) = m+1. \end{cases}$$

Here one uses that if $i(\psi) < m+1$, then ψ coincides with precisely one of the characters $\tilde{\chi}, \tilde{\chi}^\sigma$ on $H_m^{i(\psi)}$ and does not coincide with the other even on the last filtration step H_m^m (because $\chi \neq \chi^\sigma$ on $T_{w,m,0}^m$). The theorem follows if we put these values into (4.25). ☐

4.5. **Relation to strata.** We will freely use the terminology of intertwining from [2] §11 and of strata and cuspidal inducing data from [2] Chapter 4. From results of Section 4.3 we deduce that R_χ is irreducible, cuspidal and contains an unramified stratum. First we have the following general result.

Proposition 4.22. *Let $m \geqslant 0$ and let Ξ be a ZK-representation, which restriction to K is the inflation of an irreducible K_m-representation ξ, which does not contain the trivial character on N_m^m. Then the $G(F)$-representation $\Pi_\Xi = \mathrm{c-Ind}_{ZK}^{G(F)} \Xi$ is irreducible, cuspidal and admissible. If $m > 0$, it contains an unramified simple stratum $(\mathfrak{M}, m, \alpha)$ for some $\alpha \in t^{-m}\mathfrak{M}$. Moreover, $\ell(\Pi_\Xi) = m$ and Π_Ξ does not contain an essentially scalar stratum. In particular, for any character ϕ of F^*, one has $0 < \ell(\Pi_\Xi) \leqslant \ell(\phi\Pi_\Xi)$.*

Corollary 4.23. *Let $(E/F, \chi)$ be a minimal pair, such that χ has level $m \geqslant 0$. The representation R_χ is irreducible, cuspidal and admissible. Assume $m > 0$. Then the representation R_χ contains an unramified simple stratum. In particular, $\ell(R_\chi) = m$ and R_χ is unramified. Moreover, for any character ϕ of F^*, one has $0 < \ell(R_\chi) \leqslant \ell(\phi R_\chi)$.*

Proof. All assumptions of Proposition 4.22 are satisfied for the ZK-representation Ξ_χ and the corresponding K_m-representation ξ_χ by Corollary 4.12 and Proposition 4.10. ☐

Proof of Proposition 4.22. Irreducibility and cuspidality of Π_Ξ follow from [2] Theorem 11.4, which assumptions are satisfied due to irreducibility of Ξ and Lemma 4.24. Then admissibility follows from irreducibility (cf. e.g. [2] 10.2 Corollary). Now assume $m > 0$. To contain a stratum is a priori defined with respect to a choice of an additive character. So fix some $\psi \in F^\vee$ of level 1 (i.e., $\psi|_{\mathcal{O}_F}$ non-trivial, $\psi|_{t\mathcal{O}_F}$ trivial). Then [2] 12.5 Proposition gives us an isomorphism (here we use $m > 0$):

$$t^{-m}\mathfrak{M}/t^{-m+1}\mathfrak{M} \xrightarrow{\sim} (K^m/K^{m+1})^\vee = (K_m^m)^\vee, \quad a + t^{-m+1}\mathfrak{M} \mapsto \psi_a|_{K^m},$$

where ψ_a is given by $\psi_a(x) = \psi(\mathrm{tr}_{\mathfrak{M}}(a(x-1)))$, where $\mathrm{tr}_{\mathfrak{M}}$ is the trace map $\mathfrak{M} \to \mathcal{O}_F$. Explicitly, if $a = t^{-m}\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in t^{-m}\mathfrak{M}$ and $x = 1 + t^m \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in K_m^m$, then

$$\psi_a(x) = \psi(a_1x_1 + a_2x_3 + a_3x_2 + a_4x_4). \tag{4.26}$$

We show that Π_Ξ contains an unramified simple stratum. Therefore, note that Π_Ξ contains the inflation to K of the K_m-representation ξ. Thus it is enough to show that for any $\alpha \in t^{-m}\mathfrak{M}$, such that ψ_α is contained in ξ on K_m^m, the stratum $(\mathfrak{M}, m, \alpha)$ is unramified simple. As in [2] 13.2, for $\alpha \in t^{-m}\mathfrak{M}$ we can write $\alpha = t^{-m}\alpha_0$ with $\alpha_0 \in \mathfrak{M}$ and let $f_\alpha(T) \in \mathcal{O}_F[T]$ be the characteristic polynomial of α_0. Let $\tilde{f}_\alpha(T)$ be its reduction modulo t. By definition, $(\mathfrak{M}, m, \alpha)$ is unramified simple if and only if $\tilde{f}_\alpha(T)$ is irreducible in $k[T]$, or equivalently, if and only if $\alpha_0 \mod t \in G(k)$ is not triangularizable.

Let now $\alpha = t^{-m}\alpha_0$ be arbitrary such that ξ contains ψ_α on K_m^m. It is enough to show that α_0 mod t is not triangularizable. Suppose it is. Then there is some $\beta = t^{-m}\begin{pmatrix} \beta_1 & \beta_2 \\ 0 & \beta_4 \end{pmatrix} \in t^{-m}\mathfrak{M}$ such that $g\alpha g^{-1} \equiv \beta \mod t^{-m+1}\mathfrak{M}$, i.e., $\psi_{g\alpha g^{-1}}$ and ψ_β coincide on K_m^m. By Lemma 4.25, ψ_β also occurs in ξ on K_m^m and (4.26) shows that $\psi_\beta|_{N_m^m}$ is the trivial character of N_m^m. This is a contradiction to our assumption that ξ does not contain the trivial character on N_m^m. This contradiction shows that Π_Ξ contains an unramified simple stratum. As an unramified simple stratum is fundamental, [2] 12.9 Theorem shows that $\ell(\Pi_\Xi) = m$.

Suppose now $(\mathfrak{M}, m', \alpha')$ is some essentially scalar stratum contained in Π_Ξ. It has to intertwine with the previously found unramified simple stratum $(\mathfrak{M}, m, \alpha)$ contained in Π_Ξ (cf. [2] 12.9). As essentially scalar strata are fundamental, [2] 12.9 Lemma 2 implies $m' = m$. But in this case the above argumentation shows that $(\mathfrak{M}, m, \alpha')$ is unramified simple and hence not essentially scalar. Finally, Theorem [2] 13.3 implies the last statement of the proposition. □

Lemma 4.24. *Let Ξ, Ξ' be two ZK-representations, which restrictions to K are inflations of K_m-representations ξ, ξ'. Assume that ξ does not contain the trivial character on N_m^m. An element $g \in G(F) \smallsetminus ZK$ never intertwines Ξ with Ξ'.*

Proof. The property of intertwining only depend on the double coset $ZKgZK$ of g. By Cartan decomposition, a set of representatives of these cosets is given by the diagonal matrices $\{m_\alpha = t^{(0,\alpha)} \colon \alpha \in \mathbb{Z}_{\geqslant 0}\}$ (cf. e.g. [2] 7.2.2). Assume $\alpha > 0$. Then $ZK \cap {}^{m_\alpha}(ZK) = ZK \cap m_\alpha ZK m_\alpha^{-1}$ contains the subgroup $N^m = \begin{pmatrix} 1 & t^m\mathcal{O}_F \\ & 1 \end{pmatrix}$, on which Ξ does not contain the trivial character, and on the other hand we have

$$^{m_\alpha}\Xi'\begin{pmatrix} 1 & g \\ & 1 \end{pmatrix} = \Xi'(m_\alpha^{-1}\begin{pmatrix} 1 & g \\ & 1 \end{pmatrix} m_\alpha) = \Xi'\begin{pmatrix} 1 & t^\alpha g \\ & 1 \end{pmatrix},$$

i.e., ${}^{m_\alpha}\Xi'$ restricted to N^m is the trivial representation (as $\alpha > 0$ and Ξ' is trivial on K^{m+1}). Hence

$$\mathrm{Hom}_{ZK\cap {}^{m_\alpha}(ZK)}(\Xi, {}^{m_\alpha}\Xi') \subseteq \mathrm{Hom}_{N^m}(\Xi, {}^{m_\alpha}\Xi') = 0.$$ □

Lemma 4.25. *Let $a \in t^{-m}\mathfrak{M}$, $g \in K$. If ψ_a occurs in ξ on K_m^m, then $\psi_{gag^{-1}}$ occurs in ξ on K_m^m.*

Proof. For $x \in K_m^m$ one has:

$$\psi_{gag^{-1}}(x) = \psi(\mathrm{tr}_{\mathfrak{M}}(gag^{-1}(x-1))) = \psi(\mathrm{tr}_{\mathfrak{M}}(ag^{-1}(x-1)g)) = \psi(\mathrm{tr}_{\mathfrak{M}}(a(g^{-1}xg-1))) = \psi_a(g^{-1}xg).$$

Let V denote the space in which ξ acts. For simplicity we write $x.v$ instead of $\xi(x)(v)$ for $x \in K_m, v \in V$. Let $v \in V$, such that $x.v = \psi_a(x)v$ for all $x \in K_m^m$. Then for all $x \in K_m^m$ we have:

$$g^{-1}x.(g.v) = g^{-1}xg.v = \psi_a(g^{-1}xg)v = \psi_{gag^{-1}}(x)v.$$

Thus $x.(g.v) = \psi_{gag^{-1}}(x)(g.v)$, i.e., on the linear span of $g.v$ any element $x \in K_m^m$ acts as the scalar $\psi_{gag^{-1}}(x)$. In particular, $\psi_{gag^{-1}}$ occurs in ξ on K_m^m. □

Proposition 4.26. *The map $R\colon \mathbb{P}_2^{\mathrm{nr}}(F) \to \mathscr{A}_2^{\mathrm{nr}}(F)$ from Theorem 4.3 is injective.*

Proof. Let $(E/F, \chi_1), (E/F, \chi_2)$ be two non-isomorphic admissible pairs. By Corollary 4.23 we may assume that χ_1, χ_2 have the same level and by Lemma 4.4 we may assume that χ_1, χ_2 coincide on F^*. Twisting by a central character, we may assume that both pairs are minimal. The last statement of Corollary 4.14 shows $\Xi_{\chi_1} \not\cong \Xi_{\chi_2}$. It remains to show that this implies $R_{\chi_1} \not\cong R_{\chi_2}$. Frobenius reciprocity and Mackey formula show that

$$\begin{aligned} \mathrm{Hom}_{G(F)}(R_{\chi_1}, R_{\chi_2}) &= \bigoplus_{g \in ZK\backslash G(F)/ZK} \mathrm{Hom}_{ZK \cap {}^g(ZK)}(\Xi_{\chi_1}, {}^g\Xi_{\chi_2}) \\ &= \mathrm{Hom}_{ZK}(\Xi_{\chi_1}, \Xi_{\chi_2}), \end{aligned}$$

where the second equality follows from Lemma 4.24. As $\Xi_{\chi_1}, \Xi_{\chi_2}$ are irreducible (by Corollary 4.12) and unequal, the Hom-space is zero. □

4.6. **Relation to cuspidal inducing data.** Now we want to compare our construction to the construction in [2] §19 of representations attached to minimal pairs. For the convenience of the reader and to have appropriate notations, we briefly recall their set up ([2] §15,§19). Let ψ be some fixed (additive) character of F of level one. Let $\psi_E = \psi \circ \mathrm{tr}_{E/F}$, $\psi_{\mathfrak{M}} = \psi \circ \mathrm{tr}_{\mathfrak{M}}$. Let $(E/F, \chi)$ be a minimal pair. Let $m > 0$ be the level of χ. Let $\alpha \in \mathfrak{p}_E^{-m}$ be such that $\chi(1+x) = \psi_E(\alpha x)$ for $x \in \mathfrak{p}_E^{\lfloor \frac{m}{2} \rfloor + 1}$. Choose an F-embedding $E \hookrightarrow M_{22}(F)$ such that $E^* \subseteq ZK$ (not to be confused with ι_E from the beginning of Section 4.1). Then $(\mathfrak{M}, m, \alpha)$ is an unramified simple stratum. Let then

$$J_\alpha = E^* K^{\lfloor \frac{m+1}{2} \rfloor};$$

this is an open subgroup of ZK. Moreover, via the embedding of E into $M_{22}(F)$, α defines ([2] 12.5) a character ψ_α of $K^{\lfloor \frac{m}{2} \rfloor + 1}$, which is trivial on K^{m+1} (thus inducing a character of $K_m^{\lfloor \frac{m}{2} \rfloor + 1}$). Let $C(\psi_\alpha, \mathfrak{M})$ be the set of isomorphism classes of all irreducible representations Λ of J_α, such that Λ contains the character ψ_α on $K^{\lfloor \frac{m}{2} \rfloor + 1}$, or equivalently (by [2] 15.3 Theorem), $\Lambda|_{K^{\lfloor \frac{m}{2} \rfloor + 1}}$ is a multiple of ψ_α.

For any $\Lambda \in C(\psi_\alpha, \mathfrak{M})$, the triple $(\mathfrak{M}, J_\alpha, \Lambda)$ is a (in our case, unramified) *cuspidal type* in $G(F)$ in the sense of [2] 15.5 Definition. An equivalent reformulation is given in terms of *cuspidal inducing data* ([2] 15.8): the cuspidal inducing datum attached to $(\mathfrak{M}, J_\alpha, \Lambda)$ is the pair $(\mathfrak{M}, \Xi)$, where $\Xi = \mathrm{Ind}_{J_\alpha}^{ZK} \Lambda$. The $G(F)$-representation $\mathrm{c-Ind}_{J_\alpha}^{G(F)} \Lambda = \mathrm{c-Ind}_{ZK}^{G(F)} \Xi$ attached to $(\mathfrak{M}, J_\alpha, \Lambda)$, resp. to $(\mathfrak{M}, \Xi)$ is then irreducible and cuspidal.

Out of the given minimal pair $(E/F, \chi)$ one constructs now the representation Λ of J_α, and thus gets a corresponding cuspidal type. We have two different cases.

Case m odd. ([2] 19.3) Then $\lfloor \frac{m}{2} \rfloor + 1 = \lfloor \frac{m+1}{2} \rfloor$. Let Λ be the character of J_α defined by

$$\Lambda|_{K^{\lfloor \frac{m+1}{2} \rfloor}} = \psi_\alpha, \quad \Lambda|_{E^*} = \chi \tag{4.27}$$

(this is a consistent definition, as one sees from $\mathrm{tr}_{\mathfrak{M}}|_{\mathcal{O}_E} = \mathrm{tr}_E|_{\mathcal{O}_E}$, $E^* \cap K^{\lfloor \frac{m+1}{2} \rfloor} = U_E^{\lfloor \frac{m+1}{2} \rfloor}$).

Case $m > 0$ even. ([2] 15.6, 19.4) Let $J_\alpha^1 = J_\alpha \cap K^1 = U_E^1 K^{\lfloor \frac{m+1}{2} \rfloor}$, $H_\alpha^1 = U_E^1 K^{\lfloor \frac{m}{2} \rfloor + 1}$. Then $J_\alpha^1 \supsetneq H_\alpha^1$. Let θ be the character of H_α^1 defined (as in the odd case) by

$$\theta(ux) = \chi(u)\psi_\alpha(x), \quad u \in U_E^1, x \in K^{\lfloor \frac{m}{2} \rfloor + 1}.$$

Let η be the unique irreducible (q-dimensional) J_α^1-representation containing θ. Let μ_M denote the group of roots unity of a field M and let $\tilde{\eta}$ be the unique irreducible representation of $\mu_E/\mu_F \ltimes J_\alpha^1$ such that $\tilde{\eta}|_{J_\alpha^1} \cong \eta$ and $\mathrm{tr}\,\tilde{\eta}(\zeta u) = -\theta(u)$ for all $u \in H_\alpha^1, \zeta \in \mu_E/\mu_F \smallsetminus \{1\}$. Then $\tilde{\eta}$ factors through a representation of $\mu_E/\mu_F \ltimes J_\alpha^1/\ker(\theta)$. Let ν be the representation of $E^* \ltimes J_\alpha^1/\ker(\theta)$ which arises by inflation from $\tilde{\eta}$ via the surjection induced by $E^* \twoheadrightarrow E^*/F^*U_E^1 \cong \mu_E/\mu_F$. Let $\tilde{\chi}$ be the character of $E^* \ltimes J_\alpha^1/\ker(\theta)$, which is χ on E^* and trivial on $J_\alpha^1/\ker(\theta)$. Define the $E^* \ltimes J_\alpha^1/\ker(\theta)$-representation $\tilde{\Lambda} = \tilde{\chi} \otimes \nu$. It factors through the surjection $E^* \ltimes J_\alpha^1/\ker(\theta) \twoheadrightarrow J_\alpha/\ker(\theta)$, $(e,j) \mapsto ej \mod \ker(\theta)$, hence it is an inflation of a representation Λ_1 of $J_\alpha/\ker(\theta)$. Take Λ to be the inflation of Λ_1 to J_α.

Let then in both cases $(\mathfrak{M}, \Theta_\chi)$ be the corresponding cuspidal inducing datum, i.e.,

$$\Theta_\chi = \mathrm{Ind}_{J_\alpha}^{ZK} \Lambda. \tag{4.28}$$

Thus we attached a cuspidal inducing datum to χ and now the $G(F)$-representation π_χ from (4.1) is defined in [2] 19.4.2 as

$$\pi_\chi = \mathrm{c} - \mathrm{Ind}_{ZK}^{G(F)} \Theta_\chi = \mathrm{c} - \mathrm{Ind}_{J_\alpha}^{G(F)} \Lambda.$$

Proposition 4.27. *Let $(E/F, \chi)$ be a minimal pair. Then $R_\chi \cong \pi_\chi$.*

Using Proposition 4.27, we can prove our main result.

Proof of Theorem 4.3. By Lemma 4.4 we can assume that $(E/F, \chi)$ is minimal in the first statement of the theorem. If $\ell(\chi) = 0$, then the first statement follows essentially from [10] Theorem 1.1(i). If $\ell(\chi) > 0$, then the first statement follows from Corollary 4.23 and the part about the central character follows from Lemma 4.4.

To show $R_\chi \cong \pi_\chi$ we can assume by Lemma 4.4 (along with the fact that $\phi\pi_\chi = \pi_{\phi_E\chi}$) that $(E/F, \chi)$ is minimal. Then $R_\chi \cong \pi_\chi$ follows from Proposition 4.27. Now bijectivity of (4.2) follows from bijectivity of (4.1). ∎

Proof of Proposition 4.27. Let m be the level of χ. If $m = 0$, the proposition follows essentially from [10] Theorem 1.1(i) and [2] 19.1. Assume $m > 0$. The unramified representation R_χ is induced from the cuspidal inducing datum (ZK, Ξ_χ). As the map (4.1) in the tame parametrization theorem is surjective, there is some character χ' such that $(E/F, \chi')$ is minimal and $R_\chi \cong \pi_{\chi'}$. By Corollary 4.23, $\ell(\chi') = m$. One deduces $\Xi_\chi \cong \Theta_{\chi'}$ (e.g. by the same reasoning as in the proof of Lemma 4.24). We have to show that $\chi = \chi'$ or $\chi = (\chi')^\sigma$. A comparison of the central characters shows $\chi|_{F^*} = \chi'|_{F^*}$. Thus it remains to show that $\chi|_{U_E} = \chi'|_{U_E}$ or $\chi|_{U_E} = (\chi')^\sigma|_{U_E}$. The K-representation $\Xi_\chi|_K$ is inflated from the K_m-representation ξ_χ. Note that the image of U_E in K_m is a non-split torus H_m, as considered in Theorem 4.13. Thus $\chi|_{U_E}, \chi^\sigma|_{U_E}$ are the unique characters among all U_E-characters of level m, which satisfy condition (i) resp. (ii) of Corollary 4.14 if m odd resp. even. Thus it is enough to show that $\Theta_{\chi'}|_K$ characterizes $\chi'|_{U_E}$ in the same way. This is the content of Lemma 4.28. ∎

Lemma 4.28. *Let χ be a character of E^* of level $m > 0$ such that $(E/F, \chi)$ is a minimal pair.*

(i) *If m is odd, the representation $\Theta_\chi = \mathrm{Ind}_{J_\alpha}^{ZK}\Lambda$ (cf. (4.28)) contains the character χ on E^* (exactly once) and does not contain all the characters χ' of E^*, which satisfy $\chi'|_{U_E} \neq \chi|_{U_E}$ and $\chi'|_{F^*U_E^1} = \chi|_{F^*U_E^1}$.*

(ii) *If m is even, the representation $\Theta_\chi = \mathrm{Ind}_{J_\alpha}^{ZK}\Lambda$ (cf. (4.28)) does not contain the character χ on E^* and it contains all the characters χ' of E^*, which satisfy $\chi'|_{U_E} \neq \chi|_{U_E}$ and $\chi'|_{F^*U_E^1} = \chi|_{F^*U_E^1}$.*

Proof. Let first $m > 0$ be arbitrary and let χ' be a character of E^*, satisfying $\chi'|_{F^*U_E^1} = \chi|_{F^*U_E^1}$. Mackey formula and Frobenius reciprocity show:

$$\mathrm{Hom}_{E^*}(\chi', \Theta_\chi) = \bigoplus_{g\in E^*\backslash ZK/J_\alpha} \mathrm{Hom}_{E^*\cap {}^gJ_\alpha}(\chi', {}^g\Lambda).$$

Let $g \in ZK$. We claim that $\mathrm{Hom}_{E^*\cap {}^gJ_\alpha}(\chi', {}^g\Lambda) = 0$, unless $g \in J_\alpha$. Indeed, we have $E^* \cap {}^gJ_\alpha \supseteq U_E^{\lfloor \frac{m}{2}\rfloor+1}$ and $\Lambda|_{K^{\lfloor \frac{m}{2}\rfloor+1}}$ is a multiple of ψ_α, hence ${}^g\Lambda|_{K^{\lfloor \frac{m}{2}\rfloor+1}}$ is a multiple of $\psi_{g^{-1}\alpha g}$. Moreover, $\chi'|_{U_E^{\lfloor \frac{m}{2}\rfloor+1}} = \chi|_{U_E^{\lfloor \frac{m}{2}\rfloor+1}} = \psi_\alpha$. Thus if $\mathrm{Hom}_{E^*\cap {}^gJ_\alpha}(\chi', {}^g\Lambda) \neq 0$, then g normalizes the character ψ_α of $U_E^{\lfloor \frac{m}{2}\rfloor+1}$. Thus Proposition 4.29 shows our claim.

The claim implies that $\mathrm{Hom}_{E^*}(\chi', \Theta_\chi) = \mathrm{Hom}_{E^*}(\chi', \Lambda)$. In particular, if m is odd, we are done, because then Λ is one-dimensional and $\Lambda|_{E^*} = \chi$. Assume m is even. By construction, Λ arises by an inflation process from the $E^* \ltimes J_\alpha^1/\ker(\theta)$-representation $\tilde{\Lambda} = \tilde{\chi}\otimes\nu$, where $\tilde{\chi}$ agrees with χ on E^* and is trivial on $J_\alpha^1/\ker(\theta)$. So, it is enough to prove the following claim: $\nu|_{E^*}$ does not contain the trivial character of U_E, but it contains all non-trivial characters of U_E, which are trivial on $U_FU_E^1$. The restriction of ν to E^* is the inflation via $E^* \twoheadrightarrow E^*/F^*U_E^1 \cong \mu_E/\mu_F$ of the restriction to μ_E/μ_F of the $\mu_E/\mu_F \ltimes J_\alpha^1$-representation $\tilde{\eta}_\theta$. In particular, $\nu|_{U_FU_E^1}$ is trivial. Now [2] 19.4 Proposition shows that $\tilde{\eta}_\theta|_{\mu_E/\mu_F} = \mathrm{Reg}_{\mu_E/\mu_F} - 1_{\mu_E/\mu_F}$, and the claim follows. □

The following proposition is an improvement of a part of the Intertwining theorem [2] 15.1. Also Lemma 4.30 below improves [2] Lemma 16.2

Proposition 4.29. *Let $g \in ZK$. Then g normalizes the character ψ_α of $U_E^{\lfloor \frac{m}{2}\rfloor+1}$ if and only if $g \in J_\alpha$.*

Proof. We can assume $g \in K$. Let X be the appropriate quotient of $t^{-m}\mathfrak{M}/t^{-\lfloor \frac{m}{2}\rfloor}\mathfrak{M}$ such that the following diagram commutes

$$\begin{array}{ccc} t^{-m}\mathfrak{M}/t^{-\lfloor \frac{m}{2}\rfloor}\mathfrak{M} & \xrightarrow{\sim} & (K_m^{\lfloor \frac{m}{2}\rfloor+1})^\vee \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & (U_E^{\lfloor \frac{m}{2}\rfloor+1}/U_E^{m+1})^\vee \end{array} \tag{4.29}$$

where the upper horizontal map is $\alpha \mapsto \psi_\alpha$ with ψ_α as in [2] 12.5, and the right vertical map is restriction of characters. Let $Y \subseteq \mathfrak{M}$ be such that $t^{-m}Y \subseteq t^{-m}\mathfrak{M}$ is the preimage in $t^{-m}\mathfrak{M}$ of the kernel of the left vertical map. Then g normalizes $\psi_\alpha|_{U_E^{\lfloor \frac{m}{2}\rfloor+1}}$ if and only if the images of $g^{-1}\alpha g$ and α in X coincide, i.e., if the following equation holds true in $t^{-m}\mathfrak{M}$:

$$g^{-1}\alpha g \equiv \alpha \mod t^{-\lfloor \frac{m}{2}\rfloor}\mathfrak{M} + t^{-m}Y.$$

Then the result follows from Lemma 4.30 applied to $k = \lfloor \frac{m+1}{2} \rfloor$. □

Lemma 4.30. *Write $\alpha = t^{-m}\alpha_0$. With notations as in the proof of Proposition 4.29, for any $1 \leqslant k \leqslant \lfloor \frac{m+1}{2} \rfloor$, we have*

$$g^{-1}\alpha_0 g \equiv \alpha_0 \mod t^k\mathfrak{M} + Y \tag{4.30}$$

in $\mathfrak{M}$ if and only of $g \in U_E + t^k\mathfrak{M}$.

Proof. The 'if' part is immediate. To prove the other part, we use induction on k (as in [2] 16.2 Lemma). Let $k \geqslant 2$ and assume (4.30). By induction hypothesis, $g \in U_E + t^{k-1}\mathfrak{M}$. We can write $g = g_1(1 + t^{k-1}g_0)$ with $g_1 \in U_E$. Thus (as $\alpha_0 \in \mathcal{O}_E$) we obtain from (4.30):

$$t^{k-1}\alpha_0 g_0 \equiv t^{k-1}g_0\alpha_0 \mod t^k\mathfrak{M} + Y.$$

Thus $t^{k-1}(\alpha_0 g_0 - g_0\alpha_0) = y + t^k m \in \mathfrak{M}$ for some $y \in Y$, $m \in \mathfrak{M}$. We deduce $y = t^{k-1}y'$ with $y' \in \mathfrak{M}$ and $\alpha_0 g_0 - g_0\alpha_0 = y' + tm$. We claim that $y' \in Y + t\mathfrak{M}$. Indeed, this claim is equivalent to $\psi_{t^{-m}y'}|_{U_E^m/U_E^{m+1}} \equiv 1$. But for $u \in \mathfrak{M}$ we have:

$$\psi_{t^{-m}y'}(1 + t^m u) = \psi(\mathrm{tr}_{\mathfrak{M}}(y'u)) = \psi_{t^{-m}y}(1 + t^{m-(k-1)}u) = 1,$$

where the last equality holds as long $m - (k-1) \geqslant \lfloor \frac{m}{2} \rfloor + 1$, or equivalently, $k \leqslant \lfloor \frac{m+1}{2} \rfloor$, which is satisfied by assumption of the Lemma. This shows our claim. From it we deduce $\alpha_0 g_0 \equiv g_0\alpha_0$ mod $Y + t\mathfrak{M}$, i.e., by induction hypothesis, $g_0 \in \mathcal{O}_E + t\mathfrak{M}$. Thus we are reduced to the case $k = 1$. We handle this case explicitly. The result remains unaffected if we replace the embedding $j\colon E \hookrightarrow M_2(F)$ by a conjugate one. As all such embeddings are $G(F)$-conjugate, we can assume that $j(\mathcal{O}_E) \mod t \subseteq \mathfrak{M}/t\mathfrak{M}$ is generated as a k-algebra by a matrix $\beta = \begin{pmatrix} & -b \\ 1 & -a \end{pmatrix}$ for some $a, b \in k$ such that the characteristic polynomial $T^2 + aT + b$ is irreducible in $k[T]$ (cf. e.g. [2] 5.3). Then $\alpha = t^{-m}\alpha_0$ with $\alpha_0 \mod t = x + y\beta$ for some $x, y \in k$ and $j(\mathcal{O}_E) = \mathcal{O}_F[\alpha_0]$. After adding and multiplying by some central elements (which does not affect the condition 4.30), we can assume that either $\mathrm{char}(k) > 2$ and there is a $D \in k^* \smallsetminus k^{*,2}$ such that $\alpha_0 = \begin{pmatrix} & 1 \\ D & \end{pmatrix}$ or that $\mathrm{char}(k) = 2$ and there is a $D \in k$ such that $T^2 + T + D \in k[T]$ is irreducible and $\alpha_0 = \begin{pmatrix} & D \\ 1 & 1 \end{pmatrix}$. We have to show that if $g \in K$ and (4.30) holds for g, α_0 and $k = 1$, then $g \in \mathcal{O}_E + t\mathfrak{M}$.

Assume first $\mathrm{char}(k) > 2$. The upper horizontal map in diagram (4.29) induces the isomorphism

$$t^{-m}\mathfrak{M}/t^{-m+1}\mathfrak{M} \xrightarrow{\sim} (K_m^m)^\vee,$$

which shows that

$$\begin{aligned} Y + t\mathfrak{M}/t\mathfrak{M} &= \{t^m\beta\colon \beta \in t^{-m}\mathfrak{M} \text{ and } \psi(\mathrm{tr}_{\mathfrak{M}}(\beta(e-1))) = 1 \text{ for all } e \in 1 + t^m\mathcal{O}_E\}/t\mathfrak{M} \\ &= \{\begin{pmatrix} B_1 & B_2 \\ -B_2 D & -B_1 \end{pmatrix} : B_1, B_2 \in k\} \end{aligned}$$

(the last equality is an easy computation). Now let $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \in G(k)$ (we can work modulo t). Then condition (4.30) translates into

$$\frac{1}{\det(g)}\left(\begin{array}{cc} g_3g_4 - g_1g_2D & -g_2^2D + g_4^2 \\ g_1^2D - g_3^2 & g_1g_2D - g_3g_4 \end{array}\right) = g^{-1}\alpha_0 g \stackrel{!}{=} \left(\begin{array}{cc} & 1 \\ D & \end{array}\right) + \left(\begin{array}{cc} B_1 & B_2 \\ -B_2D & -B_1 \end{array}\right)$$

for some $B_1, B_2 \in k$. In particular, we must have

$$\begin{cases} \frac{1}{\det(g)}(g_4^2 - g_2^2D) = 1 + B_2 \\ \frac{1}{\det(g)}(g_1^2D - g_3^2) = (1 - B_2)D. \end{cases}$$

Computing B_2 from the first equation and inserting it in the second, gives us

$$\frac{1}{\det(g)}(g_1^2 - g_3^2 - g_2^2D^2 + g_4^2D) = 2D,$$

which is equivalent to

$$D(g_1 - g_4)^2 = (g_3 - g_2D)^2.$$

If both side are non-zero, on the left side we have a non-square in k^* and on the right side we have a square, which is a contradiction. Thus both sides are zero, i.e. $g_1 = g_4$, $g_3 = g_2D$, i.e., $g \in U_E$ mod t, finishing the proof in the case $\mathrm{char}(k) > 2$.

Assume now $\mathrm{char}(k) = 2$. Analogously to the previous case we deduce

$$Y + t\mathfrak{M}/t\mathfrak{M} = \left\{\left(\begin{array}{cc} B_1 & B_1 + B_3D \\ B_3 & B_1 \end{array}\right) : B_1, B_3 \in k\right\}.$$

A similar computation as above implies that for $g \in G(k)$ satisfying condition (4.30) we must have

$$\begin{aligned} \det(g)^{-1}(g_1g_2 + g_2g_3 + g_3g_4D) &= B_1 \\ \det(g)^{-1}(g_1^2 + g_1g_3 + g_3^2D) &= 1 + B_3 \\ \det(g)^{-1}(g_2^2 + g_2g_4 + g_4^2D) &= B_1 + B_3D + D. \end{aligned}$$

with some $B_1, B_3 \in k$. Putting the first and the second equation into the third and bringing some terms together shows

$$g_2^2 + g_3^2D^2 = (g_1^2 + g_4^2)D + g_2(g_4 + g_1 + g_3) + g_3D(g_4 + g_1).$$

Add $2g_3^2D = 0$ to the right side of this equation and let $A = g_2 + g_3D$ and $B = g_1 + g_3 + g_4$. The equation is then equivalent to

$$A^2 + B^2D + AB = 0.$$

Suppose $B \neq 0$. Dividing by B^2, we obtain $(A/B)^2 + (A/B) + D = 0$, which is a contradiction to irreducibility of $T^2 + T + D \in k[T]$, as $A/B \in k$. Thus $B = 0$ and we deduce also $A = 0$, which finishes the proof also in the case $\mathrm{char}(k) = 2$. □

References

[1] Boyarchenko M.: *Deligne-Lusztig constructions for unipotent and p-adic groups*, preprint, arXiv:1207.5876, 2012.

[2] Bushnell C.J., Henniart G.: *The Local Langlands Correspondence for* GL_2, Springer, 2006.

[3] Bushnell C.J., Kutzko P.C.: *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Ann. of Math. Studies, vol. 129, Princeton University Press, 1993.

[4] Bruhat F., Tits J.: *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5-251.

[5] Bruhat F., Tits J.: *Groupes réductifs sur un corps local II. Schémas en groupes. Existence d'une donnée radicielle valuée.*, Inst. Hautes Études Sci. Publ. Math. **60** (1984), 197-376.

[6] Boyarchenko M, Weinstein J.: *Maximal Varieties and the local Langlands correspondence for $GL(n)$*, J. Amer. Math. Soc. **29** (2016), no. 1, 177-236.

[7] Chan C.: *Deligne-Lusztig Constructions for Division Algebras and the Local Langlands Correspondence*, Adv. Math. **294** (2016), 332-383.

[8] Deligne P.: *La conjecture de Weil II*, Publ. Math. IHES, (52):137-252, 1980.

[9] Deligne P., Lusztig G.: *Representations of reductive groups over finite fields*, Ann. of Math. (2), **103**(1), 103-161, 1976.

[10] Ivanov A.: *Cohomology of affine Deligne-Lusztig varieties for GL_2*, J. of Alg. **383** (2013), 42-62.

[11] Ivanov A.: *Ramified automorphic induction and zero-dimensional affine Deligne-Lusztig varieties*, preprint 2015, arXiv:1512.07530

[12] Lusztig G.: *Some remarks on the supercuspidal representations of p-adic semisimple groups*, in 'Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, pp. 171-175., Amer. Math. Soc., Providence, R.I., 1979.

[13] Lusztig G.: *Representations of reductive groups over finite rings*, Represent. Theory **8** (2004), 1-14.

[14] Milne J.: *Étale cohomology*, Princeton University Press, 1980.

[15] Pappas G., Rapoport M., *Twisted loop groups and their affine flag varieties*, Adv. in Math. **219**, no. 1, (2008), 118-198.

[16] Rapoport M.: *A guide to the reduction modulo p of Shimura varieties*, Astérisque **298** (2005), 271-318.

[17] Stasinski A.: *Extended Deligne-Lusztig varieties for general and special linear groups*, Adv. in Math. **226**, no. 3, 2011, 2825-2853.

[18] Yu J.-K.: *Smooth models associated to concave functions in Bruhat-Tits theory*, preprint (2002).

[19] Zhu X.: *Affine Grassmannians and the geometric Satake in mixed characteristic*, preprint, 2014, arXiv:1407.8519

Technische Universität München, Boltzmannstr. 3, 85748 Garching, Germany