COHOMOLOGY OF AFFINE DELIGNE-LUSZTIG VARIETIES FOR GL_2

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ABSTRACT. In this paper we study affine Deligne-Lusztig varieties $X_w(b)$ for GL_2 and their étale coverings. At first, we compute them explicitly, then we determine associated representations of a certain locally-compact group, the group of rational points of the σ-stabilizer of b, in their étale cohomology. Further, we study these representations by determining morphisms into the irreducible representations of the given group. In particular all cuspidal representations of level 0 of GL_2 of a local field and of its inner form, which is the group of units of a quaternion algebra, occur in the cohomology.

1. Introduction

Let k be a field with q elements, and let $\bar{k}$ be an algebraic closure of k. Let $k \subset k' \subset \bar{k}$ be the quadratic extension of k in $\bar{k}$. Let σ denote the Frobenius morphism of $\bar{k}/k$. Put $F = k((t))$ and $L = \bar{k}((t))$. We extend σ to the Frobenius morphism of L/F by setting $\sigma(t) = t$. Write $\mathfrak{o} = \bar{k}[[t]]$, $\mathfrak{o}_F = k[[t]]$ and $\mathfrak{p}$, $\mathfrak{p}_F$ for valuation rings of L resp. F and for their maximal ideals. Denote the valuation on L by v_L.

Let G be a split connected reductive group over k and let T be a split maximal torus in G. For a coroot $\lambda \in X_*(T)$ we write t^λ for the image of t under $\lambda\colon \mathbb{G}_m \to T$. Write W and $\tilde{W} = X_*(T) \rtimes W$ for the finite and the extended affine Weyl groups attached to T.

Fix a Borel subgroup B containing T, such that $B = TU$ with U unipotent. Let I resp. I_1 be the preimage of $B(\bar{k})$ resp. $U(\bar{k})$ under the projection $G(\mathfrak{o}) \twoheadrightarrow G(\bar{k})$. Then I is an Iwahori subgroup in $G(L)$. Let $X = G(L)/I$ be the affine flag manifold and $\dot{\mathrm{X}}$ its covering $\dot{\mathrm{X}} = G(L)/I_1$. The group $G(L)$ acts by left translation on X and $\dot{\mathrm{X}}$. The Bruhat decomposition implies that $G(L)$ is the union of the double cosets IwI, where $w \in \tilde{W}$. Put $T_1 := T(\mathfrak{o}) \cap I_1$ and set $\tilde{W}_1 = N_G(T)(L)/T_1$. Then $G(L) = \coprod_{\dot{\mathrm{w}} \in \tilde{W}_1} I_1 \,\dot{\mathrm{w}}\, I_1$ (Lemma 3.1). Following [8], the affine Deligne-Lusztig variety $X_w(b)$ attached to $b \in G(L)$ and $w \in \tilde{W}$ is the locally closed subset of X, endowed with its reduced induced sub-Ind-scheme structure, defined by

$$X_w(b) = \{xI \in G(L)/I\colon x^{-1}b\sigma(x) \in IwI\}.$$

In analogy with [2], for $b \in G(L)$ and $\dot{\mathrm{w}} \in \tilde{W}_1$, we let $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ be the locally closed subset

$$\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b) = \{xI_1 \in G(L)/I_1\colon x^{-1}b\sigma(x) \in I_1 \,\dot{\mathrm{w}}\, I_1\}$$

of $\dot{\mathrm{X}}$, endowed with its reduced induced sub-Ind-scheme structure. If $w \in \tilde{W}$ is the image of $\dot{\mathrm{w}}$, then the obvious projection map $\pi\colon \dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b) \to X_w(b)$ is a finite étale torsor under the group $T(w)^\sigma_{\mathrm{aff}}/T_1 = \{a \in T(\mathfrak{o})\colon a^{-1}\,\mathrm{ad}(w)(\sigma(a)) \in T_1\}/T_1$. The group

$$J_b = \{g \in G(L)\colon g^{-1}b\sigma(g) = b\}$$

acts by left multiplication on $X_w(b)$ and $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$, and π is J_b-equivariant.

Our first goal is to study these varieties in the case $G = GL_2$. In [10] Reuman determined the set $\{w \in \tilde{W}\colon X_w(b) \neq \emptyset\}$ for reductive groups of semisimple rank 1 and 2, and in particular for GL_2. We determine (in §2 and §3) the precise structure of $X_w(b)$ and $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ as schemes locally of finite type over k. If $X_w(b) \neq \emptyset$, there are J_b- and $T(w)^\sigma_{\mathrm{aff}}/T_1$-equivariant (the actions on the right side are defined later) $\bar{k}$-isomorphisms:

$$X_w(b) \cong \coprod_{J_b/K_b^{(m)}} \mathbb{A}^{d_{b,w}} \times S(b,w) \quad \text{and} \quad \dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b) \cong \coprod_{J_b/K_b^{(m)}} \mathbb{A}^{d_{b,w}} \times \dot{\mathrm{S}}(b,w),$$

where the numbers $m \in \{0,1\}, d_{b,w} \geq 0$ depend on b and w (cf. Section 3.3 for explicit formulas). Let $b_1 = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$ and $w_a := wb_1^{-v_L(\det(b))} \in W_a$, where W_a is the affine Weyl group of $T \cap G_{\mathrm{der}}$ in G_{der}, which is a Coxeter group equipped with a length function. Then we have:

b	$\ell(w_a)$	J_b	$K_b^{(m)}$	$S(b,w)$	$\dot{\mathrm{S}}(b,w)$
1	0 or odd	$G(F)$	$b_1^m K b_1^{-m}$	$X_{w_{\mathrm{fin}}}$	$\dot{\mathrm{X}}_{\dot{\mathrm{w}}_{\mathrm{fin}}}$
$t^{(0,\alpha)}, \alpha > 0$	α	$T(F)$	$T(\mathfrak{o}_F)$	$\{pt\}$	$\coprod_{T(k)} \{pt\}$
$t^{(0,\alpha)}, \alpha > 0$	$\ell(w_a) - \alpha > 0$ odd	$T(F)$	$T(\mathfrak{o}_F)$	$\mathbb{G}_m$	$\coprod_{k^*} \mathbb{G}_m$
b_1	even	D^*	U_D	$\{pt\}$	$\coprod_{\mu_{q^2-1}} \{pt\}$

TABLE 1. Affine Deligne-Lusztig Varieties for GL_2

In this table $K = G(\mathfrak{o}_F)$, $\dot{\mathrm{w}}_{\mathrm{fin}} \in G(k)$ lies over the image w_{fin} of w in W and $X_{w_{\mathrm{fin}}}$, $\dot{\mathrm{X}}_{\dot{\mathrm{w}}_{\mathrm{fin}}}$ denote the corresponding finite Deligne-Lusztig varieties for $G = GL_2$. We have $w_a \in W_a$ since $X_w(b) \neq \emptyset$. Further, D denotes the central division algebra over F of dimension 4 and U_D the group of units in its valuation ring.

Since $X_w(b)$ and $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ depend only on the σ-conjugacy class and also (essentially) only on the class modulo center of b, all essential cases are presented in the table above (every element of $G(L)$ is σ-conjugate to one of the b's in the table multiplied by a central element).

Our second goal is to determine the representations of the locally profinite groups J_b in the cohomology of these varieties for $G = GL_2$. In analogy to [2], for any G and for a character θ of $T(w)_{\mathrm{aff}}^{\sigma}/T_1$ define $R_b^{\theta}(w)_{\mathrm{aff}}$ to be the virtual J_b-representation

$$R_b^{\theta}(w)_{\mathrm{aff}} = \sum_i (-1)^i H_c^i(\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b), \overline{\mathbb{Q}_l})_{\theta}.$$

For $w_{\mathrm{fin}} \in W$, let $R^{\theta}(w_{\mathrm{fin}})$ denote the virtual representation of $G(k)$ defined in [2] as the alternating sum of the $G(k)$-representations $H_c^i(\dot{\mathrm{X}}_{\dot{\mathrm{w}}_{\mathrm{fin}}}, \overline{\mathbb{Q}_l})_{\theta}$, where θ is a character of $T(w_{\mathrm{fin}})^{\sigma} = \{t \in T(\bar{k}) \colon \mathrm{ad}(w_{\mathrm{fin}})\sigma(t) = t\}$. Let now $G = GL_2$ and assume without loss of generality that T is the diagonal torus. For any $w \in \tilde{W}$ with image $w_{\mathrm{fin}} \in W$, we have:

$$T(w_{\mathrm{fin}})^{\sigma} = T(w)_{\mathrm{aff}}^{\sigma}/T_1 \cong \begin{cases} T(k) & \text{if } w_{\mathrm{fin}} = 1, \\ k'^* & \text{if } w_{\mathrm{fin}} \neq 1, \end{cases}$$

the second isomorphism being given by reduction modulo $\mathfrak{p}$. For $m \in \{0,1\}$ let $p_m \colon K_b^{(m)} \to G(\bar{k})$ be conjugation with b_1^m composed with reduction modulo $\mathfrak{p}$. For $b = 1$ it maps $K_1^{(m)}$ onto $G(k)$, for $b = t^{(0,\alpha)}$ $(\alpha > 0)$ it maps $K_b^{(m)} = T(\mathfrak{o}_F)$ onto $T(k)$ and for $b = b_1$, $p_m(K_b^{(m)})$ projects onto k'^*, embedded in $G(\bar{k})$ by $a \mapsto \mathrm{diag}(a, a^q)$. If π is some representation of $\mathrm{im}(p_m)$, then we write $\pi_{K_b^{(m)}}$ for the $K_b^{(m)}$-representation obtained by inflation via p_m.

Theorem 1.1. *Assume $b \in G(L), w \in \tilde{W}$ are such that $X_w(b) \neq \emptyset$. Let θ be a character of $T(w)_{\mathrm{aff}}^{\sigma}/T_1$.*

(i) *Let w_{fin} be the image of w in W. Then:*

$$R_1^{\theta}(w)_{\mathrm{aff}} = \text{c-Ind}_{K_1^{(m)}}^{G(F)} R^{\theta}(w_{\mathrm{fin}})_{K_1^{(m)}}.$$

(ii) *Let* $b = t^{(0,\alpha)}$ *with* $\alpha > 0$. *Then:*

$$R_b^\theta(w)_{\mathrm{aff}} = \begin{cases} \text{c-Ind}\,_{T(\mathfrak{o}_F)}^{T(F)}\, \tilde{\theta}_{T(\mathfrak{o}_F)} & \text{if } \ell(w_a) = \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{\theta}$ *is either* θ *or* θ^v, *where* $1 \neq v \in W$.

(iii) *Let* $b = b_1$. *Then* $T(w)^\sigma_{\mathrm{aff}}/T_1 \cong k'^*$ *and we have:*

$$R_{b_1}^\theta(w)_{\mathrm{aff}} = \text{c-Ind}\,_{U_D}^{D^*}\, \theta_{U_D}.$$

Part (ii) was already studied for GL_n for any n. In particular, He proved in [5] Cor 11.11 that $T(\mathfrak{o}_F)$ acts trivial on the Borel-Moore homology of $X_w(b)$ for b regular element in $T(L)$ and for any $w \in \tilde{W}$. Also, for such b's, Zbarsky determined in [12] the representations on the Borel-Moore homology of $X_w(b)$ for SL_2 and SL_3.

Let us now restrict attention to the case $b = 1$. The representations of $J_1 = G(F)$ contained in $R_1^\theta(w)_{\mathrm{aff}}$ are compact inductions from K (or its conjugate) to $G(F)$ of all irreducible representations of $G(k)$. Our last goal is to study them (Theorem 4.3) by determining the homomorphisms into the smooth irreducible representations of $G(F)$, which are classified in [1]. In particular, if there are non-zero homomorphisms from $\text{c-Ind}\,_K^{G(F)}\, \rho_K$ into π (with π, ρ irreducible) then π is of level zero (for a definition see [1] 12.6) and π and ρ are (with two exceptions) of the same "type", that means, either both characters, or both twisted Steinberg representations or both principal series or both cuspidal. For a precise statement see Theorem 4.3. A similar picture is obtained for $b = b_1$ (Theorem 4.10).

The representations $\text{c-Ind}\,_K^{G(F)}\, \rho_K$ are very big, in particular they are not admissible (cf. Remark 4). Dividing out the center (i.e. changing to PGL_2) as presented in §3.4 and Remarks 3(i), 5, one obtains admissible representations. The representations obtained in this way for $b = 1$ resp. $b = b_1$ are more convenient: in particular we get all cuspidal representations of $G(F)$ resp. D^* of level 0, whose central character is trivial on $t^{(1,1)}$.

2. Affine Deligne-Lusztig varieties for GL_2

From now on we permanently assume $G = GL_2$. Without loss of generality we choose T to be the diagonal torus and B the Borel subgroup of upper triangular matrices.

2.1. **Further preliminaries.**

2.1.1. *Affine flag manifold.* We regard X, $\dot{\mathrm{X}}$ as Ind-schemes in the usual way. The surjection $v_L \circ \det\colon G(L) \to \mathbb{Z}$ induces surjections $X \twoheadrightarrow \mathbb{Z}$, $\dot{\mathrm{X}} \twoheadrightarrow \mathbb{Z}$ and $\tilde{W} \twoheadrightarrow \mathbb{Z}$. We denote all these maps by η_G. The fibers of $\eta_G\colon X \twoheadrightarrow \mathbb{Z}$ resp. $\eta_G\colon \dot{\mathrm{X}} \twoheadrightarrow \mathbb{Z}$ are the connected components of X resp. $\dot{\mathrm{X}}$. Recall that W_a is the affine Weyl group of $T \cap G_{\mathrm{der}}$ in $G_{\mathrm{der}} = SL_2$. There is a natural exact sequence

$$1 \to W_a \to \tilde{W} \xrightarrow{\eta_G} \mathbb{Z} \to 1.$$

The group W_a is a Coxeter group and we denote by $\ell(w)$ the length function on it. We fix a splitting of the above exact sequence by sending 1 to b_1, where

$$b_1 := \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}.$$

Note that b_1^{2m} is central in $G(L)$ for $m \in \mathbb{Z}$.

2.1.2. *Bruhat-Tits building.* We write $\mathfrak{B}_\infty$ for the Bruhat-Tits building for $G_{\mathrm{der}}(L)$ and $\mathfrak{B}_1$ for the Bruhat-Tits building for $G_{\mathrm{der}}(F)$, where $G_{\mathrm{der}} = SL_2$. Then $\mathfrak{B}_1$ is in a natural way a full subcomplex of $\mathfrak{B}_\infty$. The choices of $T_{\mathrm{der}} := T \cap G_{\mathrm{der}}$ and $I_{\mathrm{der}} := I \cap G_{\mathrm{der}}(L)$ determine an apartment A_M and a base alcove C^0 lying in A_M. Further, W_a acts simply transitively on the set of alcoves in A_M. We fix an identification of W_a with this set by letting W_a act on C^0.

Furthermore, $\mathfrak{B}_\infty$ admits a description in terms of $k[[t]]$-lattices. Its vertices correspond to homothety classes of $k[[t]]$-lattices in $\bar{k}((t))^2$. Two vertices are connected by an alcove (1-dimensional simplex) if and only if there are representing lattices $\mathfrak{L}, \mathfrak{L}'$ with $t\mathfrak{L} \subsetneq \mathfrak{L}' \subsetneq \mathfrak{L}$. We denote the alcoves in A_M by C^i ($i \in \mathbb{Z}$), where the i-th alcove C^i is represented by the lattice chain $\mathfrak{o} \oplus t^{i+1}\mathfrak{o} \subsetneq \mathfrak{o} \oplus t^i\mathfrak{o}$. This is compatible with the definition of C^0. Under the above identification of W_a with A_M, $\begin{pmatrix} t^{-i} & 0 \\ 0 & t^i \end{pmatrix}$ corresponds to C^{2i} and $\begin{pmatrix} 0 & t^{-i} \\ -t^i & 0 \end{pmatrix}$ corresponds to C^{2i-1} for $i \in \mathbb{Z}$. In particular, if $wC^0 = C^r$, then $\ell(w) = |r|$.

Via the above description $G(L)$ acts naturally on $\mathfrak{B}_\infty$, but this action is not type-preserving. We denote by $H \subseteq G(L)$ the stabilizer of $\eta_G^{-1}(0)$ in X, i.e. the subgroup of all matrices with valuation of determinant equal 0. Using the inclusion $G_{\mathrm{der}}(L)/I_{\mathrm{der}} \xrightarrow{\sim} H/I \hookrightarrow G(L)/I = X$, the $\bar{k}$-points of $\eta_G^{-1}(0) \subset X$ can canonically be identified with the alcoves in $\mathfrak{B}_\infty$ and with right cosets H/I. We will use this implicitly. Notice that H is the maximal subgroup of $G(L)$, such that the above inclusion $\{\text{alcoves in } \mathfrak{B}_\infty\} \hookrightarrow X(\bar{k})$ is H-equivariant.

For $m \in \{0,1\}$ denote by P_m the vertex of $\mathfrak{B}_\infty$ represented by the lattice $\mathfrak{o} \oplus t^m\mathfrak{o}$. Then P_0, P_1 are the two vertices of C^0.

2.1.3. *Vertex of departure.* $\mathfrak{B}_\infty$ and $\mathfrak{B}_1$ are trees, i.e. connected one-dimensional simplicial complexes without cycles. A gallery with fixed first and last alcoves and minimal length is unique. Define the *first vertex* of a minimal gallery of positive length to be the unique vertex of its first alcove which is not a vertex of the second alcove. The relative position of two different alcoves in $\mathfrak{B}_\infty$ is determined by the length of the minimal gallery connecting them and the type of its first vertex.

Let $\mathfrak{C}$ be a full connected subcomplex of $\mathfrak{B}_\infty$ and D be an alcove in $\mathfrak{B}_\infty$, which is not contained in $\mathfrak{C}$. Since $\mathfrak{B}_\infty$ is a tree and $\mathfrak{C}$ is connected, there is a unique gallery $\Gamma_{D,\mathfrak{C}}$ of minimal length in $\mathfrak{B}_\infty$, containing a vertex P_D in $\mathfrak{C}$, whose first alcove is D. This vertex P_D is unique.

Definition 1. *Let $\mathfrak{C}$, D, $\Gamma_{D,\mathfrak{C}}$ and P_D be as above. We call P_D the vertex of departure for D from $\mathfrak{C}$, and set*
$$d_{\mathfrak{C}}(D) := 1 + \ell(\Gamma_{D,\mathfrak{C}}),$$
where $\ell(\Gamma)$ denotes the length of a gallery Γ.

In the situation as above, if additionally $m \in \{0,1\}$ denote by $\mathfrak{C}^{(m)}$ the set of all vertices in $\mathfrak{C}$ with type m, and for $n > 0$ and P a vertex in $\mathfrak{C}$ set
$$D^n_{\mathfrak{C}}(P) := \left\{ D\colon \begin{array}{c} D \text{ is an alcove in } \mathfrak{B}_\infty \text{ having } P \text{ as} \\ \text{vertex of departure from } \mathfrak{C} \text{ and } d_{\mathfrak{C}}(D) = n \end{array} \right\}.$$

Assume for a moment, $\mathfrak{C} = \{C^0\}$ is the subcomplex consisting of the alcove C^0 and both of its vertices P_0, P_1. Then $D^n_{\{C^0\}}(P_m)$ is just the open Schubert cell inside the affine flag manifold X^{der} of G_{der}, corresponding to the element $w \in W_a$ with $wC^0 = C^{(-1)^{m+1}n}$. Thus it is a locally closed subset of $X^{\mathrm{der}} \cong \eta_G^{-1}(0)_{\mathrm{red}}$ and with the reduced induced sub-Ind-scheme structure it is a k-scheme isomorphic to $\mathbb{A}^n$. If in addition $\mathfrak{C}$ contains some other alcoves $D_1, ..., D_r$ having P_m as a vertex, then using the usual coordinates on an open Schubert cell in X^{der}, it is not hard to see that $D^n_{\mathfrak{C}}(P_m)$ is the open subset $\mathbb{A}^{n-1} \times (\mathbb{A}^1 - \{D_1, ..., D_r\})$ of $D^n_{\{C^0\}}(P_m) = \mathbb{A}^n$. If additionally $D_1, ...D_r$ lie in $\mathfrak{B}_1$, then $D^n_{\mathfrak{C}}(P_m)$ is defined over k.

2.2. **Some reductions.**

Here we reduce the general setup to few computations in $\mathfrak{B}_\infty$.

Lemma 2.1. *Let $b \in G(L)$ and $w \in \tilde{W}$.*

(i) *For every $g \in G(L)$, the map $(h, x) \mapsto (g^{-1}hg, g^{-1}x)$ gives an isomorphism of pairs between $(J_b, X_w(b))$ and $(J_{g^{-1}b\sigma(g)}, X_w(g^{-1}b\sigma(g)))$.*

(ii) *If $X_w(b) \neq \emptyset$, then $\eta_G(b) = \eta_G(w)$.*

(iii) *Let $c \in G(L)$ be central and set $v = \eta_G(c)$. Then $X_w(b) = X_{wb_1^v}(cb)$.*

Proof. (i) and (ii) are straightforward. (iii) follows from the fact that $b_1^v I = I b_1^v = Ic$. $\square$

Let $m \in \{0,1\}$. We set

$$\begin{array}{rcl} H_b & := & H \cap J_b = \ker(v_L \circ \det\colon J_b \to \mathbb{Z}), \\ K_b^{(m)} & := & \text{Stabilizer in } H_b \text{ of } P_m \text{ in } \mathfrak{B}_\infty. \end{array}$$

Let $p_m\colon K_b^{(m)} \to G(\bar{k})$ be the conjugation by b_1^m composed with reduction modulo $\mathfrak{p}$.

The σ-conjugacy classes in $G(L)$ are given by

$$\{t^{(\alpha,\beta)}\colon \alpha \leq \beta\} \cup \{t^{(\alpha,\alpha)} b_1\colon \alpha \text{ even}\},$$

which follows from [9] 1.10. From parts (i) and (iii) of Lemma 2.1, we have, up to isomorphism, only three different cases, which we will study explicitly:

(i) $b = 1$. In this case $J_1 = G(F)$ and $K_1^{(m)} = b_1^m K b_1^{-m}$ for $m \in \{0,1\}$ (recall that $K = G(\mathfrak{o}_F)$). Then $p_m(K_1^{(m)}) = G(k)$.

(ii) $b = t^{(0,\alpha)}$ with $\alpha > 0$. In this case $J_b = T(F)$ and $K_b^{(m)} = T(\mathfrak{o}_F)$ for $m \in \{0,1\}$. Then $p_m(K_b^{(m)}) = T(k)$.

(iii) $b = b_1$. Then J_{b_1} is the multiplicative group D^* of the central division algebra D over F of dimension 4, and $K_b^{(m)} = U_D$ is the unit subgroup of its valuation subring. The image of p_m is contained in $B(\bar{k})$ and the image of $p_m(U_D)$ under projection $\mathrm{pr}\colon B(\bar{k}) \twoheadrightarrow T(\bar{k})$ is $\{\mathrm{diag}(a, a^q)\colon a \in k'^*\} \cong k'^*$. We write $p'_m = \mathrm{pr} \circ p_m$.

Lemma 2.2. *Let $b \in G(L)$ and $w \in \tilde{W}$.*

(i) *The restriction of $\eta_G\colon G(L) \to \mathbb{Z}$ to J_b is surjective.*

(ii) *We have:*

$$X_w(b) \cong \coprod_{J_b/H_b} X_w(b) \cap \eta_G^{-1}(0),$$

where J_b acts on the set of these components by left multiplication.

Proof. (ii) follows from (i) using the action of J_b on $X_w(b)$. (i) has a general proof, compare for example the second page of [3]. In our case however, it can also be seen explicitly, using the knowledge of the σ-conjugacy classes in $G(L)$. $\square$

Lemma 2.2 holds also without the assumption $G = GL_2$ (in general, one has the map $\eta_G\colon G(L) \to \pi_1(G)$ defined by Kottwitz in [7], and H_b will be the kernel of the restriction $\eta_G\colon J_b \to \pi_1(G)$). Set

$$X_w^0(b) := X_w(b) \cap \eta_G^{-1}(0) \subseteq \eta_G^{-1}(0).$$

We see its points as alcoves in $\mathfrak{B}_\infty$.

Lemma 2.3. *If $xI \in H/I$ corresponds to the alcove D in $\mathfrak{B}_\infty$, and $w = w_a b_1^v$, with $w_a \in W_a$ and $v = \eta_G(b)$, then*

$$xI \in X_w^0(b) \Leftrightarrow \mathrm{inv}(D, b \cdot \sigma D) = w_a,$$

where $\mathrm{inv}(\cdot,\cdot)$ denotes the relative position map on the alcoves in $\mathfrak{B}_\infty$ and $g \cdot D$ denotes the action of $G(L)$ on the alcoves in $\mathfrak{B}_\infty$.

Proof. Since $b_1 \cdot C^0 = C^0$, we have $b \cdot \sigma D = b \cdot \sigma(x) C^0 = (b\sigma(x) b_1^{-v}) C^0$ with $b\sigma(x) b_1^{-v} \in H$. Thus:

$$\begin{array}{rcl} xI \in X_w^0(b) & \Leftrightarrow & x^{-1} b\sigma(x) \in IwI \Leftrightarrow x^{-1} b\sigma(x) b_1^{-v} \in I w_a I \\ & \Leftrightarrow & \mathrm{inv}(D, b \cdot \sigma D) = \mathrm{inv}(xC^0, b\sigma(x) b_1^{-v} C^0) = w_a. \quad \square \end{array}$$

The following lemma is shown by Reuman in [10]. We include it only for completeness.

Lemma 2.4. *(Non-emptiness of $X_w(b)$) Let $b = t^{(0,\alpha)}$ with $\alpha \geq 0$ or $b = b_1$ and $w \in \tilde{W}$. Put $w_a = w b_1^{-\eta_G(b)}$. Then $X_w(b) \neq \emptyset$ if and only if $w_a \in W_a$ and*

(a) $b = t^{(0,\alpha)}$ *and $w_a \in W_a$ with $\ell(w_a) - \alpha = 0$ or odd and positive, or*

(b) $b = b_1$ *and $\ell(w_a)$ even.*

3. Torsors over affine Deligne-Lusztig varieties and structure results

3.1. Bruhat decomposition for $\dot{\mathrm{X}}$.

(*Everything said in this and in the beginning of the next subsection holds in the more general setting of the introduction, i.e. for any connected reductive split k-group G.*) Since $T_1 = T(\mathfrak{o}) \cap I_1$ is normal in $N_G(T)(L)$, we can consider the group $\tilde{W}_1$ defined by the exact sequence:

$$1 \to T_1 \to N_G(T)(L) \to \tilde{W}_1 \to 1$$

Given two elements $x, y \in N_G(T)(L)$ lying over the same element $\dot{\mathrm{w}}$ of $\tilde{W}_1$, the ratio $x^{-1}y$ lies in $T_1 \subset I_1$. Therefore $I_1 x I_1 = I_1 y I_1$. We denote this double coset by $I_1 \,\dot{\mathrm{w}}\, I_1$.

Lemma 3.1. *We have the disjoint decomposition:*

$$G(L) = \bigcup_{\dot{\mathrm{w}} \in \tilde{W}_1} I_1 \,\dot{\mathrm{w}}\, I_1.$$

Proof. Let $\dot{\mathrm{v}}$ run through a fixed system of representatives of $\tilde{W}$ in $N_G(T)(L)$ and a through $T(\bar{k})$. Then $a\,\dot{\mathrm{v}}$ runs through a system of representatives of $\tilde{W}_1$. Since $\dot{\mathrm{v}}$ normalizes $T(\mathfrak{o})$ and since $I = I_1 \cdot T(\mathfrak{o}) = I_1 \cdot T(\bar{k})$, we have: $I \,\dot{\mathrm{v}}\, I = \bigcup_{a,b \in T(\mathfrak{o})} I_1 a \,\dot{\mathrm{v}}\, b I_1 = \bigcup_{a \in T(\mathfrak{o})} I_1 a \,\dot{\mathrm{v}}\, I_1 = \bigcup_{a \in T(\bar{k})} I_1 a \,\dot{\mathrm{v}}\, I_1$. From this and the usual Bruhat decomposition, it follows that $G(L) = \bigcup_{\dot{\mathrm{w}} \in \tilde{W}_1} I_1 \,\dot{\mathrm{w}}\, I_1$. It remains to show disjointness of two double cosets. First, $I_1 a_1 \,\dot{\mathrm{v}}_1\, I_1 = I_1 a_2 \,\dot{\mathrm{v}}_2\, I_1$ implies $I \,\dot{\mathrm{v}}_1\, I = I \,\dot{\mathrm{v}}_2\, I$, hence $\dot{\mathrm{v}}_1 = \dot{\mathrm{v}}_2$. Thus (by replacing $\dot{\mathrm{v}}_1$ by $a_1 \,\dot{\mathrm{v}}_1$) it is enough to show that if $a \in T(\mathfrak{o})$ with $I_1 \,\dot{\mathrm{v}}\, I_1 = I_1 a \,\dot{\mathrm{v}}\, I_1$, then $a \in T_1$. But then there is an $i_1 \in I_1$, such that $i_1 a \in \dot{\mathrm{v}}\, I_1 \,\dot{\mathrm{v}}^{-1}$. But clearly $i_1 a \in I$, hence $i_1 a \in I \cap \dot{\mathrm{v}}\, I_1 \,\dot{\mathrm{v}}^{-1}$. Since $\dot{\mathrm{v}}^{-1} I_1 \,\dot{\mathrm{v}} \cong I_1$ has an (infinite) filtration with subquotients isomorphic to $\bar{k}$, this intersection lies in I_1, the maximal subgroup of I satisfying this property. Hence $a \in I_1 \cap T(\mathfrak{o}) = T_1$. □

3.2. The varieties $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$.

3.2.1. *Definition.* The Frobenius σ acts on $\dot{\mathrm{X}}$. Therefore we can define:

Definition 2. *Let $b \in G(L)$ and $\dot{\mathrm{w}} \in \tilde{W}_1$. We define $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ to be the locally closed subset of $\dot{\mathrm{X}}$ given by*

$$\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b) = \{xI_1 \in G(L)/I_1 \colon x^{-1} b \sigma(x) \in I_1 \,\dot{\mathrm{w}}\, I_1\}$$

provided with the reduced induced sub-Ind-scheme structure.

Moreover, $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ is a $\bar{k}$-variety locally of finite type. The group J_b defined in the introduction acts on $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ by left multiplication. If w is the image of $\dot{\mathrm{w}}$ in $\tilde{W}$, the forgetful morphism $\dot{\mathrm{X}} \to X$ restricts to a J_b-equivariant morphism

$$\pi \colon \dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b) \to X_w(b).$$

(we write $X_w(b)$ also for the base change of $X_w(b)$ to $\bar{k}$).

Lemma 3.2. *Let $b \in G(L)$ and $\dot{\mathrm{w}} \in \tilde{W}_1$.*

- (i) *For every $g \in G(L)$, the map $(h, x) \mapsto (g^{-1}hg, g^{-1}x)$ gives an isomorphism of pairs $(J_b, \dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)) \xrightarrow{\sim} (J_{g^{-1}b\sigma(g)}, \dot{\mathrm{X}}_{\dot{\mathrm{w}}}(g^{-1}b\sigma(g)))$.*
- (ii) *If w is the image of $\dot{\mathrm{w}}$ in $\tilde{W}$, then $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b) \neq \emptyset \Leftrightarrow X_w(b) \neq \emptyset$.*
- (iii) *Let $\dot{\mathrm{w}}_1, \dot{\mathrm{w}}_2 \in \tilde{W}_1$ with the same image in $\tilde{W}$ and let τ be an element of $T(\mathfrak{o})$, such that $\dot{\mathrm{w}}_2 = \dot{\mathrm{w}}_1 \tau$ in $\tilde{W}_1$. There is a $\tau_1 \in T(\mathfrak{o})$, such that the right multiplication by τ_1 induces a J_b-equivariant isomorphism*

$$\cdot\tau_1 \colon \dot{\mathrm{X}}_{\dot{\mathrm{w}}_1}(b) \xrightarrow{\sim} \dot{\mathrm{X}}_{\dot{\mathrm{w}}_2}(b), \quad \dot{\mathrm{x}}\, I_1 \mapsto \dot{\mathrm{x}}\, \tau_1 I_1.$$

Proof. (i) is straightforward. The one direction of (ii) is obvious, the other follows from (iii), since the preimage of $X_w(b)$ in $\dot{\mathrm{X}}$ is equal to $\bigcup_{\dot{\mathrm{w}} \mapsto w} \dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$. To prove (iii) let w denote the image of $\dot{\mathrm{w}}_1$ in $\tilde{W}$. The equation $\tau_1^{-1}\,\mathrm{ad}(w)(\sigma(\tau_1)) = \mathrm{ad}(w)(\tau)$ in $T(\mathfrak{o})$ in the variable τ_1 has a solution in $T(\mathfrak{o})$ for every τ, as follows from Hilbert's Satz 90. For such a τ_1 we have $\dot{\mathrm{w}}_1\,\tau = \tau_1^{-1}\,\dot{\mathrm{w}}_1\,\sigma(\tau_1)$ in $\tilde{W}_1$. Hence if $xI_1 \in \dot{\mathrm{X}}_{\dot{\mathrm{w}}_1}(b)$, then

$$\begin{aligned} I_1(x\tau_1)^{-1}b\sigma(x\tau_1)I_1 &= I_1\tau_1^{-1}x^{-1}b\sigma(x)\sigma(\tau_1)I_1 = \tau_1^{-1}I_1x^{-1}b\sigma(x)I_1\sigma(\tau_1) = \tau_1^{-1}I_1\,\dot{\mathrm{w}}_1\,I_1\sigma(\tau_1) = \\ &= I_1\tau_1^{-1}\,\dot{\mathrm{w}}_1\,\sigma(\tau_1)I_1 = I_1\,\dot{\mathrm{w}}_1\,\tau I_1 = I_1\,\dot{\mathrm{w}}_2\,I_1. \end{aligned}$$

Hence if $xI_1 \in \dot{\mathrm{X}}_{\dot{\mathrm{w}}_1}(b)$, then $x\tau_1 I_1 \in \dot{\mathrm{X}}_{\dot{\mathrm{w}}_2}(b)$. Thus τ_1 gives a morphism from $\dot{\mathrm{X}}_{\dot{\mathrm{w}}_1}(b)$ to $\dot{\mathrm{X}}_{\dot{\mathrm{w}}_2}(b)$. Further, τ_1^{-1} defines a morphism in the opposite direction and the two are inverses of each other. The J_b-equivariance is obvious. □

3.2.2. *Torus action.* For $\dot{\mathrm{w}} \in \tilde{W}_1$, let w denote its image in $\tilde{W}$. The right action of $T(\mathfrak{o})$ on $\dot{\mathrm{X}}$ restricts to an action of the group

$$T(w)^{\sigma}_{\mathrm{aff}} = \{\tau_1 \in T(\mathfrak{o})\colon \tau_1^{-1}\,\mathrm{ad}(w)(\sigma(\tau_1)) \in T_1\}$$

on $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$. In fact, as in the proof of Lemma 3.2(iii), for $\tau_1 \in T(\mathfrak{o})$ to send $xI_1 \in \dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ to an element $x\tau_1 I_1 \in \dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$, it is necessary and sufficient that $\dot{\mathrm{w}}\,\tau = \dot{\mathrm{w}}$ in $\tilde{W}_1$, where $\mathrm{ad}(w)(\tau) = \tau_1^{-1}\,\mathrm{ad}(w)(\sigma(\tau_1))$. This is equivalent to $\tau \in T_1$ and hence to $\tau_1^{-1}\,\mathrm{ad}(w)(\sigma(\tau_1)) = \mathrm{ad}(w)(\tau) \in T_1$. Obviously, $T(w)^{\sigma}_{\mathrm{aff}} \supset T_1$ and the action factorizes through $T(w)^{\sigma}_{\mathrm{aff}}/T_1$, which is finite. Let w_{fin} denote the image of w in W. The adjoint action of $\tilde{W}$ on $T(\mathfrak{o})$ factorizes through W, hence we have only two possibilities:

$$T(w)^{\sigma}_{\mathrm{aff}}/T_1 = \begin{cases} T(k) & \text{if } w_{\mathrm{fin}} = 1, \\ \{\mathrm{diag}(a, a^q) \in T(\bar{k})\colon a \in k'^{*}\} & \text{if } w_{\mathrm{fin}} \neq 1. \end{cases}$$

In the last case, we use the identification $T(w)^{\sigma}_{\mathrm{aff}}/T_1 \cong k'^{*}$ by sending $\mathrm{diag}(a, a^q)$ to a.

Remark 1. *Since $T(\mathfrak{o})$ is abelian, the isomorphism of Lemma 3.2(iii) is also $T(w)^{\sigma}_{\mathrm{aff}}$-equivariant.*

3.3. **Structure of $X_w(b)$ and $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$.** We explain now the precise structure of the varieties $X_w(b)$ and $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$. First we introduce some notations. For $b = t^{(0,\alpha)}$ with $\alpha \geq 0$ or $b = b_1$ and $w \in \tilde{W}$, such that $w_a := wb_1^{-\eta_G(b)} \in W_a$, let $r \in \mathbb{Z}$ be such that w_a correspond to C^r (in particular, $\ell(w_a) = |r|$). Put:

$$d_{b,w} := \begin{cases} \frac{\ell(w_a)-\alpha-1}{2} & \text{if } b = t^{(0,\alpha)}, X_w(b) \neq \emptyset \text{ and } \ell(w_a) > \alpha \\ \frac{\ell(w_a)}{2} & \text{if } b = b_1 \text{ and } X_w(b) \neq \emptyset \\ 0 & \text{in all other cases,} \end{cases} \tag{3.1}$$

$$m := \begin{cases} 1 & \text{if } r - \mathrm{sign}(r)\eta_G(b) \equiv 1 \bmod 4 \\ 0 & \text{otherwise,} \end{cases}$$

$$S(b,w) := \begin{cases} \mathbb{P}^1(k) & \text{if } b = 1, w_a = 1 \\ \mathbb{P}^1 - \mathbb{P}^1(k) & \text{if } b = 1, \ell(w_a) > 0 \\ \{pt\} & \text{if } b = t^{(0,\alpha)} \text{ with } \alpha > 0, \ell(w_a) = \alpha \\ \mathbb{G}_m = \mathbb{P}^1 - \{0,\infty\} & \text{if } b = t^{(0,\alpha)} \text{ with } \alpha > 0, \ell(w_a) > \alpha \\ \{pt\} & \text{if } b = b_1. \end{cases}$$

Endow $S(b,w)$ with left $K_b^{(m)}$-action as follows: first consider the $p_m(K_b^{(m)})$-action on $S(b,w)$, which is the restriction to $p_m(K_b^{(m)})$ of the $G(\bar{k})$-action by linear transformations on $\mathbb{P}^1$ in the first, second and fourth cases above, and the trivial action in the third and the last case and then lift this to a $K_b^{(m)}$-action. This is enough to describe $X_w(b)$.

Further, let w_{fin} denote the image of w in W, and $\dot{\mathrm{w}}_{\mathrm{fin}}$ be a lift to $G(\bar{k})$. Recall that $\dot{\mathrm{X}}_{\dot{\mathrm{w}}_{\mathrm{fin}}}$ denotes the étale torsor over $X_{w_{\mathrm{fin}}}$ defined in [2]. We consider the following $p_m(K_b^{(m)})$-equivariant finite étale torsor over $S(b,w)$ with Galois group $T(w)^{\sigma}_{\mathrm{aff}}/T_1$:

$$\pi_{b,w}\colon \dot{\mathrm{S}}(b,w)\to S(b,w):=\begin{cases}\dot{\mathrm{X}}_{\dot{\mathrm{w}}_{\mathrm{fin}}}\to X_{w_{\mathrm{fin}}} & \text{if } b=1\\ T(k)\to\{pt\} & \text{if } b=t^{(0,\alpha)} \text{ with } \alpha>0, \ell(w_a)=\alpha\\ \coprod_{\zeta\in k^*}\mathbb{G}_m\to\mathbb{G}_m & \text{if } b=t^{(0,\alpha)} \text{ with } \alpha>0, \ell(w_a)>\alpha\\ \mu_{q^2-1}\to\{pt\} & \text{if } b=b_1,\end{cases}$$

where $\pi_{b,w}$ is the usual projection in the first case, the constant map in the second and in the last case, and in the third case it sends x_ζ to $\zeta^{-1}x^{-(q+1)}$ (x_ζ means the point x in the component ζ). We explain now the $p_m(K_b^{(m)})$- and $T(w)^\sigma_{\mathrm{aff}}/T_1$-action on $\dot{\mathrm{S}}(b,w)$. If

$$(b=t^{(0,\alpha)} \text{ and } w=b \text{ or } \frac{\ell(w)-\alpha+1}{2}\in\mathbb{Z} \text{ is even}) \text{ or } (b=b_1 \text{ and } \ell(w)\equiv 0 \bmod 4), \tag{3.2}$$

then the action is as follows: in the first case it is just the action of $p_m(K_1^{(m)})=G(k)$ and of $T(w)^\sigma_{\mathrm{aff}}/T_1=T(w_{\mathrm{fin}})^\sigma$ as in the finite Deligne-Lusztig theory. In the second resp. in the last case it is the multiplication action of $T(w)^\sigma_{\mathrm{aff}}/T_1=T(k)=p_m(K_b^{(m)})$ resp. of $T(w)^\sigma_{\mathrm{aff}}/T_1=k'^*=p'_m(K_{b_1}^{(m)})$ (this last has to be inflated to a $p_m(K_{b_1}^{(m)})$-action). In the third case $\mathrm{diag}(g,h).x_\zeta=(g^{-1}x)_{gh\zeta}$ for $\mathrm{diag}(g,h)\in p_m(K_b^{(m)})=T(k)$ and $x_\zeta.\mathrm{diag}(\tau,\tau^q)=(\tau^{-1}x)_{\tau^{q+1}\zeta}$ for $\tau\in k'^*\cong T(w)^\sigma_{\mathrm{aff}}/T_1$, in particular both actions induce via the determinant the same action of k^* on $\pi_0(\dot{\mathrm{S}}(b,w))$. If condition (3.2) is not satisfied, then the action of $p_m(K_b^{(m)})$ is the same as above, but the action of $T(w)^\sigma_{\mathrm{aff}}/T_1$ is twisted by the adjoint action of $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ on $T(w)^\sigma_{\mathrm{aff}}/T_1$. Finally, we lift the $p_m(K_b^{(m)})$-action to an $K_b^{(m)}$-action.

Reduction modulo $\mathfrak{p}$ gives natural embeddings $p_m(K_b^{(m)})\hookrightarrow G(\bar{k})\hookleftarrow T(w)^\sigma_{\mathrm{aff}}/T_1$. The "right" notion of an isomorphism between triples $(\dot{\mathrm{S}}_i,\alpha_i,\beta_i)$ ($i\in\{1,2\}$) where $p_m(K_b^{(m)})\xrightarrow{\alpha_i}\mathrm{Aut}(\dot{\mathrm{S}}_i)\xleftarrow{\beta_i}T(w)^\sigma_{\mathrm{aff}}/T_1$ are the actions, should be an isomorphism $\phi\colon \dot{\mathrm{S}}_1\xrightarrow{\sim}\dot{\mathrm{S}}_2$ together with an automorphism of $G(\bar{k})$, inducing automorphisms of $p_m(K_b^{(m)})$ and $T(w)^\sigma_{\mathrm{aff}}/T_1$, compatible with ϕ, α_i and β_i. This reflects the fact that both actions have the same origin, being induced from the action of $G(L)$ on X.

Theorem 3.3. *Let $b=t^{(0,\alpha)}$ with $\alpha\geq 0$ or $b=b_1$. Let $w\in\tilde{W}$, such that $X_w(b)\neq\emptyset$. Then $w_a:=wb_1^{-\eta_G(b)}\in W_a$.*

(i) *(Structure of $X_w(b)$) Let the alcove C^r of A_M correspond to w_a. Then there are J_b-equivariant k-isomorphisms:*

$$X_w(b)\cong\coprod_{J_b/K_b^{(m)}}\mathbb{A}^{d_{b,w}}\times S(b,w),$$

(ii) *(Structure of $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$) Let now $\dot{\mathrm{w}}\in\tilde{W}_1$ be a preimage of w in $\dot{\mathrm{W}}$. Then there is a J_b- and $T(w)^\sigma_{\mathrm{aff}}/T_1$-equivariant isomorphism*

$$\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)\cong\coprod_{J_b/K_b^{(m)}}\mathbb{A}_{\bar{k}}^{d_{b,w}}\times\dot{\mathrm{S}}(b,w),$$

The J_b-equivariant morphism $\pi\colon\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)\to X_w(b)$ is finite étale with Galois group $T(w)^\sigma_{\mathrm{aff}}/T_1$, equal to the disjoint sum of the morphisms

$$\mathrm{id}_{\mathbb{A}_{\bar{k}}^{d_{b,w}}}\times\pi_{b,w}\colon\mathbb{A}_{\bar{k}}^{d_{b,w}}\times\dot{\mathrm{S}}(b,w)\to\mathbb{A}_{\bar{k}}^{d_{b,w}}\times S(b,w).$$

Remark 2. *An easy computation shows that the non-emptiness and the dimension results fit into the conjectures 9.4.1 of* [4].

Proof. (i): Lemma 2.1(ii) implies $X_w(b) \neq \emptyset \Rightarrow w_a \in W_a$. By Lemma 2.2 we are reduced to compute $X_w^0(b) := X_w(b) \cap \eta_G^{-1}(0)$, which can, at least set-theoretically, by Lemma 2.3 be identified with the set of alcoves D in $\mathfrak{B}_\infty$, such that the relative position of D and $b \cdot \sigma D$ is w_a.

Case $b = 1$. Clearly, $\mathfrak{B}_1$ is exactly the subcomplex of $\mathfrak{B}_\infty$ stabilized by σ. Therefore, $X_1^0(1)$ is the disjoint union of points, indexed by alcoves in $\mathfrak{B}_1$. Since H_1 acts on alcoves in $\mathfrak{B}_1$ with stabilizer of C^0 equal $I \cap H_1$, we have:

$$X_1^0(1) = \coprod_{\text{alcoves in } \mathfrak{B}_1} \{pt\} = \coprod_{H_1/I\cap H_1} \{pt\} = \coprod_{H_1/K} \coprod_{K/I\cap H_1} \{pt\}.$$

But $\coprod_{K/I\cap H_1}\{pt\}$ is together with its K-action isomorphic to $\mathbb{P}^1(k)$ with the K-action induced from the action of $G(k)$ by linear transformations. Now assume $1 \neq w \in W_a$. In the situation of 2.1.3 consider $\mathfrak{C} = \mathfrak{B}_1$. Let D be an alcove in $\mathfrak{B}_\infty$, not contained in $\mathfrak{B}_1$. There is a unique gallery $\Gamma_{\mathfrak{B}_1,D}$ of minimal length, equal $d_{\mathfrak{B}_1}(D) - 1$, having D as first alcove and containing a vertex in $\mathfrak{B}_1$. Its image under σ is again a gallery, containing the same vertex in $\mathfrak{B}_1$ and it is easy to see that the composed gallery $\Gamma_{D,\sigma D} := (\Gamma_{\mathfrak{B}_1,D}, \sigma\Gamma_{\mathfrak{B}_1,D}^{-1})$ is still minimal and connects D with σD. The length of this gallery is $2d_{\mathfrak{B}_1}(D) - 1$, and hence the relative position of D and σD is $C^{(-1)^m(2d_{\mathfrak{B}_1}(D)-1)}$, where $m \in \{0,1\}$ is the type of the first vertex of $\Gamma_{D,\sigma D}$ (note that $Gamma_{D,\sigma D}$ is minimal and $\ell(\Gamma_{D,\sigma D}) > 0$, hence its first vertex is well-defined). In particular, $\ell(\Gamma_{D,\sigma D})$ is odd, which implies that if $D \in X_w^0(1)$, then $\ell(w)$ is odd. In particular, if $0 \neq \ell(w)$ is even, then $X_w(b) = \emptyset$ and if $\ell(w)$ is odd, then from the above description follows:

$$X_w^0(1) = \coprod_{P \in \mathfrak{B}_1^{(m)}} D_{\mathfrak{B}_1}^{\frac{\ell(w)+1}{2}}(P),$$

with m as in (3.1). Now $H_1 = G(F) \cap H$ acts naturally on $\mathfrak{B}_1^{(m)}$ and the stabilizer of P_m is $K_1^{(m)}$, hence canonically $H_1/K_1^{(m)} = \mathfrak{B}_1^{(m)}$. By definition, $X_w(1)$ is given the reduced induced sub-Ind-scheme structure and as in 2.1.3, $D_{\mathfrak{B}_1}^{\frac{\ell(w)+1}{2}}(P) \cong \mathbb{A}^{\frac{\ell(w)-1}{2}} \times (\mathbb{P}^1 - \mathbb{P}^1(k))$, where $(\mathbb{P}^1 - \mathbb{P}^1(k))$ represents the alcoves having P_m as a vertex and not lying in $\mathfrak{B}_1$. Since $K_1^{(m)}$ is the stabilizer of P_m in H_1, it acts on this $\mathbb{P}^1 - \mathbb{P}^1(k)$, and this action is the inflation via $p_m \colon K_1^{(m)} \twoheadrightarrow G(k)$ of the action of $G(k)$ on $\mathbb{P}^1 - \mathbb{P}^1(k)$ by linear transformations.

Case $b = t^{(0,\alpha)}$ **with** $\alpha > 0$. The element $t^{(0,\alpha)}$ acts on A_M by shifting everything α alcoves to the right. In particular, if D is an alcove in A_M, then the relative position of D and $b\sigma D$ is C^α or $C^{-\alpha}$, which corresponds to an element $w_a \in W_a$ of length α.

Consider now the situation as in 2.1.3 with $\mathfrak{C} = A_M$ and any alcove D, not contained in A_M. Let P_D be the vertex of departure for D from A_M and denote by Γ_{α,P_D} the minimal gallery connecting P_D with its translate by b (i.e. if $\alpha = 1$, this is the gallery with one alcove having P_D and bP_D as vertices, and if $\alpha > 1$, this is the unique gallery of minimal length, having P_D as first vertex and bP_D as last vertex). Then it is easy to see that the composed gallery $\Gamma_{D,b\sigma D} := (\Gamma_{D,A_M}, \Gamma_{\alpha,P_D}, b\sigma\Gamma_{D,A_M}^{-1})$ is defined, and is the minimal gallery connecting D with $b\sigma D$. We have $\ell(\Gamma_{D,b\sigma D}) = 2d_{A_M}(D) + \alpha - 1$, and in particular $\ell(\Gamma_{D,b\sigma D}) - \alpha$ is odd.

From all these we have for w_a with $w_aC^0 = C^\alpha$:

$$X_w^0(b) = \coprod_{r\in\mathbb{Z}} \{C^{2r}\} = \coprod_{T(F)/T(\mathfrak{o}_F)} \{pt\}$$

and analogously for $w_aC^0 = C^{-\alpha}$. The rest of the proof, concerning the case $\ell(w_a) > \alpha$ works the same way as in the case $b = 1$, once one remarks that $A_M^{(m)}$ is naturally acted on by $H_b = T(F)$ with $K_b^{(m)} = T(\mathfrak{o}_F)$ being the stabilizer of P_m (and also of any other point of $A_M^{(m)}$).

Case $b = b_1$. Take $\mathfrak{C} = \{C^0\}$ (the full connected subcomplex with one alcove). b_1 acts on it by interchanging the both vertices P_0 and P_1. The stabilizer of P_m is $H_{b_1} = K_{b_1}^{(m)} = U_D$. The arguments are similar to the both cases above and one obtains the whole open Schubert cell

$D^{d_{b,w}}_{\{C^0\}}(P_m)$ as $X^0_w(b_1)$, if $\ell(w_a)$ is even and the empty set otherwise. This completes the proof of (i).

(ii): Using the J_b-action on $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ and the J_b-equivariant morphism $\pi\colon \dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b) \to X_w(b)$ we obtain from part (i) of the theorem that

$$\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b) = \coprod_{J_b/K_b^{(m)}} \pi^{-1}(\mathbb{A}_{\bar{k}}^{d_{b,w}} \times S(b,w)).$$

Thus we have to compute $\pi^{-1}(\mathbb{A}_{\bar{k}}^{d_{b,w}} \times S(b,w)) \subseteq \dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$. We do this first in the case $b = 1$ and $\ell(w) > 0$ (since $b = 1$ and $X_w(b) \neq \emptyset$, we have $w_a = w$) where $\mathbb{A}_{\bar{k}}^{d_{b,w}} \times S(1,w) = D_{\mathfrak{B}_1}^{\frac{\ell(w)+1}{2}}(P_m)$. Write $wC^0 = C^r$ and assume $m = 0$ (the case $m = 1$ is similar). Then either $r = -2i+1$ with $i = 2n-1 > 0$ odd, or $r = 2i-1$ with $i = 2n > 0$ even. For $v \in W_a$ with $vC^0 = C^{-i}$ we have $D_{\mathfrak{B}_1}^{\frac{\ell(w)+1}{2}}(P_0) \subseteq IvI/I$. Consider first the case $i = 2n-1$ odd, i.e. $v = \begin{pmatrix} 0 & t^{n-1} \\ t^{1-n} & 0 \end{pmatrix}$. By Lemma 3.2(iii) we can assume $\dot{\mathrm{w}} = w = \begin{pmatrix} 0 & -t^{i-1} \\ t^{1-i} & 0 \end{pmatrix}$. Define affine coordinates on the preimage IvI/I_1 in $\dot{\mathrm{X}}$ of IvI/I by fixing the isomorphism : $\psi_{1,v}\colon \mathbb{A}_{\bar{k}}^{2n-1} \times \mathbb{G}_m^2 \xrightarrow{\sim} IvI/I_1$,

$$\psi_{1,v}((c_j)_{j=0}^{2n-2}, r, s) = \begin{pmatrix} r(t^{1-n}c_0 + \dots + t^{n-1}c_{2n-2}) & st^{n-1} \\ rt^{1-n} & 0 \end{pmatrix} I_1.$$

Write $c = \sum_{j=0}^{2n-2} c_j t^{j-(n-1)}$. We have $\pi\left(\begin{pmatrix} rc & st^{n-1} \\ rt^{1-n} & 0 \end{pmatrix} I_1\right) = \begin{pmatrix} c & t^{n-1} \\ t^{1-n} & 0 \end{pmatrix} I$. Then $xI_1 := \begin{pmatrix} rc & st^{n-1} \\ rt^{1-n} & 0 \end{pmatrix} I_1 \in \pi^{-1}(D_{\mathfrak{B}_1}^{\frac{\ell(w)+1}{2}}(P_0))$ in $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(1) \Leftrightarrow I_1 x^{-1}\sigma(x) I_1 = I_1 \,\dot{\mathrm{w}}\, I_1$, which by a computation is equivalent to

$$r^{-1}s^q(c_0^q - c_0)^{-1} = 1 \quad \text{ and } \quad s^{-1}r^q(c_0^q - c_0) = 1.$$

which, after eliminating $s = r^q(c_0^q - c_0)$, is equivalent to $r^{q^2-1}(c_0^q - c_0)^{q-1} = 1$. This last equation defines in the r-c_0-plane a curve together with $p_0(K_1^{(0)}) = G(k)$-action isomorphic to the finite Deligne-Lusztig variety $\dot{\mathrm{X}}_{\dot{\mathrm{w}}_{\mathrm{fin}}}$ for G over k with $G(k)$-action on it. The projection onto $S(b,w) = X_{w_{\mathrm{fin}}}$ is given by $(r, c_0) \mapsto c_0$. The same is true for the other case $i = 2n > 0$ even, with $v = t^{(n,-n)}$, but xI_1 is then represented by $\begin{pmatrix} st^n & rc \\ 0 & rt^{-n} \end{pmatrix}$. In the first case the right $T(w)^{\sigma}_{\mathrm{aff}}/T_1 = T(w_{\mathrm{fin}})^{\sigma}$-action on our curve is given by $(r, c_0).\mathrm{diag}(\tau, \tau^q) = (\tau r, c_0)$ and in the second case by $(r, c_0).\mathrm{diag}(\tau, \tau^q) = (\tau^q r, c_0)$ and the $p_m(K_1^{(0)})$-actions are equal in both cases. This explains the twist of $T(w)^{\sigma}_{\mathrm{aff}}/T_1$-action by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ used in the definition of $\dot{\mathrm{S}}(b,w)$.

All other cases can be computed similar to the case above. Let us for example work out the case $b = t^{(0,\alpha)}$ with $\alpha > 0$ and $\ell(w_a) > \alpha$. In this case $\mathbb{A}_{\bar{k}}^{d_{b,w}} \times S(b,w) = D_{A_M}^{\frac{\ell(w_a)-\alpha+1}{2}}(P_m)$. Write $w_a C^0 = C^r$ and assume $m = 0$ (the case $m = 1$ is similar). Then either $r = -\alpha - 2i + 1$ with $i = 2n-1 > 0$ odd, or $r = \alpha + 2i - 1$ with $i = 2n > 0$ even. Consider only the first of this cases, the second being similar. We can assume $\dot{\mathrm{w}} = \begin{pmatrix} 0 & -t^{\alpha+i-1} \\ t^{1-i} & 0 \end{pmatrix}$. Let $vC^0 = C^{-i}$, then $D_{A_M}^{\frac{\ell(w_a)-\alpha+1}{2}}(P_0)$ is the open subset in IvI/I on which $c_0 \neq 0$. Then $xI_1 := \psi_{v,1}(c, r, s) \in \pi^{-1}(D_{A_M}^{\frac{\ell(w_a)-\alpha+1}{2}}(P_0))$ and $b\sigma(x)I_1$ are in relative position $\dot{\mathrm{w}}$ if and only if $I_1 x^{-1} b\sigma(x) I_1 = I_1 \,\dot{\mathrm{w}}\, I_1$ which is equivalent to

$$r^q s^{-1} c_0^q = 1 \quad \text{ and } \quad r^{-1} s^q c_0^{-q} = 1.$$

After eliminating $s = r^q c_0^q$, this is equivalent to $r^{q^2-1} c_0^{q^2-q} = 1$. The locus of this equation in the affine plane with coordinates r, c_0 is the union of $q-1$ disjoint curves

$$C_b^\zeta = \mathrm{Spec}(\bar{k}[r,c_0]/(r^{q+1}c_0^q - \zeta)) \xleftarrow{\sim} \mathbb{G}_m = \mathrm{Spec}(\bar{k}[z,z^{-1}]) \quad \text{with } \zeta \in k^*,$$

where the isomorphism is on coordinates given by $(r, c_0) \mapsto (\zeta z^q, \zeta^{-1} z^{-(q+1)})$. The statements about the $p_m(K_b^{(m)})$- and $T(w)^\sigma_{\mathrm{aff}}$-actions can now be deduced explicitly from this description. □

3.4. PGL_2 and admissibility.

Remark that $J_b \cap Z(G(L)) = Z(G(F))$. Denote this group by Z. Then $ZK_1^{(m)}/K_1^{(m)}$ is not compact, which is the reason, why the J_b-representations in the cohomology of $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ for $b = 1, b_1$, considered in the next paragraph, are not admissible (see Remark 4). If one changes to PGL_2 (i.e. if one divides out the center), one obtains admissible representations (cf. Remark 3(i) for the case $b = 1$ and Remark 5 for the case $b = b_1$). Here we describe $X_w(b)$ and $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ for PGL_2.

The affine flag manifold X^{PGL} of PGL_2 is obtained from X by identifying all odd and all even connected components ($\eta_G^{-1}(v)$ is identified with $\eta_G^{-1}(v+2n)$ via multiplication with $t^{(n,n)}$). The affine Weyl group W_{PGL} of PGL_2 is the quotient of $\tilde{W}$ by the subgroup generated by $t^{(1,1)}$. If $w \in \tilde{W}$, write again w for its image in W_{PGL}. We denote by

$$X_w^{PGL}(b) \quad \text{and} \quad \dot{\mathrm{X}}_{\dot{\mathrm{w}}}^{PGL}(b)$$

the affine Deligne-Lusztig varieties associated to PGL_2. Then J_b acts on $X_w^{PGL}(b)$ and $X_{\dot{\mathrm{w}}}^{PGL}(b)$ through the quotient J_b/Z, which is the σ-stabilizer of b in $PGL_2(L)$. Then $X_w^{PGL}(b)$ is exactly the image of $X_w(b)$ under the projection map $X \twoheadrightarrow X^{PGL}$.

Let $b = t^{(0,\alpha)}$ with $\alpha \geq 0$ or $b = b_1$ and $w \in \tilde{W}$, such that $X_w(b) \neq \emptyset$. Extend the action of $K_b^{(0)}$ on $S(b,w)$, $\dot{\mathrm{S}}(b,w)$ to an action of $ZK_b^{(0)}$ by letting $t^{(1,1)}$ act trivial. This action of $ZK_b^{(0)}$ factorizes through $r\colon ZK_b^{(0)} \twoheadrightarrow ZK_b^{(0)}/\langle t^{(1,1)}\rangle \cong K_b^{(0)}$, and therefore through $p_0 r\colon ZK_b^{(0)} \twoheadrightarrow p_0(K_b^{(0)})$.

Proposition 3.4. *Let $b, w, \dot{\mathrm{w}}$ be as in Theorem 3.3. Then there are the following J_b-equivariant (for the action defined above) isomorphisms:*

$$\begin{aligned} X_w^{PGL}(b) &\cong \coprod_{J_b/ZK_b^{(0)}} \mathbb{A}_{\bar{k}}^{d_{b,w}} \times S(b,w), \\ \dot{\mathrm{X}}_{\dot{\mathrm{w}}}^{PGL}(b) &\cong \coprod_{J_b/ZK_b^{(0)}} \mathbb{A}_{\bar{k}}^{d_{b,w}} \times \dot{\mathrm{S}}(b,w). \end{aligned}$$

The second isomorphism is $T(w)^\sigma_{\mathrm{aff}}/T_1$-equivariant.

Proof. This follows by easy computations from Theorem 3.3. □

4. Representation Theory of J_b

For a topological group Γ we denote by $\widehat{\Gamma}$ the set of classes of smooth irreducible Γ-representations over $\overline{\mathbb{Q}_l}$. Further we define the cohomology with compact support of a colimit (which in our case is just a disjoint union) of schemes of finite type over k as the colimit of the cohomology with compact support of these schemes. With this definition, the cohomology with compact support commutes with colimits.

4.1. Definition of $R_b^\theta(w)_{\mathrm{aff}}$.

The groups $T(w)^\sigma_{\mathrm{aff}}/T_1$ and J_b act on $H_c^r(\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b), \overline{\mathbb{Q}_l})$ by transport of the structure. As in [2] 1.8, since $T(w)^\sigma_{\mathrm{aff}}/T_1$ is abelian and since its action and that of J_b commute (the first is a right and the second a left action), $H_c^r(\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b), \overline{\mathbb{Q}_l})$ decomposes as the direct sum of the

J_b-subrepresentations $H_c^r(\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b), \overline{\mathbb{Q}_l})_\theta$ on which $T(w)_{\mathrm{aff}}^\sigma/T_1$ acts through the character $\theta \in \widehat{T(w)_{\mathrm{aff}}^\sigma/T_1}$. The following definition is the analogon in the affine case of [2] 1.9.

Definition 3. *Let $b \in G(L)$ and $\dot{\mathrm{w}} \in \tilde{W}_1$ with image w in $\tilde{W}$, such that $X_w(b) \neq \emptyset$. Let $\theta \in \widehat{T(w)_{\mathrm{aff}}^\sigma/T_1}$ be a character. For $*$ either empty or PGL, define the virtual J_b-representation*

$$R_b^\theta(w)_{\mathrm{aff},*} := \sum_r (-1)^r H_c^r(\dot{\mathrm{X}}^*_{\dot{\mathrm{w}}}(b), \overline{\mathbb{Q}_l})_\theta$$

In the second case J_b acts as explained in subsection 3.4.

This virtual representation is in fact independent of the choice of $\dot{\mathrm{w}}$ lying over w by Lemma 3.2 (which has an obvious extension to the case $* = PGL$).

Lemma 4.1. *Let $b \in G(L)$ and $\dot{\mathrm{w}} \in \tilde{W}_1$ with image w in $\tilde{W}$, such that $X_w(b) \neq \emptyset$. Define*

$$K_{b,*}^{(m)} = \begin{cases} K_b^{(m)} & \text{if } * = \text{empty} \\ ZK_b^{(0)} & \text{if } * = PGL. \end{cases}$$

Let $\theta \in \widehat{T(w)_{\mathrm{aff}}^\sigma/T_1}$. Then

$$H_c^r(\dot{\mathrm{X}}^*_{\dot{\mathrm{w}}}(b), \overline{\mathbb{Q}_l})_\theta = \text{c-Ind}\,_{K_{b,*}^{(m)}}^{J_b} H_c^{r-2d_{b,w}}(\dot{\mathrm{S}}(b,w), \overline{\mathbb{Q}_l}(d_{b,w}))_\theta.$$

where $K_{b,*}^{(m)}$-action on $\dot{\mathrm{S}}(b,w)$ is the inflation of the $p_m(K_b^{(m)})$-action, and the $d_{b,w}$ in the brackets denotes the twist, defining the action of the Galois group of $\bar{k}/k$.

Proof. It follows by taking the θ-isotypic components from

$$\begin{aligned} H_c^r(\dot{\mathrm{X}}^*_{\dot{\mathrm{w}}}(b), \overline{\mathbb{Q}_l}) &= H_c^r(\coprod_{J_b/K_{b,*}^{(m)}} \mathbb{A}_{\bar{k}}^{d_{b,w}} \times \dot{\mathrm{S}}(b,w), \overline{\mathbb{Q}_l}) = \text{c-Ind}\,_{K_{b,*}^{(m)}}^{J_b} H_c^r(\mathbb{A}_{\bar{k}}^{d_{b,w}} \times \dot{\mathrm{S}}(b,w), \overline{\mathbb{Q}_l}) \\ &= \text{c-Ind}\,_{K_{b,*}^{(m)}}^{J_b} H_c^{r-2d_{b,w}}(\dot{\mathrm{S}}(b,w), \overline{\mathbb{Q}_l}(d_{b,w})), \end{aligned}$$

where the third equality is a consequence of the Künneth-formula. Second statement has a similar proof. □

4.2. Case $b = 1$: Representation theory of $G(F)$.

4.2.1. *Over a finite field.* The representation theory of $G = GL_2$ over a finite field is easy. We recall the irreducible representations to state our results. Let $\alpha, \beta \in \widehat{k^*}$. There are the following four types of irreducible representations of $G(k)$:

- $\rho(\theta) := \mathrm{Ind}_{B(k)}^{G(k)} \theta$, where $\theta = \alpha \otimes \beta$ is a character of the split torus $T(k) \cong k^* \times k^*$, such that $\alpha \neq \beta$ vary over $\widehat{k^*}$. These are the principal series representations.
- $\alpha_{G(k)} := \alpha \circ \det$.
- $\alpha \cdot \mathrm{St}_{G(k)}$, the q-dimensional twisted Steinberg representation defined through the following split exact sequence, where the first map is the diagonal embedding:

$$0 \to \alpha_{G(k)} \to \mathrm{Ind}_{B(k)}^{G(k)} \alpha \otimes \alpha \to \alpha \cdot \mathrm{St}_{G(k)} \to 0.$$

- $\pi(\theta')$, the cuspidal representations, where θ' varies over characters of a non-split torus in $G(k)$ with $\theta' \neq \theta'^q$. Then $\pi(\theta') \cong \pi(\theta'^q)$.

For $w_{\mathrm{fin}} \in W$ and $\theta \in \widehat{T(w_{\mathrm{fin}})^\sigma}$ Deligne and Lusztig defined in [2] the virtual representations $R^\theta(w_{\mathrm{fin}})$, which for $G = GL_2$ are for example computed in [11].

4.2.2. *Over F.* From the finite Deligne-Lusztig theory for G over k and from Lemma 4.1 we obtain (since $\pi(\theta') \cong \pi(\theta'^q)$ for cuspidal representations of $G(k)$, the ugly technical condition (3.2) is irrelevant):

Corollary 4.2. *Let $w \in \tilde{W}$, such that $X_w(1) \neq \emptyset$, $m \in \{0,1\}$ as in* (3.1)*. Write w_{fin} for the image of w in W and let $\theta \in \widehat{T(w)^\sigma_{\mathrm{aff}}/T_1} = \widehat{T(w_{\mathrm{fin}})^\sigma}$. For $*$ either empty or PGL, let $K^{(m)}_{1,*}$ be as in Lemma 4.1.*

(i) *If $w = 1$, then $T(w)^\sigma_{\mathrm{aff}}/T_1 = T(k)$, and*

$$R^\theta_1(1)_{\mathrm{aff}} = \begin{cases} \text{c-Ind}\,^{G(F)}_{K_1^{(m)}}\,\rho(\theta) & \text{if } \theta = \alpha \otimes \beta \text{ with } \alpha \neq \beta \\ \text{c-Ind}\,^{G(F)}_{K_1^{(m)}}(\alpha_{G(k)} + \alpha \cdot \mathrm{St}_{G(k)}) & \text{if } \theta = \alpha \otimes \alpha. \end{cases}$$

(ii) *If $w \neq 1$, then $T(w)^\sigma_{\mathrm{aff}}/T_1 \cong k'^*$, and*

$$R^\theta_{1,*}(w)_{\mathrm{aff},*} = \begin{cases} -\,\text{c-Ind}\,^{G(F)}_{K_{1,*}^{(m)}}\,\pi(\theta) & \text{if } \theta \neq \theta^q \\ \text{c-Ind}\,^{G(F)}_{K_{1,*}^{(m)}}(\alpha_{G(k)} - \alpha \cdot \mathrm{St}_{G(k)}) & \text{if } \theta = \theta^q, \end{cases}$$

where $\alpha(Nz) := \theta(z)$, where N denotes the norm of k'/k.

Remark 3. (i) *If $w \neq 1$ the representations $-R^\theta_1(w)_{\mathrm{aff,PGL}}$ with $\theta \neq \theta^q$ run through all cuspidal $G(F)$-representations of level zero, whose central character is trivial on $t^{(1,1)}$. In particular, they are admissible by* [1] *11.4 Theorem and 10.2 Corollary (cf. also section 3.4).*

(ii) *If $w \neq 1$ and $X_w(1) \neq \emptyset$, the $G(F)$-representations $R^\theta_1(w)_{\mathrm{aff}}$, $R^\theta_1(w)_{\mathrm{aff,PGL}}$ are independent of w, which illustrates the general principle proved by He in* [6] *Cor. 4.8 that for PGL_n, the representations occurring in the cohomology of $X_w(1)$ for some w, already occur in the cohomology of $X_w(1)$ for w in the finite Weyl group. However, by Lemma 4.1 $R^\theta_1(w)_{\mathrm{aff}}$ comming from w's with different length, differ as $\mathrm{G}(\bar{k}/k)$-representations, since the numbers $d_{1,w} = \frac{\ell(w_a)-1}{2}$ (cf.* (3.1)*), defining the Tate twist, depend on $\ell(w_a)$.*

Now we study morphisms from the representations occurring in $R^\theta_1(w)_{\mathrm{aff}}$ into smooth irreducible representations of $G(F)$. These are described in [1]. Let us first fix some notation. We write $\phi_{G(F)} := \phi \circ \det$, if $\phi \in \widehat{F^*}$, and $\phi \cdot \mathrm{St}_{G(F)}$ for the corresponding twisted Steinberg representation. If a smooth irreducible representation π of $G(F)$ contains the trivial character on a subgroup conjugate to $\ker(p_0 \colon K \twoheadrightarrow G(k))$, then the level $\ell(\pi)$ of π is 0 (*loc.cit.* 12.6). The levels of $\phi_{G(F)}$ and $\phi \cdot \mathrm{St}_{G(F)}$ resp. $\mathrm{Ind}^{G(F)}_{B(F)}\chi$ ($\chi \in \widehat{T(F)}$) are zero if and only if the level of ϕ resp. χ is zero (this follows from *loc.cit.* 12.9 Theorem and 14.2 Theorem, which characterize representations of level > 0 as such, containing fundamental strata). The restriction of a character $\phi \in \widehat{F^*}$ resp. $\chi \in$ $widehatT(F)$ of level zero to $\mathfrak{o}_F^*$ resp. $T(\mathfrak{o}_F)$ is an inflation of a character which we denote by $\bar{\phi}$ resp. $\bar{\chi}$ of k^* resp. $T(k)$. Analogously the level of a cuspidal representation ϱ is 0 if and only if its restriction to K contains an inflation of a (unique up to isomorphism) cuspidal representation $\bar{\varrho}$ of $G(k)$.

Let v denote the non-trivial element of W. Then if $\lambda = \lambda_1 \otimes \lambda_2 \in \widehat{T(k)}$, we write ${}^v\lambda := \lambda_2 \otimes \lambda_1$. For simplicity we only handle the case $m = 0$. By abuse of notation, if π is any $G(k)$-representation, we write also π for its inflation via p_0 to K.

Theorem 4.3. *Let the first row (resp. first column) entries in the table below run through $\widehat{G(F)}$ (resp. through compact inductions of $\widehat{G(k)}$) with α, $\rho(\theta)$, $\pi(\theta')$ as in 4.2.1, $\chi \in \widehat{T(F)}$, $\phi \in \widehat{F^*}$ and $\varrho \in \widehat{G(F)}$ cuspidal as above. There are no non-zero homomorphisms from any left into any upper entry, unless the level of the upper entry is 0.*

Assume that this holds and denote by $\bar{\chi}, \bar{\phi}$, resp. $\bar{\varrho}$ the corresponding representation of $k^*, T(k)$, resp. $G(k)$. The (i,j)-th entry in the table below is the space of all $G(F)$-homomorphisms from the representation in the i-th row into the one in the j-th column.

	$\mathrm{Ind}_{B(F)}^{G(F)}\chi$	$\phi_{G(F)}$	$\phi\cdot\mathrm{St}_{G(F)}$	ϱ *cuspidal*
$\text{c-Ind}\,_K^{G(F)}\rho(\theta)$	$\begin{cases}\overline{\mathbb{Q}_l} & \textit{if } \bar{\chi}=\theta \\ & \textit{or } \bar{\chi}={}^v\theta \\ 0 & \textit{otherwise}\end{cases}$	*0*	*0*	*0*
$\text{c-Ind}\,_K^{G(F)}\alpha_{G(k)}$	$\begin{cases}\overline{\mathbb{Q}_l} & \textit{if } \bar{\chi}=\alpha_{T(k)} \\ 0 & \textit{otherwise}\end{cases}$	$\begin{cases}\overline{\mathbb{Q}_l} & \textit{if } \bar{\phi}=\alpha \\ 0 & \textit{otherwise}\end{cases}$	*0*	*0*
$\text{c-Ind}\,_K^{G(F)}\alpha\cdot\mathrm{St}_{G(k)}$	$\begin{cases}\overline{\mathbb{Q}_l} & \textit{if } \bar{\chi}=\alpha_{T(k)} \\ 0 & \textit{otherwise}\end{cases}$	*0*	$\begin{cases}\overline{\mathbb{Q}_l} & \textit{if } \bar{\phi}=\alpha \\ 0 & \textit{otherwise}\end{cases}$	*0*
$\text{c-Ind}\,_K^{G(F)}\pi(\theta')$	*0*	*0*	*0*	$\begin{cases}\overline{\mathbb{Q}_l} & \textit{if } \bar{\varrho}=\pi(\theta') \\ 0 & \textit{otherwise}\end{cases}$

Table 2. Morphisms into irreducible representations

Proof. First of all, if $\pi\in\widehat{G(k)}$ and $\rho\in\widehat{G(F)}$ with

$$0\neq\mathrm{Hom}_{G(F)}(\text{c-Ind}\,_K^{G(F)}\pi,\rho)=\mathrm{Hom}_K(\pi,\rho),$$

then ρ must contain the trivial character on $\ker(p_0\colon K\twoheadrightarrow G(k))$, since π is inflated from $G(k)$, and hence is trivial on this kernel. Therefore the statement about the levels follows and we can assume the levels of the representations in the upper row are 0. In particular, $\phi|_{\mathfrak{o}_F^*}$ resp. $\chi|_{T(\mathfrak{o}_F)}$ are inflations from k^* resp. $T(k)$ of characters $\bar{\phi}$, $\bar{\chi}$. Now we compute the places (i,j) with $1\leq i,j\leq 3$.

Lemma 4.4. *Let π be any $G(k)$-representation and $\mu\in\widehat{T(F)}$ of level 0, such that $\mu|_{T(\mathfrak{o}_F)}$ is induced from $\bar{\mu}\in\widehat{T(k)}$. Then*

$$\mathrm{Hom}_{G(F)}(\text{c-Ind}\,_K^{G(F)}\pi,\mathrm{Ind}_{B(F)}^{G(F)}\mu)=\mathrm{Hom}_{B(k)}(\pi,\bar{\mu}).$$

Proof of the lemma. Frobenius reciprocity implies:

$$\mathrm{Hom}_{G(F)}(\text{c-Ind}\,_K^{G(F)}\pi,\mathrm{Ind}_{B(F)}^{G(F)}\mu)=\mathrm{Hom}_K(\pi,\mathrm{Ind}_{B(F)\cap K}^{K}\mu)=\mathrm{Hom}_{B(F)\cap K}(\pi,\mu),$$

where for the first equality we used the Mackey formula and $B(F)\cdot K=G(F)$. Since inflation commutes with taking homomorphisms, we obtain:

$$\mathrm{Hom}_{B(F)\cap K}(\pi,\mu)=\mathrm{Hom}_{B(F)\cap K}(\inf_{B(k)}^{B(F)\cap K}\pi,\inf_{B(k)}^{B(F)\cap K}\bar{\mu})=\mathrm{Hom}_{B(k)}(\pi,\bar{\mu}).\qquad\square$$

Apply the lemma to $\pi=\mathrm{Ind}_{B(k)}^{G(k)}\lambda$ for some $T(k)$-character $\lambda=\lambda_1\otimes\lambda_2$:

$$\begin{aligned}\mathrm{Hom}_{G(F)}(\text{c-Ind}\,_K^{G(F)}\pi,\mathrm{Ind}_{B(F)}^{G(F)}\mu)=\mathrm{Hom}_{B(k)}(\pi,\bar{\mu}) &= \mathrm{Hom}_{B(k)}(\mathrm{Ind}_{B(k)}^{G(k)}\lambda,\bar{\mu})\\ &= \mathrm{Hom}_{B(k)}(\lambda\oplus\mathrm{Ind}_{T(k)}^{B(k)}{}^v\lambda,\bar{\mu}) \qquad (4.1)\\ &= \mathrm{Hom}_{T(k)}(\lambda,\bar{\mu})\oplus\mathrm{Hom}_{T(k)}({}^v\lambda,\bar{\mu}),\end{aligned}$$

where v is the non-trivial element of W. In particular we obtain the (1,1)-entry of the table by taking $\lambda=\theta$, $\mu=\chi$. Put $\phi_{T(F)}:=\phi\otimes\phi\in\widehat{T(F)}$, $\alpha_{T(k)}:=\alpha\otimes\alpha\in\widehat{T(k)}$. Then

$$\phi_{G(F)}|_K\oplus\phi\cdot\mathrm{St}_{G(F)}|_K\cong\mathrm{Ind}_{B(F)}^{G(F)}\phi_{T(F)}|_K\quad\text{and}\quad\alpha_{G(k)}\oplus\alpha\cdot\mathrm{St}_{G(k)}\cong\mathrm{Ind}_{B(k)}^{G(k)}\alpha_{T(k)},\qquad(4.2)$$

(the first equality uses that K is compact and hence its representation theory is semisimple). Also from Lemma 4.4 we have:

$$\mathrm{Hom}_{G(F)}(\text{c-Ind}\,_K^{G(F)}\alpha_{G(k)},\mathrm{Ind}_{B(F)}^{G(F)}\chi)=\mathrm{Hom}_{B(k)}(\alpha_{G(k)},\bar{\chi})=\mathrm{Hom}_{T(k)}(\alpha_{T(k)},\bar{\chi})\qquad(4.3)$$

which implies the (2,1)-entry. This, the second formula of (4.2) and (4.1) applied to $\pi = \operatorname{Ind}_{B(k)}^{G(k)} \alpha_{T(k)}$ and $\mu = \chi$ imply also the $(3,1)$-entry. From (4.1) and the first formula of (4.2) the entries $(1,2)$ and $(1,3)$ follow.

Now we investigate the four squares in the middle of the table. Let us write (i,j) for the Hom-space standing in the (i,j)-place. Together with (4.2) and Lemma 4.4, an application of the Mackey formula as in (4.1) gives:

$$\begin{aligned}(2,2)\oplus(2,3)\oplus(3,2)\oplus(3,3) &= \operatorname{Hom}_{T(k)}(\alpha_{T(k)}, \bar\phi_{T(k)}) \oplus \operatorname{Hom}_{T(k)}(\alpha_{T(k)}, \bar\phi_{T(k)}) \\ &= \begin{cases} \overline{\mathbb{Q}_l}^2 & \text{if } \alpha = \bar\phi, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

In particular, we can assume that $\alpha = \bar\phi$. It will suffice to show that $(2,2)$, $(2,2)\oplus(2,3)$ and $(2,2)\oplus(3,2)$ are one-dimensional. This is done by a (three) one-line computation(s) using the same techniques as above and the formula $\phi_{G(F)}|_K = \inf_{G(k)}^K \bar\phi_{G(k)}$.

Let $\pi(\theta') \in \widehat{G(k)}$ be cuspidal. If $\lambda \in \widehat{T(F)}$ and $\bar\lambda \in \widehat{T(k)}$, such that $\lambda|_{T(\mathfrak{o}_F)} = \inf_{T(k)}^{T(\mathfrak{o}_F)} \bar\lambda$, then Lemma 4.4 implies:

$$\operatorname{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} \pi(\theta'), \operatorname{Ind}_{B(F)}^{G(F)} \lambda) = \operatorname{Hom}_{B(k)}(\pi(\theta'), \bar\lambda) \subseteq \operatorname{Hom}_{U(k)}(\pi(\theta'), \bar\lambda),$$

where $U(k)$ is the unipotent radical of $B(k)$. Now $\bar\lambda$ is trivial on $U(k)$ and $\pi(\theta')$ does not contain the trivial character on $U(k)$, since it is cuspidal. Hence the last expression is equal 0. Therefore $(4,1) = 0$ follows by taking $\lambda = \chi$ and $(4,2), (4,3) = 0$ follow from (4.2) by taking $\lambda = \phi_{T(F)}$. This finishes the proof for the first three columns.

Now we consider the last column. ϱ is cuspidal and of level 0, hence $\varrho = \text{c-Ind}_{ZK}^{G(F)} \Lambda$ where $\Lambda|_K = \bar\varrho$ is an inflation of a (unique up to isomorphism) cuspidal representation $\bar\varrho$ of $G(k)$ (this follows from [1] 14.5 Theorem and 11.5 Theorem). If $0 \neq \operatorname{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} \pi, \varrho) = \operatorname{Hom}_K(\pi, \varrho)$, for $\pi \in \widehat{G(k)}$, then π and $\bar\varrho$ both occur in ϱ, hence by *loc.cit.* 11.1 Proposition 1 they must intertwine in $G(F)$. Thus by *loc.cit.* 11.5 Lemma, $\pi \cong \bar\varrho$ and in particular π is cuspidal. Hence $(1,4) = (2,4) = (3,4) = 0$. The next lemma finishes the proof. □

Lemma 4.5. *Let $\pi \in \widehat{G(k)}$ be cuspidal and denote again by π its inflation to K. Let Λ be some extension of π to ZK. Then*

$$\operatorname{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} \pi, \text{c-Ind}_{ZK}^{G(F)} \Lambda) = \overline{\mathbb{Q}_l}.$$

Proof. Write $I^{(n)} := K \cap t^{(0,n)} K t^{(0,-n)}$, and ${}^{(n)}\pi(g) := \pi(t^{(0,-n)} g t^{(0,n)})$ for $n \geq 0$ and $g \in t^{(0,n)} K t^{(0,-n)}$ (in particular $I^{(0)} = K$, ${}^{(0)}\pi = \pi$). Frobenius reciprocity and the Mackey formula imply:

$$\begin{aligned}\operatorname{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} \pi, \text{c-Ind}_{ZK}^{G(F)} \Lambda) &= \operatorname{Hom}_K(\pi, (\text{c-Ind}_{ZK}^{G(F)} \Lambda)|_K) \\ &= \operatorname{Hom}_K(\pi, \bigoplus_{n\geq 0} \operatorname{Ind}_{I^{(n)}}^K {}^{(n)}\pi) \\ &= \operatorname{End}_K(\pi) \oplus \bigoplus_{n\geq 1} \operatorname{Hom}_{I^{(n)}}(\pi, {}^{(n)}\pi).\end{aligned}$$

Now Schur's lemma for the compact group K implies $\operatorname{End}_K(\pi) = \overline{\mathbb{Q}_l}$, since π is irreducible, and we have to show $\operatorname{Hom}_{I^{(n)}}(\pi, {}^{(n)}\pi) = 0$ for $n \geq 1$. In fact, consider the subgroup $N = \{\begin{pmatrix} 1 & 0 \\ at^n & 1 \end{pmatrix} : a \in k\}$ of $I^{(n)}$. Since N is in the kernel of $K \twoheadrightarrow G(k)$ and since π is inflated from $G(k)$, the restriction of π to N is trivial. But ${}^{(n)}\pi(\begin{pmatrix} 1 & 0 \\ at^n & 1 \end{pmatrix}) = \pi(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix})$ and π is cuspidal, hence ${}^{(n)}\pi$ does not contain the trivial character on N, and the claim follows. □

Remark 4. *Let π be any representation of $G(k)$. The representation* $\text{c-Ind}_K^{G(F)}\pi$ *is very big. More precise, put* $I^{(n)} := K \cap t^{(0,n)}Kt^{(0,-n)}$ *(in particular,* $I^{(0)} = K$ *and* $I^{(1)} = I$*) and* ${}^{(n)}\pi(g) := \pi(t^{(0,-n)}gt^{(0,n)})$ *for* $g \in t^{(0,n)}Kt^{(0,-n)}$*. Then the Mackey formula implies:*

$$(\text{c-Ind}_K^{G(F)}\pi)|_K \cong \bigoplus_{\mathbb{Z}}\bigoplus_{n\geq 0}\text{Ind}_{I^{(n)}}^K {}^{(n)}\pi.$$

In particular, $\text{c-Ind}_K^{G(F)}\pi$ *is not admissible: its restriction to K contains $\mathbb{Z}$ copies of π.*

4.3. **Case b regular semisimple.**

Proposition 4.6. *Let* $b = t^{(0,\alpha)}$ *with* $\alpha > 0$*,* $w \in \tilde{W}$*, such that* $X_w(b) \neq \emptyset$ *and let* $w_a \in W_a$ *with* $w = w_a b_1^\alpha$ *and m as in* (3.1)*. Let* $\theta \in \widehat{T(w)^\sigma_{\text{aff}}/T_1}$*.*

(i) *If* $\ell(w_a) = \alpha$*, then* $T(w)^\sigma_{\text{aff}}/T_1 = T(k)$*. Let $\tilde\theta$ be equal θ if condition* (3.2) *is true, and equal* θ^v *otherwise, where* $1 \neq v \in W$*. Write* $\tilde\theta_{T(\mathfrak{o}_F)} = \inf_{T(k)}^{T(\mathfrak{o}_F)}\tilde\theta$ *(inflation via p_0). Then*

$$R_b^\theta(w)_{\text{aff}} = \text{c-Ind}_{T(\mathfrak{o}_F)}^{T(F)}\tilde\theta_{T(\mathfrak{o}_F)}.$$

(ii) *If* $\ell(w_a) > \alpha$*, then* $T(w)^\sigma_{\text{aff}}/T_1 \cong k'^*$*. If* $\theta = \theta^q$*, denote again by θ the character induced on k^* via* $\det\colon T(w)^\sigma_{\text{aff}}/T_1 \to k^*$ *and by* $\theta_{T(\mathfrak{o}_F)}$ *its inflation via* $T(\mathfrak{o}_F) \xrightarrow{p_m} T(k) \xrightarrow{\det} k^*$ *to* $T(\mathfrak{o}_F)$ *(note that* $\theta_{T(\mathfrak{o}_F)}$ *is independent of m). Then:*

$$\begin{aligned} R_b^\theta(w)_{\text{aff}} &= 0,\\ H_c^r(\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b),\overline{\mathbb{Q}_l})_\theta &= \begin{cases}\text{c-Ind}_{T(\mathfrak{o}_F)}^{T(F)}\theta_{T(\mathfrak{o}_F)}(\frac{\ell(w)-\alpha-1}{2}) & \textit{if } r = \ell(w)-\alpha \textit{ and } \theta = \theta^q\\ \text{c-Ind}_{T(\mathfrak{o}_F)}^{T(F)}\theta_{T(\mathfrak{o}_F)}(\frac{\ell(w)-\alpha-3}{2}) & \textit{if } r = \ell(w)-\alpha+1 \textit{ and } \theta = \theta^q\\ 0 & \textit{otherwise,}\end{cases}\end{aligned}$$

where $\dot{\mathrm{w}}$ *is a preimage of w in* $\tilde{W}_1$*.*

Proof. Denote by $\dot{\mathrm{w}}$ a preimage of w in $\tilde{W}_1$. Let first $\ell(w_a) = \alpha$. By Theorem 3.3, $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ is a disjoint union of points with stabilizer of a point equal T_1 in $T(F)$. As in Lemma 4.1, we have

$$H_c^0(\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b),\overline{\mathbb{Q}_l}) = \text{c-Ind}_{T(\mathfrak{o}_F)}^{T(F)}\text{Ind}_{T_1}^{T(\mathfrak{o}_F)}1_{\overline{\mathbb{Q}_l}} = \bigoplus_{\theta\in\widehat{T(k)}}\text{c-Ind}_{T(\mathfrak{o}_F)}^{T(F)}\theta_{T(\mathfrak{o}_F)},$$

as $T(F)$-representations. If condition (3.2) is true (i.e. $w_aC^0 = C^\alpha$), $K_b^{(m)} = T(\mathfrak{o}_F) = T(w)^\sigma_{\text{aff}}$ act in the same way on $\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b)$ and hence

$$R_b^\theta(w)_{\text{aff}} = H_c^0(\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b),\overline{\mathbb{Q}_l})_\theta = \text{c-Ind}_{T(\mathfrak{o}_F)}^{T(F)}\theta.$$

If (3.2) fails to be true (i.e. $w_aC^0 = C^{-\alpha}$), the action of $T(\mathfrak{o}_F)$ is the same and the torus action is twisted by the non-trivial element of W. This proves (i). Assume now $\ell(w_a) > \alpha$. The statement about $R_b^\theta(w)_{\text{aff}}$ follows from the statement about $H_c^r(\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b),\overline{\mathbb{Q}_l})_\theta$. First of all we have:

$$H_c^r(\dot{\mathrm{S}}(b,w),\overline{\mathbb{Q}_l})_\theta = \begin{cases}\theta_{T(\mathfrak{o}_F)} & \text{if } \theta = \theta^q \text{ and } r = 1\\ \theta_{T(\mathfrak{o}_F)}(-1) & \text{if } \theta = \theta^q \text{ and } r = 2\\ 0 & \text{otherwise,}\end{cases} \tag{4.4}$$

where (-1) means the Tate twist. In fact, one applies the Mayer-Vietoris sequence to the inclusions $\mathbb{G}_m \hookrightarrow \mathbb{P}^1 \hookleftarrow \{0,\infty\}$ to compute the cohomology of $\mathbb{G}_m$. In both degrees ($r = 1,2$) with non-vanishing cohomology, the representations $H_c^r(\mathbb{G}_m,\overline{\mathbb{Q}_l})$ of the stabilizers in $p_m(K_b^{(m)}) = T(k)$ and in $T(w)^\sigma_{\text{aff}}$ of a (hence any) connected component of $\dot{\mathrm{S}}(b,w) = \coprod_{k^*}\mathbb{G}_m$ are trivial. Therefore $H_c^{1,2}(\coprod_{k^*}\mathbb{G}_m,\overline{\mathbb{Q}_l}) = \bigoplus_{\theta\in\widehat{k^*}}\theta$ as k^*-representation (the k^*-action on the connected components is defined as the quotient of the action of $T(k)$ or, equivalently, of $T(w)^\sigma_{\text{aff}}/T_1$).

Now (4.4) follows, since for any $\theta \in \widehat{T(w)^{\sigma}_{\mathrm{aff}}/T_1}$, we have: θ factors through $\det\colon T(w)^{\sigma}_{\mathrm{aff}}/T_1 \twoheadrightarrow k^* \Leftrightarrow \theta =$ $theta^q$. From (4.4) and Lemma 4.1 the second statement in (ii) follows. $\square$

4.4. **Case $b = b_1$.**

Recall that $J_{b_1} = D^*$ where D is the central quaternion algebra over F, and $K^{(m)}_{b_1} = U_D$ is the group of units in the valuation ring of D. Recall that p'_m is p_m composed with projection $B(\bar{k}) \twoheadrightarrow T(\bar{k})$ and that for $m \in \{0, 1\}$ and $w \in \tilde{W}$ with non-trivial image in W, the images of $U_D \xrightarrow{p'_m} T(\bar{k}) \hookleftarrow T(w)^{\sigma}_{\mathrm{aff}}/T_1$ coincide. If $\theta \in \widehat{T(w)^{\sigma}_{\mathrm{aff}}/T_1}$, denote by θ_{U_D} the inflation of θ via p'_m to U_D.

Proposition 4.7. *Let $w \in \tilde{W}$, such that $X_w(b_1) \neq \emptyset$. Then $T(w)^{\sigma}_{\mathrm{aff}}/T_1 \cong k'^*$. Let $\theta \in \widehat{T(w)^{\sigma}_{\mathrm{aff}}/T_1}$. Then*

$$R^{\theta}_{b_1}(w)_{\mathrm{aff}} = \text{c-Ind}\,^{D^*}_{U_D}\, \theta_{U_D}.$$

Proof. Every w with $X_w(b_1) \neq \emptyset$ has non-trivial image in W, hence the first statement. As in part (i) of Proposition 4.6, it follows that $R^{\theta}_{b_1}(w)_{\mathrm{aff}} = H^0_c(\dot{\mathrm{X}}_{\dot{\mathrm{w}}}(b_1), \overline{\mathbb{Q}_l})_{\theta}$ is either $\text{c-Ind}\,^{D^*}_{U_D}\, \theta_{U_D}$ or $\text{c-Ind}\,^{D^*}_{U_D} (\theta^q)_{U_D}$, depending on the condition (3.2). The next lemma finishes the proof, since $(\theta^q)_{U_D}(u) = \theta_{U_D}(b_1 u b_1^{-1})$ for $u \in U_D$. $\square$

Lemma 4.8. *Let χ be a character of U_D, and set $\chi'(u) = \chi(b_1 u b_1^{-1})$ for $u \in U_D$. Then:* $\text{c-Ind}\,^{D^*}_{U_D}\, \chi \cong \text{c-Ind}\,^{D^*}_{U_D}\, \chi'$.

Proof. The functions $f_i\colon D^* \to \overline{\mathbb{Q}_l}$, with $\mathrm{supp}(f_i) = U_D b_1^i$ and $f_i(u b_1^i) = \chi(u)$ for $u \in U_D$, resp. $f'_i\colon D^* \to \overline{\mathbb{Q}_l}$, with $\mathrm{supp}(f'_i) = U_D b_1^i$ and $f'_i(u b_1^i) = \chi'(u)$ define a basis of $\text{c-Ind}\,^{D^*}_{U_D}\, \chi$ resp. of $\text{c-Ind}\,^{D^*}_{U_D}\, \chi'$. An element $h = v b_1^r$ with $v \in U_D$ acts on f_i by $h.f_i = \chi(b_1^{i-r} v b_1^{r-i}) f_{i-r}$, and analogously on f'_i. It is easy to see that $f_i \mapsto f'_{i+1}$ gives a D^*-isomorphism between the two representations. $\square$

The lemma also implies that $\text{c-Ind}\,^{D^*}_{U_D}\, \theta_{U_D}$ depends only on θ and not on m.

If $\phi \in \widehat{F^*}$, then $\phi_D := \phi \circ \det$ is a character of D^*, and every character of D^* is obtained by this construction. Let π be a smooth irreducible representation of D^*. The *level* $\ell(\pi)$ of π is the least non-negative integer n, such that π is trivial on $(n+1)$-units U_D^{n+1} of D.

Lemma 4.9. *Let $\pi \in \widehat{D^*}$ containing the trivial character on U_D^n $(n \geq 1)$. Then $\ell(\pi) < n$.*

Proof. Let $\ell(\pi) = m$. Then π is trivial on $U_D^{(m+1)}$, hence $\pi|_{U_D^m} \cong \bigoplus_i \psi_i$, where ψ_i are characters. By [1] 11.1 Proposition 1, ψ_i intertwine in D^*, and since U_D^m is normal in D^*, it is equivalent to ψ_i's being conjugate by elements of D^*. If $m \geq n$, then one of ψ_i's would be trivial, and since they are all conjugate, each of them would be trivial, i.e. π would be trivial on U_D^m, which would imply $\ell(\pi) < m$, a contradiction. $\square$

The representations of D^* of level 0 and dimension > 1 are parametrized by admissible pairs $(E/F, \chi)$ of level zero in the following way: for the unramified extension E/F of degree two contained in L, fix an embedding $E^* \hookrightarrow D^*$ (all such are conjugate in D^*). Extend any $\chi \in \widehat{E^*}$ of level zero by triviality to a character Ξ of $E^* U_D^1$. The map $\chi \mapsto \pi(\Xi) := \text{c-Ind}\,^{D^*}_{E^* U_D^1}\, \Xi$ defines an one-to-one bijection between the set of all characters of E^* with $\chi|_{U_E} \neq \chi^q|_{U_E}$ of level zero modulo the equivalence relation $\chi \sim \chi^q$ and all irreducible representations of D^* of level zero and dimension > 1 (this is [1] 54.2 Proposition).

Theorem 4.10. *Let $\theta \in \widehat{k'^*}$ and put $\theta_{U_D} := \inf^{U_D}_{k'^*} \theta$ (inflation via p'_0). Let $\pi \in \widehat{D^*}$.*

(i) *If $\theta = \theta^q$, then*

$$\mathrm{Hom}_{D^*}(\text{c-Ind}\,^{D^*}_{U_D}\, \theta_{U_D}, \pi) = \begin{cases} \overline{\mathbb{Q}_l} & \textit{if } \pi \textit{ is a character of level } 0 \textit{ and } \pi|_{U_D} = \theta_{U_D}, \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) *If $\theta \neq \theta^q$, then*

$$\mathrm{Hom}_{D^*}(\text{c-Ind}\,_{U_D}^{D^*}\theta_{U_D},\pi) = \begin{cases} \overline{\mathbb{Q}_l} & \text{if } \pi = \pi(\Xi) \text{ with } \Xi|_{U_D} = \theta_{U_D} \text{ or } \Xi^q|_{U_D} = \theta_{U_D}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First of all we have

$$\mathrm{Hom}_{D^*}(\text{c-Ind}\,_{U_D}^{D^*}\theta_{U_D},\pi) = \mathrm{Hom}_{U_D}(\theta_{U_D},\pi). \tag{4.5}$$

It follows that if the space on the left is non-zero, then π must contain the trivial character on U_D^1. Hence Lemma 4.9 implies $\ell(\pi) = 0$.

Let first π be a character. Then the Hom-space above is non-zero if and only if $\theta_{U_D} = \pi|_{U_D}$. Now, θ_{U_D} factors as $\theta \circ p_0' \colon U_D \to k'^* \to \overline{\mathbb{Q}_l}^*$, and since π has level 0, it factors as $\bar{\pi} \circ p \circ \det \colon U_D \to U_F \xrightarrow{p} k^* \xrightarrow{\bar{\pi}} \overline{\mathbb{Q}_l}$. Since $N_{k'/k} \circ p_0' = p \circ \det \colon U_D \to k^*$, and since p_0' is an epimorphism, it follows that $\theta = \bar{\pi} \circ N_{k'/k}$. In particular, θ factors through the norm, which is equivalent to $\theta = \theta^q$.

Now assume $\pi = \pi(\Xi) = \text{c-Ind}\,_{E^*U_D^1}^{D^*}\Xi$ with Ξ trivial on U_D^1 and $\Xi|_{U_D} \neq \Xi^q|_{U_D}$. We have $U_D \subseteq E^*U_D^1$ and both are normal in D^*. The $(E^*U_D^1, U_D)$-double cosets in D^* are just left E^*U_D-cosets and there are exactly two of them, represented by 1 and b_1. Hence the Mackey formula implies: $\pi|_{U_D} = (\text{c-Ind}\,_{E^*U_D^1}^{D^*}\Xi)|_{U_D} = \Xi|_{U_D} \oplus \Xi^q|_{U_D}$. From this and (4.5) the theorem follows. $\square$

Remark 5. *With the same assumptions as in Proposition 4.7, taking additionally $w \neq b_1$ and $\theta \neq \theta^q$, let Ξ be a character on $E^*U_D^1$, trivial on U_D^1 and on the uniformizer $t^{(1,1)}$ of E and such that $\Xi|_{U_E} = \theta \circ p$, where $p \colon U_E \to k'^*$ is the reduction modulo $\mathfrak{p}$ map. From Lemma 4.1(ii), using $ZU_D = E^*U_D = E^*U_D^1$ we obtain (condition* (3.2) *is irrelevant since $\pi(\Xi) = \pi(\Xi^q)$):*

$$R_{b_1}^{\theta}(w)_{\mathrm{aff,PGL}} = \pi(\Xi).$$

In particular, these representations are admissible.

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References

[1] Bushnell C.J. and Henniart G.: *Local Langlands conjecture for GL_2*, Springer, 2006.

[2] Deligne P. and Lusztig G.: *Representations of reductive groups over finite fields*, Ann. of Math. **103** (1976), 103-161.

[3] Görtz U., Haines T.J., Kottwitz R.E. and Reuman D.C.: *Dimension of certain affine Deligne-Lusztig varieties*, Ann. Scient. Ec. Norm. Sup. 4^e serie, t. **39**, 2006, 467-511.

[4] Görtz U., Haines T.J., Kottwitz R.E. and Reuman D.C.: *Affine Deligne-Lusztig varieties in affine flag varieties*, Compositio Math. **146** (2010), 1339-1382.

[5] He X.: *Minimal length elements in conjugacy classes of extended affine Weyl group*, preprint, arXiv:1004.4040 [math.RT], 2010.

[6] He X.: *On some partitions of an affine flag variety*, arXiv:0910.5026 [math.RT], 2009.

[7] Kottwitz R.: *Isocrystals with additional structure II*, Comp. Math. **109** (1997). 255-339.

[8] Rapoport M.: *A positivity property of the Satake isomorphism*, Manuscripta math. **101** (2000), 153-166.

[9] Rapoport M. and Richartz M.: *On the classification and specialization of F-isocrystals with additional structure*, Comp. Math. **103** (1996) 153-181.

[10] Reuman D.: *Determining whether certain affine Deligne-Lusztig sets are empty*, Dissertation, Univ. of Chicago, arXiv:math/0211434v1 [math.RT], 2002.

[11] Tazehkand H. A.: *Deligne-Lusztig character theory for the general linear groups of rank 2*, Master Thesis, Simon Fraser Univ. 2007, http://ir.lib.sfu.ca/handle/1892/4241.

[12] Zbarsky B.: *On some stratifications of affine Deligne-Lusztig varieties for SL_3*, Dissertation, Univ. of Chicago, arXiv:0906.0186v1 [math.RT], 2009.

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