RAMIFIED AUTOMORPHIC INDUCTION AND ZERO-DIMENSIONAL AFFINE DELIGNE-LUSZTIG VARIETIES

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Abstract. To any connected reductive group G over a non-archimedean local field F and to any maximal torus T of G, we attach a family of extended affine Deligne-Lusztig varieties (and families of torsors over them) over the residue field of F. This construction generalizes affine Deligne-Lusztig varieties of Rapoport, which are attached only to unramified tori of G. Via this construction, we can attach to any maximal torus T of G and any character of T a representation of G. This procedure should conjecturally realize the automorphic induction from T to G.

For $G = \mathrm{GL}_2$ in the equal characteristic case, we prove that our construction indeed realizes the automorphic induction from at most tamely ramified tori. Moreover, if the torus is purely tamely ramified, then the varieties realizing this correspondence turn out to be (quite complicate) combinatorial objects: they are zero-dimensional and reduced, i.e., just disjoint unions of points.

1. Introduction

Let G be a connected reductive group over a finite field $\mathbb{F}_q$ and let σ be the Frobenius over $\mathbb{F}_q$. Then there is a natural correspondence, which to any pair (T, χ) consisting of a maximal $\mathbb{F}_q$-torus T of G and a character χ of $T(\mathbb{F}_q)$ in general position, associates an irreducible representation $\pm R_T^{\chi}$ of $G(\mathbb{F}_q)$. Moreover, $\pm R_T^{\chi}$ is cuspidal, whenever T is anisotropic modulo the center of G. This was conjectured in 1968 by MacDonald and proven in 1976 by Deligne and Lusztig in their celebrated paper [DL76]. They defined R_T^{χ} as the alternating sum of the ℓ-adic cohomology with compact support of an étale cover of a Deligne-Lusztig variety. This last is just the variety of all Borel subgroups of G, which are in a fixed relative position with their σ-translates.

Let now G be a connected reductive group over a non-archimedean local field F with residue field $\mathbb{F}_q$ and let $\ell \neq \mathrm{char}(\mathbb{F}_q)$ be a prime. For simplicity (to avoid dealing with endoscopy phenomena, etc.) let us assume here that $G = \mathrm{GL}_n$. The local Langlands correspondence states that there is a natural bijection between irreducible admissible representations of $G(F)$ and a certain class of n-dimensional representations of the Weil group of F. It was shown by geometric methods in [HT99] and [LRS93]. In a series of papers, Bushnell, Henniart and Kutzko could make this correspondence more explicit in the tamely ramified case by parametrizing both sides by admissible pairs, see for example [BH06], [BK93] and [BH05]. To give an admissible pair is essentially the same as to give a maximal F-torus T of G, which is anisotropic modulo center and a smooth $\overline{\mathbb{Q}}_\ell^\times$-character χ of $T(F)$, satisfying certain conditions. Thus, basically, this explicit construction associates a supercuspidal representation R_T^{χ} to the given pair (T, χ), like in the classical Deligne-Lusztig theory. This is a special case of the more general principle of *automorphic induction* for G.

Let G again be arbitrary. Roughly, there are two types of geometric objects attached to G, in the cohomology of which one tried to realize the automorphic induction:

(i) Varieties (or rigid or adic spaces) over $\mathrm{Spec}\, F$ equipped with integral models over $\mathrm{Spec}\, \mathcal{O}_F$ and special fibers over $\mathbb{F}_q$.

(ii) Reduced varieties over $\mathbb{F}_q$.

Constructions of type (ii) are purely in characteristic p, i.e., over $\mathbb{F}_q$, and only the reduced structure is relevant. Up to now, constructions of type (ii) only existed for unramified tori of G (except for a construction by Stasinski for GL_n and SL_n, see below), which was a serious drawback. This article contributes to the automorphic induction over local fields by introducing a new construction of type (ii), which works for all tori and all reductive groups (in the equal characteristic case). For $G = \mathrm{GL}_2$ in the equal characteristic case we prove that our construction indeed realizes the ℓ-adic automorphic induction for all at most tamely ramified tori. Here a very intriguing phenomenon occurs: the constructed varieties attached to the totally tamely ramified torus of GL_2 turn out to be zero-dimensional and reduced, more precisely, they are just discrete unions of $\mathbb{F}_q$-rational points and the automorphic induction is realized in their zeroth cohomology groups $\mathrm{H}^0_c(-, \overline{\mathbb{Q}}_\ell)$ with coefficients in the constant sheaf $\overline{\mathbb{Q}}_\ell$.

We recall some of the existing unramified constructions of type (ii). A first such construction was suggested in 1977 by Lusztig [Lus79]. Its variants were studied by Boyarchenko, Boyarchenko-Weinstein and Chan in [Boy12], [BW16], [Cha16]. A different, but apparently related approach via higher level covers of Rapoport's affine Deligne-Lusztig varieties was studied by the author in [Iva16]. The nature of all these constructions is strongly related to the classical Deligne-Lusztig construction explained in the beginning. In particular, if $\breve{F}$ denotes the completion of the unramified closure of F, σ the Frobenius of $\breve{F}/F$, and $b \in G(\breve{F})$ is some element, then an affine Deligne-Lusztig variety attached to these data can be seen as the subvariety of the affine flag manifold of G, consisting of all Iwahori subgroups of $G(\breve{F})$ being in a fixed relative position with their $b\sigma$-translates.

Main construction. We will define the *extended affine Deligne-Lusztig varieties* and torsors naturally attached to them in Section 2 below. Roughly, the construction goes as follows. Let F be a non-archimedean local field. Let G be a connected reductive group over F. Let $\mathfrak{T}$ be a maximal F-torus of G. Let $\breve{E}/F$ be the completion of the maximal unramified extension of the splitting field E of $\mathfrak{T}$ and let Σ be a set of continuous F-automorphisms of $\breve{E}$, such that $\breve{E}^\Sigma = F$. Let $\underline{w}$ be a map from Σ to the set of all possible relative positions of Iwahori subgroups of $G(\breve{E})$ and let $b \in G(\breve{E})$. Then the extended affine Deligne-Lusztig set attached to $\underline{w}$ and b is the subset $X_{\underline{w}}(b)$ of the affine flag manifold $\mathscr{F}$ of $G_{\breve{E}}$ consisting of all Iwahori subgroups, whose relative position to their $b\gamma$-translate is equal to $\underline{w}(\gamma)$ for all $\gamma \in \Sigma$.

Now we turn to torsors over $X_{\underline{w}}(b)$. Let I be some Σ-stable Iwahori subgroup of $G(\breve{E})$ (as G is residually quasi-split over F by [BT87], such I always exists). By a level f we essentially mean a congruence subgroup I^f of I. Attached to such f, there is a natural cover $\mathscr{F}^f \twoheadrightarrow \mathscr{F}$ of the affine flag manifold. Then to any lift $\underline{w}_f$ of $\underline{w}$ to a function into an appropriate space of relative positions of level f, we naturally attach a subset $X^f_{\underline{w}_f}(b)$ of $\mathscr{F}^f$, which lies over $X_{\underline{w}}(b)$. In many cases, $X_{\underline{w}}(b)$ and $X^f_{\underline{w}_f}(b)$ can be given a scheme structure, turning them into reduced schemes locally of finite type over a finite extension of $\mathbb{F}_q$. Moreover, we obtain two natural commuting group actions

$$J_b(F) \circlearrowright X^f_{\underline{w}_f}(b) \circlearrowleft \tilde{I}_{f,\underline{w}_f} \twoheadrightarrow \mathfrak{T}(F).$$

Here J_b is the Σ-stabilizer of b, i.e., the algebraic group over F defined by

$$J_b(R) := \{g \in G(R \otimes_F \breve{E}) \colon g^{-1} b\gamma(g) = b \ \forall \gamma \in \Sigma\}$$

for an F-algebra R, and $\tilde{I}_{f,\underline{w}_f}$ is a certain subgroup of $G(\breve{E})$, which depends on $\underline{w}_f$ (but not on b) and admits $\mathfrak{T}(F)$ as a natural quotient, if $\underline{w}_f$ is appropriate. Further, $X^f_{\underline{w}_f}(b)$ is in a natural way a torsor over $X_{\underline{w}}(b)$ under a certain subquotient of $\tilde{I}_{f,\underline{w}_f}$, which is an algebraic group of finite type over a finite extension of $\mathbb{F}_q$.

Comment of the scheme structure. The affine flag manifold $\mathscr{F}$ and its covers $\mathscr{F}^f$ are Ind-schemes resp. Ind perfect algebraic spaces if F has equal resp. mixed characteristics by [PR08] Theorem 1.4 resp. [Zhu14] Theorem 1.4. Thus $X_{\underline{w}}(b)$ and its covers $X^f_{\underline{w}_f}(b)$ can be given structures of sub-Ind-schemes resp. sub-Ind perfect algebraic spaces, whenever they can be shown to be locally closed in $\mathscr{F}$ or $\mathscr{F}^f$. In either case one can attach to them ℓ-adic cohomology groups with compact support.

The construction explained above generalizes the unramified construction from [Iva16], i.e., if one chooses $\mathfrak{T}$ to be an unramified maximal torus of G (i.e., $\breve{E} = \breve{F}$) and Σ to be the set with one element containing only the Frobenius of $\breve{F}/F$, then the corresponding Iwahori-level variety $X_{\underline{w}}(b)$ will be just the affine Deligne-Lusztig variety of Rapoport, and the torsors $X^f_{\underline{w}_f}(b)$ over it will be precisely the torsors defined in [Iva16].

We wish to point out that in 2011 Stasinski made in [Sta11] the first attempt to define some varieties (of type (ii)) attached to ramified tori. He worked over finite rings $\mathbb{F}_q[t]/(t^r)$ and was interested in the representation theory of the finite group $G(\mathbb{F}_q[t]/(t^r))$. For $G = \mathrm{GL}_n, \mathrm{SL}_n$, he constructed extended Deligne-Lusztig varieties (hence our choice of terminology) attached to $G(\mathbb{F}_q[t]/(t^r))$ and any maximal torus in $G = \mathrm{GL}_n, \mathrm{SL}_n$. This construction is technically involved, and, in particular, works a priori only for $G = \mathrm{GL}_n, \mathrm{SL}_n$. Also, there are issues about defining higher-level torsors. Nevertheless, the main ideas of his and our constructions seem to coincide: in both cases, one defines a variety by fixing the relative position with respect to many automorphisms of an extension of F, and not only with respect to the Frobenius. Moreover, the first example of a zero-dimensional variety (attached to $\mathrm{GL}_2(\mathbb{F}_q[t]/(t^2))$), realizing interesting representations occurs in [Sta11].

Affine Deligne-Lusztig induction. We can use the ℓ-adic cohomology of $X^f_{\underline{w}_f}(b)$ to define the following map, which we call the *affine Deligne-Lusztig induction*:

$$R = R_{f,\underline{w}_f,b} \colon \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\tilde{I}_{f,\underline{w}_f}/I^f) \to \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(J_b(F)), \quad \chi \mapsto \sum_i (-1)^i \mathrm{H}^i_c(X^f_{\underline{w}_f}(b), \overline{\mathbb{Q}}_\ell)[\chi] \tag{1.1}$$

between the categories of smooth $\overline{\mathbb{Q}}_\ell$-representations. In particular, whenever $\underline{w}_f$ is such that $\mathfrak{T}(F)$ is a quotient of $\tilde{I}_{f,\underline{w}_f}$, characters of $\mathfrak{T}(F)$ (of level bounded by f) induce, after inflation to $\tilde{I}_{f,\underline{w}_f}$, representations of $J_b(F)$. We formulate the following conjecture here only for GL_n and $b = 1$, in which case $J_1(F) = G(F)$. For more general reductive groups G we expect that a similar statement is true, but that some endoscopy phenomena occur.

Conjecture 1.1. *Let $G = \mathrm{GL}_n$ and $b = 1$. The collection of maps* (1.1) *satisfies the following properties:*

- (A) *If $\mathfrak{T}$ is anisotropic modulo the center of G, and χ is a character of $\mathfrak{T}(F)$ in sufficiently general position, then there are $f, \underline{w}_f$, such that $\tilde{I}_{f,\underline{w}_f} \twoheadrightarrow \mathfrak{T}(F)$ and $R_{f,\underline{w}_f,1}(\chi)$ is an irreducible supercuspidal representation of $G(F)$.*
- (B) *The map $\chi \mapsto R_{f,\underline{w}_f,1}(\chi)$ in (A) is injective up to Galois conjugation.*
- (C) *The map $\chi \mapsto R_{f,\underline{w}_f,1}(\chi)$ in (A) coincides with the realization of the automorphic induction constructed via cuspidal types by Bushnell, Kutzko and others (see [BK93]).*

Further, we expect the following two facts to be true:

(D) If $\mathfrak{T}$ is unramified, then $X^f_{\underline{w}_f}(1)$ is isomorphic (up to a possible unessential defect) to the reduction of the open affinoid in the Lubin-Tate perfectoid space constructed by Boyarchenko and Weinstein in [BW16].

(E) If $\mathfrak{T}$ is purely tamely ramified, then $\underline{w}_f$ can be chosen such that $X^f_{\underline{w}_f}(1)$ is a zero-dimensional reduced scheme (disjoint union of points).

The evidence for Conjecture 1.1 and expectations (D) and (E) is build up mainly on the at most tamely ramified GL_2-case (see Theorem 1.2 below), and the analogy with the classical Deligne-Lusztig induction. We discuss some further evidence below in this introduction.

Case GL_2 in characteristic $p > 0$. Assume now $\mathrm{char}\, F = p > 0$. Let $\mathbb{P}_2(F)$ be the set of all isomorphism classes of admissible pairs $(E/F, \chi)$ attached to at most tamely ramified quadratic extensions E/F. Note that if $\mathfrak{T} \subseteq G$ is a torus with $\mathfrak{T}(F) = E^\times$, then characters of $\mathfrak{T}(F)$ in general position up to Galois conjugation are in 1:1-correspondence with the subset of minimal pairs. Let $\mathscr{A}_2^{\mathrm{tame}}(F)$ be the set of isomorphism classes of irreducible supercuspidal representations of $G(F)$, which are additionally assumed to be unramified if $\mathrm{char}\, F = 2$. Then the tame parametrization theorem ([BH06] 20.2 Theorem) shows the existence of a certain well-behaved bijection

$$\mathbb{P}_2(F) \xrightarrow{\sim} \mathscr{A}_2^{\mathrm{tame}}(F), \quad (E/F, \chi) \mapsto \pi_\chi. \tag{1.2}$$

The following theorem shows Conjecture 1.1 and expectation (E) for GL_2 in the positive characteristic case for all at most tamely ramified tori in GL_2.

Theorem 1.2 (rough statement; cf. Theorem 4.2 and [Iva16] Theorem 4.3)**.** *Let $G = \mathrm{GL}_2$. Let $\mathfrak{T}$ be a non-split maximal torus of G. Let $\breve{E}/F$ be the completion of the unramified closure of the splitting field E of $\mathfrak{T}$. Then there are choices of $\Sigma, f, \underline{w}_f$ such that the corresponding maps $R = R_{f,\underline{w}_f,1}$ from* (1.1) *realize the automorphic induction from $\mathfrak{T}$ to G. Moreover, they induce a bijection*

$$\mathbb{P}_2(F) \xrightarrow{\sim} \mathscr{A}_2^{\mathrm{tame}}(F), \quad (E/F, \chi) \mapsto R_\chi, \tag{1.3}$$

and one has $R_\chi \cong \pi_\chi$, i.e., the maps (1.3) *and* (1.2) *coincide. In the case of the ramified torus, $\underline{w}_f$ can be chosen such that the varieties $X^f_{\underline{w}_f}(1)$ are disjoint unions of $\mathbb{F}_q$-points.*

If $\mathfrak{T}$ is tamely ramified with splitting field E, it suffices to work with E instead of $\breve{E}$ to obtain the same results (see Remark 3.11(i)). A similar statement is not true if $\mathfrak{T}$ is unramified.

Ramified torus in GL_2. Besides the construction explained above, the main result of this article is Theorem 4.2 (the tamely ramified case of Theorem 1.2; the unramified case was proven in [Iva16]). Its proof is analogous to the proof in the unramified case [Iva16] Theorem 4.3. Roughly speaking, there are three instruments used in the proof:

(1) a trace formula, which in the present case reduces, due to zero-dimensionality, to (quite involved) point-counting arguments.

(2) the theory of elementary modifications of characters of $E^\times$, developed in Section 4.4. It allows to prove our second main result, Theorem 4.18. It gives a precise description of the restriction of a certain cuspidal type (from which R_χ is induced) to an $E^\times$-representation. From this we deduce the injectivity of (1.3)

(3) the theory of cuspidal types of Bushnell, Kutzko, Henniart and others. We need it to compare R_χ with π_χ and, in particular, to show surjectivity of (1.3).

Some evidence for Conjecture 1.1. Let now $G = \mathrm{GL}_n$ and let $\mathfrak{T}$ be a maximal F-torus of G. Here are some heuristic reasons, justifying Conjecture 1.1 and the expectations (D) and (E):

- If $G = \mathrm{GL}_2$ and $\mathfrak{T}$ is at most tamely ramified, (A),(B),(C),(E) hold in the positive characteristic case (see Theorem 4.2 below and [Iva16] Theorem 4.3)
- The construction is completely analogous to the classical Deligne-Lusztig induction.
- (D) becomes evident for $G = \mathrm{GL}_2$, b superbasic by looking at the explicit defining equations (see [Iva16] Section 3.6).
- Assume $\mathfrak{T}$ is unramified, and let $\Sigma := \{\sigma\}$ consist only of the Frobenius, i.e., the corresponding varieties are torsors of level f over the usual affine Deligne-Lusztig varieties $X_w(1)$, where w is an element of the extended affine Weyl group $\tilde{W}$ of G. By [GH15] Proposition 2.2.1, if w is contained in a finite Weyl subgroup of $\tilde{W}$, then essentially, $X_w(1) \cong \coprod_{g \in G(F)/G(\mathcal{O}_F)} gX_w$, where the union is taken over translates of classical Deligne-Lusztig varieties X_w. This implies a similar decomposition for level-f-covers $X^f_{\underline{w}_f}(1) = \coprod_{g \in G(F)/G(\mathcal{O}_F)} gY^f_{\underline{w}_f}$ with $Y^f_{\underline{w}_f}$ of finite type. If Z denotes the center of $G(F)$, we deduce
$$\mathrm{H}^*_c(X^f_{\underline{w}_f}(1), \overline{\mathbb{Q}}_\ell)[\chi] = \mathrm{c}-\mathrm{Ind}_{ZG(\mathcal{O}_F)}^{G(F)} \mathrm{H}^*_c(Y^f_{\underline{w}_f}, \overline{\mathbb{Q}}_\ell)[\chi].$$
On the other side, it follows from the theory of cuspidal types (see [BK93]) that any supercuspidal representation is compactly induction from a cuspidal type, thus we have, in particular, a natural family of supercuspidal representations of $G(F)$, which are all of the form $\mathrm{c}-\mathrm{Ind}_{ZG(\mathcal{O}_F)}^{G(F)} \Xi$, where Ξ is some cuspidal inducing datum. Thus if the torus is unramified, the conjecture should boil down to statements about smooth representations of the group $ZG(\mathcal{O}_F)$, which is compact modulo center. For $n = 2$, this holds also for the tamely ramified torus and is part of our strategy of the proof of Theorem 1.2.
- Concerning expectation (E), we remark that if E/F is tamely ramified and some lift σ' of the Frobenius of $\breve{F}/F$ lies in Σ and $\underline{w}(\sigma') = 1$, then the extended affine Deligne-Luszstig variety $X_{\underline{w}}(1)$ of Iwahori level is a disjoint union of points (for any G).

Outline. In Section 2 we define the extended affine Deligne-Lusztig varieties and their covers for arbitrary connected reductive groups. In Section 3 we compute certain of those varieties for $G = \mathrm{GL}_2$ explicitly. Based on these computations, in Section 4, we state and prove our main results about tamely ramified automorphic induction for GL_2. The proofs of all results from Section 4, which contain any trace computations, are postponed to Section 5.

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2. Extended affine Deligne-Lusztig varieties of higher level

2.1. **Preliminaries.**

2.1.1. *Basic notation.* Let F be a non-archimedean local field with residue field k with q elements. Let $\breve{F}$ be the completion of a maximal unramified extension of F with residue field $\bar{k}$, which is an algebraic closure of k. Let E/F be a finite separable extension of F, such that $\breve{E} := E\breve{F}$ is the completion of a Galois extension of F. We denote by u a uniformizer of E and by $k_E \subseteq \bar{k}$ its residue field. For a Galois extension M/L we denote by $\mathrm{Gal}_{M/L}$ its Galois group.

2.1.2. *Group theoretic data.* Let G be a connected reductive group over F. Let S_0 be a maximal split torus in $G_{\breve{F}}$. By [BT84] 5.1.12, S_0 can be chosen to be defined over F. Let $T_0 := \mathscr{Z}_{G_{\breve{F}}}(S_0)$ be the centralizer of S_0. By Steinberg's theorem, $G_{\breve{F}}$ is quasi-split, hence T_0 is a maximal torus. Then the base change $T := T_0 \otimes_{\breve{F}} \breve{E}$ is a maximal torus of $G_{\breve{E}}$. Let S' be a maximal $\breve{E}$-split subtorus of T, containing $S := S_0 \otimes_{\breve{F}} \breve{E}$. We consider the root system $\Phi := \Phi(G_{\breve{E}}, S')$. For $a \in \Phi$, we write U_a for the corresponding root subgroup of $G_{\breve{E}}$. Moreover, we write $U_0 := T$.

2.1.3. *Bruhat-Tits buildings.* For any finite extension L of F or $\breve{F}$, let $\mathscr{B}_L$ denote the Bruhat-Tits building of G over L. It always exists by [BT72], [BT84]§4, [Rou77] Chap. 5 and [MSVM14]. If $L \subseteq M$ are two such extensions such that M/L is Galois, then $\mathrm{Gal}_{M/L}$ acts on $\mathscr{B}_M$. Moreover, there is a unique embedding of $\mathscr{B}_L$ into $\mathscr{B}_M$ in the sense of [Rou77] Definition 2.5.1. Indeed, the centralizer T of S_0 is a maximal torus, hence abelian, and its derived group is trivial. This allows to apply [Rou77] Theorem 2.5.6, to show that there is a unique such embedding. Note that if M/L is ramified, then $\mathscr{B}_L$, $\mathscr{B}_M^{\mathrm{Gal}_{M/L}}$ are not equal as simplicial complexes. However, if M/L is Galois tamely ramified, then $\mathscr{B}_L = \mathscr{B}_M^{\mathrm{Gal}_{M/L}}$ as subsets, as follows from [Rou77] 5.1.1 (see also [Pra01]).

2.1.4. *Apartments and Galois-stable alcoves.* Let $\mathscr{A}_{S'}$ be the apartment of $\mathscr{B}_{\breve{E}}$ corresponding to S'. Via the embedding $\mathscr{B}_{\breve{F}} \hookrightarrow \mathscr{B}_{\breve{E}}$ it contains the apartment $\mathscr{A}_{S_0}$ of $\mathscr{B}_{\breve{F}}$ corresponding to S_0. The restriction of any root a in Φ to S is non-trivial (indeed, otherwise U_a would lie in the centralizer $\mathscr{Z}_{G_{\breve{E}}}(S)$ of S, but taking the centralizer commutes with base change, hence $\mathscr{Z}_{G_{\breve{E}}}(S) = \mathscr{Z}_{G_{\breve{F}}}(S_0) \otimes_F \breve{F} = T$. This leads to a contradiction). This means that $\mathscr{A}_{S_0}$ is not contained in a wall of $\mathscr{A}_{S'}$. Further, by [BT87] Theorem 4.1, G is residually quasi-split over F, i.e., there exists an alcove $\underline{a}_0$ in $\mathscr{A}_{S_0}$, which is fixed by all continuous automorphisms of $\breve{F}/F$. Let now $\underline{a}$ be some alcove of $\mathscr{A}_{S'}$ which contains a point x_0 of $\underline{a}_0$ in its interior. As x_0 is $\mathrm{Gal}_{\breve{E}/F}$-stable, and as $\underline{a}$ is the unique alcove of $\mathscr{B}_{\breve{E}}$ containing x_0, it follows that $\underline{a}$ is also $\mathrm{Gal}_{\breve{E}/F}$-stable. Let I denote the associated $\mathrm{Gal}_{\breve{E}/F}$-stable Iwahori subgroup of $G(\breve{E})$.

2.1.5. *Filtrations on root subgroups.* Let x be a vertex of $\underline{a}$. Let $\tilde{\mathbb{R}} := \mathbb{R} \cup \{r+ \colon r \in \mathbb{R}\} \cup \{\infty\}$ be the ordered monoid as in [BT72] 6.4.1. Bruhat-Tits theory (see [BT72] 6.2.1, 6.4.1 and especially 6.2.3 e)) provides a filtration (depending on x) on root subgroups $U_a(\breve{E})$ for all $a \in \Phi$: for $r \in \tilde{\mathbb{R}}$, we denote the r-th filtration step by $U_a(\breve{E})_{x,r}$. If $U_a \cong \mathbb{G}_a$, then this filtration is up to some shift (depending on x) equal to the usual valuation filtration on $U_a(\breve{E}) \cong \breve{E}$.

2.1.6. *Filtration on the torus.* We choose an admissible schematic filtration on tori in the sense of Yu [Yu02] §4. This gives a filtration $U_0(\breve{E})_r = T(\breve{E})_r$ on $T(\breve{E})$. If G satisfies the condition (T) from [Yu02] 4.7.1 (in particular, if G is either simply connected or adjoint or split over a tamely ramified extension), then this filtration is independent of the choice of the admissible schematic filtration and coincides with the Moy-Prasad filtration [Yu02] Lemma 4.7.4.

2.1.7. *Smooth models of root subgroups.* Let $f \colon \Phi \cup \{0\} \to \tilde{\mathbb{R}}_{\geqslant 0} \smallsetminus \{\infty\}$ be a concave function, i.e., $f(\sum_i a_i) \geqslant \sum_i f(a_i)$, for all $a_i \in \Phi \cup \{0\}$, such that $\sum_i a_i \in \Phi \cup \{0\}$ (see [BT72] 6.4). Following [Yu02], let $G(\breve{E})_{x,f}$ be the subgroup of $G(\breve{E})$ generated by $U_a(\breve{E})_{x,f(a)}$ for $a \in \Phi \cup \{0\}$. We refer to f as a *level* and to $G(\breve{E})_{x,f}$ as the corresponding *level subgroup*. By [Yu02] Theorem 8.3, there is a unique smooth model $\underline{G}_{x,f}$ of $G_{\breve{E}}$ over $\mathcal{O}_{\breve{E}}$ such that $\underline{G}_{x,f}(\mathcal{O}_{\breve{E}}) = G(\breve{E})_{x,f}$. Moreover, if $G(\breve{E})_{x,f}$ is stable under the Frobenius of $\breve{E}/E$, then $\underline{G}_{x,f}$ descends to a group scheme over $\mathcal{O}_E$ (see [Yu02] Section 9.1).

For a concave function f, we write $I^f := G(\breve{E})_{x,f}$. Let f_I denote the concave function such that $G(\breve{E})_{x,f_I} = I$. If $f \geqslant f_I$, then $I^f \subseteq I$.

2.1.8. *Loop groups and covers of the affine flag variety.* To have a common notation for the equal and mixed characteristic cases, for a k_E-algebra R set

$$\mathbb{W}(R) := \begin{cases} R \mathbin{\widehat{\otimes}}_{k_E} \mathcal{O}_E & \text{if } \operatorname{char}(F) > 0 \\ W(R) \otimes_{W(k_E)} \mathcal{O}_F & \text{if } \operatorname{char}(F) = 0, \end{cases}$$

where $W(R)$ denotes the p-typical Witt-ring of R. The ring $W(R)$ behaves well in the mixed characteristic case only if R is a perfect k-algebra. Let LG be the loop group of G, i.e., the functor on the category of k_E-algebras,

$$LG \colon R \mapsto G(\mathbb{W}(R)[u^{-1}]).$$

Let $f \geqslant f_I$ be some level, such that I^f is stable under the Frobenius of $\breve{E}/E$. Let $L^+\underline{G}_{x,f}$ be the functor on the category of k_E-algebras,

$$L^+\underline{G}_{x,f} \colon R \mapsto \underline{G}_{x,f}(\mathbb{W}(R)).$$

Let

$$\mathscr{F}^f := LG/L^+\underline{G}_{x,f}$$

be the quotient of fpqc-sheaves on the category of k_E-algebras in the equal characteristic case resp. on the category of perfect k_E-algebras in the mixed characteristic case. By [PR08] Theorem 1.4 resp. [Zhu14] Theorem 1.4 it is represented by an Ind-k_E-scheme resp. Ind-perfect algebraic space of Ind-finite type over k_E. Its $\bar{k}$-points are $\mathscr{F}^f(\bar{k}) = G(\breve{E})/G(\breve{E})_{x,f}$. Moreover, if $g \geqslant f$ are two concave functions satisfying the above assumptions, then there is a natural projection $\mathscr{F}^g \twoheadrightarrow \mathscr{F}^f$. We write $\mathscr{F}$ instead of $\mathscr{F}^{f_I}$. This is just the affine flag manifold of G.

2.1.9. *Actions on $\mathscr{F}^f$.* Let $f \geqslant f_I$ be some level. By construction, LG acts on $\mathscr{F}^f$ by left multiplication. In particular, $G(\breve{E}) = LG(\bar{k})$ acts on the $\bar{k}$-valued points $\mathscr{F}^f(\bar{k}) = G(\breve{E})/G(\breve{E})_{x,f}$. Assume now that I^f is normal in I and let Z be the center of G. Then $Z(\breve{E})I$ acts by right multiplication on $\mathscr{F}^f(\bar{k})$.

2.1.10. *Extended affine Weyl group and Iwahori-Bruhat decomposition.* Let $\mathscr{N}_{S'}$ be the normalizer of S' in G. Let $\tilde{W} := \mathscr{N}_{S'}(\breve{E})/(\mathscr{N}_{S'}(\breve{E}) \cap I)$ be the extended affine Weyl group of $G_{\breve{E}}$ associated with S'. Let $W := \mathscr{N}_{S'}(\breve{E})/T(\breve{E})$ be the finite Weyl group. Then $\tilde{W}$ sits in the short exact sequence

$$0 \to X_*(T_{\breve{E}})_{\mathrm{Gal}_{\breve{E}}} \to \tilde{W} \to W \to 0$$

(here $\mathrm{Gal}_{\breve{E}}$ denotes the absolute Galois group of $\breve{E}$). The Iwahori-Bruhat decomposition states that

$$G(\breve{E}) = \coprod_{w \in \tilde{W}} I\dot{w}I,$$

where $\dot{w}$ is any lift of w to $\mathscr{N}_T(\breve{E})$.

2.1.11. *Double coset decomposition at level f.* Let $f \geqslant f_I$ be some fixed level. Consider the set of double cosets

$$D_{G_{\breve{E}},f} := I^f \backslash G(\breve{E})/I^f.$$

If $g \geqslant f$, then there is a natural projection $D_{G_{\breve{E}},g} \twoheadrightarrow D_{G_{\breve{E}},f}$. In particular, we have the natural projection $D_{G_{\breve{E}},f} \twoheadrightarrow D_{G_{\breve{E}},f_I} = I\backslash G(\breve{E})/I \cong \tilde{W}$. For many w, f the fiber $D_{G,f}(w)$ of this projection can be given the structure of a finite-dimensional affine variety over $\bar{k}$. This will be discussed in detail in a future work. Below we only need the case $G = \mathrm{GL}_2$, where an explicit parametrization can be given (see Section 3.1.8).

2.1.12. *Relative position.* Let f be as in Section 2.1.11. We define the map

$$\mathrm{inv}^f : \mathscr{F}^f(\bar{k}) \times \mathscr{F}^f(\bar{k}) \to D_{G_{\breve{E}},f}$$

on $\bar{k}$-points by $\mathrm{inv}^f(xI^f, yI^f) = w_f$, where w_f is the unique double I^f-coset containing $x^{-1}y$.

2.2. Extended affine Deligne-Lusztig varieties of higher level.

2.2.1. *Main definition.*

Definition 2.1. Let G be a connected reductive group over F. Fix the following data:

- a finite separable extension E/F, such that $\breve{E} = E\breve{F}$ is the completion of a Galois extension of F
- a set Σ of continuous F-automorphisms of $\breve{E}$ such that $\breve{E}^{\Sigma} = F$.
- a concave function $f : \Phi \cup \{0\} \to \tilde{\mathbb{R}}_{\geqslant 0} \setminus \{\infty\}$, such that I^f is Σ-stable
- a function $\underline{w}_f : \Sigma \to D_{G_{\breve{E}},f}$
- an element $b \in G(\breve{E})$.

We define the *extended affine Deligne-Lusztig set* $X^f_{\underline{w}_f}(b)$ attached to $(G, E/F, f, \Sigma, \underline{w}_f, b)$ as the subset

$$X^f_{\underline{w}_f}(b) := \{x \in \mathscr{F}^f(\bar{k}) : \mathrm{inv}^f(x, b\gamma(x)) = \underline{w}_f(\gamma)\ \forall \gamma \in \Sigma\} \subseteq \mathscr{F}^f(\bar{k}).$$

2.2.2. *Left action by the Σ-stabilizer of b.* For $b \in G(\breve{E})$, let J_b be the Σ-stabilizer of b, i.e., the algebraic group over F defined by

$$J_b(R) := \{g \in G(R \otimes_F \breve{E}) : g^{-1} b \gamma(g) = b\ \forall \gamma \in \Sigma\}$$

for any F-algebra R. Then $J_b(F)$ acts on $X^f_{\underline{w}_f}(b)$ by left multiplication for any f and any $\underline{w}_f$. If $g \geqslant f$ and $\underline{w}_f$ equals $\underline{w}_g$ composed with the natural projection $D_{G_{\breve{E}},g} \twoheadrightarrow D_{G_{\breve{E}},f}$, then the natural projection $\mathscr{F}^g \twoheadrightarrow \mathscr{F}^f$ restricts to a map $X^g_{\underline{w}_g}(b) \to X^f_{\underline{w}_f}(b)$ and the $J_b(F)$-actions are compatible.

2.2.3. *Right action on $X^f_{\underline{w}_f}(b)$ by the stabilizer of $\underline{w}_f$.* If I^f is normal in I, then $Z(\breve{E})I/I^f$ acts on $D_{G_{\breve{E}},f}$ by left and right multiplication and we obtain a (right) action of $Z(\breve{E})I/I^f$ on the set of maps $\psi : \Sigma \to D_{G_{\breve{E}},f}$ by $(\psi, i) \mapsto \psi.i$, where $(\psi.i)(\gamma) := i^{-1}\psi(\gamma)\gamma(i)$ for any $i \in Z(\breve{E})I/I^f$, $\gamma \in \Sigma$. This inflates to an action of $Z(\breve{E})I$ on the same set.

Lemma 2.2. *Let $(G, E/F, \Sigma, f, \underline{w}_f, b)$ be as in Definition 2.1. Assume that $f \geqslant f_I$ and that I^f is normal in I. For $i \in Z(\breve{E})I$, the map $xI^f \mapsto xiI^f$ defines an isomorphism $X^f_{\underline{w}_f}(b) \xrightarrow{\sim} X^f_{\underline{w}_f.i}(b)$.*

Proof. Since I^f is normal in I and $Z(\breve{E})$ is the center of $G(\breve{E})$, we have $iI^f = I^f i$. As a consequence, the map $\mathscr{F}^f \to \mathscr{F}^f$ given by $xI^f \mapsto xiI^f$ is well-defined. Let $xI^f \in X^f_{\underline{w}_f}(b)$. Then for each $\gamma \in \Sigma$, one has $\mathrm{inv}_f(xI^f, b\gamma(x)I^f) = \underline{w}_f(\gamma)$, or equivalently, $x^{-1}b\gamma(x) \in I^f\underline{w}_f(\gamma)I^f$. We deduce for each $\gamma \in \Sigma$:

$$(xi)^{-1}b\gamma(xi) = i^{-1}x^{-1}b\gamma(x)\gamma(i) \in i^{-1}I^f\underline{w}_f(\gamma)I^f\gamma(i) = I^f i^{-1}\underline{w}_f(\gamma)\gamma(i)I^f = I^f(\underline{w}_f.i)(\gamma)I^f,$$

where the second equality uses normality of I^f in I. Thus $xiI^f \in X^f_{\underline{w}_f.i}(b)$. Hence $xI^f \mapsto xiI^f$ defines a map $X^f_{\underline{w}_f}(b) \to X^f_{\underline{w}_f.i}(b)$. Analogously, one shows that $xI^f \mapsto xi^{-1}I^f$ defines a map in other direction. Obviously, these two maps are inverse to each other. □

Thus if I^f is normal in I, the group

$$\tilde{I}_{f,\underline{w}_f} := \{i \in Z(\breve{E})I \colon \underline{w}_f.i = \underline{w}_f\}$$

acts on $X^f_{\underline{w}_f}(b)$ by right multiplication. We have the subgroup $I_{f,\underline{w}_f} := \tilde{I}_{f,\underline{w}_f} \cap I$. It is clear that $I_{f,\underline{w}_f} \supseteq I^f$ and that the right $\tilde{I}_{f,\underline{w}_f}$-action on $X^f_{\underline{w}_f}(b)$ factors through an action of $\tilde{I}_{f,\underline{w}_f}/I^f$.

Lemma 2.3. *Let $\underline{w} \colon \Sigma \to D_{G,f_I}$ be the composition of $\underline{w}_f$ with the natural projection $D_{G,f} \to D_{G,f_I}$. Then $X^f_{\underline{w}_f}(b)$ is a $I_{f,\underline{w}_f}$-torsor over the underlying Iwahori-level set (resp. variety, if a variety structure is provided) $X^{f_I}_{\underline{w}}(b)$.*

Proof. Let $p \colon X^f_{\underline{w}_f}(b) \to X^{f_I}_{\underline{w}}(b)$ denote the natural projection. Pick a point $xI^f \in X^f_{\underline{w}_f}(b)$ with image $p(xI^f) = xI \in X^{f_I}_{\underline{w}}(b)$. By Lemma 2.2, $I_{f,\underline{w}_f}/I^f$ acts on $X^f_{\underline{w}_f}(b)$ and this action stabilizes the fibers of p, as $I_{f,\underline{w}_f} \subseteq I$. We have to show that $i \mapsto xiI^f$ defines a bijection $I_{f,\underline{w}_f}/I^f \to p^{-1}(p(xI^f))$. If $xiI^f = xjI^f$ for $i,j \in I_{f,\underline{w}_f}$, then $iI^f = jI^f$, or with other words, the images of i,j in $I_{f,\underline{w}_f}/I^f$ are equal. This shows injectivity. Let $yI^f \in p^{-1}(p(xI^f))$, that is $xI = yI$. Then $i := x^{-1}y \in I$ and it remains to show that $i \in I_{f,\underline{w}_f}$. But $yI^f = xiI^f$, and hence by Lemma 2.2, $yI^f \in X^f_{\underline{w}_f.i}(b)$. But by assumption $yI^f \in X^f_{\underline{w}_f}(b)$, hence $\underline{w}_f.i = \underline{w}_f$, that is $i \in I_{f,\underline{w}_f}$. This finishes the proof. □

2.2.4. *Scheme structure.* Being subsets of $\bar{k}$-points of the Ind-schemes resp. Ind-perfect algebraic spaces $\mathscr{F}^f$, the sets $X^f_{\underline{w}_f}(b)$ can be equipped with the same kind of structure, when they can be shown to be locally closed. This is for example done by Rapoport for affine Deligne-Lusztig varieties, i.e., in the case $E = F$, $\Sigma = \{\sigma\}$, where σ is the Frobenius automorphism of $\breve{F}/F$ and $f = f_I$. Moreover, in that case he has shown that these varieties are also locally of finite type over k.

Here we investigate a sufficient condition, under which $X^f_{\underline{w}_f}(b)$ is locally closed in $\mathscr{F}^f$ and hence can be endowed with the induced reduced sub-Ind-scheme resp. sub-Ind perfect algebraic space structure. For a level subgroup $I^f \subseteq I$ and $\dot{w} \in G(\breve{E})$, the action of I^f on $I\dot{w}I/I^f$ by left multiplication factors through the action of a quotient, which is a finite-dimensional algebraic group over k (indeed, any subgroup $J \subseteq I^f \cap \dot{w}I^f\dot{w}^{-1}$, which is normal in I^f acts trivially on C^m_v).

Proposition 2.4. *Let $(G, E/F, \Sigma, f, \underline{w}_f, b)$ be as in Definition 2.1 with $f \geqslant f_I$. Assume that Σ is finite and contains a lift of a power of the Frobenius of $\breve{F}/F$. Assume that the action of a finite-dimensional quotient of I^f by left multiplication on $I\dot{\underline{w}}_f(\gamma)I/I^f$ possesses a geometric quotient in the sense of Mumford for any $\gamma \in \Sigma$, where $\dot{\underline{w}}_f(\gamma)$ is any preimage of $\underline{w}_f(\gamma)$ in $G(\breve{E})$. Then the subset*

$X^f_{\underline{w}_f}(b) \subseteq \mathscr{F}^f$ *is locally closed, and hence can be equipped with the induced reduced sub-Ind-scheme resp. sub-Ind perfect algebraic space structure.*

Proof. We write $X^{\Sigma,f}_{\underline{w}_f}(b)$ for $X^f_{\underline{w}_f}(b)$. Let $\underline{w}$ be the composition of $\underline{w}_f$ with the projection $D_{G_{\breve{E}},f} \twoheadrightarrow \tilde{W}$. If $\tilde{\sigma}$ is a lift of a power of the Frobenius of $\breve{F}/F$ to a continuous F-automorphism of $\breve{E}$, then [HV11] Corollary 6.5 shows that $X^{\{\tilde{\sigma}\},f_I}_{\underline{w}(\tilde{\sigma})}(b)$ is a locally closed subset of $\mathscr{F}$. Moreover, it is a scheme locally of finite type over a finite extension of k_E. Now, $X^{\Sigma,f_I}_{\underline{w}}(b)$ is the subset of $X^{\{\tilde{\sigma}\},f_I}_{\underline{w}(\tilde{\sigma})}(b)$ cut out by the finitely many locally closed conditions $x^{-1}b\gamma(x) \in \underline{w}(\gamma)$. This shows that $X^{\Sigma,f_I}_{\underline{w}}(b)$ is locally closed and locally of finite type over k_E.

Consider the preimage $\tilde{X}$ of $X^{\Sigma,f_I}_{\underline{w}}(b)$ under $\mathscr{F}^f \twoheadrightarrow \mathscr{F}$. The projection $\beta\colon LG \twoheadrightarrow \mathscr{F}^f$ admits sections locally for the étale topology. In the equal characteristic case this follows from [PR08] Theorem 1.4 and in the mixed characteristic case it is contained in the proof of [Zhu14] Lemma 1.3. Let $U \to \tilde{X}$ be étale, such that there is a section $s\colon U \to \beta^{-1}(U)$ of β. Consider the composition of the two morphisms

$$\psi\colon U \to \prod_{\gamma\in\Sigma} \beta^{-1}(U) \times U \to \prod_{\gamma\in\Sigma} \mathscr{F}^f,$$

where the first is given by $x \mapsto (s(x)^{-1}, b\gamma(x))_{\gamma\in\Sigma}$ and the second is just the componentwise restriction of the left multiplication action of $G(\breve{E})$ on $\mathscr{F}^f$. As U lies over $\tilde{X}$, this composed morphism factors through the inclusion $\prod_{\gamma\in\Sigma} I\dot{\underline{w}}_f(\gamma)I/I^f \subseteq \prod_{\gamma\in\Sigma}\mathscr{F}^f$. Let $\pi_\gamma\colon I\dot{\underline{w}}_f(\gamma)I/I^f \to D_{G_{\breve{E}},f}(\underline{w}(\gamma))$ denote the geometric quotient with respect to the left multiplication action by I^f. Let $\pi = \prod_{\gamma\in\Sigma}\pi_\gamma$. Then the composition

$$U \xrightarrow{\psi} \prod_{\gamma\in\Sigma} I\underline{w}_f(\gamma)I/I^f \overset{\pi}{\twoheadrightarrow} \prod_{\gamma\in\Sigma} D_{G_{\breve{E}},f}(\underline{w}(\gamma)). \tag{2.1}$$

is independent of the choice of the section s. Moreover, it sends a $\bar{k}$-point x to the tuple $(I^f x^{-1}b\gamma(x)I^f)_{\gamma\in\Sigma}$. Thus, étale locally, $X^{\Sigma,f}_{\underline{w}_f}(b)$ is just the preimage of a $\bar{k}$-point under the composite morphism (2.1). This finishes the proof. □

The condition about the existence of geometric quotients is satisfied in many cases. This will be studied in detail in a future work.

3. GL_2, tamely ramified case: geometry

From here and until the end of the article we set $G = \mathrm{GL}_2$ *and assume that* $\operatorname{char} F = p > 0$ *and* $p \neq 2$. After fixing some notation in Section 3.1, we study some extended affine Deligne-Lusztig varieties of Iwahori level in Section 3.2 and of higher levels in Section 3.3.

3.1. Some preliminaries in the GL_2-case.

3.1.1. *Basic notation.* Let t be an uniformizer of F, i.e., $F = k((t))$ and $\breve{F} = \bar{k}((t))$, with $\operatorname{char} k \neq 2$. Let E/F be a totally tamely ramified degree 2 extension and let $\breve{E} := E\breve{F}$. We can find an uniformizer $u \in E$ such that $u^2 = t$. Then $E = k((u))$, $\breve{E} = \bar{k}((u))$. For an algebraic extension M of F, we denote by $\mathcal{O}_M$ resp. $\mathfrak{p}_M$ its ring of integers resp. its maximal ideal. We denote by $U_M := \mathcal{O}_M^\times$ the units of $\mathcal{O}_M$, and for $n \geqslant 0$ we denote by U_M^n the n-units of M. We have $\mathcal{O}_E = k[\![u]\!]$, $\mathcal{O}_{\breve{E}} = \bar{k}[\![u]\!]$. We denote by v_t the t-adic valuation on F, normalized such that $v_t(t) = 1$ and extend it to a valuation of $\breve{E}$. Analogously, we denote by v_u the u-adic valuation on $\breve{E}$ normalzed such that $v_u(u) = 1$. The Galois group of $\breve{E}/F$ is generated by the two commuting elements σ, τ given by $\sigma(\sum_i a_i u^i) = \sum_i a_i^q u_i$ and $\tau(\sum_i a_i u^i) = \sum_i (-1)^i a_i u^i$. We set $\Sigma := \{\sigma, \tau\}$[1].

3.1.2. *Level subgroups.* We use the standard Iwahori subgroup $I \subseteq G(\breve{E})$ and the filtration of it given for $m \geqslant 0$ by

$$I^m := \begin{pmatrix} 1 + \mathfrak{p}_{\breve{E}}^{m+1} & \mathfrak{p}_{\breve{E}}^{m} \\ \mathfrak{p}_{\breve{E}}^{m+1} & 1 + \mathfrak{p}_{\breve{E}}^{m+1} \end{pmatrix} \subseteq I := \begin{pmatrix} \mathcal{O}_{\breve{E}}^{\times} & \mathcal{O}_{\breve{E}} \\ \mathfrak{p}_{\breve{E}} & \mathcal{O}_{\breve{E}}^{\times} \end{pmatrix}.$$

We write $\mathscr{F}$ for the affine flag manifold of $G_{\breve{E}}$ and $\mathscr{F}^m$ for its cover corresponding to I^m (see Section 2.1.8).

3.1.3. *Subgroups of* $G(F)$. Consider the $\mathcal{O}_F$-subalgebra

$$\mathfrak{J} := \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix}$$

of $M_2(\mathcal{O}_F)$. Then the units $U_{\mathfrak{J}}$ of $\mathfrak{J}$ form a compact subgroup of $G(F)$. Note that $U_{\mathfrak{J}} = I^{\Sigma}$ is the Iwahori subgroup of $G(F)$. Further, we fix the embedding of F-algebras

$$\iota\colon E \hookrightarrow M_2(F), \qquad \iota(u) = \varpi := \begin{pmatrix} & 1 \\ t & \end{pmatrix}$$

(here and further, omitted entries are zeros). Via ι we consider $E^{\times}$ as a subgroup of $G(F)$. The center of $G(F)$ is $\iota(F^{\times})$. Usually we omit ι from the notation and write $E^{\times} \subseteq G(F)$, etc. We have $U_E = U_{\mathfrak{J}} \cap E^{\times}$.

3.1.4. *Root subgroups.* The extended set of roots $\Phi \cup \{0\}$ consists of three elements. Denote by $+$ resp. $-$ the positive resp. the negative root. For $* \in \Phi \cup \{0\}$, we denote by

$$e_*\colon U_* \to G$$

the embedding of the root subgroup. Thus, for $a \in \breve{E}$, $e_+(a) = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$, $e_0(c,d) = \begin{pmatrix} c & \\ & d \end{pmatrix}$, etc.

3.1.5. *Slices of positive loops.* Consider the additive group $\mathbb{G}_a$ over $\breve{E}$. The group $\mathbb{G}_a(\breve{E})$ has a filtration by subgroups $\mathbb{G}_a(\breve{E})_\lambda := u^\lambda \bar{k}[\![u]\!]$ for $\lambda \in \mathbb{Z}_{\geqslant 0}$. There is a unique smooth model $\mathbb{G}_{a,\lambda}$ of $\mathbb{G}_a$ over $\mathcal{O}_{\breve{E}}$, such that $\mathbb{G}_{a,\lambda}(\mathcal{O}_{\breve{E}}) = \mathbb{G}_a(\breve{E})_\lambda$. For any $\mu \leqslant \lambda$, there exists a unique morphism $\mathbb{G}_{a,\lambda} \to \mathbb{G}_{a,\mu}$, inducing the natural embedding $u^\lambda k[\![u]\!] \hookrightarrow u^\mu k[\![u]\!]$ (see [BT84] Section 1.7). Let L^+ denote the positive loop group functor from $\bar{k}[\![u]\!]$-schemes to $\bar{k}$-schemes. For non-negative integers $\mu \leqslant \lambda$, we define

$$L_{[\mu,\lambda]}\mathbb{G}_a := L^+\mathbb{G}_{a,\mu}/L^+\mathbb{G}_{a,\lambda+1}.$$

This is a smooth $\bar{k}$-group of finite type and we have canonically $L_{[\mu,\lambda]}\mathbb{G}_a(\bar{k}) = u^\mu \bar{k}[u]/u^{\lambda+1}\bar{k}[u]$. Replacing $\mathbb{G}_a$ by $\mathbb{G}_m$ and using the filtration on $\mathbb{G}_m(\breve{E}) = \bar{k}((u))^{\times}$ given by $\mathbb{G}_m(\breve{E})_0 = \bar{k}[\![u]\!]^{\times}$, $\mathbb{G}_m(\breve{E})_\lambda = 1 + u^\lambda \bar{k}[\![u]\!]$ for $\lambda > 0$, we obtain in exactly the same way the $\bar{k}$-groups $L_{[\mu,\lambda]}\mathbb{G}_m$. All these groups uniquely descend to smooth group schemes over k.

[1] the more canonical choice of all Frobenius lifts $\Sigma' := \{\sigma, \sigma\tau\}$ was suggested to the author by P. Scholze. At least in the cases we study in this article, this choice will lead to the same results as Σ from the text.

Let now $0 \leqslant \mu \leqslant \lambda \leqslant \lambda'$. We have the natural projection, which comes from reduction mod $u^{\lambda+1}$:

$$p_{\lambda,\lambda'} \colon L_{[\mu,\lambda']}\mathbb{G}_a \to L_{[\mu,\lambda]}\mathbb{G}_a.$$

Moreover, there are group-theoretic sections

$$s_{\lambda,\lambda'} \colon L_{[\mu,\lambda]}\mathbb{G}_a \to L_{[\mu,\lambda']}\mathbb{G}_a$$

of $p_{\lambda,\lambda'}$, sending $\sum_{i=\mu}^{\lambda} a_i u^i$ to $\sum_{i=\mu}^{\lambda} a_i u^i + \sum_{i=\lambda+1}^{\lambda'} 0u^i$. Then the image of $s_{\lambda,\lambda'}$ is a closed subgroup scheme of $L_{[\mu,\lambda']}\mathbb{G}_a$ and we denote it by $L^{\leqslant\lambda}_{[\mu,\lambda']}\mathbb{G}_a$. For $a \in L_{[\mu,\lambda']}\mathbb{G}_a$, we use the shortcut notation $a|_\lambda := s_{\lambda,\lambda'}(p_{\lambda,\lambda'}(a))$.

3.1.6. *Schubert cells.* Let $\tilde{W}$ denote the extended affine Weyl group of $G_{\breve{E}}$ relative to the diagonal torus (as in Section 2.1.10). Let $v \in \tilde{W}$ and let $\dot{v} \in G(\breve{E})$ be a lift. We denote by $C_v = I\dot{v}I/I \subseteq \mathscr{F}$ the Schubert cell attached to v. There is a parametrization (depending of $\dot{v}$) of C_v given by:

$$\psi_{\dot{v}} \colon L_{[\mu,\mu+\ell(v)-1]}\mathbb{G}_a \xrightarrow{\sim} C_v, \quad a \mapsto e_{\pm}(a)\dot{v}I,$$

where $\mu \in \{0,1\}$ and the sign in $e_{\pm}$ depend on v, and $\ell(v)$ is the length of v. E.g., for $\dot{v} = \begin{pmatrix} & u^{-k} \\ u^k & \end{pmatrix}$ resp. $\dot{v} = \begin{pmatrix} & u^{-k} \\ u^{k+1} & \end{pmatrix}$, this parametrization is given by:

$$\psi_{\dot{v}} \colon L_{[1,\ell(v)]}\mathbb{G}_a \xrightarrow{\sim} C_v, \quad a \mapsto e_{-}(a)\dot{v}I, \tag{3.1}$$

where $a = \sum_{i=1}^{\ell(v)} a_i u^i$ (note that $\ell(v) = 2k-1$ resp. $\ell(v) = 2k$).

3.1.7. *Schubert cells in higher levels.* For $m \geqslant 0$, let $\mathrm{pr}_m \colon \mathscr{F}^m \to \mathscr{F}$ be the natural projection. Let $v \in \tilde{W}$ with lift $\dot{v}$ to $G(\breve{E})$. Let $C_v^m := \mathrm{pr}_m^{-1}(C_v)$. We give a parametrization of C_v^m for $v, \dot{v}$ as in (3.1) (for other $v \in \tilde{W}$ the parametrization is defined similarly). There is a well-defined injective morphism $L_{[1,\ell(v)+m]}\mathbb{G}_a \to C_v^m$ given by $a \mapsto e_{-}(a)\dot{v}I$. Using it we get a diagram

$$\begin{array}{ccccc} L^{\leqslant\ell(v)}_{[1,\ell(v)+m]}\mathbb{G}_a & \hookrightarrow & L_{[1,\ell(v)+m]}\mathbb{G}_a & \hookrightarrow & C_v^m \\ & \nwarrow{\scriptstyle\sim} & \uparrow & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle \mathrm{pr}_m} \\ & & L_{[1,\ell(v)]}\mathbb{G}_a & \xrightarrow{\sim} & C_v \end{array}$$

where the lower horizontal map is $\psi_{\dot{v}}$, the left vertical map is $s_{\ell(v),\ell(v)+m}$, and the section s to pr_m is defined such that the diagram commutes. As $C_v^m \twoheadrightarrow C_v$ is a I/I^m-torsor, s induces the trivialization isomorphism $C_v \times I/I^m \xrightarrow{\sim} C_v^m$ given by $x, i \mapsto s(x)i$. Using a parametrization of I/I^m, we obtain the following explicit parametrization of C_v^m (depending on $\dot{v}$):

$$\begin{array}{rcl} \psi_{\dot{v}}^m \colon L^{\leqslant\ell(v)}_{[1,\ell(v)+m]}\mathbb{G}_a \times L_{[0,m]}\mathbb{G}_m^2 \times L_{[0,m-1]}\mathbb{G}_a \times L_{[1,m]}\mathbb{G}_a & \xrightarrow{\sim} & C_v^m = I\dot{v}I/I^m \\ (a, C, D, A, B) & \mapsto & e_{-}(a)\dot{v}e_0(C,D)e_{+}(A)e_{-}(B)I^m \end{array} \tag{3.2}$$

3.1.8. *Spaces of double cosets.* Let $m \geqslant 0$ be an integer and let $\dot{w} = \begin{pmatrix} & u^{-n} \\ u^n & \end{pmatrix}$ with $n > 0$ with image $w \in \tilde{W}$. An explicit parametrization of the set of double cosets $D_{G_{\breve{E}},m}(w) = I^m \backslash IwI/I^m$ is given by

$$\begin{array}{rcl}\phi^m_{\dot w}\colon L_{[0,m]}\mathbb{G}_m^2(\bar k)\times L_{[1,m]}\mathbb{G}_a^2(\bar k) & \xrightarrow{\sim} & D_{G_{\breve E},m}(w)=I^m\backslash IwI/I^m\\ ((C,D),(E,B)) & \mapsto & I^m e_-(E)\dot w e_0(C,D)e_-(B)I^m.\end{array}\tag{3.3}$$

We use this parametrization to give the set $D_{G_{\breve E},m}(w)$ the structure of a smooth variety over k (which is in fact isomorphic to an affine space).

Lemma 3.1. *The natural projection $p\colon C_v^m = I\dot wI/I^m \twoheadrightarrow I^m\backslash I\dot wI/I^m = D_{G_{\breve E},m}(w)$ is a geometric quotient.*

Proof. Let a, C, D, A, B be the coordinates on C_v^m given by (3.2) and let C', D', E', B' be the coordinates on $D_{G_{\breve E},m}(w)$ given by (3.3). Then p is given by $E' \mapsto a + u^{2n+1}CD^{-1}A \mod u^{m+1}$, $C' \mapsto C$, $D' \mapsto D$, $B' \mapsto B$. In particular, p is a separable morphism of varieties.

The action of I^m factors through an appropriate quotient, which is finite dimensional over k (indeed, any subgroup $J \subseteq I^m \cap \dot wI^m\dot w^{-1}$, which is normal in I^m acts trivially on C_v^m), thus taking this quotient instead of I^m, we may replace I^m by some finite-dimensional algebraic group over k. Now p is surjective with fibers being precisely the orbits of the action, and both, C_v^m and $D_{G_{\breve E},m}(w)$ are isomorphic to affine spaces. Now [Bor91] Proposition 6.6 finishes the proof. □

3.1.9. *Bruhat-Tits buildings.* (cf. Section 2.1.3) For any finite extension M of F or $\breve F$, the Bruhat-Tits building $\mathscr{B}_M$ of G over M is an one-dimensional simplicial complex and it carries a $\mathrm{Gal}_{M/F}$-action if M/F is Galois. We identify the subcomplex $\mathscr{B}_{\breve E}^{\langle\sigma\rangle}$ of $\mathscr{B}_{\breve E}$ with $\mathscr{B}_E$. Moreover, as $\breve E/\breve F$ is tamely ramified, the embedding $\mathscr{B}_{\breve F} \hookrightarrow \mathscr{B}_{\breve E}$ identifies $\mathscr{B}_{\breve F}$ with $\mathscr{B}_{\breve E}^{\langle\tau\rangle}$ as subsets. The simplicial complex $\mathscr{B}_{\breve E}^{\langle\tau\rangle}$ is obtained from $\mathscr{B}_{\breve F}$ by adding an extra vertex in the middle of each alcove. Thus any alcove of $\mathscr{B}_{\breve F}$ 'contains' two alcoves of $\mathscr{B}_{\breve E}^{\langle\tau\rangle}$. Any vertex of $\mathscr{B}_{\breve F}$ has an associated type in $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$, which is defined as the t-valuation modulo 2 of the determinant of the lattice, representing it. Similarly, we attach to any vertex of $\mathscr{B}_{\breve E}$ its *relative type* in $\frac{1}{2}\mathbb{Z}/\mathbb{Z} = v_t(\breve E^\times)/v_t(\breve F^\times)$, defined as the class modulo $\mathbb{Z}$ of the t-valuation of the determinant of the representing lattice. The same considerations also apply to the relationship between the σ-stable subcomplexes $\mathscr{B}_F \rightsquigarrow \mathscr{B}_E^{\langle\tau\rangle} \subseteq \mathscr{B}_E$.

3.1.10. *Vertex of departure.* In the proofs below we have to use the simple combinatorics of the tree $\mathscr{B}_{\breve E}$. Therefore, following [Reu02] we introduce the notion of the vertex of departure. Let $\mathscr{C} \subseteq \mathscr{B}_{\breve E}$ be a connected non-empty subcomplex. For any alcove C of $\mathscr{B}_{\breve E}$, which is not contained in $\mathscr{C}$, there is a unique gallery $\Gamma = (C_0, C_1, \dots, C_d)$ of minimal length d, such that $C_0 = C$ and C_d is not contained in $\mathscr{C}$ and has a (unique) vertex which is contained in $\mathscr{C}$. This vertex of C_d is called the *vertex of departure* of C from $\mathscr{C}$. The same considerations can also be applied to $\mathscr{B}_E$ and a connected subcomplex.

3.1.11. *Connected components of $\mathscr{F}$.* It is well-known ([PR08] Theorem 5.1) that $v_u \circ \det\colon G(\breve E) \to \mathbb{Z}$ induces an isomorphism

$$\pi_0(v_u \circ \det)\colon \pi_0(\mathscr{F}) \xrightarrow{\sim} \mathbb{Z}.$$

Denote the connected component of $\mathscr{F}$ corresponding to the integer i by $\mathscr{F}^{(i)}$. Note that the alcoves of $\mathscr{B}_{\breve E}$ can be identified with $\bar k$-points of $\mathscr{F}^{(0)}$ and that there are (non-canonical) isomorphisms $\mathscr{F}^{(0)} \xrightarrow{\sim} \mathscr{F}^{(i)}$. Further, denote by $\mathscr{F}^{\equiv i}$ the preimage under $v_u \circ \det$ in $\mathscr{F}$ of $2\mathbb{Z} + i$. This divides

$\mathscr{F}$ in two disjoint sub-Ind-schemes $\mathscr{F} = \mathscr{F}^{\equiv 0} \dot\cup \mathscr{F}^{\equiv 1}$. Note that the subgroup $G(F)$ of $G(\breve{E})$ acts transitively (by left multiplication) on the set of connected components of $\mathscr{F}^{\equiv i}$ and that the action of $\begin{pmatrix} & 1 \\ u & \end{pmatrix} \in G(\breve{E})$ interchanges the two parts $\mathscr{F}^{\equiv 0}$ and $\mathscr{F}^{\equiv 1}$ of $\mathscr{F}$.

3.1.12. *Commutation relations.* We will need the following relations: for $a, b \in \breve{E}$ with $N := 1+ab \neq 0$, we have

$$\begin{aligned} e_+(a)e_-(b) &= e_-(bN^{-1})e_+(aN)e_0(N, N^{-1}) \\ e_-(b)e_+(a) &= e_0(N^{-1}, N)e_+(aN)e_-(bN^{-1}). \end{aligned} \tag{3.4}$$

3.2. **Structure of $X_{\underline{w}}(1)$ at the Iwahori-level in some cases.**

We write $X_{\underline{w}}(1)$ instead of $X_{\underline{w}}^{f_I}(1)$.

Notation 3.2. Let $w := \begin{pmatrix} & u^{-n} \\ u^n & \end{pmatrix} \in \tilde{W}$ with some integer $n > 0$ and let $\underline{w}\colon \Sigma \to \tilde{W}$ be defined by $\underline{w}(\sigma) = 1$ and $\underline{w}(\tau) = w$. Further, we set

$$\dot{v} = \dot{v}(w) := \begin{cases} \begin{pmatrix} & u^{-k} \\ u^k & \end{pmatrix} & \text{if } n = 2k-1 > 0 \text{ is odd,} \\ \begin{pmatrix} & u^{-k} \\ u^{k+1} & \end{pmatrix} & \text{if } n = 2k > 0 \text{ is even.} \end{cases}$$

In both cases let v be the image of $\dot{v}$ in $\tilde{W}$ and let D_w^τ be the set of k-rational points of C_v lying in the locus $a_1 \neq 0$ with respect to the coordinates (3.1). In particular, D_w^τ is just a finite discrete union of k-rational points.

It will follow from the proof of Proposition 3.4 (or can be seen directly) that D_w^τ is stable under the left multiplication action by $U_{\mathfrak{J}}$ on $\mathscr{F}$.

Remark 3.3. If in Notation 3.2, n is odd, then C_v is contained in the connected component $\mathscr{F}^{(0)}$ of $\mathscr{F}$, i.e., its points can be seen as alcoves in $\mathscr{B}_{\breve{E}}$. Moreover, they all are k-rational, hence lie in $\mathscr{B}_E$. Let $P_{1/2}$ be the vertex of the base alcove (= the alcove corresponding to I) of $\mathscr{B}_E$ with relative type $\frac{1}{2}$ (see Section 3.1.9). Then D_w^τ corresponds to the set of the alcoves contained in $\mathscr{B}_E$, having relative position v to the base alcove and having $P_{1/2}$ as the vertex of departure from $\mathscr{B}_E^{\langle\tau\rangle}$.

Proposition 3.4. *Let $\underline{w}$ be as in Notation 3.2. There is an isomorphism*

$$X_{\underline{w}}(1) \cong \coprod_{g \in G(F)/U_{\mathfrak{J}}} gD_w^\tau$$

equivariant for the left $G(F)$-action. In particular, $X_{\underline{w}}(1)$ is a zero-dimensional reduced k-variety, containing only k-rational points.

Proof. Let first n be odd. The natural action of $G(F)$ on $X_{\underline{w}}(1)$ induces a transitive action of $G(F)$ on the set

$$\{X_{\underline{w}}(1) \cap \mathscr{F}^{(2i)} \colon i \in \mathbb{Z}\}$$

of subsets of $X_{\underline{w}}(1)$. This follows by taking any element $g \in G(F)$ with $v_u(\det(g)) = 2$. The stabilizer of $\mathscr{F}^{(0)}$ in $G(F)$ is

$$H := (v_u \circ \det)^{-1}(0) \subseteq G(F),$$

where $v_u \circ \det$ is the map $v_u \circ \det\colon G(\breve{E}) \mapsto \mathbb{Z}$. We deduce

$$X_{\underline{w}}(1) \cap \mathscr{F}^{\equiv 0} = \coprod_{g \in G(F)/H} g.(X_{\underline{w}}(1) \cap \mathscr{F}^{(0)}). \tag{3.5}$$

The $\bar{k}$-rational points of $X_{\underline{w}}(1) \cap \mathscr{F}^{(0)}$ can be identified with alcoves in $\mathscr{B}_{\breve{E}}$, which satisfy two conditions (defined by $\underline{w}(\sigma) = 1$ and $\underline{w}(\tau) = w$) on the relative position with respect to their σ- resp. τ-translate. The σ-condition simply assures that each of the alcoves contained in $X_{\underline{w}}(1)$ is σ-stable, i.e., is contained in $\mathscr{B}_E$. Let $(\mathscr{B}_E^{\langle\tau\rangle})^{(1/2)}$ be the set of all vertices of $\mathscr{B}_E^{\langle\tau\rangle}$ of relative type $\frac{1}{2}$. For an alcove C of $\mathscr{B}_E$, which is not contained in $\mathscr{B}_E^{\langle\tau\rangle}$, let $\Gamma_{C,\tau}$ denote the unique minimal gallery connecting C with $\mathscr{B}_E^{\langle\tau\rangle}$. Taking into account the types of the involved vertices, we deduce (exactly as in [Iva13]) that

$$X_{\underline{w}}(1) \cap \mathscr{F}^{(0)} = \coprod_{P \in (\mathscr{B}_E^{\langle\tau\rangle})^{(1/2)}} \left\{ C\colon \begin{array}{c} C \text{ is an alcove in } \mathscr{B}_E \text{ with vertex of departure from} \\ \mathscr{B}_E^{\langle\tau\rangle} \text{ equal to } P \text{ and length of } \Gamma_{C,\tau} \text{ equal to } n-1 \end{array} \right\}, \tag{3.6}$$

with n as in Notation 3.2. Let $P_{1/2}$ be the vertex of type $\frac{1}{2}$ of the base alcove of $\mathscr{B}_E$. Observe that for $P = P_{1/2}$, the set of alcoves C on the right hand side of (3.6) is simply D_w^τ (cf. Remark 3.3).

Now, $(\mathscr{B}_E^{\langle\tau\rangle})^{(1/2)}$ can be canonically identified with the set of alcoves in $\mathscr{B}_F$ (see Section 3.1.9). The natural action of H on $\mathscr{B}_F$ induces a transitive action of H on the set of alcoves of $\mathscr{B}_F$, and the stabilizer of the base alcove in $\mathscr{B}_F$ is precisely $U_{\mathfrak{J}} \subseteq H$. Combining these observations, we obtain a natural H-equivariant bijection

$$H/U_{\mathfrak{J}} \cong (\mathscr{B}_E^{\langle\tau\rangle})^{(1/2)}, \quad hU_{\mathfrak{J}} \mapsto hP_{1/2}. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we deduce

$$X_{\underline{w}}(1) \cap \mathscr{F}^{\equiv 0} = \coprod_{g \in G(F)/H} g.\left(X_{\underline{w}}(1) \cap \mathscr{F}^{(0)}\right) = \coprod_{g \in G(F)/H} g.\left(\coprod_{h \in H/U_{\mathfrak{J}}} hD_w^\tau\right) = \coprod_{g \in G(F)/U_{\mathfrak{J}}} gD_w^\tau.$$

It remains to show that $X_{\underline{w}}(1) \cap \mathscr{F}^{\equiv 1} = \varnothing$. This can be done as follows: let $h = \begin{pmatrix} & 1 \\ u & \end{pmatrix}$. By Lemma 2.2, the map $xI \mapsto xhI$ defines an isomorphism

$$X_{\underline{y}}(1) \cap \mathscr{F}^{\equiv 1} \xrightarrow{\sim} X_{\underline{y}.h}(1) \cap \mathscr{F}^{\equiv 0} \tag{3.8}$$

for any $\underline{y}\colon \Sigma \to \tilde{W}$, where $(\underline{y}.h)(\gamma) := h^{-1}\underline{y}(\gamma)\gamma(h)$ for $\gamma \in \Sigma$. Thus it is enough to show that $X_{\underline{w}.h}(1) \cap \mathscr{F}^{\equiv 0} = \varnothing$, where $(\underline{w}.h)(\sigma) = 1$, $(\underline{w}.h)(\tau) = h^{-1}w\tau(h) = \begin{pmatrix} & u^{n-1} \\ u^{1-n} & \end{pmatrix} \in \tilde{W}$. This follows from (3.5) and $X_{\underline{w}.h}(1) \cap \mathscr{F}^{(0)} = \varnothing$. This last follows from the combinatorics of $\mathscr{B}_E$ as $n-1$ is even: one has to use the fact that a vertex P of $\mathscr{B}_E$ of relative type 0 cannot be the vertex of departure from $\mathscr{B}_E^{\langle\tau\rangle}$ for a non-τ-stable alcove C of $\mathscr{B}_E$, as all alcoves having P as a vertex lie in $\mathscr{B}_E^{\langle\tau\rangle}$.

Let now n be even. Applying the isomorphism (3.8) with h replaced by h^{-1}, we reduce to determining $X_{\underline{y}}(1)$ with $\underline{y}(\sigma) = 1$ and $\underline{y}(\tau) = \begin{pmatrix} & u^{n-1} \\ u^{1-n} & \end{pmatrix}$, where we can proceed exactly as in the case n odd (after replacing n by $-n$). □

3.3. **Structure of $X^m_{\underline{w}_m}(1)$ in some cases.**

Continuing with notations from preceding sections, we now study higher level covers of $X_{\underline{w}}(1)$. For $x \in G(\breve{E})$ we denote the image of x in $D_{G_{\breve{E}},m}$ again by x, if no ambiguity can occur.

Notation 3.5. Let n, w, $\underline{w}$ be as in Notation 3.2. We define the lift $\dot{w} \in G(\breve{E})$ of w by

$$\dot{w} := \begin{cases} \begin{pmatrix} & (-1)^k u^{1-2k} \\ (-1)^{k+1} u^{2k-1} & \end{pmatrix} & \text{if } n = 2k-1 > 0 \text{ is odd,} \\ \begin{pmatrix} & (-1)^k u^{-2k} \\ (-1)^k u^{2k} & \end{pmatrix} & \text{if } n = 2k > 0 \text{ is even.} \end{cases}$$

Moreover, let $m \geqslant 1$ be an odd integer. Let $\underline{w}_m \colon \Sigma \to D_{G_{\breve{E}},m}$ be the lift of $\underline{w}$ defined by

$$\underline{w}_m(\sigma) := 1, \qquad \underline{w}_m(\tau) := \dot{w}. \tag{3.9}$$

We have $J_1(F) = G(F)$ and hence by Section 2.2 we obtain the group actions

$$G(F) \circlearrowright X^m_{\underline{w}_m}(1) \circlearrowleft \tilde{I}_{m,\underline{w}_m}/I^m. \tag{3.10}$$

Lemma 3.6. *Let $m \geqslant 1$ be an odd integer. There is an isomorphism*

$$\tilde{I}_{m,\underline{w}_m}/I^m \xrightarrow{\sim} \left\{ \begin{pmatrix} i_1 & i_2 \\ 0 & \tau(i_1) \end{pmatrix} : i_1 \in E^\times/U_E^{m+1}, i_2 \in E/\mathfrak{p}_E^m, v_u(i_2) \geqslant v_u(i_1) \right\} \subset Z(\breve{E})I/I^m.$$

In particular, there is a surjection induced by the projection onto the diagonal part

$$\tilde{I}_{m,\underline{w}_m}/I^m \twoheadrightarrow E^\times/U_E^{m+1}, \tag{3.11}$$

under which $I_{m,\underline{w}_m}/I^m$ maps onto U_E/U_E^{m+1}.

Proof. The proof is an easy computation. □

Recall from Section 3.1.3 that we see $E^\times$ as a subgroup of $G(F)$. This defines a left multiplication action of $E^\times$ on $X^m_{\underline{w}_m}(1)$ (do not confuse this $E^\times$ with the quotient $E^\times$ of $\tilde{I}_{m,\underline{w}_m}$ acting on the right).

Definition 3.7. With notation from Notations 3.2,3.5, we define the discrete subscheme $Y^m_{\dot{w}}$ of $C^m_v \subseteq \mathscr{F}^m$ as follows. Let $a = \sum_{i=1}^{\ell(v)} a_i u^i \in L^{\leqslant \ell(v)}_{[1,\ell(v)+m]}\mathbb{G}_a(\bar{k})$ be as used in the parametrization (3.2) of C^m_v. Put $R := u^{-1}(\tau(a) - a) \mod u^{m+1}$. We define $Y^m_{\dot{w}}$ to be the subscheme of C^m_v defined in coordinates $\psi^m_{\dot{v}}$ from (3.2) by the following conditions:

$$\begin{array}{rcl} a, A, C & & \text{are } k\text{-rational} \\ a_1 & \neq & 0 \quad \text{(in particular, } R \text{ is invertible)} \\ B & = & C\tau(C)^{-1}u^n \\ D & = & R^{-1}\tau(C)(1 + C\tau(C)^{-1}Au^n - C^{-1}\tau(C)\tau(A)u^n) \end{array} \tag{3.12}$$

(both last equations take place in $k[u]/(u^{m+1})$). In particular, $Y^m_{\dot w}$ is just a finite discrete union of k-rational points. Moreover, let $y_i := e_0(u^i, (-u)^i)$ and define $\tilde Y^m_{\dot w} \subseteq \mathscr{F}^m$ to be (disjoint) union

$$\tilde Y^m_{\dot w} := \coprod_{i\in\mathbb{Z}} Y^m_{\dot w}\cdot y_i.$$

It will follow from the proof of Theorem 3.9 that the right multiplication action of I/I^m on C^m_v restricts to an action of $I_{m,\underline{w}_m}/I^m$ on $Y^m_{\dot w}$, which in turn extends to a right $\tilde I_{m,\underline{w}_m}/I^m$-action on $\tilde Y^m_{\dot w}$.

Remark 3.8. The varieties $Y^m_{\dot w}, \tilde Y^m_{\dot w}$ depend on $\dot w$, not only on w, but the choice of the lift $\dot w$ of w is not essential: another choices would give either empty varieties or varieties isomorphic to those attached to $\dot w$. The full study of these choices is not relevant for the goals of this article, so we restrict our attention to our choice $\dot w$.

Theorem 3.9. *Let $m \geqslant 1$ be an odd integer. With notation as in Definition 3.7 assume that $m \leqslant \ell(w) = 2n-1$. Then $\tilde Y^m_{\dot w}$ (resp. $Y^m_{\dot w}$) is invariant under the left $E^\times U_{\mathfrak{J}}$- (resp. $U_{\mathfrak{J}}$-)action and the right $\tilde I_{m,\underline{w}_m}/I^m$- (resp. $I_{m,\underline{w}_m}/I^m$-)action and there is an isomorphism*

$$X^m_{\underline{w}_m}(1) \cong \coprod_{g\in G(F)/E^\times U_{\mathfrak{J}}} g\tilde Y^m_{\dot w}$$

equivariant for the left $G(F)$- and right $\tilde I_{m,\underline{w}_m}/I^m$-actions. In particular, $X^m_{\underline{w}_m}(1)$ is a zero-dimensional reduced k-variety, containing only k-rational points.

Proof. We claim that $X^m_{\underline{w}_m}(1) \cong \coprod_{g\in G(F)/U_{\mathfrak{J}}} gY^m_{\dot w}$. As the natural projection $\mathscr{F}^m \twoheadrightarrow \mathscr{F}$ restricts to a $G(F)$-equivariant projection $\mathrm{p}_m\colon X^m_{\underline{w}_m}(1) \to X_{\underline{w}}(1)$, Proposition 3.4 shows

$$X^m_{\underline{w}_m}(1) \cong \coprod_{g\in G(F)/U_{\mathfrak{J}}} \mathrm{p}_m^{-1}(gD^\tau_w) = \coprod_{g\in G(F)/U_{\mathfrak{J}}} g.\mathrm{p}_m^{-1}(D^\tau_w).$$

Now, Lemma 3.10 implies that $\mathrm{p}_m^{-1}(D^\tau_w) = X^m_{\underline{w}_m}(1)\cap C^m_v = Y^m_{\dot w}$, hence the isomorphism claimed in the theorem. As $\mathrm{p}_m^{-1}(D^\tau_w) \subseteq X^m_{\underline{w}_m}(1)$ is stable under the right $I_{m,\underline{w}_m}$- and left $U_{\mathfrak{J}}$-actions, the above shows that $Y^m_{\dot w}$ also is. As $\tilde I_{m,\underline{w}_m} = \coprod_i I_{m,\underline{w}_m} y_i$, with $y_i := e_0(u^i, (-u)^i)$ the theorem now follows from Lemma 3.12. ☐

Lemma 3.10. *Let $\dot x I^m = \psi^m_v(a, C, D, A, B)$ be a point of C^m_v. Assume $m \leqslant \ell(w) = 2n-1$.*

(i) *Let $R := u^{-1}(\tau(a) - a) \mod u^{m+1}$. Then*

$$\mathrm{inv}^m(\dot x I^m, \tau(\dot x I^m)) = \dot w \Leftrightarrow \begin{cases} a_1 & \neq 0 \quad \text{(i.e., } R \text{ is invertible)} \\ B & = u^n C\tau(C)^{-1} \\ D & = R^{-1}\tau(C)(1 + u^n C\tau(C)^{-1}A + u^n(-1)^n C^{-1}\tau(C)\tau(A)) \end{cases} \tag{3.13}$$

(the equations on the right hand side take place in $k[u]/u^{m+1}$).

(ii) *Suppose, $\dot x I^m$ satisfies the equations on the right hand side of* (3.13). *Then:*

$$\mathrm{inv}^m(\dot x I^m, \sigma(\dot x I^m)) = 1 \Leftrightarrow a, A, B, C, D \text{ are } k\text{-rational.}$$

Proof. Choose some lifts of a, A, B, C, D to elements of $k[\![u]\!]$. We denote them by the same letters. (i): A computation shows that the I-double coset of $\dot x^{-1}\tau(\dot x)$ is equal to the I-double coset of the element $e_+(u^{-n}R)$, and $\dot w$ lies in this double coset if and only if R is invertible. This is clearly necessary for the left hand side of part (i) to hold. Thus we can assume in the following that $a_1 \neq 0$, i.e., that R is invertible. In $G(\breve{E})$ one easily computes (independently of the parity of n):

$$\dot{v}^{-1}e_-(-a)e_-(\tau(a))\tau(\dot{v}) = e_-(u^nR^{-1})e_0(R,R^{-1})\dot{w}e_-((-1)^{n+1}u^nR^{-1}). \tag{3.14}$$

In the rest of the proof we write $x \sim y$ to express that x, y lie in the same I^m-double coset. Using (3.14) we compute:

$$\begin{aligned} \dot{x}^{-1}\tau(\dot{x}) &\sim e_-(-B)e_+(-A)e_0(C^{-1},D^{-1})\cdot[e_-(u^nR^{-1})e_0(R,R^{-1})\cdot\dot{w}\cdot e_-((-1)^{n+1}u^nR^{-1})]\cdot\ldots \\ \ldots &\cdot e_0(\tau(C),\tau(D))e_+(\tau(A))e_-(\tau(B)) \\ &\sim e_-(-B)e_+(-A)e_-(u^nCD^{-1}R^{-1})\cdot\dot{w}\cdot e_0(D^{-1}R^{-1}\tau(C),C^{-1}R\tau(D))\ldots \\ \ldots &\; e_-((-1)^{n+1}u^n\tau(C)\tau(D)^{-1}R^{-1})e_+(\tau(A))e_-(\tau(B)). \end{aligned} \tag{3.15}$$

Let $N := 1 - u^nCD^{-1}R^{-1}A$. We apply formulas (3.4) to deduce:

$$\begin{aligned} I^me_-(-B)e_+(-A)e_-(u^nCD^{-1}R^{-1}) &= I^me_-(-B+u^nCD^{-1}R^{-1}N^{-1})e_+(-AN)e_0(N,N^{-1}) \\ &= I^me_-(-B+u^nCD^{-1}R^{-1})e_+(-AN)e_0(N,N^{-1}), \end{aligned} \tag{3.16}$$

where the last equation is true, since $u^nN^{\pm1} \equiv u^n \mod u^{m+1}$, which in turn follows from $2n-1 \geqslant m$. Noting that the product of the last three matrices in the last expression in (3.15) is equal to τ applied to the inverse of the product of the first three (use $\tau(R) = R$), we deduce from (3.15) and (3.16):

$$\begin{aligned} \dot{x}^{-1}\tau(\dot{x}) &\sim e_-(-(B-u^nCD^{-1}R^{-1}))e_+(-NA)e_0(N,N^{-1})\cdot\dot{w}\cdot\ldots \\ \ldots &\; e_0(\tau(C)D^{-1}R^{-1},C^{-1}\tau(D)R)e_0(\tau(N)^{-1},\tau(N))e_+(\tau(NA))\ldots \\ \ldots &\; e_-(\tau(B-u^nCD^{-1}R^{-1})). \end{aligned}$$

Now we bring the term $e_+(-NA)$ to the right side of $\dot{w}$, without modifying the other terms and it can be canceled there, as it lands in I^m and I^m is normal in I. Here we again used $2n-1 \geqslant m$. Analogously, we cancel the term $e_+(\tau(NA))$ by bringing it to the left side of $\dot{w}$. Now put the three e_0-terms together and obtain

$$\begin{aligned} \dot{x}^{-1}\tau(\dot{x}) &\sim e_-(-(B-u^nCD^{-1}R^{-1}))\cdot\dot{w}\cdot\ldots \\ \ldots &\; e_0(\tau(C)D^{-1}R^{-1}N^{-1}\tau(N)^{-1},C^{-1}\tau(D)RN\tau(N))e_-(\tau(B-u^nCD^{-1}R^{-1})) \end{aligned} \tag{3.17}$$

The left hand side of (3.13) is equivalent to $\dot{x}^{-1}\tau(\dot{x}) \sim \dot{w}$, which by (3.17) and Section 3.1.8 is equivalent to

$$\begin{aligned} B-u^nCD^{-1}R^{-1} &\equiv 0 \mod u^{m+1} \\ \tau(C)D^{-1}R^{-1}N^{-1}\tau(N)^{-1} &\equiv 1 \mod u^{m+1} \\ C^{-1}\tau(D)RN\tau(N) &\equiv 1 \mod u^{m+1}. \end{aligned} \tag{3.18}$$

Using $\tau^2 = 1$ and $\tau(R) = R$, we see that the second and the third equations are equivalent. Hence the third can be ignored. Assume first $n \geqslant m+1$. Then it is trivial to see that (3.18) is equivalent to the right hand side of (3.13). Assume now $m \geqslant n$. Then, as $n \geqslant m+1-n > 0$ and $N \equiv 1 \mod u^n$, the second equation of (3.18) shows

$$D \equiv \tau(C)R^{-1} \mod u^{m+1-n}. \tag{3.19}$$

Using this and $N = 1 - u^n CD^{-1}R^{-1}A$ it is now easy to deduce the equivalence of (3.18) and the right hand side of (3.13).

(ii): The implication '$\Leftarrow$' is immediate. We prove '$\Rightarrow$'. Assume $\dot{x}^{-1}\sigma(\dot{x}) \in I^m$. In particular, the I-double coset of $\dot{x}^{-1}\sigma(\dot{x})$ is I. This is equivalent to a being k-rational, and we deduce $\dot{v}^{-1}e_-(\sigma(a)-a)\sigma(\dot{v}) = 1$. Setting $G := 1 + AB$, we compute

$$\begin{aligned} \dot{x}^{-1}\sigma(\dot{x}) &= e_-(-B)e_+(-A)e_0(C^{-1}, D^{-1})e_0(\sigma(C), \sigma(D))e_+(\sigma(A))e_-(\sigma(B)) \\ &= \begin{pmatrix} C^{-1}\sigma(C)\sigma(G) - D^{-1}\sigma(D)\sigma(B)A & C^{-1}\sigma(C)\sigma(A) - D^{-1}\sigma(D)A \\ D^{-1}\sigma(D)G\sigma(B) - C^{-1}\sigma(C)B\sigma(G) & D^{-1}\sigma(D)G - C^{-1}\sigma(C)B\sigma(A) \end{pmatrix}. \end{aligned} \tag{3.20}$$

We have to show that B, C, D resp. A are σ-stable mod u^{m+1} resp. mod u^m. If $n \geqslant m+1$, we have $B \equiv 0 \mod u^{m+1}$ and $G \equiv 1 \mod u^{m+1}$ by assumption and part (i), and the claimed equivalence is trivial. Assume $m \geqslant n$. By assumption and as $n \geqslant m+1-n > 0$, we know that

$$\begin{aligned} G &= 1 + AB \equiv 1 + u^n C\tau(C)^{-1}A \mod u^{m+1} \\ B &= u^n C\tau(C)^{-1} \equiv 0 \mod u^n, \end{aligned}$$

and we deduce from (3.20)

$$C^{-1}\sigma(C)\sigma(1 + u^n C\tau(C)^{-1}A) \equiv 1 + D^{-1}\sigma(D)u^n\sigma(C\tau(C)^{-1})A \mod u^{m+1} \tag{3.21}$$

$$C^{-1}\sigma(C)\sigma(A) \equiv D^{-1}\sigma(D)A \mod u^m \tag{3.22}$$

$$D^{-1}\sigma(D)(1 + u^n C\tau(C)^{-1}A) \equiv 1 + C^{-1}\sigma(C)u^n C\tau(C)^{-1}\sigma(A) \mod u^{m+1} \tag{3.23}$$

$$\begin{aligned} &D^{-1}\sigma(D)(1 + u^n C\tau(C)^{-1}A)\sigma(u^n C\tau(C)^{-1}) \equiv \dots \\ &\qquad \dots \equiv C^{-1}\sigma(C)u^n C\tau(C)^{-1}(1 + u^n\sigma(C)\sigma(\tau(C))^{-1}\sigma(A)) \mod u^{m+1}. \end{aligned} \tag{3.24}$$

From (3.21), (3.23) and $m \geqslant n$, we deduce $C \equiv \sigma(C) \mod u^n$ and $D \equiv \sigma(D) \mod u^n$. Using this and $m \geqslant n$, we deduce from (3.22) that $A \equiv \sigma(A) \mod u^n$. Using these congruences and $n \geqslant m+1-n > 0$, we may replace $\sigma(A), \sigma(C), \sigma(D)$ by A, C, D in all terms which are $\equiv 0 \mod u^n$ in equations (3.21)-(3.24). Then (3.21) simplifies to $C^{-1}\sigma(C) \equiv 1 \mod u^{m+1}$ and (3.23) to $D^{-1}\sigma(D) \equiv 1 \mod u^{m+1}$. Using this, we deduce $\sigma(A) \equiv A \mod u^m$ from equation (3.22). The σ-stability of B follows by assumption and (3.13). This finishes the proof of the lemma. ☐

Remark 3.11.

(i) Lemma 3.10(ii) shows, that one could have started directly with E/F and $\Sigma = \{\tau\}$, instead of $\breve{E}/F$ and $\Sigma = \{\sigma, \tau\}$ as in the text, to obtain the same results. However, the approach in the text seems to the author to be more flexible.

(ii) The computations in the proof of Lemma 3.10 get significantly simpler under the stronger assumption $n \geqslant m+1$. However, it is the 'hardest' case $m = 2n-1$ of this theorem, which is necessary to realize the automorphic induction in a pure way, see Theorems 4.2, 4.32.

Lemma 3.12. *$\tilde{Y}^m_{\dot{w}}$ is stable under the left action of $E^\times U_{\mathfrak{J}}$.*

Proof. As $Y^m_{\dot{w}}$ is $U_{\mathfrak{J}}$-stable (see the proof of Theorem 3.9), $\tilde{Y}^m_{\dot{w}}$ also is. As $E^\times U_{\mathfrak{J}}$ is generated by $U_{\mathfrak{J}}$ and ϖ (ϖ as in Section 3.1.3), it is enough to show $\varpi \tilde{Y}^m_{\dot{w}} = \tilde{Y}^m_{\dot{w}}$, which in turn follows from $\varpi Y^m_{\dot{w}} = Y^m_{\dot{w}} y_1$. Let $\mathrm{pr}_m \colon \mathscr{F}^m \twoheadrightarrow \mathscr{F}$ denote the natural projection. Lemma 3.13 shows $\varpi D^\tau_w = D^\tau_w y_1$. Using ϖ-(resp. y_1-)equivariance of pr_m and ϖ-(resp. y_1-)invariance of $X^m_{\underline{w}_m}(1)$, we deduce from this

$$\begin{aligned} \varpi Y^m_{\dot{w}} &= \varpi(\mathrm{pr}_m^{-1}(D^\tau_w) \cap X^m_{\underline{w}_m}(1)) = \mathrm{pr}_m^{-1}(\varpi D^\tau_w) \cap X^m_{\underline{w}_m}(1) = \mathrm{pr}_m^{-1}(D^\tau_w y_1) \cap X^m_{\underline{w}_m}(1) \\ &= (\mathrm{pr}_m^{-1}(D^\tau_w) \cap X^m_{\underline{w}_m}(1))y_1 = Y^m_{\dot{w}} y_1. \quad \square \end{aligned}$$

Lemma 3.13. *Let $\psi_{\dot{v}}(a) \in C_v$ be a point. Write $a = ua'$ and assume that $v_u(a') = 0$. The point $\varpi\psi_{\dot{v}}(a)y_1^{-1}$ of $\mathscr{F}$ (with y_1 as in Definition 3.7) lies in C_v. Moreover,*

$$\varpi\psi_{\dot{v}}(a)y_1^{-1} = \psi_{\dot{v}}(ua'^{,-1}).$$

Proof. A computation shows that the I-cosets $\varpi e_-(a)\dot{v}y_1^{-1}I$ and $e_-(ua'^{,-1})I$ coincide. $\square$

4. Representation Theory

Recall that $G = \mathrm{GL}_2$ and $\mathrm{char}\, k \neq 2$. We use the notation from Section 3. Further, we fix a prime $\ell \neq \mathrm{char}\, k$. All representations considered below are smooth $\overline{\mathbb{Q}}_\ell$-representations.

4.1. **Some preparations.**

4.1.1. *Filtrations on $U_{\mathfrak{J}}$ and U_E.* Recall the $\mathcal{O}_F$-algebra $\mathfrak{J}$ from Section 3.1.3. Then

$$U^n_{\mathfrak{J}} := 1 + \varpi^n \mathfrak{J} = \begin{pmatrix} 1 + \mathfrak{p}_F^{\lfloor \frac{n+1}{2} \rfloor} & \mathfrak{p}_F^{\lfloor \frac{n}{2} \rfloor} \\ \mathfrak{p}_F^{\lfloor \frac{n}{2} \rfloor + 1} & 1 + \mathfrak{p}_F^{\lfloor \frac{n+1}{2} \rfloor} \end{pmatrix}$$

for $n \geqslant 0$ form a filtration of $U^0_{\mathfrak{J}} := U_{\mathfrak{J}}$ by open subgroups. Note that via ι we have $U^n_{\mathfrak{J}} \cap E^\times = U^n_E$.

4.1.2. *Some notation.* For a locally compact group H, we denote by $H^\vee$ the set of all smooth $\overline{\mathbb{Q}}_\ell^\times$-valued characters of H. For an additive character ψ of F, we let $\psi_E := \psi \circ \mathrm{tr}_{E/F}$ be the corresponding character of E, where $\mathrm{tr}_{E/F}$ is the trace of E/F. Let $\mathfrak{M} := M_2(\mathcal{O}_F)$. We denote by $\psi_{\mathfrak{M}} := \psi \circ \mathrm{tr}_{\mathfrak{M}}$ the corresponding character of $\mathfrak{M}$. For a character ϕ of $F^\times$ we set $\phi_E := \phi \circ \mathrm{N}_{E/F}$ be the corresponding character of $E^\times$, where $\mathrm{N}_{E/F}$ is the norm of E/F. For a $G(F)$-representation π we denote by $\phi\pi$ the $G(F)$-representation $g \mapsto \phi(\det(g))\pi(g)$.

4.1.3. *Characters of $U_{\mathfrak{J}}$.* Let ψ be an additive character of F of level 1 (i.e., $\psi(\mathfrak{p}_F) = 1$, but ψ non-trivial on $\mathcal{O}_F$). Let $0 \leqslant k < r \leqslant 2k+1$ be integers. By [BH06] 12.5 Proposition we have isomorphisms

$$\varpi^{-r}\mathfrak{J}/\varpi^{-k}\mathfrak{J} \xrightarrow{\sim} (U^{k+1}_{\mathfrak{J}}/U^{r+1}_{\mathfrak{J}})^\vee, \qquad a + \varpi^{-k}\mathfrak{J} \mapsto \psi_{\mathfrak{M},a}|_{U^{k+1}_{\mathfrak{J}}}, \tag{4.1}$$

where $\psi_{\mathfrak{M},a}$ denotes the function $x \mapsto \psi_{\mathfrak{M}}(a(x-1))$ and $\mathfrak{M}$ is as in Section 4.1.2.

4.1.4. *Admissible pairs.* Let χ be a character of $E^\times$. The *level* $\ell(\chi)$ of χ is the least integer $m \geqslant 0$, such that $\chi|_{U_E^{m+1}}$ is trivial. The pair $(E/F, \chi)$ is said to be *admissible* ([BH06] 18.2) if $\chi|_{U_E^1}$ does not factor through the norm map $\mathrm{N}_{E/F}$. An admissible pair $(E/F, \chi)$ with χ of level m is called *minimal*, if $\chi|_{U_E^m}$ does not factor through $\mathrm{N}_{E/F}$. Note that if $(E/F, \chi)$ is minimal, then $\ell(\chi)$ is odd. Two pairs $(E/F, \chi)$, $(E/F, \chi')$ are said to be F-isomorphic if there is some $\gamma \in \mathrm{Gal}_{E/F}$ such that $\chi' = \chi \circ \gamma$. We denote by $\mathbb{P}_2^{\mathrm{tr}}(F)$ the set of isomorphism classes of all admissible pairs attached to the tamely ramified extension E/F.

4.1.5. *Supercuspidal representations.* Denote by $\mathscr{A}_2^{\mathrm{tr}}(F)$ the set of all isomorphism classes of irreducible supercuspidal representations of $G(F)$, which are not unramified (i.e., are not attached to an unramified stratum. We use the definition of *unramified* from [BH06] 20.1, see also 20.3 Lemma). The ramified part of the tame parametrization theorem ([BH06] 20.2 Theorem) states the existence of a certain bijection

$$\pi\colon \mathbb{P}_2^{\mathrm{tr}}(F) \xrightarrow{\sim} \mathscr{A}_2^{\mathrm{tr}}(F) \qquad (E/F, \chi) \mapsto \pi_\chi. \tag{4.2}$$

4.1.6. *Bushnell-Henniart construction of* π_χ. We recall the construction of π_χ from [BH06]§15,19. By twisting with a character of $F^\times$, it is enough to construct π_χ for minimal pairs. Fix an additive character ψ of F of level one. Let $(E/F, \chi)$ be a minimal admissible pair with χ of odd level $m = 2n-1 \geqslant 1$. Choose an element $\beta \in \mathfrak{p}_E^{-m}$ such that

$$\chi(1+x) = \psi_E(\beta x) \quad \text{for all } x \in \mathfrak{p}_E^n. \tag{4.3}$$

Via ι we see β as an element of $M_2(F)$. Then $(\mathfrak{J}, m, \beta)$ is a *ramified simple stratum* (see [BH06] 13.1). Via (4.1), β defines a character ψ_β of $U_{\mathfrak{J}}^n$, which is trivial on $U_{\mathfrak{J}}^{m+1}$. Let Λ be the character of $J_\beta := E^\times U_{\mathfrak{J}}^n$ defined by

$$\Lambda|_{U_{\mathfrak{J}}^n} := \psi_\beta, \quad \Lambda|_{E^\times} := \chi$$

(by (4.3) this is a consistent definition, as $\mathrm{tr}_{\mathfrak{M}}|_{\mathcal{O}_E} = \mathrm{tr}_{E/F}|_{\mathcal{O}_E}$ and $E \cap U_{\mathfrak{J}}^n = U_E^n$). Then $(\mathfrak{J}, J_\beta, \Lambda)$ is a *cuspidal type* in $G(F)$ attached to $(E/F, \chi)$ (see [BH06] 15.5). The *cuspidal inducing datum* attached to this cuspidal type is the pair $(U_{\mathfrak{J}}, \Theta_\chi)$, where $\Theta_\chi := \mathrm{c}-\mathrm{Ind}_{J_\beta}^{E^\times U_{\mathfrak{J}}} \Lambda$. Then π_χ is defined to be the compact induction

$$\pi_\chi := \mathrm{c}-\mathrm{Ind}_{J_\beta}^{G(F)} \Lambda = \mathrm{c}-\mathrm{Ind}_{E^\times U_{\mathfrak{J}}}^{G(F)} \Theta_\chi.$$

The isomorphism class of π_χ is independent of the choices of ι, ψ and β. We work with the fixed choice of ι, but ψ and β can be arbitrary.

4.1.7. *Cohomology.* For a scheme X over k we denote by $\mathrm{H}_c^*(X, \overline{\mathbb{Q}}_\ell)$ the ℓ-adic cohomology of X with compact support.

4.2. Automorphic induction from the ramified torus of GL_2.

Let $m \geqslant 1$ be an odd integer. Let χ be character of $E^\times$ of level m. Let $\underline{w}_m$ be as in Notation 3.5. By inflation via (3.11), χ determines a character of $\tilde{I}_{m,\underline{w}_m}/I^m$ and hence we can consider the χ-isotypic subspace $\mathrm{H}_c^*(X_{\underline{w}_m}^m(1), \overline{\mathbb{Q}_\ell})[\chi]$ of the cohomology of $X_{\underline{w}_m}^m(1)$. Analogously, we can consider the χ-isotypic subspace in the cohomology of $\tilde{Y}_{\dot{w}}^m$.

Definition 4.1. Let $(E/F,\chi)$ be a minimal pair of odd level $m \geqslant 1$. Let w, n, $\underline{w}$ be as in Notation 3.2 such that $\ell(w) = 2n-1 \geqslant m$ and take $\underline{w}_m$ as in Notation 3.5 lying over $\underline{w}$. Define $R_{\chi,n}$ to be the $G(F)$-representation

$$R_{\chi,n} := \mathrm{H}^0_c(X^m_{\underline{w}_m}(1), \overline{\mathbb{Q}_\ell})[\chi]$$

and $\Xi_{\chi,n}$ to be the $E^\times U_{\mathfrak{J}}$-representation

$$\Xi_{\chi,n} := \mathrm{H}^0_c(\tilde{Y}^m_{\dot{w}}, \overline{\mathbb{Q}_\ell})[\chi].$$

For an arbitrary admissible pair $(E/F,\chi)$ such that $\chi = \phi\chi'$ with $(E/F,\chi')$ minimal we define $R_{\chi,n} := \phi R_{\chi',n}$, $\Xi_{\chi,n} := \phi\Xi_{\chi',n}$. If $m = 2n-1$, write

$$R_\chi := R_{\chi,n} \quad \text{and} \quad \Xi_\chi := \Xi_{\chi,n}.$$

We also denote by V_χ the space in which Ξ_χ acts.

As $X^m_{\underline{w}_m}(1)$ is zero-dimensional, its cohomology in all positive degrees vanishes, and Definition 4.1 is compatible with (1.1). The following theorem is our main result.

Theorem 4.2. *Let $(E/F,\chi)$ be an admissible pair. The representation R_χ is irreducible, cuspidal, ramified, has level $\ell(\chi)$ and central character $\chi|_{F^\times}$. Moreover, R_χ is isomorphic to π_χ, i.e., the map*

$$R\colon \mathbb{P}^{\mathrm{tr}}_2(F) \to \mathscr{A}^{\mathrm{tr}}_2(F) \qquad (E/F,\chi) \mapsto R_\chi \tag{4.4}$$

coincides with the map π_χ from (4.2) and is, in particular, a bijection.

We believe that R_χ for non-minimal pairs also occurs naturally in the zeroth cohomology of $X^m_{\underline{w}_m}(1)$ with m even. After necessary preparations, Theorem 4.2 is shown in Sections 4.6, 4.7. We wish to point out, that the injectivity of (4.4) follows from the results of Section 4.5 and essentially does not use cuspidal types and the isomorphism $R_\chi \cong \pi_\chi$. We need them to prove surjectivity of (4.4). From Theorem 3.9 we deduce:

Lemma 4.3. *Let $(E/F,\chi)$ be an admissible pair. Then*

$$R_{\chi,n} = \mathrm{c}-\mathrm{Ind}^{G(F)}_{E^\times U_{\mathfrak{J}}} \Xi_{\chi,n}.$$

Proof. It follows from Theorem 3.9 and the commutativity of the left and the right group actions on $X^m_{\underline{w}_m}(1)$. ☐

In Section 4.8 we also study the representations $R_{\chi,n}$ for $n \geqslant m+1$, where m is the (odd) level of χ. We determine the structure of $R_{\chi,n}$ and give a recipe how to reconstruct χ (up to τ-conjugacy) from $R_{\chi,n}$.

4.3. Unipotent traces.

From now on and until the end of Section 4.7 we assume $2n-1 = m$.

Lemma 4.4. *The central character of R_χ is $\chi|_{F^\times}$. The subgroup $U^{m+1}_{\mathfrak{J}}$ acts trivially in V_χ and V_χ has dimension $(q-1)q^{n-1}$.*

Proof. Elements of $F^\times$ act on $X^m_{\underline{w}_m}(1)$ in the same way from the left and from the right. As R_χ is the χ-isotypic component of $\mathrm{H}^0_c(X^m_{\underline{w}_m}(1), \overline{\mathbb{Q}}_\ell)$, the first statement of the lemma follows. The proof of the second statement is given in Section 5.3. ☐

By Lemma 4.4 we can consider Ξ_χ as a $E^\times U_{\mathfrak{J}}/U_{\mathfrak{J}}^{m+1}$-representation. Let N_n be the finite subgroup of $E^\times U_{\mathfrak{J}}/U_{\mathfrak{J}}^{m+1}$, equipped with a descending filtration by subgroups N_n^i for $1 \leqslant i \leqslant n+1$ defined by

$$N_n^i := \begin{pmatrix} 1 & 0 \\ \mathfrak{p}_F^i & 1 \end{pmatrix} \Big/ \begin{pmatrix} 1 & 0 \\ \mathfrak{p}_F^{n+1} & 1 \end{pmatrix} \subseteq N_n := \begin{pmatrix} 1 & 0 \\ \mathfrak{p}_F & 1 \end{pmatrix} \Big/ \begin{pmatrix} 1 & 0 \\ \mathfrak{p}_F^{n+1} & 1 \end{pmatrix} \subseteq U_{\mathfrak{J}}/U_{\mathfrak{J}}^{m+1}.$$

Proposition 4.5. *As N_n-representations one has*

$$\Xi_\chi \cong \mathrm{Ind}_1^{N_n} 1 - \mathrm{Ind}_{N_n^n}^{N_n} 1 \cong \bigoplus_{\substack{\psi \in N_n^\vee \\ \psi|_{N_n^n} \text{ non-trivial}}} \psi.$$

In particular, Ξ_χ does not contain the trivial character on N_n^n.

Proof. If A is a finite abelian group and $B \subseteq A$ is a subgroup, then the traces of elements of $A \smallsetminus B$ in the induced representation $\mathrm{Ind}_B^A 1$ are equal to 0 and the traces of elements of B are equal to the index of B in A. This allows to compute the traces on the right hand side in the proposition. The proposition follows from Lemma 4.6 by comparing the traces of N_n-representations on the left and the right side. □

Lemma 4.6. *For $g \in N_n$ we have:*

$$\mathrm{tr}(g; \Xi_\chi) = \begin{cases} (q-1)q^{n-1} & \text{if } g = 1 \\ -q^{n-1} & \text{if } g \in N_n^n \smallsetminus \{1\} \\ 0 & \text{if } g \in N_n \smallsetminus N_n^n. \end{cases} \tag{4.5}$$

Proof. The proof is given in Section 5.3. □

Corollary 4.7. *Ξ_χ is irreducible as B-representation, where $B \subseteq U_{\mathfrak{J}}$ is the subgroup consisting of lower triangular matrices.*

Proof. The proof is the same as the proof of [Iva16] Corollary 4.12 (using Proposition 4.5 instead of [Iva16] Proposition 4.10). □

4.4. **Some character theory.**

In this section we work relative to a fixed character χ of $E^\times$ of the odd level $m = 2n-1 \geqslant 1$. We write $\chi^\tau := \chi \circ \tau$. Moreover, in this section (in the proof of Proposition 4.10 and in Lemma 4.17) it will be convenient to use the following notation: for two elements $x, y \in E$ and an integer a, the writing $x = y + O(u^a)$ will just indicate that $x \equiv y \mod u^a$ (and the same notation with E, u replaced by F, t).

4.4.1. *Admissibility of $(E/F, \chi)$.*

Lemma 4.8. *The following hold:*

(i) $\chi|_{U_E^m}$ *does not factor through the norm $N_{E/F}$.*
(ii) $\chi|_{U_E^m} \neq \chi^\tau|_{U_E^m}$.

Proof. First we show (ii): Assume $\chi(1+u^m x) = \chi(1-u^m x)$ for all $x \in k$. As $(1-u^m x)^{-1} \in (1+u^m x)U_E^{2m}$ and as χ has level $m \geqslant 1$, we deduce $1 = \chi((1+u^m x)^2) = \chi(1+u^m 2x)$ for all $x \in k$. As $\mathrm{char}\, E \neq 2$, we obtain a conradiction to our assumption $\ell(\chi) = m$. Now we deduce

(i) from (ii): assume that $\chi|_{U_E^m}$ factors through the norm, i.e., $\chi = \chi' \circ N_{E/F}$ on U_E^m. Then $\chi^\tau(x) = \chi'(\mathrm{N}_{E/F}((\tau(x))) = \chi'(N_{E/F}(x)) = \chi(x)$, which contradicts (ii). □

4.4.2. *Filtration on U_E.* We have the disjoint decomposition

$$U_E = U_F U_E^{m+1} \cup \bigcup_{\alpha=0}^{n-1} (U_F U_E^{2\alpha+1} \smallsetminus U_F U_E^{2\alpha+3}).$$

Note that $U_F U_E^{2\alpha+1} = U_F U_E^{2\alpha}$.

4.4.3. *Index of coincidence for characters.*

Definition 4.9. For a character θ of $E^\times$, which coincides with χ on $F^\times U_E^{m+1}$, we define the integer $i(\theta) = i_\chi(\theta)$ to be the smallest integer $i \geqslant 0$, such that $\theta|_{F^\times U_E^i} = \chi|_{F^\times U_E^i}$ or $\theta|_{F^\times U_E^i} = \chi^\tau|_{F^\times U_E^i}$.

Observe that $0 \leqslant i(\theta) \leqslant m+1$ and $i(\theta)$ is always even.

4.4.4. *Modifications of characters.* Fix some integer $0 \leqslant \alpha < n$. Consider the k-algebra

$$R_\alpha := \mathcal{O}_E/\mathfrak{p}_E^{m-2\alpha} = k[u]/(u^{m-2\alpha}).$$

The τ-invariants of it are $R_\alpha^{\langle\tau\rangle} = k[t]/(t^{n-\alpha})$. Consider the subset

$$R_\alpha^{\langle\tau\rangle,\prime} := \{s \in R_\alpha^{\langle\tau\rangle} : s \equiv \pm 1 \mod u^{m+1-2(2\alpha+1)}\}$$

of $R_\alpha^{\langle\tau\rangle}$ (note that $R_\alpha^{\langle\tau\rangle,\prime} = R_\alpha^{\langle\tau\rangle}$ if $2\alpha+1 \geqslant n$, or equivalently, $\alpha \geqslant \lfloor \frac{n}{2} \rfloor$).

Proposition 4.10. *Let $0 \leqslant \alpha < n$. Let $s \in R_\alpha^{\langle\tau\rangle,\prime}$. There is a unique character χ_s of $F^\times U_E^{2\alpha+1}$, such that the following hold:*

(i) *χ_s coincides with χ on $F^\times U_E^{m+1}$.*
(ii) *if $\alpha < \lfloor \frac{n}{2} \rfloor$, then χ_s coincides on $F^\times U_E^{2(2\alpha+1)}$ with $\chi \circ \tau^i$, where $s \equiv (-1)^i \mod u$.*
(iii) *$\chi_s(1+u^{2\alpha+1}h) = \chi(1+u^{2\alpha+1}hs)$ for all $h \in \mathcal{O}_F$.*

Conversely, let θ be a character of $F^\times U_E^{2\alpha+1}$, which coincides with χ or χ^τ on $F^\times U_E^{\min\{m+1, 2(2\alpha+1)\}}$. Then there is a unique $s \in R_\alpha^{\langle\tau\rangle,\prime}$ such that $\theta = \chi_s$.

Note that the expression $\chi(1+u^{2\alpha+1}hs)$ in (iii) is well-defined: Indeed, χ is trivial on U_E^{m+1}, and on the other hand if $\tilde{s}_1, \tilde{s}_2 \in \mathcal{O}_F = k[\![t]\!] \subseteq k[\![u]\!]$ represent the same element s in $R_\alpha^{\langle\tau\rangle,\prime}$, then $\tilde{s}_1 \equiv \tilde{s}_2 \mod u^{m-2\alpha}$, hence $1+u^{2\alpha+1}h\tilde{s}_1 \equiv 1+u^{2\alpha+1}h\tilde{s}_2 \mod u^{m+1}$.

Proof. Consider the subset

$$U_E^{2\alpha+1,\prime} := \{x \in U_E^{2\alpha+1} : \exists h \in \mathcal{O}_F \text{ with } x \equiv 1+u^{2\alpha+1}h \mod U_E^{m+1}\} \subseteq U_E^{2\alpha+1}.$$

Lemma 4.11. *Any element $x \in F^\times U_E^{2\alpha+1}$ can be written as $x = z_x x'$ with $z_x \in U_F$, $x' \in U_E^{2\alpha+1,\prime}$. Moreover, modulo U_E^{m+1}, z_x, x' are uniquely determined by x and if $x = \sum_{i\geqslant 0} x_i u^i \in U_E^{2\alpha+1}$, then $z_x \equiv \sum_{i\geqslant 0} x_{2i}u^{2i} \mod u^{m+1}$.*

Proof. Multiplying by an element in $F^\times$, we can assume $x \in U_E^{2\alpha+1}$. Write $x = 1 + \sum_{i=2\alpha+1}^m x_i u^i + O(u^{m+1})$. As x' has to lie in $U_E^{2\alpha+1,\prime} \subseteq U_E^{2\alpha+1}$, also z_x must lie in $U_E^{2\alpha+1}$. Thus we seek for two elements $z_x := 1 + \sum_{i=\alpha+1}^{n-1} z_{2i}u^{2i} + O(u^{m+1})$ and $x' = 1 + \sum_{i=\alpha}^{n-1} y_{2i+1}u^{2i+1} + O(u^{m+1})$ which have to satisfy

$$\left(1+\sum_{i=\alpha+1}^{n-1} z_{2i}u^{2i}+O(u^{m+1})\right)\left(1+\sum_{i=\alpha}^{n-1} y_{2i+1}u^{2i+1}+O(u^{m+1})\right)=1+\sum_{i=2\alpha+1}^{m} x_i u^i+O(u^{m+1}).$$

Comparing the parity of the degrees we see that $z_{2i}=x_{2i}$. Further, a computation shows that y_i's satisfying this equation exist and are uniquely determined by the x_i's. □

Let now $s \in R_\alpha^{\langle\tau\rangle,\prime}$. For $x \in F^\times U_E^{2\alpha+1}$ with decomposition $x = z_x x'$ according to Lemma 4.11, set

$$\chi_s(x) := \chi(z_x)\chi(1+u^{2\alpha+1}sh) \quad \text{where } x' \equiv 1+u^{2\alpha+1}h \mod U_E^{m+1} \text{ with } h \in \mathcal{O}_F.$$

We show that χ_s is a character of $F^\times U_E^{2\alpha+1}$. Let $x, y \in F^\times U_E^{2\alpha+1}$ with decompositions $x = z_x x'$, $y = z_y y'$ as in Lemma 4.11 and let $x' = 1+u^{2\alpha+1}h_x$, $y' = 1+u^{2\alpha+1}h_y$ (up to some elements in U_E^{m+1}). Write $A := u^{2\alpha+1}(h_x+h_y)$, $B := u^{2(2\alpha+1)}h_x h_y$. We compute

$$\begin{aligned}\chi_s(x)\chi_s(y) &= \chi(z_x z_y(1+u^{2\alpha+1}sh_x)(1+u^{2\alpha+1}sh_y))\\ &= \chi(z_x z_y(1+sA+s^2B))\\ &= \chi(z_x z_y(1+sA+B)),\end{aligned}$$

the last equation being true, as $s^2 \equiv 1 \mod u^{m+1-2(2\alpha+1)}$. We have

$$x'y' = 1+A+B.$$

As $h_x, h_y \in \mathcal{O}_F$, Lemma 4.11 implies $z_{x'y'} = 1+B$ (up to elements in U_E^{m+1}). We deduce

$$(x'y')' = x'y'z_{x'y'}^{-1} = 1+A-AB(1+B)^{-1}.$$

Now, $xy = z_x z_y z_{x'y'}(x'y')'$ is the decomposition of xy according to Lemma 4.11 and we compute

$$\chi_s(xy) = \chi(z_x z_y z_{x'y'})\chi(1+s(A-AB(1+B)^{-1})).$$

If $2\alpha+1 \geqslant n$, we have $B \in (u^{m+1})$, hence all terms containing B can be ignored and we deduce $\chi_s(x)\chi_s(y) = \chi_s(xy)$. Assume $2\alpha+1 < n$. Let $\operatorname{sgn}(s) := \pm 1$, if $s \equiv \pm 1 \mod u$. From the above, $s \equiv (-1)^{\operatorname{sgn}(s)} \mod u^{m+1-2(2\alpha+1)}$ and $B \equiv 0 \mod u^{2(2\alpha+1)}$ we deduce

$$\begin{aligned}\chi_s(xy) &= \chi(z_x z_y(1+B)(1+sA-(-1)^{\operatorname{sgn}(s)}AB(1+B)^{-1}))\\ &= \chi(z_x z_y(1+sA+B+sAB-(-1)^{\operatorname{sgn}(s)}AB))\\ &= \chi(z_x z_y(1+sA+B)).\end{aligned}$$

This shows that χ_s is a character. Now, χ_s satisfies (i) and (iii) by definition. Let us show (ii). Therefore, assume $\alpha < \lfloor \frac{n}{2} \rfloor$. As $s \in R_\alpha^{\langle\tau\rangle,\prime}$, we may write $s = (-1)^{\operatorname{sgn}(s)} + u^{m+1-2(2\alpha+1)}s'$ for some $s' \in R_\alpha^{\langle\tau\rangle}$. Let $x \in F^\times U_E^{2(2\alpha+1)}$. Write $x = z_x x'$ with $z_x \in F^\times$, $x' \in U_E^{2(2\alpha+1)}$. To compute $\chi_s(x)$, we write $x' \equiv 1+u^{2\alpha+1}h \mod U_E^{m+1}$ for some $h \in \mathcal{O}_F$. As $x' \in U_E^{2(2\alpha+1)}$, we have further $h = u^{2\alpha+1}h'$ for some $h' \in \mathcal{O}_E$. By definition of χ_s we compute:

$$\begin{aligned}\chi_s(x) &= \chi(z_x)\chi(1+u^{2\alpha+1}sh)\\ &= \chi(z_x)\chi(1+u^{2\alpha+1}((-1)^{\mathrm{sgn}(s)}+u^{m+1-2(2\alpha+1)}s')h'u^{2\alpha+1})\\ &= \chi(z_x)\chi(1+u^{2(2\alpha+1)}((-1)^{\mathrm{sgn}(x)}+u^{m+1-2(2\alpha+1)}s')h')\\ &= \chi(z_x)\chi(1+(-1)^{\mathrm{sgn}(x)}u^{2(2\alpha+1)}h')\\ &= \chi(z_x)\chi(1+(-1)^{\mathrm{sgn}(x)}u^{2\alpha+1}h) = \chi(z_x)\chi(\tau^{\mathrm{sgn}(s)}(x')) = (\chi\circ\tau^{\mathrm{sgn}(s)})(x),\end{aligned}$$

where the fourth equality follows as χ is trivial on U_E^{m+1}. This finishes the proof of the first part of the proposition. For the converse statement, one shows by a simple computation that the map $s \mapsto \chi_s$ from $R_\alpha^{\langle\tau\rangle,\prime}$ to characters of $F^\times U_E^{2\alpha+1}$ is injective. This completes the proof, as the number of elements in $R_\alpha^{\langle\tau\rangle,\prime}$ coincides with the number of characters θ of $F^\times U_E^{2\alpha+1}$, which are equal to χ or χ^τ on $F^\times U_E^{\min\{m+1,2(2\alpha+1)\}}$ (if $2\alpha+1 \geqslant n$, then there are $q^{n-\alpha}$ those, otherwise there are $2q^{\alpha+1}$). ☐

4.4.5. *Compatibility with changing α.* Let $0 \leqslant \alpha < n$. Let θ be a character of $E^\times$, coinciding on $F^\times U_E^{\min\{m+1,2(2\alpha+1)\}}$ with χ or χ^τ. By Proposition 4.10, there is some $s(\theta,\alpha) \in R_\alpha^{\langle\tau\rangle,\prime}$ such that $\theta|_{F^\times U_E^{2\alpha+1}} = \chi_{s(\theta,\alpha)}$. This construction is compatible with changing the level α.

Lemma 4.12. *Let $0 \leqslant \alpha_1 \leqslant \alpha_2 < n$. Let θ be a character of $E^\times$ coinciding on $F^\times U_E^{\min\{m+1,2(2\alpha_1+1)\}}$ with χ or χ^τ. Under the natural projection $R_{\alpha_1}^{\langle\tau\rangle} \to R_{\alpha_2}^{\langle\tau\rangle}$, $s(\theta,\alpha_1)$ maps to $s(\theta,\alpha_2)$.*

Proof. Write $s_i := s(\theta,\alpha_i)$. Let $\overline{s}_1$ denote the image of s_1 in $R_{\alpha_2}^{\langle\tau\rangle,\prime}$. On $F^\times U_E^{2\alpha_2+1}$ we have

$$\theta(1+u^{2\alpha_2+1}h) = \chi_{s_1}(1+u^{2\alpha_2+1}h) = \chi(1+u^{2\alpha_2+1}\overline{s}_1 h).$$

Thus on $F^\times U_E^{2\alpha_2+1}$ we have $\chi_{s_2} = \theta = \chi_{\overline{s}_1}$. By the uniqueness statement in Proposition 4.10 we have $\overline{s}_1 = s_2$. ☐

4.4.6. *Elementary modifications and distances.*

Definition 4.13. For $s \in R_\alpha^{\langle\tau\rangle,\prime}$, we call the character χ_s of $F^\times U_E^{2\alpha+1}$ constructed in Proposition 4.10 an *elementary modification* of χ. Let θ be character of $E^\times$ coinciding with χ on $F^\times U_E^{m+1}$. Set

$$\alpha_\theta := \min\{\alpha\colon 0 \leqslant \alpha < n, 2(2\alpha+1) \geqslant i(\theta)\},$$

i.e., α_θ is the smallest integer such that θ restricted to $F^\times U_E^{2\alpha_\theta+1}$ is an elementary modification of χ. We define the *distance* from χ to θ to be the (uniquely determined by Proposition 4.10) element $s(\theta) := s(\theta,\alpha_\theta) \in R_{\alpha_\theta}^{\langle\tau\rangle,\prime}$, such that θ coincides on $F^\times U_E^{2\alpha_\theta+1}$ with $\chi_{s(\theta)}$.

As $i(\theta)$ is an even integer $\leqslant m+1 = 2n$, it follows easily that in any case $\alpha_\theta \leqslant \lfloor\frac{n}{2}\rfloor$. (Moreover, one has $\alpha_\theta = \lfloor\frac{i(\theta)}{4}\rfloor$, but we will not use this). Further, $\alpha_\chi = \alpha_{\chi^\tau} = 0$ and $s(\chi) = 1$ and $s(\chi^\tau) = -1$.

4.4.7. *Quadratic distance.* Let $0 \leqslant \alpha < n$. There is the norm map

$$\mathrm{N}_{\tau,\alpha}\colon R_\alpha \to R_\alpha^{\langle\tau\rangle} \quad s \mapsto s\tau(s).$$

Lemma 4.14. *The image $\mathrm{im}(N_{\tau,\alpha})$ of $\mathrm{N}_{\tau,\alpha}$ consists of precisely such elements $s \in R_\alpha^{\langle\tau\rangle}$ for which $(-1)^{v_t(s)} \cdot s_{v_t(s)}$ is a square in $k^\times$, where $s_{v_t(s)}$ denotes the leading coefficient of s.*

Proof. This follows immediately from Lemma 4.15. ☐

Lemma 4.15. *Let $x \in k[\![t]\!] \smallsetminus \{0\}$ with leading coefficient $x_{v_t(x)} \in k^\times$. Then x is*

- *a square of an element of $k[\![t]\!]$ if and only if $v_t(x)$ is even and $x_{v_t(x)}$ is a square in $k^\times$,*
- *a square of an element of $k[\![u]\!]$ if and only if $x_{v_t(x)}$ is a square in $k^\times$,*
- *in the image of the norm map $\mathrm{N}_{E/F}$ if and only if $(-1)^{v_t(x)}x_{v_t(x)}$ is a square in $k^\times$.*

Proof. This is well-known. ☐

Consider the following subset of $R_\alpha^{\langle\tau\rangle,\prime}$:

$$Q_\alpha := \{s \in R_\alpha^{\langle\tau\rangle,\prime} \colon s^2 - 1 \in \mathrm{im}(\mathrm{N}_{\tau,\alpha})\} \smallsetminus \{\pm 1\}.$$

Definition 4.16. Let θ be a character of $E^\times$ coinciding with χ on $F^\times U_E^{m+1}$. We say that the distance from χ to θ is *properly quadratic* if $s(\theta) \in Q_{\alpha_\theta}$, with $s(\theta)$ as in Definition 4.13.

4.4.8. *Structure of Q_α.* Set $R_n^{\langle\tau\rangle} := \{1\}$. Let pr_α be the natural projection

$$\mathrm{pr}_\alpha \colon R_\alpha^{\langle\tau\rangle} \cong k[t]/t^{n-\alpha} \to k[t]/t^{n-\alpha-1} \cong R_{\alpha+1}^{\langle\tau\rangle}.$$

Lemma 4.17. *Let $0 \leqslant \alpha \leqslant n-1$. An element $s \in R_\alpha^{\langle\tau\rangle} \smallsetminus \{\pm 1\}$ lies in Q_α if and only if either*

- *$\alpha \geqslant \lfloor \frac{n}{2} \rfloor$, $s \not\equiv \pm 1 \mod u$ and $s^2 - 1 \mod u$ is a square in $k^\times$, or*
- *$s \equiv \pm 1 \mod u$, i.e., $s = \pm 1 + t^j s_0 + O(t^{j+1})$ for some $s_0 \in k^\times$, $\max\{1, n-(2\alpha+1)\} \leqslant j \leqslant n - \alpha - 1$ and $\pm(-1)^j 2 s_0$ is a square in $k^\times$.*

Moreover, the following hold:

(i) *Let $0 \leqslant \alpha \leqslant n-2$. The preimage of 1 (resp. -1) under the composed map $Q_\alpha \hookrightarrow R_\alpha^{\langle\tau\rangle} \to R_{\alpha+1}^{\langle\tau\rangle}$ contains precisely $\frac{q-1}{2}$ elements.*

(ii) *Assume $0 \leqslant \alpha \leqslant n-2$. Let $s_0 \in R_\alpha^{\langle\tau\rangle,\prime}$ with $\mathrm{pr}_\alpha(s_0) \neq \pm 1$. Then $\sharp\,\mathrm{pr}_\alpha^{-1}(\mathrm{pr}_\alpha(s_0)) = q$ and we have the equivalence $s_0 \in Q_\alpha \Leftrightarrow \mathrm{pr}_\alpha^{-1}(\mathrm{pr}_\alpha(s_0)) \subseteq Q_\alpha$.*

(iii) *We have $\sharp Q_{n-1} = \frac{q-3}{2}$.*

Proof. The description of Q_α follows by an easy computation from Lemma 4.15. (i),(ii) follow from this description (along with $\sharp k^{\times,2} = \frac{q-1}{2}$).

(iii): Note that $Q_{n-1} = \{s \in k \colon s^2 - 1$ is a square in $k\} \smallsetminus \{\pm 1\}$. Consider the affine curve $C : s^2 - 1 = y^2$ over k and let $\overline{C}$ be the unique smooth projective curve over k containing C as an open subset. We have $\sharp(\overline{C}(k) \smallsetminus C(k)) = 2$. Further, $\overline{C}$ is a smooth quadric in $\mathbb{P}^2$ over a finite field, hence isomorphic to $\mathbb{P}^1$, i.e., $\sharp\overline{C}(k) = q+1$. We deduce $\sharp C(k) = q-1$. Now (iii) follows from the fact that the map $C(k) \smallsetminus \{(\pm 1, 0)\} \to Q_{n-1}$ given by $(s,y) \mapsto s$ is surjective and two-to-one. ☐

4.5. **Restriction to the ramified torus $E^\times \subseteq G(F)$.**

For a finite finite group H, let $\langle,\rangle_H$ denote the inner product on the set of class functions of H. For a character θ let $\langle \theta, \Xi_\chi \rangle_{E^\times}$ denote the multiplicity of θ in Ξ_χ.

Theorem 4.18. *Let $(E/F, \chi)$ be a minimal pair of odd level $m \geqslant 1$. A character θ of $E^\times$ can only occur in Ξ_χ, if θ coincides with χ on $F^\times U_E^{m+1}$. In this case we have*

$$\langle \theta, \Xi_\chi \rangle_{E^\times} = \begin{cases} 1 & \text{if } \theta = \chi \text{ or } \theta = \chi^\tau \text{ or the distance from } \chi \text{ to } \theta \text{ is properly quadratic} \\ 0 & \text{otherwise.} \end{cases}$$

We prove this theorem below. First we investigate the restriction of Ξ_χ to U_E. Note that $s(\theta)$ from Definition 4.13 is in exactly the same way also defined for characters θ of U_E, which coincide with χ on $U_F U_E^{m+1}$.

Proposition 4.19. *Let $(E/F, \chi)$ be a minimal pair of odd level $m \geqslant 1$. A character θ of U_E can only occur in $\Xi_\chi|_{U_E}$, if θ coincides with χ on $U_F U_E^{m+1}$. In this case we have*

$$\langle \theta, \Xi_\chi \rangle_{U_E} = \begin{cases} 1 & \text{if } \theta = \chi \text{ or } \chi^\tau \\ 2 & \text{if } \theta \neq \chi, \chi^\tau \text{ and } s(\theta) \in Q_{\alpha_\theta} \\ 0 & \text{otherwise.} \end{cases}$$

The main ingredient in the proof of Proposition 4.19 is the following trace computation.

Proposition 4.20. *Let $0 \leqslant \alpha < n$. Let $g \in U_F U_E^{2\alpha+1} \smallsetminus U_F U_E^{2\alpha+3}$. Then*

$$\mathrm{tr}(g; \Xi_\chi) = 2q^\alpha \sum_{s \in Q_\alpha} \chi_s(g) + q^\alpha(\chi(g) + \chi^\tau(g)).$$

Proof. We can write $g = zg'$ for $z \in U_F U_E^{m+1}$ and $g' = \iota(1 + u^{2\alpha+1} y)$ with $y \in U_F$. We have the following computation, where the first equality follows from Lemma 4.4, which shows that z acts in Ξ_χ as the scalar $\chi(z)$, and the second equality follows from Proposition 5.12 applied to g':

$$\begin{aligned} \mathrm{tr}(g; \Xi_\chi) &= \chi(z)\mathrm{tr}(g'; \Xi_\chi) \\ &= \chi(z)\left(q^\alpha(\chi(g') + \chi^\tau(g')) + 2q^\alpha \sum_{s \in Q_\alpha} \chi(1 + u^{2\alpha+1} sy) \right) \\ &= q^\alpha(\chi(g) + \chi^\tau(g)) + 2q^\alpha \sum_{s \in Q_\alpha} \chi(z)\chi(1 + u^{2\alpha+1} sy), \end{aligned}$$

Now, $\chi(z)\chi(1 + u^{2\alpha+1} sy) = \chi_s(g)$ with χ_s as in Proposition 4.10 and hence we are done. □

Proof of Proposition 4.19. As $U_F U_E^{m+1}$ acts in V_χ by $\chi|_{U_F U_E^{m+1}}$, the first statement is clear. Assume $\theta|_{U_F U_E^{m+1}} = \chi|_{U_F U_E^{m+1}}$. Now, U_E^{m+1} acts trivially in V_χ, thus we can equivalently consider V_χ as a U_E/U_E^{m+1}-representation. The filtration from Section 4.4.2 induces a disjoint decomposition

$$U_E/U_E^{m+1} = Z \cup \bigcup_{\alpha=0}^{n-1} (H^\alpha \smallsetminus H^{\alpha+1}),$$

where $H^\alpha := U_F U_E^{2\alpha+1}/U_E^{m+1}$ and $Z := H^n = U_F U_E^{m+1}/U_E^{m+1}$. We have $\sharp H^\alpha = (q-1)q^{m-\alpha}$ for $0 \leqslant \alpha \leqslant n$. For $0 \leqslant \alpha < n$ set

$$S_\alpha := \sum_{g \in H^\alpha \smallsetminus H^{\alpha+1}} \theta(g^{-1})\mathrm{tr}(g; \Xi_\chi).$$

Then the trace computation Proposition 4.20 shows that $S_\alpha = (q-1)q^{m-1} S'_\alpha$ with

$$S'_\alpha := 2 \sum_{s \in Q_\alpha} (q\langle \theta, \chi_s \rangle_{H^\alpha} - \langle \theta, \chi_s \rangle_{H^{\alpha+1}}) + q\langle \theta, \chi + \chi^\tau \rangle_{H^\alpha} - \langle \theta, \chi + \chi^\tau \rangle_{H^{\alpha+1}},$$

and using Lemma 4.4 we deduce ($H^0 = U_E/U_E^{m+1}$)

$$\begin{aligned}\langle\theta,\Xi_\chi\rangle_{H^0} &= \frac{1}{\sharp H^0}\left(\sum_{g\in Z}\theta(g^{-1})\mathrm{tr}(g;\Xi_\chi)+\sum_{\alpha=0}^{n-1}S_\alpha\right)=\frac{1}{\sharp H^0}\left(\sharp Z\cdot\dim V_\chi+\sum_{\alpha=0}^{n-1}S_\alpha\right)\\ &= \frac{1}{q}\left(q-1+\sum_{\alpha=0}^{n-1}S'_\alpha\right).\end{aligned}$$

Now the proposition follows from Lemma 4.21, by considering the five cases $i(\theta)=0$, $0<i(\theta)<m+1$ and $s(\theta)\in Q_{\alpha_\theta}$, $0<i(\theta)<m+1$ and $s(\theta)\notin Q_{\alpha_\theta}$, $i(\theta)=m+1$ and $s(\theta)\in Q_{\alpha_\theta}$, $i(\theta)=m+1$ and $s(\theta)\notin Q_{\alpha_\theta}$. □

Lemma 4.21. (i) *Assume $0\leqslant\alpha\leqslant n-2$. Then*

$$S'_\alpha=\begin{cases}0 & \text{if } i(\theta)\leqslant 2\alpha+1\\ q & \text{if } i(\theta)=2\alpha+2 \text{ and } s(\theta,\alpha)\in Q_\alpha\\ -q & \text{if } i(\theta)=2\alpha+2 \text{ and } s(\theta,\alpha)\notin Q_\alpha\\ 0 & \text{if } i(\theta)\geqslant 2\alpha+3.\end{cases}$$

(ii) *For $\alpha=n-1$ we have*

$$S'_{n-1}=\begin{cases}1 & \text{if } i(\theta)\leqslant m=2n-1\\ q+1 & \text{if } i(\theta)=m+1=2n \text{ and } s(\theta,n-1)\in Q_{n-1}\\ -q+1 & \text{if } i(\theta)=m+1=2n \text{ and } s(\theta,n-1)\notin Q_{n-1}.\end{cases}$$

Proof. Let pr_α be as in Section 4.4.8. (i): Assume $0\leqslant\alpha\leqslant n-2$.

Case $i(\theta)\leqslant 2\alpha+1$. Then on H^α resp. on $H^{\alpha+1}$ the character θ is equal to exactly one of the characters χ or χ^τ (as $\alpha\leqslant n-2$ and $\chi\neq\chi^\tau$ on H^{n-1} by Lemma 4.8). Assume that this character is χ (the other case is similar). Thus $\langle\theta,\chi_s\rangle_{H^\alpha}=0$ for all $s\in Q_\alpha$, $\langle\theta,\chi+\chi^\tau\rangle_{H^\alpha}=\langle\theta,\chi+\chi^\tau\rangle_{H^{\alpha+1}}=1$ and by Lemma 4.12 we have

$$\langle\theta,\chi_s\rangle_{H^{\alpha+1}}=\begin{cases}1 & \text{if } \mathrm{pr}_\alpha(s)=1\\ 0 & \text{otherwise.}\end{cases}\tag{4.6}$$

By Lemma 4.17(i), the first case happens for exactly $(q-1)/2$ elements in $s\in Q_\alpha$. Altogether we obtain $S'_\alpha=2(0-\frac{q-1}{2})+q-1=0$.

Case $i(\theta)=2\alpha+2$. The character θ coincides on H^α neither with χ nor with χ^τ, hence $\langle\theta,\chi+\chi^\tau\rangle_{H^\alpha}=0$. As $2\alpha+3\leqslant 2(n-2)+3=m$ by assumption, θ coincides on $H^{\alpha+1}$ with precisely one of the characters χ or χ^τ and hence $\langle\theta,\chi+\chi^\tau\rangle_{H^{\alpha+1}}=1$. As $2(2\alpha+1)\geqslant 2\alpha+2=i(\theta)$, the quantity $s(\theta,\alpha)\in R_\alpha^{\langle\tau\rangle,\prime}$ is well-defined. Thus

$$\sum_{s\in Q_\alpha}q\langle\theta,\chi_s\rangle_{H^\alpha}=\begin{cases}q & \text{if } s(\theta,\alpha)\in Q_\alpha\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, (4.6) holds also in this case, and again there are precisely $(q-1)/2$ elements of Q_α with image 1 in $R_{\alpha+1}^{\langle\tau\rangle}$. From this we deduce the result.

Case $i(\theta)\geqslant 2\alpha+3$. Then $\langle\theta,\chi+\chi^\tau\rangle_{H^\alpha}=\langle\theta,\chi+\chi^\tau\rangle_{H^{\alpha+1}}=0$. Assume first $2(2\alpha+1)\geqslant i(\theta)$ (in particular, $\alpha>0$). By Proposition 4.10 there is a unique $s(\theta,\alpha)\in R_\alpha^{\langle\tau\rangle,\prime}$ such that θ coincides with $\chi_{s(\theta,\alpha)}$ on H^α. Hence

$$\sum_{s\in Q_\alpha} q\langle\theta,\chi_s\rangle_{H^\alpha} = \begin{cases} q & \text{if } s(\theta,\alpha)\in Q_\alpha \\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, note that θ coincides with χ_s on $H^{\alpha+1}$ if and only if $s \in \mathrm{pr}_\alpha^{-1}(\mathrm{pr}_\alpha(s(\theta,\alpha)))$. Thus using Lemma 4.17(ii) we deduce

$$\sum_{s\in Q_\alpha} \langle\theta,\chi_s\rangle_{H^{\alpha+1}} = \begin{cases} q & \text{if } s(\theta,\alpha)\in Q_\alpha \\ 0 & \text{otherwise.}\end{cases}$$

In any case we compute $S'_\alpha = 0$. Finally, assume that $i(\theta) > 2(2\alpha+1)$, i.e., $i(\theta) \geqslant 2(2\alpha+2)$ as $i(\theta)$ is even. Thus θ does not coincide with χ or χ^τ on $U_F U_E^{\min\{m+1,2(2\alpha+1)\}}/U_E^{m+1} = H^{2\alpha+1}$. On the other hand, for $s \in R_\alpha^{\langle\tau\rangle,\prime}$, the character χ_s coincides by definition with χ or χ^τ on $H^{2\alpha+1}$. Thus θ does not coincide with any of the characters χ_s on $H^{2\alpha+1}$ and from $2\alpha+1 \leqslant 2\alpha+3 \leqslant 4\alpha+3$ we deduce

$$\langle\theta,\chi_s\rangle_{H^\alpha} = \langle\theta,\chi_s\rangle_{H^{\alpha+1}} = 0,$$

and hence also $S'_\alpha = 0$.

(ii): **Case** $i(\theta) \leqslant m$**.** Then θ coincides with exactly one of the characters χ, χ^τ on H^{n-1}. Thus $\langle\theta,\chi+\chi^\tau\rangle_{H^{n-1}} = 1$, $\langle\theta,\chi+\chi^\tau\rangle_Z = 2$, and $\langle\theta,\chi_s\rangle_{H^{n-1}} = 0$, $\langle\theta,\chi_s\rangle_Z = 1$ for all $s \in Q_\alpha$. Using Lemma 4.17(iii) we compute

$$S'_{n-1} = 2(q\cdot 0 - \frac{q-3}{2}) + (q\cdot 1 - 2) = 1.$$

Case $i(\theta) = m+1$**.** Then $\langle\theta,\chi+\chi^\tau\rangle_{H^{n-1}} = 0$, $\langle\theta,\chi+\chi^\tau\rangle_Z = 2$, $\langle\theta,\chi_s\rangle_Z = 1$ for all $s \in Q_{n-1}$. Moreover, $s(\theta, n-1)$ is well-defined and

$$\sum_{s\in Q_{n-1}} q\langle\theta,\chi_s\rangle_{H^{n-1}} = \begin{cases} q & \text{if } s(\theta,n-1)\in Q_{n-1} \\ 0 & \text{otherwise.}\end{cases}$$

Again we conclude by Lemma 4.17(iii). □

Proof of Theorem 4.18. Let ϕ be any one of the two characters of $E^\times$ satisfying $\phi(U_E) = 1$ and $\phi(t) = \chi(t)^{-1}$. Consider the $E^\times$-representation $\phi\Xi_\chi$ given by $(\phi\Xi_\chi)(e) = \phi(e)\Xi_\chi(e)$. By construction, it is trivial on the subgroup $\langle t, U_E^{m+1}\rangle$ of $E^\times$, and we consider it as a representation of the finite group $E^\times/\langle t, U_E^{m+1}\rangle \cong U_E/U_E^{m+1} \times \langle u\rangle/\langle u^2\rangle$. Let θ be a character of $E^\times$. Then $\langle\theta,\Xi_\chi\rangle_{E^\times} = 0$, unless θ coincides with χ on $F^\times U_E^{m+1}$. Assume this holds. Then $\phi\theta$ also factors through a character of $U_E/U_E^{m+1} \times \langle u\rangle/\langle u^2\rangle$ and its multiplicity in Ξ_χ can be computed as follows:

$$\langle\theta,\Xi_\chi\rangle_{E^\times} = \langle\phi\theta,\phi\Xi_\chi\rangle_{U_E/U_E^{m+1}\times\langle u\rangle/\langle u^2\rangle} = \frac{1}{2(q-1)q^m} \sum_{g\in U_E/U_E^{m+1}\times\langle u\rangle/\langle u^2\rangle} \theta(g^{-1})\mathrm{tr}(g;\phi\Xi_\chi).$$

Let $\lambda(\theta) \in \{0,1\}$ be such that $\theta(u) = (-1)^{\lambda(\theta)}\chi(u)$ and let $\mathrm{sgn}(\chi)$ be 0 if χ is even, and 1 otherwise. We deduce from the above and from Proposition 4.22:

$$\langle\theta,\Xi_\chi\rangle_{E^\times} = \frac{1}{2}(\langle\theta,\Xi_\chi\rangle_{U_E/U_E^{m+1}} + (-1)^{\lambda(\theta)}\langle\theta,\chi+(-1)^{\mathrm{sgn}(\chi)}\chi^\tau\rangle_{U_E/U_E^{m+1}}).$$

Now Theorem 4.18 follows from Proposition 4.19 by a simple case-by-case study. For example, if $\theta = \chi$, then $\langle \theta, \Xi_\chi \rangle_{U_E} = 1$ by Proposition 4.19, $\lambda(\theta) = 0$ and $\langle \theta, \chi^\tau \rangle_{U_E} = 0$. Hence the above computation shows

$$\langle \theta, \Xi_\chi \rangle_{E^\times} = \frac{1}{2}(1 + (-1)^0 \langle \theta, \chi + (-1)^{\mathrm{sgn}(\chi)} \chi^\tau \rangle_{U_E/U_E^{m+1}}) = \frac{1}{2}(1+1) = 1.$$

If $\theta = \chi^\tau$, then $\langle \theta, \Xi_\chi \rangle_{U_E} = 1$ by Proposition 4.19, $\theta(u) = \chi^\tau(u) = \chi(-u) = (-1)^{\mathrm{sgn}(\chi)}\chi(u)$, i.e., $\lambda(\theta) = \mathrm{sgn}(\chi)$, and $\langle \theta, \chi \rangle_{U_E} = 0$. Hence the above computation shows

$$\langle \theta, \Xi_\chi \rangle_{E^\times} = \frac{1}{2}(1 + (-1)^{\lambda(\theta)} \langle \theta, \chi + (-1)^{\mathrm{sgn}(\chi)} \chi^\tau \rangle_{U_E/U_E^{m+1}}) = \frac{1}{2}(1+1) = 1.$$

The other cases, i.e., θ coincides with χ (resp. with χ^τ) on U_E, but not on $E^\times$, θ does not coincide with χ or χ^τ on U_E and the distanse from χ to θ is properly quadratic (resp. is not properly quadratic), follow by similar computations. □

Proposition 4.22. *Let $g \in E^\times$ with $v_u(g) = 1$. Then*

$$\mathrm{tr}(g; \Xi_\chi) = \chi(g) + \chi^\tau(g).$$

Proof. The proof is given in Section 5.5. □

Corollary 4.23. *The character χ can be reconstructed from the $E^\times$-representation $\Xi_\chi|_{E^\times}$.*

Proof. By Lemma 4.4, $\Xi_\chi|_{E^\times}$ determines $\chi|_{F^\times U_E^{m+1}}$ uniquely. Consider the map

$$f\colon A := \{\theta \in (E^\times)^\vee : \theta|_{F^\times U_E^{m+1}} = \chi|_{F^\times U_E^{m+1}}\} \twoheadrightarrow \{\theta' \in U_E^\vee : \theta'|_{U_F U_E^{m+1}} = \chi|_{U_F U_E^{m+1}}\},$$

given by restricting characters of $E^\times$ to U_E. It is surjective and 2-to-1. By Proposition 4.19 and Theorem 4.18, χ and χ^τ are the two unique elements among all elements $\theta \in A$, with the following property: θ occurs in Ξ_χ, but the unique element of $f^{-1}(f(\theta)) \smallsetminus \{\theta\}$ does not occur in Ξ_χ. □

4.6. **Relation to strata, cuspidality.**

Using the unipotent traces computed in Section 4.3, we show the first part of Theorem 4.2. We use the terminology of intertwining and strata from [BH06]§11 and Chapter 4. The following is analogous to [Iva16] Proposition 4.22 and Corollary 4.23. Recall the notation N_n, N_n^n from Section 4.3. Let N resp. N^n denote the preimage of N_n resp. N_n^n under the natural projection $U_{\mathfrak{J}} \twoheadrightarrow U_{\mathfrak{J}}/U_{\mathfrak{J}}^{m+1}$.

Proposition 4.24. *Let $m \geqslant 0$. Let Ξ be an irreducible $E^\times U_{\mathfrak{J}}$-representation, which is trivial on $U_{\mathfrak{J}}^{m+1}$ and does not contain the trivial character on N^n. Then the $G(F)$-representation $\Pi_\Xi = \mathrm{c-Ind}_{E^\times U_{\mathfrak{J}}}^{G(F)} \Xi$ is irreducible, cuspidal and admissible. Moreover, it contains a ramified simple stratum $(\mathfrak{J}, m, \alpha)$ for some $\alpha \in \varpi^{-m}\mathfrak{J}$. One has $\ell(\Pi_\Xi) = \frac{m}{2}$. For any character ϕ of $F^\times$ one has $0 < \ell(\Pi_\Xi) \leqslant \ell(\phi\Pi_\Xi)$.*

From this we can deduce the first statement of Theorem 4.2.

Corollary 4.25. *Let $(E/F, \chi)$ be a minimal pair. The representation R_χ is irreducible, cuspidal and admissible. It contains a ramified simple stratum and is, in particular, ramified. Moreover, $\ell(R_\chi) = \frac{\ell(\chi)}{2}$ and for any character ϕ of $F^\times$ one has $0 < \ell(R_\chi) \leqslant \ell(\phi R_\chi)$.*

Proof. The assumptions of Proposition 4.24 are satisfied for the $E^\times U_{\mathfrak{J}}$-representation Ξ_χ by Corollary 4.7 and Proposition 4.5. □

Proof of Proposition 4.24. Irreducibility and cuspidality of Ξ follow from [BH06] Theorem 11.4, which assumptions are satisfied due to Lemma 4.26. To contain a stratum is defined with respect to an additive character. So fix some character ψ of F of level 1. Make the isomorphism (4.1) explicit for $k = m-1$, $r = m$:

$$\varpi^{-m}\mathfrak{J}/\varpi^{1-m}\mathfrak{J} \xrightarrow{\sim} (U_{\mathfrak{J}}^m/U_{\mathfrak{J}}^{m+1})^\vee.$$

An element of $\varpi^{-m}\mathfrak{J}/\varpi^{1-m}\mathfrak{J}$ resp. of $U_{\mathfrak{J}}^m/U_{\mathfrak{J}}^{m+1}$ is represented by a matrix $a = \begin{pmatrix} & a_2t^{-n} \\ a_3t^{1-n} & \end{pmatrix}$ resp. $x = \begin{pmatrix} & x_2t^{n-1} \\ x_3t^n & \end{pmatrix}$ with $a_2, a_3, x_2, x_3 \in k$ and $\psi_{\mathfrak{M},a}(x) = \psi(a_2x_3 + a_3x_2)$. The restriction of Ξ to $U_{\mathfrak{J}}^m$ factors through a representation of the abelian group $U_{\mathfrak{J}}^m/U_{\mathfrak{J}}^{m+1}$, thus it decomposes as a sum of characters, each of which is of the form $\psi_{\mathfrak{M},a}|_{U_{\mathfrak{J}}^m}$ for some $a \in \varpi^{-m}\mathfrak{J}$. With other words, for each a, such that $\psi_{\mathfrak{M},a}|_{U_{\mathfrak{J}}^m}$ is contained in Ξ, the ramified stratum $(\mathfrak{J}, m, a)$ occurs in Π_Ξ. By definition, a ramified stratum is simple, if and only it is fundamental, i.e., the coset $a + \varpi^{1-m}\mathfrak{J}$ does not contain a nilpotent element of $\mathfrak{M}$. Thus to show that Π_Ξ contains a ramified simple stratum it is enough to show the following claim.

Claim. Let $a \in \varpi^{-m}\mathfrak{J}$. Assume $\psi_{\mathfrak{M},a}|_{U_{\mathfrak{J}}^m}$ occurs in Ξ. Then $a + \varpi^{1-m}\mathfrak{J}$ does not contain nilpotent elements of $\mathfrak{M}$, or with other words $a_2, a_3 \neq 0$ (with notations as above).

Proof of the claim. Assume $a_2 = 0$, then the restriction of $\psi_{\mathfrak{M},a}$ to the subgroup N^n of $U_{\mathfrak{J}}^m$ is the trivial character, which contradicts our assumptions on Ξ. Thus $a_2 \neq 0$. Assume $a_3 = 0$. As $\varpi \in E^\times U_{\mathfrak{J}}$, the character $\psi_{\varpi a \varpi^{-1}}$ also occurs in Ξ (proof as in [Iva16] Lemma 4.25). But $\varpi a \varpi^{-1} = \begin{pmatrix} & a_3t^{-n} \\ a_2t^{1-n} & \end{pmatrix}$ and we deduce a contradiction as in the already proven part. □

Thus we have shown that Π_Ξ contains a ramified fundamental stratum of the form $(\mathfrak{J}, m, a)$. Then [BH06] Theorem 12.9 shows that $\ell(\Pi_\Xi) = \frac{m}{2}$. Furthermore, if an essentially scalar stratum would be contained in Π_Ξ, then by [BH06] Section 12.9, it would have to intertwine with $(\mathfrak{J}, m, a)$. But by [BH06] 13.2 Proposition, no fundamental stratum of the form $(\mathfrak{M}, r, b)$ can intertwine with the fundamental ramified stratum $(\mathfrak{J}, m, a)$. Thus no essentially scalar stratum is contained in Π_Ξ and [BH06] 13.3 Theorem shows the last statement of the proposition. □

Lemma 4.26. *Let Ξ be an irreducible $E^\times U_{\mathfrak{J}}$-representation, which is trivial on $U_{\mathfrak{J}}^{m+1}$ and does not contain the trivial character on N^n. An element $g \in G(F)$ intertwines Ξ if and only if $g \in E^\times U_{\mathfrak{J}}$.*

Proof. The double $E^\times U_{\mathfrak{J}}$-cosets in $G(F)$ are represented by diagonal matrices with entries t^α, 1 for $\alpha \geqslant 0$. The rest of the proof works exactly as in [Iva16] Lemma 4.24. □

4.7. **Relation to cuspidal inducing data.**

We relate the representations R_χ, π_χ to each other. The following proposition finishes the proof of Theorem 4.2.

Proposition 4.27. *Let $(E/F, \chi)$ be an admissible pair. Then $R_\chi \cong \pi_\chi$.*

Proof. By twisting both sides with a character of $F^\times$, we can assume that $(E/F, \chi)$ is a minimal pair. By construction of π_χ and Lemma 4.3, it is enough to show that $\Xi_\chi \cong \Theta_\chi$ (Θ_χ is as in Section 4.1.6). From Corollary 4.25 and the proof of Proposition 4.24 it follows that there is a simple (ramified) stratum $(\mathfrak{J}, m, \beta)$ such that $\Xi_\chi|_{U_{\mathfrak{J}}^m}$ contains ψ_β. By [BH06] 15.8 Exercise it follows that

$(E^\times U_{\mathfrak{J}}, \Xi_\chi)$ is a cuspidal inducing datum in $G(F)$, i.e., there is some χ' with $\Xi_\chi \cong \Theta_{\chi'}$. By the last statement of Corollary 4.25, $(E/F, \chi')$ has to be minimal. By Lemma 4.28, $\Theta_{\chi'}|_{E^\times} \cong \Xi_{\chi'}|_{E^\times}$. Thus $\Xi_{\chi'}|_{E^\times} \cong \Xi_\chi|_{E^\times}$. Now, by Corollary 4.23, χ is (up to τ-conjugacy) uniquely determined by Ξ_χ, and we deduce $\chi' = \chi$ or $\chi' = \chi^\tau$. As $\Theta_\chi \cong \Theta_{\chi^\tau}$, the proposition follows. □

Lemma 4.28. *Let $(E/F, \chi)$ be a minimal pair. We have $\Theta_\chi|_{E^\times} \cong \Xi_\chi|_{E^\times}$.*

Proof. The proof is given in Section 5.6. □

4.8. **Small level case.**

Let χ be a character of $E^\times$ of (odd) level $m \geqslant 1$. Let $n \geqslant m+1$ be an integer. Then χ defines a character $\tilde{\chi}$ of the group $E^\times U_{\mathfrak{J}}^n$ by composition

$$\tilde{\chi}\colon E^\times U_{\mathfrak{J}}^n \twoheadrightarrow E^\times U_{\mathfrak{J}}^n / U_{\mathfrak{J}}^n \cong E^\times / U_E^n \twoheadrightarrow E^\times / U_E^{m+1} \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times. \tag{4.7}$$

Proposition 4.29. *Let χ be a character of $E^\times$ of odd level $m \geqslant 1$ and let $n \geqslant m+1$. Then*

$$\mathrm{R}_{\chi,n} \cong \mathrm{c-Ind}_{E^\times U_{\mathfrak{J}}^n}^{G(F)} \tilde{\chi}.$$

Proof of Proposition 4.29. By Lemma 4.3 it is enough to show $\Xi_{\chi,n} \cong \mathrm{c-Ind}_{E^\times U_E^n}^{E^\times U_{\mathfrak{J}}} \tilde{\chi}$. To do so, it is enough to show that the traces of each element of $E^\times U_{\mathfrak{J}}$ in both spaces agree. Modulo center, which acts by $\chi|_{F^\times}$ in both spaces, any element of $E^\times U_{\mathfrak{J}}$ is represented by an element in $U_{\mathfrak{J}} \cup \varpi U_{\mathfrak{J}}$, thus we can restrict to elements lying in this union. The required trace computations are covered by Lemmas 4.30 and 4.31. □

Lemma 4.30. *Let $g \in U_{\mathfrak{J}}$. Precisely one of the following cases occurs:*

- (i) *$g \in U_F U_{\mathfrak{J}}^n$. Then $\mathrm{tr}(g; \Xi_{\chi,n})[\chi]) = (q-1)q^{n-1}\tilde{\chi}(g)$. In particular, $U_{\mathfrak{J}}^n$ acts trivial in $\Xi_{\chi,n})[\chi]$ and U_F acts through the character $\chi|_{F^\times}$.*
- (ii) *$g \in U_{\mathfrak{J}} \smallsetminus U_F U_{\mathfrak{J}}^n$ is conjugate to an element x of $U_F U_E^{2\alpha+1} U_{\mathfrak{J}}^n \smallsetminus U_F U_E^{2\alpha+3} U_{\mathfrak{J}}^n$, such that $2\alpha + 2 \leqslant n$. Then*
$$\mathrm{tr}(g; \Xi_{\chi,n})[\chi]) = q^{2\alpha+1}(\tilde{\chi}(x) + \tilde{\chi}^\tau(x)).$$
- (iii) *$g \in U_{\mathfrak{J}} \smallsetminus U_F U_{\mathfrak{J}}^n$ is not conjugate to an element of $U_E U_{\mathfrak{J}}^n$. Then $\mathrm{tr}(g; \Xi_{\chi,n})[\chi]) = 0$.*

Let $g \in \varpi U_{\mathfrak{J}}$. Precisely one of the following two cases can occur:

- (i)′ *g is not conjugate to an element of $E^\times U_{\mathfrak{J}}^n$. Then $\mathrm{tr}(g; \Xi_{\chi,n})[\chi]) = 0$.*
- (ii)′ *g is conjugate to an element x of $E^\times U_{\mathfrak{J}}^n$. Then $\mathrm{tr}(g; \Xi_{\chi,n})[\chi]) = \tilde{\chi}(x) + \tilde{\chi}^\tau(x)$.*

Proof. The proof is given in Section 5.7. □

Lemma 4.31. *Lemma 4.30 holds with $\Xi_{\chi,n}$ replaced by $\mathrm{c-Ind}_{E^\times U_E^n}^{E^\times U_{\mathfrak{J}}} \tilde{\chi}$.*

Proof. The lemma follows by an explicit computation using the Mackey-formula in a way very similar to the proof of Lemma 4.28. We omit the details. □

Using notations from Definition 4.16 we have the following structure result.

Theorem 4.32. *Let χ be a character of $E^\times$ of odd level $m \geqslant 1$ and let $n \geqslant m+1$. Let θ be a character of level $\geqslant m$. There are no non-zero maps from $R_{\chi,n}$ to R_θ, unless θ coincides with χ on $F^\times U_E^{m+1}$. In this case, we have*

$$\mathrm{Hom}_{G(F)}(R_{\chi,n},R_\theta)=\begin{cases}\overline{\mathbb{Q}}_\ell & \text{if } \theta=\chi, \text{ or } \theta=\chi^\tau, \text{ or the distance from } \theta \text{ to } \chi \text{ is properly quadratic}\\ 0 & \text{otherwise.}\end{cases}$$

In particular, the character χ can be reconstructed from $R_{\chi,n}$.

Proof. By Lemma 4.33 we may assume that θ and χ coincide on $F^\times U_{\mathfrak{J}}^{m+1}$. Thus by our assumption on θ, $(E/F,\chi)$ is a minimal pair and we compute

$$\begin{aligned}\mathrm{Hom}_{G(F)}(R_{\chi,n},R_\theta) &= \mathrm{Hom}_{E^\times U_{\mathfrak{J}}}(\Xi_{\chi,n},\Xi_\theta)\\ &= \mathrm{Hom}_{E^\times U_{\mathfrak{J}}^n}(\tilde{\chi},\Xi_\theta)\\ &= \mathrm{Hom}_{E^\times}(\tilde{\chi},\Xi_\theta)\\ &= \mathrm{Hom}_{E^\times}(\tilde{\chi},\textstyle\bigoplus_{\theta'}\theta'),\end{aligned}$$

where θ' runs through the set of all characters of $E^\times$ coinciding with θ on $F^\times U_E^{m+1}$, such that either $\theta'=\theta$, or $\theta'=\theta^\tau$, or the distance from θ to θ' is properly quadratic. Above, the first equality follows from Lemma 4.33. The second is Frobenius reciprocity and Proposition 4.29. The third follows, as $n\geqslant m+1$ and hence $\tilde{\chi}$, Ξ_θ are trivial on $U_{\mathfrak{J}}^n$. Finally, the forth equality follows from Theorem 4.18. The above computation shows the statement of the theorem about $\mathrm{Hom}_{G(F)}(R_{\chi,n},R_\theta)$. It remains to show that χ can be reconstructed from $R_{\chi,n}$. First, the above considerations characterize m as the greatest odd integer, such that there are non-zero maps from $R_{\chi,n}$ to R_θ for some θ of level m. The rest follows as in the proof of Corollary 4.23. $\square$

Lemma 4.33. *Let θ be a character of $E^\times$ of odd level $\ell(\theta)\geqslant m$. If θ does not coincide with χ on $F^\times U_E^{m+1}$, then there are no non-zero morphisms between $R_{\chi,n}$ and R_θ. Assume θ coincides with χ on $F^\times U_E^{m+1}$ (in particular, $\ell(\theta)=m$). Then*

$$\mathrm{Hom}_{G(F)}(R_{\chi,n},R_\theta)=\mathrm{Hom}_{E^\times U_{\mathfrak{J}}}(\Xi_{\chi,n},\Xi_\theta).$$

Proof. Applying twice the Frobenius reciprocity and once the Mackey formula, we see by Lemma 4.3

$$\mathrm{Hom}_{G(F)}(R_{\chi,n},R_\theta)=\bigoplus_{g\in E^\times U_{\mathfrak{J}}\backslash G(F)/E^\times U_{\mathfrak{J}}}\mathrm{Hom}_{E^\times U_{\mathfrak{J}}\cap {}^g(E^\times U_{\mathfrak{J}})}({}^g\Xi_{\chi,n},\Xi_\theta),$$

where ${}^g(E^\times U_{\mathfrak{J}})=g(E^\times U_{\mathfrak{J}})g^{-1}$ and ${}^g\Xi_{\chi,n}(x)=\Xi_{\chi,n}(g^{-1}xg)$. The set $E^\times U_{\mathfrak{J}}\backslash G(F)/E^\times U_{\mathfrak{J}}$ is represented by the diagonal matrices $e_0(t^\alpha,1)$ for $\alpha\geqslant 0$. Let $g=e_0(t^\alpha,1)$ with $\alpha>0$. We show that the summand on the right side corresponding to g vanish. Note that $E^\times U_{\mathfrak{J}}\cap {}^g(E^\times U_{\mathfrak{J}})\supseteq e_-(\mathfrak{p}_F^{\frac{m+1}{2}})$. On the one hand, Proposition 4.5 shows that Ξ_θ does not contain the trivial character on $e_-(\mathfrak{p}_F^{\frac{m+1}{2}})$. On the other hand, $e_-(\mathfrak{p}_F^{\frac{m+1}{2}+\alpha})\subseteq U_{\mathfrak{J}}^{m+1}$ as $\alpha>0$. As $U_{\mathfrak{J}}^{m+1}$ is normal in $E^\times U_{\mathfrak{J}}$, and $\tilde{\chi}$ is trivial on $U_{\mathfrak{J}}^{m+1}$, we see by Proposition 4.29 that $\Xi_{\chi,n}$ is trivial on $U_{\mathfrak{J}}^{m+1}$ and, in particular, on $e_-(\mathfrak{p}_F^{\frac{m+1}{2}+\alpha})$. As

$${}^g\Xi_{\chi,n}(e_-(x))=\Xi_{\chi,n}(e_-(t^\alpha x)),$$

we deduce that ${}^g\Xi_{\chi,n}$ is trivial on $e_-(\mathfrak{p}_F^{\frac{m+1}{2}})$. The claim follows, and hence

$$\mathrm{Hom}_{G(F)}(R_{\chi,n},R_\theta)=\mathrm{Hom}_{E^\times U_{\mathfrak{J}}}(\Xi_{\chi,n},\Xi_\theta).$$

It remains to show that this space is 0, unless θ coincides with χ on $F^\times U_E^{m+1}$. Asssume $\mathrm{Hom}_{E^\times U_{\mathfrak{J}}}(\Xi_{\chi,n},\Xi_\theta) \neq 0$. By comparing central characters, we see that $\theta|_{F^\times} = \chi|_{F^\times}$. As above, we see that $\Xi_{\chi,n}$ is trivial on $U_{\mathfrak{J}}^{m+1}$. Hence Ξ_θ is too. We deduce $\ell(\theta) = m$. This shows our claim. □

5. Trace computations

In this section we use notations from Sections 3 and 4, especially from Notations 3.2, 3.5 and Definition 3.7. For $g \in G(F)$ we always write $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$.

5.1. Left and right group actions on $X^m_{\underline{w}_m}(1)$.

To apply a trace formula in what follows, we make here the actions (3.10) explicit using the coordinates $\psi^m_{\dot v}$ from (3.2). It is clear that I acts on $C^m_v \subseteq \mathscr{F}^m$ by left multiplication. The following proposition describes this action.

Proposition 5.1. *Let $\dot v$,v,n be as in Notation 3.2 and $m \geqslant 1$ odd (we do not assume $2n-1 \geqslant m$ here). Let $\dot x I^m = \psi^m_{\dot v}(a,C,D,A,B)$ be a point of C^m_v. Then $g \in I$ acts on $\dot x I^m$ by*

$$\begin{aligned} g.\dot x I^m &= \psi^m_{\dot v}(g.a|_n, \frac{\det(g)}{g_2a+g_1}CN^{-1}, (g_2a+g_1)DN, \dots \\ &\qquad \dots AN + h(g,a)\frac{(g_2a+g_1)^2DN^2}{\det(g)C}, B + u^{n+1}\frac{g_2}{(g_2a+g_1)}CD^{-1}N^{-1}), \end{aligned}$$

where

$$\begin{aligned} g.a &:= \frac{g_4a+g_3}{g_2a+g_1} \in L_{[1,n+m]}\mathbb{G}_a(\bar k) \quad \textit{and } \cdot|_n \textit{ is as in Section 3.1.5} \\ N &:= 1 + u^{n+1}\frac{g_2}{g_2a+g_1}CD^{-1}A \\ h(g,a) &:= u^{-(n+1)}(g.a - g.a|_n) \in L_{[0,m-1]}\mathbb{G}_a(\bar k). \end{aligned}$$

Proof. First, observe that the expressions in the proposition are well-defined, as $v_u(g_1) = v_u(g_4) = 0$, $v_u(g_2) \geqslant 0$, $v_u(g_3) > 0$ and $v_u(a) > 0$. We compute in $G(\breve E)$ (with $a, g.a, C, D$ replaced by some representatives in $\bar k[\![u]\!]$):

$$ge_-(a)\dot v e_0(C,D) = e_-(g.a)\dot v e_0(\frac{\det(g)C}{g_2a+g_1}, (g_2a+g_1)D)e_-(u^{n+1}\frac{g_2CD^{-1}}{g_2a+g_1}).$$

Further, using (3.4) we see that

$$e_-(u^{n+1}\frac{g_2CD^{-1}}{g_2a+g_1})e_+(A)e_-(B) = e_0(N^{-1},N)e_+(AN)e_-(B+u^{n+1}\frac{g_2CD^{-1}}{g_2a+g_1}N^{-1}),$$

with N as in the proposition. Combining the two last computations we see:

$$\begin{aligned} g.\dot x I^m &= ge_-(a)\dot v e_0(C,D)e_+(A)e_-(B) \\ &= e_-(g.a)\dot v e_0(\frac{\det(g)C}{g_2a+g_1}, (g_2a+g_1)D)e_-(u^{n+1}\frac{g_2CD^{-1}}{g_2a+g_1})e_+(A)e_-(B) \qquad (5.1) \\ &= e_-(g.a)\dot v e_0(\frac{\det(g)C}{g_2a+g_1}, (g_2a+g_1)D)e_0(N^{-1},N)e_+(AN)e_-(B+u^{n+1}\frac{g_2CD^{-1}}{g_2a+g_1}). \end{aligned}$$

Now the only thing we have to do, is to replace $g.a \in L_{[1,n+m]}\mathbb{G}_a(\bar{k})$ in the last expression by an element in $L^{\leqslant n}_{[1,n+m]}\mathbb{G}_a(\bar{k})$. Therefore, note that $g.a$ and $g.a|_n$ have the same image in $L_{[1,n]}\mathbb{G}_a(\bar{k})$, i.e., $g.a - g.a|_n$ is divisible by u^{n+1} and $h(g,a)$ is well-defined as an element of $L_{[0,m-1]}\mathbb{G}_a(\bar{k})$ and that

$$e_-(g.a)\dot{v} = e_-(g.a|_n)e_-(g.a - g.a|_n)\dot{v} = e_-(g.a|_n)\dot{v}e_+(h(g,a)).$$

Combining this and (5.1) finishes the proof of the proposition. □

We compute $h(g,a)$ from Proposition 5.1 in some cases of interest for us. We point out, that later we need to know $h(g,a)$ only modulo u^n (cf. Proposition 5.6).

Lemma 5.2. *Let n,m be as in Proposition 5.1. Assume $m \leqslant 2n-1$. Let $g \in U_{\mathfrak{J}}$ and $a \in L^{\leqslant n}_{[1,n+m]}\mathbb{G}_a(k)$ with $a_1 \neq 0$.*

(i) *For $g \in U^{m+1}_{\mathfrak{J}}$ we have $v_u(h(g,a)) \geqslant n$.*

Let $g = \iota(1 + yu^{2\alpha+1})$ with $0 \leqslant \alpha \leqslant n-1$ and $y \in U_F$. Write $a = ua'$

(ii) *If $\alpha \geqslant \lfloor \frac{n}{2} \rfloor$, then $h(g,a) = u^{2\alpha+1-n}y(1-a'^{,2})(1-u^{2\alpha+1}ya')$.*

(iii) *If $0 \leqslant \alpha < \lfloor \frac{n}{2} \rfloor$ and $a' = \pm 1 + u^{n-2\alpha-1}b$ for some $b \in L^{\leqslant 2\alpha}_{[0;m+2\alpha+1]}\mathbb{G}_a(k)$, then $h(g,a) = \frac{y(\mp 2b - u^{n-2\alpha-1}b^2)}{1 \pm u^{2\alpha+1}y + u^n yb}$.*

Proof. In any of the three cases, a simple calculation shows $g.a|_n = a$ (only this case is of interest for us, cf. (5.2)). Now (i) is an easy computation. For (ii) and (iii) we compute

$$u^{n+1}h(g,a) = g.a - g.a|_n = g.a - a = \frac{a + u^{2\alpha+2}y}{1 + u^{2\alpha}ya} - a = \frac{u^{2\alpha+2}y(1-a'^{,2})}{1 + u^{2\alpha+1}ya'}.$$

From this the lemma follows. □

Let ϖ be as in Section 3.1.3 and $y_1 := e_0(u,-u)$. Left multiplication by ϖ composed with right multiplication by y_1^{-1} defines an automorphism $\tilde{\beta}_\varpi$ of $\tilde{Y}^m_{\dot{w}}$. By (the proof of) Lemma 3.12, $\tilde{\beta}_\varpi$ restricts to an automorphism

$$\beta_\varpi\colon Y^m_{\dot{w}} \xrightarrow{\sim} Y^m_{\dot{w}} \quad \text{given by} \quad \dot{x}I^m \mapsto \varpi\dot{x}y_1^{-1}I^m.$$

Proposition 5.3. *Let $\dot{v}$, v, n, D^τ_w be as in Notation 3.2 and $2n-1 \geqslant m \geqslant 1$ odd. Let $\psi^m_{\dot{v}}(\pm u, C, D, A, B)$ be a point of $Y^m_{\dot{w}}$ lying over $\pm u \in D^\tau_w$. Then*

$$\beta_\varpi(\psi^m_{\dot{v}}(\pm u, C, D, A, B)) = \psi^m_{\dot{v}}(\pm u, \mp CM^{-1}, \mp DM, -AM, B),$$

where $M := 1 - 2u^nC\tau(C)^{-1}A$.

Proof. Write $\dot{x}I^m = \psi^m_{\dot{v}}(\pm u, C, D, A, B)$. Using formulas (3.4) we compute

$$\begin{aligned}\beta_\varpi(\dot{x}I^m) &= e_-(\pm u)\dot{v}e_0(\mp 1, \mp 1)e_-(\mp u^n)e_0(C,D)e_+(-A)e_-(-B)I^m = \\ &= e_-(\pm u)\dot{v}e_0(\mp CM'^{,-1}, \mp DM')e_+(-AM')e_-(\mp u^nCD^{-1} - B),\end{aligned}$$

where $M' := 1 \pm u^nCD^{-1}A$. Now $a = \pm u$ gives $R = u^{-1}(\tau(a) - a) = u^{-1}(\mp u - (\pm u)) = \mp 2$ and as $\dot{x}I^m \in Y^m_{\dot{w}}$, we have $D^{-1} \equiv R\tau(C)^{-1} \equiv \mp 2\tau(C)^{-1} \mod u^n$ and $B = u^nC\tau(C)^{-1}$. This shows on the one hand $M' = M$, and on the other hand $\mp u^nCD^{-1} - B = 2B - B = B$. □

Now we make the right $I_{m,\underline{w}_m}/I^m$-action on $Y^m_{\dot{w}}$ explicit.

Proposition 5.4. *Let $\dot{v}$,v, n be as in Notation 3.2 and $m \geqslant 1$ odd (we do not assume $2n-1 \geqslant m$ here). Let $\psi_{\dot{v}}^m(a,C,D,A,B)$ be a point of C_v^m. Then $i = \begin{pmatrix} i_1 & \\ & \tau(i_1) \end{pmatrix}\begin{pmatrix} 1 & i_2 \\ & 1 \end{pmatrix} \in I_{m,\underline{w}_m}/I^m$ acts on $\psi_{\dot{v}}^m(a,C,D,A,B)$ by*

$$\psi_{\dot{v}}^m(a,C,D,A,B).i \quad = \psi_{\dot{v}}^m(a, Ci_1H^{-1}, D\tau(i_1)H, i_1^{-1}\tau(i_1)H^2A + i_2H, i_1\tau(i_1)^{-1}BH^{-1}),$$

where $H = 1 + i_1\tau(i_1)^{-1}i_2B \in \bar{k}[u]/u^{m+1}$ (note that i_2 is only determined $\mod u^m$, but $B \equiv 0 \mod u$).

Proof. The proof is a computation similar to (and simpler as) the proof of Proposition 5.1. □

5.2. **Generalities on the trace formula.**

We use the following trace formula due to Boyarchenko.

Lemma 5.5 ([Boy12] Lemma 2.12)**.** *Let X be a separated scheme of finite type over a finite field $\mathbb{F}_Q$ with Q elements, on which a finite group A acts on the right. Let $g\colon X \to X$ be an automorphism of X, which commutes with the action of A. Let $\psi\colon A \to \overline{\mathbb{Q}}_\ell^*$ be a character of A. Assume that $\mathrm{H}_c^i(X)[\psi] = 0$ for $i \neq i_0$ and Frob_Q acts on $\mathrm{H}_c^{i_0}(X)[\psi]$ by a scalar $\lambda \in \overline{\mathbb{Q}}_\ell^*$. Then*

$$\mathrm{Tr}(g^*, \mathrm{H}_c^{i_0}(X)[\psi]) = \frac{(-1)^{i_0}}{\lambda \cdot \sharp A} \sum_{a \in A} \psi(a) \cdot \sharp S_{g,a},$$

where $S_{g,a} = \{x \in X(\overline{\mathbb{F}_q}) \colon g(\mathrm{Frob}_Q(x)) = x \cdot a\}$.

We adapt Lemma 5.5 to our situation.

Proposition 5.6. *Let $n \geqslant 1$, $m \geqslant 1$ two integers with $m \leqslant 2n+1$. Let χ be a character of $E^\times$ of level m. Let $g \in U_{\mathfrak{J}}$. Then*

$$\mathrm{tr}(g; \mathrm{H}_c^0(\tilde{Y}_{\dot{w}}^m)[\chi]) = \sum_{i_1 \in U_E/U_E^{m+1}} \sharp S_{g,i_1}\chi(i_1),$$

where S_{g,i_1} is empty, unless $\det(g) \equiv i_1\tau(i_1) \mod u^{m+1}$, in which case it is the set of solutions of the equations

$$g_2a^2 + (g_1 - g_4)a - g_3 \equiv 0 \mod u^{n+1} \tag{5.2}$$
$$\tau(i_1)(1 + u^n h(g,a)R^{-1}) \equiv g_2a + g_1 \mod u^{m+1} \tag{5.3}$$

in $a \in L_{[1,n+m]}^{\leqslant n}\mathbb{G}_a(k)$ (with $a_1 \neq 0$), where

$$h(g,a) = u^{-(n+1)}(g.a - g.a|_{u^n}) \in L_{[0,m-1]}\mathbb{G}_a(\bar{k})$$
$$R = u^{-1}(\tau(a) - a). \tag{5.4}$$

Lemma 5.7. *Let χ be a character of $E^\times$. We have $\mathrm{H}_c^0(\tilde{Y}_{\dot{w}}^m)[\chi] \cong \mathrm{H}_c^0(Y_{\dot{w}}^m)[\chi|_{U_E}]$.*

Proof. The proof is the same as in [Iva16] Lemma 4.5. □

Lemma 5.8. *Let $n \leqslant s \leqslant 2n$ be positive integers. Let $f \in k[u]/(u^s)$ and let $h\colon k[u]/(u^n) \to k[u]/(u^{s-n})$ be some map. Then for $x \in k[u]/(u^s)$ we have*

$$x = f + u^n h(x \mod u^n) \quad \Leftrightarrow \quad x = f + u^n h(f \mod u^n)$$

(both equalities take place in $k[u]/(u^s)$).

Proof. This is trivial. $\square$

Proof of Proposition 5.6. The action of g on $\tilde{Y}^m_{\ddot{w}}$ fixes $Y^m_{\ddot{w}}$. By Lemma 5.7 we have

$$\mathrm{tr}(g; \mathrm{H}^0_c(\tilde{Y}^m_{\ddot{w}})[\chi]) = \mathrm{tr}(g; \mathrm{H}^0_c(Y^m_{\ddot{w}})[\chi|_{U_E}]).$$

We have $\sharp I_{m,\underline{w}_m}/I^m = (q-1)q^{2m}$. Applying Lemma 5.5 to the left action of $U_{\mathfrak{J}}$ and the right action of $I_{m,\underline{w}_m}/I^m$ on $Y^m_{\ddot{w}}$ and Frob_q (this is possible, as only the zeroth cohomology is non-vanishing, and as the Frobenius acts as a scalar in H^0_c), we deduce

$$\mathrm{tr}(g; \mathrm{H}^0_c(\tilde{Y}^m_{\ddot{w}})[\chi]) = \frac{1}{(q-1)q^{2m}} \sum_{i \in I_{m,\underline{w}_m}/I^m} \sharp S_{g,i}\chi(i),$$

where $S_{g,i}$ is the set of points $y \in Y^m_{\ddot{w}}$ with $g.y = y.i$ (note that any point in $Y^m_{\ddot{w}}$ has coordinates in k, hence Frobenius acts trivial). Further, note that a point of $Y^m_{\ddot{w}}$ is uniquely determined by its coordinates a, C, A (cf. Definition 3.7). Write $i = \begin{pmatrix} i_1 & \\ & \tau(i_1) \end{pmatrix}\begin{pmatrix} 1 & i_2 \\ & 1 \end{pmatrix}$ with $i_1 \in U_E/U_E^{m+1}$, $i_2 \in k[u]/u^m$. As the determinant is multiplicative, we see that $S_{g,i} = \varnothing$, unless $\det(g) \equiv \det(i) = i_1\tau(i_1) \mod u^{m+1}$. Assume this holds. By Propositions 5.1 and 5.4, we see that $\sharp S_{g,i}$ is equal to the number of solutions of the equations

$$g.a|_n \equiv a \mod u^{n+1} \tag{5.5}$$

$$\frac{\det(g)}{g_2 a + g_1} C N^{-1} \equiv C i_1 H^{-1} \mod u^{m+1} \tag{5.6}$$

$$AN + h(g,a)\frac{(g_2 a + g_1)^2 D N^2}{\det(g) C} \equiv i_1^{-1}\tau(i_1) H^2 A + i_2 H \mod u^m \tag{5.7}$$

in the variables $a = \sum_{i=1}^n a_i u^i + \sum_{i=n+1}^m 0u^i \in L^{\leqslant n}_{[1,n+m]}\mathbb{G}_a(k)$ (with $a_1 \neq 0$), $C \in (k[u]/u^{m+1})^\times$ and $A \in k[u]/u^m$, where

$$\begin{aligned} B &= u^n C\tau(C)^{-1} \\ D &= R^{-1}\tau(C)(1 + u^n C\tau(C)^{-1}A - u^n C^{-1}\tau(C)\tau(A)) \end{aligned}$$

(as we are in $Y^m_{\ddot{w}}$; here $R = u^{-1}(\tau(a) - a)$) and $h(g,a)$ and

$$\begin{aligned} N &= 1 + u^{n+1}\frac{g_2}{g_2 a + g_1} C D^{-1} A \equiv 1 + u^{n+1}\frac{g_2}{g_2 a + g_1} R C \tau(C)^{-1} A \mod u^{m+1} \\ H &= 1 + i_1\tau(i_1)^{-1} i_2 B = 1 + u^n i_1 \tau(i_1)^{-1} i_2 C \tau(C)^{-1} \end{aligned}$$

are as in Propositions 5.1 and 5.4. As the character χ of $I_{m,\underline{w}_m}/I^m$ is inflated from a character of U_E/U_E^{m+1} (again denoted by χ), we see that

$$\sum_{i \mapsto i_1} \sharp S_{g,i}\chi(i) = \sharp S'_{g,i_1}\chi(i_1),$$

where i varies through all elements of $I_{m,\underline{w}_m}/I^m$ lying over i_1 and $\sharp S'_{g,i_1}$ is the number of solutions of equations (5.5), (5.6), (5.7) in the variables a, C, A, i_2. It is enough to show that $\sharp S'_{g,i_1} =$

$(q-1)q^{2m}\sharp S_{g,i_1}$. If $n \geqslant m+1$, then $N, H \equiv 1 \mod u^{m+1}$ and the proof is immediate. Assume $n \leqslant m \leqslant 2n$. We cancel C in (5.6) and insert the condition on the determinant to bring it to the form

$$\frac{\tau(i_1)}{g_2 a + g_1} H \equiv N \mod u^{m+1}. \tag{5.8}$$

By replacing N by $\frac{\tau(i_1)}{g_2 a+g_1} H$ in (5.7) and canceling the invertible term H we see that the equations (5.5), (5.6), (5.7) are equivalent to the three equations (5.5), (5.8) and

$$i_2 \equiv \frac{h(g,a)\tau(i_1)HD}{i_1 C} + \frac{\tau(i_1)A}{g_2 a + g_1} - \frac{\tau(i_1)HA}{i_1} \mod u^m. \tag{5.9}$$

Using $H \equiv 1 \mod u^n$ and $D \equiv R^{-1}\tau(C) \mod u^n$ equation (5.9) implies:

$$i_2 \equiv \frac{h(g,a)\tau(i_1)\tau(C)}{i_1 RC} + \frac{\tau(i_1)}{g_2 a + g_1} A - \frac{\tau(i_1)A}{i_1} \mod u^n \tag{5.10}$$

(the right hand side does not depend on i_2). We can replace i_2 occurring in the term H in (5.8) by the right hand side of (5.10) and hence our three original equations (5.5), (5.6), (5.7) are equivalent to (5.5),

$$\begin{aligned} \tau(i_1)(1 + u^n(\frac{h(g,a)}{R} + \frac{i_1}{g_2 a + g_1} C\tau(C)^{-1}A - C\tau(C)^{-1}A)) \equiv \\ \equiv (g_2 a + g_1) + u^{n+1} g_2 RC\tau(C)^{-1}A \mod u^{m+1} \end{aligned} \tag{5.11}$$

and (5.9). By Lemma 5.8 applied to $x = i_2$, equation (5.9) is just an expression of i_2 in terms of g, i_1, a, C, A, hence it can be ignored and we see that $\sharp S'_{g,i_1}$ is the number of solutions of (5.5) and (5.11) in the variables a, C, A.

Now, (5.11) implies $\tau(i_1) \equiv g_2 a + g_1 \mod u^n$. Applying Lemma 5.8 to $x = \tau(i_1)$, we see that (5.11) is equivalent to

$$\begin{aligned} \tau(i_1) \quad + \quad & u^n((g_2 a + g_1)h(g,a)R^{-1} + (g_2\tau(a) + g_1)C\tau(C)^{-1}A - (g_2 a + g_1)C\tau(C)^{-1}A) \equiv \\ \equiv \quad & (g_2 a + g_1) + u^{n+1} g_2 RC\tau(C)^{-1}A \mod u^{m+1}. \end{aligned} \tag{5.12}$$

Inserting on the right hand side $R = u^{-1}(\tau(a) - a)$, we immediately see that (5.12) is equivalent to (5.3). Moreover, (5.5) is immediately seen to be equivalent to (5.2). As in (5.2), (5.3) neither C, nor A occur, and as C lives in $(k[u]/u^{m+1})^\times$ and A lives in $k[u]/u^m$, we deduce that $\sharp S'_{g,i_1} = (q-1)q^{2m}\sharp S_{g,i_1}$. □

We now examine solutions of the equation (5.2) in $a \in L^{\leqslant n}_{[1,n+m]}\mathbb{G}_a(k)$ (with $a_1 \neq 0$). Recall that via the embedding ι (see Section 3.1.3) we have the subgroups $U_F U^n_{\mathfrak{J}} \subseteq U_E U^n_{\mathfrak{J}} \subseteq U_{\mathfrak{J}}$.

Lemma 5.9. *Let $g \in U_{\mathfrak{J}}$. Precisely one of the following cases occur:*

- (i) *$g \in U_F U^n_{\mathfrak{J}}$. Then (5.2) has precisely $(q-1)q^{n-1}$ solutions.*
- (ii) *$g \in U_{\mathfrak{J}} \smallsetminus U_F U^n_{\mathfrak{J}}$ is conjugate in $U_{\mathfrak{J}}$ to an element of $U_E U^n_{\mathfrak{J}}$. In this case (5.2) has precisely $2q^{v_u(g_3)-1}$ solutions.*
- (iii) *$g \in U_{\mathfrak{J}} \smallsetminus U_F U^n_{\mathfrak{J}}$ is not conjugate in $U_{\mathfrak{J}}$ to an element of $U_E U^n_{\mathfrak{J}}$. Then (5.2) has no solutions.*

Proof. Assume (5.2) has a solution a. As $g \in U_{\mathfrak{J}}$, the integers $v_u(g_2), v_u(g_3), v_u(g_1 - g_4)$ are even. As $a_1 \neq 0$, we have $v_u(a) = 1$. We deduce that $v_u((g_1 - g_4)a)$ is odd and $v_u(g_2 a^2)$, $v_u(g_3)$ are even. Thus by Lemma 5.10 we are either in the case $g \in U_F U_{\mathfrak{J}}^n$ of the lemma, where each of these three integers is $\geqslant n+1$ and each element of $L_{[1,n+m]}^{\leqslant n}\mathbb{G}_a(k)$ solves equation (5.2), or we are forced to have $v_u(g_3) = v_u(g_2) + 2 < n+1$ and $v_u(g_3) \leqslant v_u(g_1 - g_4) + 1$ (this last is, using parity, equivalent to $v_u(g_3) \leqslant v_u(g_1 - g_4)$). In the last case write $g_2 = g_2' u^{v_u(g_2)}$, $g_3 = g_3' u^{v_u(g_2)+2}$, $a = a'u$ and $g_1 - g_4 = g_{1,4}' u^{v_u(g_2)+2}$ with $g_2', g_3', a' \in k[\![t]\!]^\times$ and $g_{1,4}' \in k[\![t]\!]$. After canceling $u^{v_u(g_3)} = u^{v_u(g_2)+2}$, (5.2) is equivalent to

$$g_2' a'^{,2} + g_{1,4}' a' u - g_3' \equiv 0 \mod u^{n+1-v_u(g_3)}, \tag{5.13}$$

where $n + 1 - v_u(g_3) \geqslant 1$. Reducing modulo u, we deduce $a_1^2 \equiv \frac{g_3'}{g_2'} \mod u$, which shows that $\frac{g_3'}{g_2'}$ mod u must be a square of an element of $k^\times$, or, equivalently (cf. Lemma 4.15), that $\frac{g_3}{tg_2} \in k[\![t]\!]^\times$ is a square. Thus by Lemma 5.10 we deduce that we must be in case (ii) of the lemma and that in case (iii) there are no solutions. In case (ii) with notations as above, we have to determine how many solutions in $a' = a_1 + a_2 u + \cdots + a_n u^{n-1}$ equation (5.13) has. Using induction, one now easily deduces that there are exactly two possibilities for a_1, exactly 1 possibility for each $a_2, \ldots, a_{n+1-v_u(g_3)}$ and exactly q possibilities for each $a_{n+2-v_u(g_3)}, \ldots, a_n$. □

Lemma 5.10. *Let $g \in U_{\mathfrak{J}}$ and $n \geqslant 1$. Then*

(i) $g \in U_F U_{\mathfrak{J}}^n \Leftrightarrow v_u(g_2) \geqslant n-1,\ v_u(g_3) \geqslant n+1,\ v_u(g_1 - g_4) \geqslant n.$

(ii) *$g \in U_{\mathfrak{J}} \smallsetminus U_F U_{\mathfrak{J}}^n$ and g is conjugate to an element of $U_E U_{\mathfrak{J}}^n$ if and only if $v_u(g_3) = v_u(g_2) + 2 < n+1$, $v_u(g_3) \leqslant v_u(g_1 - g_4)$ and $\frac{g_3}{tg_2} \in k[\![t]\!]^\times$ is a square of an element in $k[\![u]\!]^\times$*

Proof. (i): is an easy computation (use that $v_u(g_j)$ is always even). (ii): In the $\mathcal{O}_F$-algebra $\mathfrak{J}$ the subset $\varpi^n \mathfrak{J}$ form a two-sided ideal and $U_{\mathfrak{J}}/U_{\mathfrak{J}}^n = (\mathfrak{J}/\varpi^n\mathfrak{J})^\times$. Assume $g \in U_{\mathfrak{J}} \smallsetminus U_F U_{\mathfrak{J}}^n$ and $v_u(g_3) = v_u(g_2) + 2 < n+1$, $v_u(g_3) \leqslant v_u(g_1 - g_4)$ and $\frac{g_3}{tg_2} \in k[\![t]\!]^\times$ is a square of an element in $k[\![u]\!]^\times$. We replace $U_{\mathfrak{J}}$ (resp. $\mathfrak{J}$) by $U_{\mathfrak{J}}/U_{\mathfrak{J}}^n$ (resp. $\mathfrak{J}/\varpi^n\mathfrak{J}$) and g by its image there. We show that g is conjugate to an element of $U_E/U_E^n = U_E/U_E \cap U_{\mathfrak{J}}^n$. Replace g by the difference of g and the scalar matrix with entries $\frac{1}{2}(g_1 + g_4)$. Thus we can assume that g has trace zero and we must show that there is some $b \in \mathcal{O}_F$ such that g is conjugate in $\mathfrak{J}/\varpi^n\mathfrak{J}$ to the image of $\begin{pmatrix} & b \\ tb & \end{pmatrix}$. Consider $r_{y,\lambda}$ from Lemma 5.15. Note that

$$r_{y,\lambda} \begin{pmatrix} & b \\ tb & \end{pmatrix} r_{y,\lambda}^{-1} = \begin{pmatrix} b\lambda t & by^{-1}(1-\lambda^2 t) \\ byt & -b\lambda t \end{pmatrix}$$

By our assumptions we can write $g_2 = t^\alpha g_2'$, $g_3 = t^{\alpha+1} g_3'$, $g_1 = -g_4 = t^{\alpha+1} g_1'$ with $\alpha + 1 \leqslant \lfloor \frac{n}{2} \rfloor$ and $g_2', g_3' \in k[\![t]\!]^\times$. Thus we can conclude, if we find appropriate $y \in U_F/U_F^{\lfloor \frac{n+1}{2} \rfloor}$, $\lambda \in \mathcal{O}_F/\mathcal{O}_F^{\lfloor \frac{n}{2} \rfloor}$ and $b = b_0 t^\alpha \in \mathcal{O}_F$ with $b_0 \in U_F$ such that

$$\begin{aligned} b_0 \lambda &\equiv g_1' \mod t^{\lfloor \frac{n+1}{2} \rfloor - (\alpha+1)} \\ b_0 y &\equiv g_3' \mod t^{\lfloor \frac{n}{2} \rfloor - \alpha} \\ b_0 y^{-1}(1 - \lambda^2 t) &\equiv g_2' \mod t^{\lfloor \frac{n}{2} \rfloor - \alpha} \end{aligned} \tag{5.14}$$

Using the first and the second equations to eliminate b_0 and λ, the only remaining equation is

$$y^2 \equiv g_3' g_2'^{,-1}(1 - g_1'^{,2} g_3'^{,-2} y^2 t) \mod t^{\lfloor \frac{n}{2} \rfloor - \alpha}$$

This equation has a solution in y by Hensel's lemma and our assumption on $\frac{g_3}{tg_2}$. The other direction in (ii) is an immediate computation. $\square$

5.3. **Traces of unipotent elements.**

In Sections 5.3-5.6 we assume $m = 2n-1$.

Proof of Lemma 4.4. We use notations of Proposition 5.6. Let $g \in U_{\mathfrak{J}}^{m+1}$. Thus $v_u(g_1-1), v_u(g_2), v_u(g_4-1) \geqslant 2n = m+1$ and $v_u(g_3) \geqslant m+3$. This, Proposition 5.6 and Lemma 5.2(i) show that $\sharp S_{g,i_1} = 0$ for $i_1 \in U_E/U_E^{m+1} \smallsetminus \{1\}$. Lemma 5.9 implies $\sharp S_{g,1} = (q-1)q^{n-1}$. Proposition 5.6 shows $\mathrm{tr}(g;\Xi_\chi) = (q-1)q^{n-1}$. $\square$

Proof of Lemma 4.6. We use notations from Proposition 5.6. The case $g=1$ of Lemma 4.6 follows from Lemma 4.4. Write $\delta := \lfloor\frac{n+1}{2}\rfloor - \lfloor\frac{n}{2}\rfloor$. For $0 \leqslant \alpha \leqslant \lfloor\frac{n+1}{2}\rfloor - 1$ consider the subgroup

$$A_\alpha := \{1 + u^{(n-\delta)+2\alpha+1}y \colon y \text{ is } \tau\text{-invariant}\}$$

of $U_E^{(n-\delta)+2\alpha+1}/U_E^{m+1}$, and let $A_{\lfloor\frac{n+1}{2}\rfloor} := \{1\} \subseteq U_E/U_E^{m+1}$.

Lemma 5.11. *Let $g \in N_n \smallsetminus \{1\}$. If $g \notin N_n^{\lfloor\frac{n}{2}\rfloor+1}$, then $S_{g,i_1} = \varnothing$ for all $i_1 \in U_E/U_E^{m+1}$. Otherwise, let $g \in N_n^{\lfloor\frac{n}{2}\rfloor+1+\alpha} \smallsetminus N_n^{\lfloor\frac{n}{2}\rfloor+2+\alpha}$ for some $0 \leqslant \alpha \leqslant \lfloor\frac{n+1}{2}\rfloor - 1$ and $i_1 \in U_E/U_E^{m+1}$. Then*

$$\sharp S_{g,i_1} = \begin{cases} c(\alpha) & \text{if } i_1 \in A_\alpha \smallsetminus A_{\alpha+1} \\ 0 & \text{otherwise,} \end{cases}$$

where $c(\alpha)$ depends only on α, not on i_1. Moreover, $c(\lfloor\frac{n+1}{2}\rfloor - 1) = q^{n-1}$.

Proof. If $g \in N_n \smallsetminus N_n^{\lfloor\frac{n}{2}\rfloor+1}$, then g is not conjugate to an element of $U_E U_{\mathfrak{J}}^n$ by Lemma 5.10, so $S_{g,i_1} = \varnothing$ for all $i_1 \in U_E/U_E^{m+1}$ by Lemma 5.9, and the first statement of the lemma follows from Proposition 5.6 (alternatively, look at equation (5.2) for g). Let $g \in N_n^{\lfloor\frac{n}{2}\rfloor+1+\alpha} \smallsetminus N_n^{\lfloor\frac{n}{2}\rfloor+2+\alpha}$ for some $0 \leqslant \alpha \leqslant \lfloor\frac{n+1}{2}\rfloor - 1$ and $i_1 \in U_E/U_E^{m+1}$. Write $g = \begin{pmatrix} 1 & \\ t^{\lfloor\frac{n}{2}\rfloor+1+\alpha}x & 1 \end{pmatrix}$ with $v_t(x) = 0$. Then equation (5.2) is trivially satisfied for each a and equation (5.3) takes the form

$$i_1 \equiv 1 + u^{(n-\delta)+2\alpha+1} x R^{-1} \mod u^{m+1} \tag{5.15}$$

(one easily computes $h(g,a) = u^{2\alpha+1-\delta}x$). Write $a = \sum_{i=1}^n a_i u^i$. Then $R = u^{-1}(\tau(a) - a) = -2(a_1 + a_3u^2 + \dots)$ and x are τ-invariant and we have $v_t(R) = v_t(x) = 0$. Hence $S_{g,i_1} = \varnothing$ unless $i_1 \in A_\alpha \smallsetminus A_{\alpha+1}$. On the other hand, from the explicit form of R, it is clear that for any $i_1 \in A_\alpha \smallsetminus A_{\alpha+1}$ the set S_{g,i_1} of solutions a of (5.15) has the same cardinality. The second statement of the lemma follows. To see the last statement, put $\alpha = \lfloor\frac{n+1}{2}\rfloor - 1$. Then $(n-\delta)+2\alpha+1 = 2n-1 = m$ and for a fixed $i_1 \in A_{\lfloor\frac{n+1}{2}\rfloor-1} \smallsetminus A_{\lfloor\frac{n+1}{2}\rfloor} = U_E^m/U_E^{m+1} \smallsetminus \{1\}$ equation (5.15) amounts to a condition on $R \mod u$, or, which is the same, on a_1. It determines a_1 uniquely and $a_2, a_3, \dots, a_n$ can be chosen arbitrarily. Thus (5.15) has exactly q^{n-1} solutions. $\square$

Now we can finish the proof of the Lemma 4.6. Let $g \in N_n \smallsetminus N_n^n$. If $g \notin N_n^{\lfloor\frac{n}{2}\rfloor+1}$, then Proposition 5.6 and the first statement in Lemma 5.11 immediately show $\mathrm{tr}(g;\Xi_\chi) = 0$. Otherwise, there is some α with $0 \leqslant \alpha < \lfloor\frac{n+1}{2}\rfloor - 1$, such that $g \in N_n^{\lfloor\frac{n}{2}\rfloor+1+\alpha} \smallsetminus N_n^{\lfloor\frac{n}{2}\rfloor+2+\alpha}$, and we deduce from Proposition 5.6 and Lemma 5.11

$$\operatorname{tr}(g;\Xi_\chi) = \sum_{i_1\in A_\alpha \smallsetminus A_{\alpha+1}} c(\alpha)\chi(i_1) = c(\alpha)\sum_{i_1\in A_\alpha}\chi(i_1) - c(\alpha)\sum_{i_1\in A_{\alpha+1}}\chi(i_1) = 0,$$

as A_α, $A_{\alpha+1}$ both are subgroups containing U_E^m/U_E^{m+1} and χ is a non-trivial character on U_E^m/U_E^{m+1}. Now assume $g \in N_n^n \smallsetminus \{1\}$. This corresponds to $\alpha = \lfloor\frac{n+1}{2}\rfloor - 1$ and $A_{\lfloor\frac{n+1}{2}\rfloor-1} = U_E^m/U_E^{m+1}$. By Proposition 5.6 and Lemma 5.11 we compute

$$\operatorname{tr}(g;\Xi_\chi) = \sum_{i_1\in U_E^m/U_E^{m+1}\smallsetminus\{1\}} q^{n-1}\chi(i_1) = -q^{n-1},$$

as χ is non-trivial on U_E^m/U_E^{m+1}. This finishes the proof of Lemma 4.6. □

5.4. **Traces of some non-split elements.**

Proposition 5.12. *Let $0 \leqslant \alpha \leqslant n-1$. Let $g = \iota(1+u^{2\alpha+1}h)$ for some $h \in U_F$. Then*

$$\operatorname{tr}(g;\Xi_\chi) = q^\alpha(\chi(g)+\chi^\tau(g)) + 2q^\alpha \cdot \sum_{\substack{i_1 = 1+u^{2\alpha+1}hs\\ s\in Q_\alpha}} \chi(i_1), \tag{5.16}$$

with Q_α as in Section 4.4.7.

Proof. This follows immediately from Proposition 5.6 and Lemma 5.13. □

Lemma 5.13. *Let α, g, h be as in Proposition 5.12. Then $S_{g,i_1} = \varnothing$, unless $i_1 = 1+u^{2\alpha+1}hs$ for some $s\in R_\alpha$. Assume this holds. Then*

$$S_{g,i_1} = \begin{cases} 2q^\alpha & \text{if } s\in Q_\alpha \\ q^\alpha & \text{if } s = \pm 1 \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Assume first $\alpha \geqslant \lfloor\frac{n}{2}\rfloor$, or equivalently $2\alpha+1\geqslant n$. In this case $\sharp S_{g,i_1}$ is equal to the number of solutions of (5.3) in the variable a. Using Lemma 5.2(ii), we see that $S_{g,i_1} = \varnothing$, unless $i_1 = 1+u^{2\alpha+1}hs$ for some $s\in R_\alpha$. Assume this holds. As h is τ-invariant, it follows that the condition $\det(g) \equiv i_1\tau(i_1) \mod u^{m+1}$ (necessary for the non-emptiness of S_{g,i_1}) is equivalent to $s\in R_\alpha^{\langle\tau\rangle}$. Thus we can assume that $i_1 = 1+u^{2\alpha+1}hs$ with $s\in R_\alpha^{\langle\tau\rangle}$. Then, using Lemma 5.2(ii), (5.3) is seen to be equivalent to

$$s \equiv (1-a'^{,2})(1-u^{2\alpha+1}ha')R^{-1} - a' \mod u^{m-2\alpha},$$

where we write $a = ua'$. By assumption $2\alpha+1\geqslant m-2\alpha$, and moreover, $R = u^{-1}(\tau(a)-a) = -(\tau(a')+a')$. Hence (5.3) is equivalent to

$$s \equiv -\frac{1+a'\tau(a')}{a'+\tau(a')} \mod u^{m-2\alpha}. \tag{5.17}$$

Assume this equation has a solution in a'. Then we deduce

$$s\tau(s) - 1 = s^2 - 1 = \left(\frac{1+a'\tau(a')}{a'+\tau(a')}\right)^2 - 1 = \mathrm{N}_{\tau,\alpha}\left(\frac{1-a'^{,2}}{a'+\tau(a')}\right)$$

in R_α. This shows that if $S_{g,i_1}\neq\varnothing$, then $s\in Q_\alpha\cup\{\pm1\}$. Conversely, assume that $s\in Q_\alpha\cup\{\pm1\}$. Write $a' = a_1 + a_2u + \cdots + a_nu^{n-1}$. We differ between three cases.

Case 1. $s \not\equiv \pm 1 \mod u$. Let $s_0 := s \mod u$. By Lemma 4.17, $s_0^2 - 1$ is a square in $k^\times$. We can rearrange the equation (5.17) and bring it to the form

$$\begin{aligned}(1 + a_1^2) \quad &+ \quad (2a_1a_3 - a_2^2)u^2 + \cdots + (2\sum_{j=0}^{i-1}(-1)^j a_{j+1}a_{2i-j+1} + (-1)^i a_{i+1}^2)u^{2i} + \ldots \\ \ldots \quad &+ \quad (2a_1a_{2\lfloor\frac{n+1}{2}\rfloor-1} + \ldots)u^{2\lfloor\frac{n+1}{2}\rfloor-2} \equiv -2s(a_1 + a_3u^2 + \cdots + a_{2\lfloor\frac{n+1}{2}\rfloor-1}u^{2\lfloor\frac{n+1}{2}\rfloor-2}) \mod u^{m-2\alpha}.\end{aligned}$$

Taking this equation modulo u, we obtain the equation $a_1^2 + 2s_0a_1 + 1 = 0$ in k. It has precisely two different solutions in a_1 as $s_0^2 - 1$ is a square in $k^\times$. Note that both solutions satisfy $a_1 \neq -s_0$ due to $s_0 \neq \pm 1$. Taking the above equation iteratively modulo $u^3, \ldots, u^{m-2\alpha}$ and using $a_1 \neq -s_0$, we see that there are exactly q possibilities to choose any of the pairs $(a_2, a_3), \ldots, (a_{m-2\alpha-1}, a_{m-2\alpha})$ and we obtain q possibilities for each of the remaining variables $a_{m-2\alpha}, \ldots, a_n$ (note that $m - 2\alpha \leqslant 2\lfloor\frac{n+1}{2}\rfloor - 1$). Altogether we obtain $2q^{n-\alpha-1}q^{n-(m-2\alpha)} = 2q^\alpha$ solutions.

Case 2. $v_u(s+1) = 2j$ or $v_u(s-1) = 2j$ with $0 < 2j < m - 2\alpha$ (note that the $v_u(s \pm 1)$ has to be even, as s is τ-invariant). We assume $v_u(s-1) = 2j$ (the other case is similar). Then we can write $s = 1 + u^{2j}s'$ for some τ-invariant unit s'. Then (5.17) is equivalent to

$$(1 + a')(1 + \tau(a')) \equiv -u^{2j}s'(a' + \tau(a')) \mod u^{m-2\alpha},$$

and we deduce that a solution a' must satisfy $v_u(1 + a') = j$ (as $s', a' + \tau(a')$ are necessarily units and $v_u(1 + a') = v_u(1 + \tau(a'))$). Set $a' = -1 + u^jb$ with some $b = \sum_{i=0}^{n-j-1} b_iu^i \in (k[u]/u^{n-j})^\times$. The number of solutions of (5.17) in a' is equal to the number of solutions of

$$(-1)^j b\tau(b) \equiv s'(2 - u^j(b + (-1)^j\tau(b))) \mod u^{m-2\alpha-2j} \tag{5.18}$$

in the variable $b \in (k[u]/u^{n-j})^\times$. Taking this equation modulo u we get the equation $(-1)^j b_0^2 \equiv 2s' \mod u$. As $s = 1 + u^{2j}s' \in Q_\alpha$, Lemma 4.17 shows that $(-1)^j 2s' \mod u$ is a square in $k^\times$, and thus this equation has exactly two solutions in b_0. Similarly as in case 1 above, taking (5.18) iteratively modulo $u^3, u^5, \ldots, u^{m-2\alpha-2j}$, we get per step exactly one condition which determines $b_2, b_4, \ldots, b_{m-2\alpha-2j-1}$ uniquely (note: the set of these conditions also can be empty). For each b_i with $i \notin \{0, 2, 4, \ldots, m - 2\alpha - 2j - 1\}$ there are q possible choices. Thus the number of solutions of (5.18) in b is equal to $2q^{(n-j-1)-(n-\alpha-j-1)} = 2q^\alpha$.

Case 3. $s = \pm 1$. Assume $s = 1$ (the other case is similar). Then (5.17) is equivalent to

$$(1 + a')(1 + \tau(a')) \equiv 0 \mod u^{m-2\alpha},$$

which in turn is equivalent to $v_u(1 + a') \geqslant \frac{m-2\alpha+1}{2} = n - \alpha$. We easily deduce that the number of solutions of this equation in a' is equal to q^α. This finishes the case $\alpha \geqslant \lfloor\frac{n}{2}\rfloor$.

Assume now $0 \leqslant \alpha < \lfloor\frac{n}{2}\rfloor$. Then $2\alpha + 1 < n$. The quantity $\sharp S_{g,i_1}$ is equal to the number of solutions of (5.2) and (5.3) in a. We again write $a = ua'$. Equation (5.2) is immediately seen to be equivalent to $a' \equiv \pm 1 \mod u^{n-2\alpha-1}$ and we write $a' = \pm 1 + u^{n-2\alpha-1}b$ for $b \in k[u]/u^{2\alpha+1}$. We deduce

$$R \equiv -(a' + \tau(a')) \equiv \mp 2 - u^{n-2\alpha-1}(b + (-1)^{n+1}\tau(b)) \mod u^{m+1}. \tag{5.19}$$

Let us denote the 'automorphic factor' $g_2a + g_1$ by

$$f := g_2a + g_1 = 1 \pm u^{2\alpha+1}h + u^nhb. \tag{5.20}$$

By Lemma 5.2(iii), the quantity $\sharp S_{g,i_1}$ is equal to the number of solutions in the variable $b \in k[u]/u^{2\alpha+1}$ of the equation

$$\tau(i_1)(1 + u^n R^{-1}\frac{h(\mp 2b - u^{n-2\alpha-1}b^2)}{f}) \equiv f \mod u^{m+1},$$

or equivalently,

$$\tau(i_1) \equiv f - u^n R^{-1} h(\mp 2b - u^{n-2\alpha-1}b^2) \mod u^{m+1}. \tag{5.21}$$

Taking this equation modulo $u^{m-2\alpha} = u^{2n-2\alpha-1}$ and using (5.19) and (5.20), we deduce that $S_{g,i_1} = \varnothing$, unless $i_1 \equiv 1 \mp u^{2\alpha+1}h \mod u^{m-2\alpha}$, or with other words, $i_1 = 1 + u^{2\alpha+1}hs$ with $s \in R_\alpha$ satisfying $s \equiv \mp 1 \mod u^{m-4\alpha-1}$. Assume that this holds. An easy computation shows now that $\det(g) \equiv i_1\tau(i_1) \mod u^{m+1}$ is equivalent to $s \in R_\alpha^{\langle\tau\rangle,\prime}$, so we also can assume this (otherwise, $S_{g,i_1} = \varnothing$). Let us write $s = \mp 1 + u^{m-4\alpha-1} \cdot (u^{2j}s_0)$, with $s_0 \in (k[u]/u^{2\alpha-2j+1})^\times$ τ-invariant with $0 \leqslant j \leqslant \alpha+1$ ($j = \alpha+1$ corresponds to $s = \pm 1$). Straightforward rearrangements of terms show that (5.21) is equivalent to

$$(\mp 2 - (b + (-1)^{n+1}\tau(b))u^{n-2\alpha-1})u^{2j}s_0 \equiv (-1)^{n+1}b\tau(b) \mod u^{2\alpha+1}. \tag{5.22}$$

If $j = \alpha+1$, then $s = \pm 1$ and (5.22) is equivalent to $b\tau(b) \equiv 0 \mod u^{2\alpha+1}$. This is equivalent to $b \equiv 0 \mod u^{\alpha+1}$, and hence (5.22) has precisely q^α solutions in b. Assume $j \leqslant \alpha$. A potential solution b of (5.22) must satisfy $b \equiv 0 \mod u^j$, hence we can write $b = u^j b'$ for a $b' \in k[u]/u^{2\alpha+1-j}$ and rewrite (5.22) as

$$(\mp 2 - (b' + (-1)^{n+j+1}\tau(b'))u^{n-2\alpha-1+j})s_0 \equiv (-1)^{n+j+1}b'\tau(b') \mod u^{2\alpha+1-2j}. \tag{5.23}$$

Assume first $S_{g,i_1} \neq \varnothing$, i.e., (5.23) has at least one solution. Taking (5.23) modulo u, we deduce that $\pm(-1)^{n+j}2s_0 \mod u$ is a square in $k^\times$, which is by Lemma 4.17 equivalent to $s \in Q_\alpha$. Thus $S_{g,i_1} \neq \varnothing$ implies $s \in Q_\alpha$. Conversely, if $s \in Q_\alpha$, we can deduce that $\sharp S_{g,i_1} = 2q^\alpha$ in the same way as in the case $\alpha \geqslant \lfloor \frac{n}{2} \rfloor$. ☐

We are convinced that there must be a more elegant proof of Lemma 5.13, but we still can not find it.

5.5. **Traces of elements in $E^\times$ with u-valuation 1.**

Proof of Proposition 4.22. Put $y_1 := e_0(u, -u)$. Consider the automorphism $\tilde{\beta}_g \colon \tilde{Y}_{\dot{w}}^m \to \tilde{Y}_{\dot{w}}^m$ given by $\tilde{\beta}_g(\dot{x}I) = g\dot{x}y_1^{-1}I^m$. Then $\tilde{\beta}_g$ induces an automorphism of $\mathrm{H}_c^0(\tilde{Y}_{\dot{w}}^m)$ and hence also an automorphism $\tilde{\beta}_g^* \colon V_\chi \to V_\chi$ of its χ-isotypic quotient. As y_1 acts in V_χ as the scalar multiplication[2] with $\chi(u)$, we have $\mathrm{tr}(g; \Xi_\chi) = \chi(u)\mathrm{tr}(\tilde{\beta}_g^*; V_\chi)$. We determine $\mathrm{tr}(\tilde{\beta}_g^*; V_\chi)$. As $v_u(g) = 1$, Lemma 3.13 shows that $gY_{\dot{w}}^m = Y_{\dot{w}}^m y_1$. With other words, $\tilde{\beta}_g$ restricts to an automorphism β_g of $Y_{\dot{w}}^m$. Further, β_g induces an automorphism β_g^* of $\mathrm{H}_c^0(Y_{\dot{w}}^m)[\chi|_{U_E}]$. Moreover, the isomorphism from Lemma 5.7 induces a commutative diagram

[2] A subtlety: we suppressed our choice of an identification of $E^\times$ with the diagonal quotient of $\tilde{I}_{m,w}$, for which we silently have chosen that u corresponds to y_1. This choice determines on the one hand that y_1 acts in V_χ by $\chi(u)$, and on the other hand, that we have to evaluate the trace formula using the identifications $\varpi \leftrightarrow u \leftrightarrow y_1 = e_0(u, -u)$.

$$\begin{array}{ccc} \mathrm{H}^0_c(Y^m_{\ddot{w}})[\chi|_{U_E}] & \xrightarrow{\sim} & \mathrm{H}^0_c(\tilde{Y}^m_{\ddot{w}})[\chi] \\ \Big\downarrow{\scriptstyle \beta_g^*} & & \Big\downarrow{\scriptstyle \tilde{\beta}_g^*} \\ \mathrm{H}^0_c(Y^m_{\ddot{w}})[\chi|_{U_E}] & \xrightarrow{\sim} & \mathrm{H}^0_c(\tilde{Y}^m_{\ddot{w}})[\chi] \end{array}$$

from which we deduce $\mathrm{tr}(\tilde{\beta}_g^*; V_\chi) = \mathrm{tr}(\beta_g^*; \mathrm{H}^0_c(Y^m_{\ddot{w}})[\chi|_{U_E}])$. Lemma 5.14 finishes the proof. □

Lemma 5.14. *Let $g \in E^\times$ with $v_u(g) = 1$. Let β_g be the automorphism of $Y^m_{\ddot{w}}$ defined by $\beta_g(\dot{x}I^m) = g\dot{x}y_1^{-1}I^m$. Write $g = g_0 u$. Then*

$$\mathrm{tr}(\beta_g^*; \mathrm{H}^0_c(Y^m_{\ddot{w}})[\chi|_{U_E}]) = \chi(g_0) + \chi^\tau(-g_0).$$

Proof. Multiplying with some central element in $U_F \subseteq U_E$ (those act as scalars in V_χ) we can assume that $g_0 = 1 + u^{2\alpha+1}h$ for some $h \in U_F$ (and $0 \leqslant \alpha < n$). We proceed analogously as in the proof of Proposition 5.6. Let $i \in I_{m,\underline{w}_m}/I^m$. A point $\dot{x}I \in Y^m_{\ddot{w}}$ can lie in the set $S_{\beta_g,i} = \{\dot{x}I \in Y^m_{\ddot{w}} \colon \beta_g(\dot{x}I^m) = \dot{x}iI^m\}$ from Lemma 5.5 only if β_g fixes its a-coordinate. By Lemma 3.13, β_u acts on the coordinate a by $a = ua' \mapsto ua'^{,-1}$. From this and Proposition 5.1 one easily deduces that $\beta_g^*(a) = a$ is equivalent to $a = \pm u$ (for any g_0) and that $h(g, \pm u) = 0$. Apply Propositions 5.1, 5.3, 5.4 to determine the actions of β_g and i on $Y^m_{\ddot{w}}$. Exactly as in the proof of Proposition 5.6 we see that

$$\mathrm{tr}(\beta_g^*; \mathrm{H}^0_c(Y^m_{\ddot{w}})[\chi|_{U_E}]) = \frac{1}{(q-1)q^{2m}} \sum_{i_1 \in U_E/U_E^{m+1}} \sharp S_{\beta_g,i_1}\chi(i_1),$$

where S_{β_g,i_1} is the set of all solutions of the equations

$$(1 \mp u^{2\alpha+1}h)(\mp C)M^{-1}N^{-1} \equiv Ci_1H^{-1} \mod u^{m+1} \tag{5.24}$$

$$-AMN \equiv i_1^{-1}\tau(i_1)H^2A + i_2H \mod u^m \tag{5.25}$$

in the variables $C \in (k[u]/u^{m+1})^\times$, $A, i_2 \in k[u]/u^m$ (the sign $\pm$ corresponds to the two possibilities $a = \pm u$), where

$$\begin{aligned} M &= 1 - 2u^nC\tau(C)^{-1}A \quad \text{as in Proposition 5.3, and} \\ N &= 1 + u^{n+1}\frac{g_2}{g_2(\pm 1) + g_1}(\mp CM^{-1})(\mp DM)^{-1}(-AM) = 1 \pm 2u^n\frac{u^{2\alpha+1}h}{1 \pm u^{2\alpha+1}h}C\tau(C)^{-1}A \\ H &= 1 + i_1\tau(i_1)^{-1}i_2B = 1 + u^ni_1\tau(i_1)^{-1}i_2C\tau(C)^{-1}. \end{aligned}$$

Canceling C in (5.24) we see that it is equivalent to

$$MNi_1 \equiv \mp(1 \mp u^{2\alpha+1}h)H \mod u^{m+1}. \tag{5.26}$$

Taking (5.26) modulo u^n, we see that $S_{\beta_g,i} = \varnothing$, unless $i_1 \equiv \mp(1 \mp u^{2\alpha+1}h) \mod u^n$. Assume the last holds. Taking equation (5.25) modulo u^n and inserting MN from (5.26) and $i_1 \equiv \mp(1 \mp u^{2\alpha+1}h)$ mod u^n we deduce

$$i_2 \equiv -A\frac{2}{1 \mp u^{2\alpha+1}h} \mod u^n.$$

This allows to compute

$$H = 1 - 2u^n \frac{1}{1 \pm u^{2\alpha+1}h} C\tau(C)^{-1}A. \tag{5.27}$$

As in the proof of Proposition 5.6 we eliminate i_2 and ignore equation (5.25). Thus $\sharp S_{\beta_g, i_1}$ is equal to the number of solutions of (5.26) in C and A. Finally, we compute $MN = H$ (this uses $i_1 \equiv \mp(1 \mp u^{2\alpha+1}h) \mod u^n$ and (5.27)) and canceling these terms in (5.26) shows that $S_{\beta_g, i_1} = \varnothing$, unless $i_1 = \mp(1 \mp u^{2\alpha+1}h)$, in which case $\sharp S_{\beta_g, i_1} = (q-1)q^{2m}$, finishing the proof. □

5.6. **Traces on the induced side.**

Proof of Lemma 4.28. In Ξ_χ and Θ_χ the central characters are $\chi|_{F^\times}$ and U_E^{m+1} acts trivial. Thus it is enough to show

$$\begin{aligned} \operatorname{tr}(g; \Xi_\chi) &= \operatorname{tr}(g; \Theta_\chi) \quad \forall\, g = \iota(1 + u^{2\alpha+1}h) \text{ with } h \in U_F \text{ and } 0 \leqslant \alpha \leqslant n-1, && (5.28) \\ \operatorname{tr}(g; \Xi_\chi) &= \operatorname{tr}(g; \Theta_\chi) \quad \forall\, g \in \varpi U_E. && (5.29) \end{aligned}$$

Lemma 5.15. *Let $n \geqslant 1$. For $y \in U_F/U_F^{\lfloor \frac{n+1}{2} \rfloor}$, $\lambda \in \mathcal{O}_F/\mathcal{O}_F^{\lfloor \frac{n}{2} \rfloor}$ the matrices*

$$r_{y,\lambda} := \begin{pmatrix} 1 & \\ & y \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix} \in U_{\mathfrak{J}}/U_{\mathfrak{J}}^n \tag{5.30}$$

for a system of representatives for the left (and right) cosets of $U_E U_{\mathfrak{J}}^n$ in $U_{\mathfrak{J}}$ and hence also of $J_\beta = E^\times U_{\mathfrak{J}}^n$ in $E^\times U_{\mathfrak{J}}$.

Proof. We have $U_{\mathfrak{J}}/U_E U_{\mathfrak{J}}^n \cong E^\times U_{\mathfrak{J}}/E^\times U_{\mathfrak{J}}^n$. The rest is an immediate computation. □

We use notations from Section 4.1.6 and compute the traces $\operatorname{tr}(g; \Theta_\chi)$. Let $g = \iota(1 + u^{2\alpha+1}h)$ be as in (5.28). Applying the Mackey formula to $\Theta_\chi = \operatorname{Ind}_{J_\beta}^{E^\times U_{\mathfrak{J}}} \Lambda$ we see

$$\operatorname{tr}(g; \Theta_\chi) = \sum_{y,\lambda} \Lambda(r_{y,\lambda} g r_{y,\lambda}^{-1}), \tag{5.31}$$

where the sum is taken over all representatives $r_{y,\lambda}$ of $E^\times U_{\mathfrak{J}}/J_\beta$ (from Lemma 5.15), such that $r_{y,\lambda} g r_{y,\lambda}^{-1} \in J_\beta = E^\times U_{\mathfrak{J}}^n$. We compute:

$$r_{y,\lambda} g r_{y,\lambda}^{-1} = \begin{pmatrix} 1 + \lambda h t^{\alpha+1} & y^{-1}h(1-\lambda^2 t)t^\alpha \\ y h t^{\alpha+1} & 1 - \lambda h t^{\alpha+1} \end{pmatrix}.$$

Write $\beta = (b + uc)u^{-m}$ with some $b, c \in \mathcal{O}_F$. Assume first $\alpha \geqslant \lfloor \frac{n}{2} \rfloor$. Then $r_{y,\lambda} g r_{y,\lambda}^{-1} \in U_{\mathfrak{J}}^n \subseteq U_E U_{\mathfrak{J}}^n$ for all $r_{y,\lambda}$ and we compute:

$$\operatorname{tr}(g; \Theta_\chi) = \sum_{y,\lambda} \psi_{\mathfrak{M},\beta}(r_{y,\lambda} g r_{y,\lambda}^{-1}) = \sum_{y,\lambda} \psi(t^{\alpha+1-n} b h(y + y^{-1}(1-\lambda^2 t))).$$

Taking some lifts of y, λ to E and setting $n := \frac{1}{2}h(y + y^{-1}(1-\lambda^2 t))u^{2\alpha+1} \in E$, we see that $\beta n + \tau(\beta n) = t^{\alpha+1-n} b h(y + y^{-1}(1-\lambda^2 t))$, i.e., using (4.3), we deduce

$$\operatorname{tr}(g; \Theta_\chi) = \sum_{y,\lambda} \psi_E(\beta n) = \sum_{y,\lambda} \chi(1 + \frac{1}{2}h(y + y^{-1}(1-\lambda^2 t))u^{2\alpha+1}).$$

This does not depend on the choice of the lifts y, λ to E, as χ is of level m. Interpreting $1 + \frac{1}{2}h(y + y^{-1}(1-\lambda^2 t))u^{2\alpha+1}$ as an element of U_E/U_E^{m+1}, we have to show that the summand $\chi(i_1)$

for $i_1 \in U_E/U_E^{m+1}$ occurs in this sum if and only if and with the same multiplicity as it occurs in the sum (5.16). Therefore, writing $i_1 = 1 + u^{2\alpha+1}hs$, it is enough to show that for a fixed $s \in R_\alpha^{\langle\tau\rangle}$ the equation

$$\frac{1}{2}(y + y^{-1}(1-\lambda^2 t)) \equiv s \mod t^{n-\alpha} \tag{5.32}$$

in the variables $y \in U_F/U_F^{\lfloor\frac{n+1}{2}\rfloor}$, $\lambda \in \mathcal{O}_F/\mathcal{O}_F^{\lfloor\frac{n}{2}\rfloor}$ is equivalent to the equation (5.17) in the variable $a' = a_1 + a_2 u + \cdots + a_n u^{n-1} \in (k[u]/u^n)^\times$. Indeed, write $a' = -b' + c'u$ with $b' = -\frac{1}{2}(\tau(a') + a') = \sum_{j=0}^{\lfloor\frac{n+1}{2}\rfloor-1} b_j t^j$ and $c'u = \frac{1}{2}(\tau(a') - a') = u\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor-1} c_j t^j$ with b', c' τ-invariant. Then (5.17) can be rewritten as

$$s \equiv \frac{1 + b'^{,2} - c'^{,2}t}{2b'} \mod t^{n-\alpha},$$

which is evidently equivalent to (5.32) (replace b' by y and c' by λ). The case $\alpha < \lfloor\frac{n}{2}\rfloor$ can be done similarly. This shows (5.28).

To show (5.29), we let $g = \iota(u(1+hu))$ for some $h \in \mathcal{O}_F$ (restriction to this case is possible after multiplication with a central element). We compute

$$\varpi^{-1} r_{y,\lambda} g r_{y,\lambda}^{-1} = \begin{pmatrix} y & h-\lambda \\ (h+\lambda)t & y^{-1}(1-\lambda^2 t) \end{pmatrix}. \tag{5.33}$$

Notice that $r_{y,\lambda} g r_{y,\lambda}^{-1} \in J_\beta = E^\times U_{\mathfrak{J}}^n$ if and only if $\varpi^{-1} r_{y,\lambda} g r_{y,\lambda}^{-1} \in E^\times U_{\mathfrak{J}}^n \cap U_{\mathfrak{J}} = U_E U_{\mathfrak{J}}^n$. By (5.33), this is the case if and only if $\lambda = 0$, $y = \pm 1$. Thus $\mathrm{tr}(g; \Theta_\chi) = \chi(g) + \chi^\tau(g)$. Together with Proposition 4.22 it shows (5.29) and thus the lemma. □

5.7. **Computation of traces in the small level case.**

In this section we assume $n \geqslant m+1$.

Proof of Lemma 4.30. Let $g \in U_{\mathfrak{J}}$. We apply Proposition 5.6. Observe first that equation (5.3) reduces to

$$\tau(i_1) \equiv g_2 a + g_1 \mod u^{m+1}. \tag{5.34}$$

(i): Then we are exactly in the case (i) of Lemma 5.9. As $v_u(a) = 1$ and $v_u(g_2) \geqslant 2\lfloor\frac{n}{2}\rfloor$, we see that $g_2 a + g_1 \equiv g_1 \mod u^n$ and hence also $g_2 a + g_1 \equiv g_1 \mod u^{m+1}$. Let $i_1 \in (k[u]/u^{m+1})^\times$. Then (5.34) simply says that either i_1 is $g_1 \mod u^{m+1}$ or $S_{g,i_1} = \varnothing$. By Proposition 5.6 we deduce

$$\mathrm{tr}(g; \mathrm{H}_c^0(\tilde{Y}_{\dot{w}})[\chi]) = (q-1)q^{n-1}\chi(g_1) = (q-1)q^{n-1}\tilde{\chi}(g),$$

showing the first statement of (i). The last statement of (i) follows immediately from the first, as $\tilde{\chi}$ is trivial on $U_{\mathfrak{J}}^n$.

(ii): Conjugating g into $U_E U_{\mathfrak{J}}^n$ and multiplying with an element of $U_F U_{\mathfrak{J}}^n$ (these elements act by part (i) as scalars), we can without loss of generality assume that $g = \iota(1 + u^{2\alpha+1}h)$ with some $h \in U_F$ and with $2\alpha + 2 = v_u(g_3) \leqslant n$. Let $i_1 \in U_E/U_E^{m+1}$. We determine $\sharp S_{g,i_1}$. First of all (5.2) is equivalent to

$$(t^\alpha h)a^2 - t^{\alpha+1}h \equiv 0 \mod u^{n+1}.$$

Write $a = a'u$ with a' invertible. This equation is equivalent to

$$a' \equiv \pm 1 \mod u^{n+1-(2\alpha+2)}. \tag{5.35}$$

Equation (5.34) takes the form

$$\tau(i_1) = 1 + u^{2\alpha+1} h a' \mod u^{m+1}.$$

Thus (5.35) and $n \geqslant m+1$ shows that either $S_{g,i_1} = \varnothing$, or $i_1 = 1 \pm u^{2\alpha+1}h$. Moreover, for each of this two choices of i_1, there are exactly $q^{v_u(g_3)-1} = q^{2\alpha+1}$ possible a's satisfying equations (5.35) and (5.34) (cf. Lemma 5.9(ii)). We obtain

$$\mathrm{tr}(g; \mathrm{H}_c^0(\tilde{Y}_{\ddot{w}})[\chi]) = q^{v_u(g_3)-1} \cdot (\tilde{\chi}(g) + \tilde{\chi}^{\tau}(g)).$$

(iii): By Lemma 5.9(iii) it is clear that $S_{g,i_1} = \varnothing$ for all i_1 in this case.

Let now $g = g_0 \varpi \in \varpi U_E$. As in the proof of Proposition 4.22 we have the automorphism $\tilde{\beta}_g$ of $\tilde{Y}_{\ddot{w}}^m$ defined by $\tilde{\beta}_g(\dot{x} I^m) = g \dot{x} y_1^{-1} I^m$, where $y_1 = e_0(u, -u)$ and its restriction β_g to $Y_{\ddot{w}}^m$. Again, we have

$$\mathrm{tr}(g; \mathrm{H}_c^0(\tilde{Y}_{\ddot{w}}^m)[\chi]) = \chi(u) \mathrm{tr}(\tilde{\beta}_g^*; \mathrm{H}_c^0(\tilde{Y}_{\ddot{w}}^m)[\chi]) = \chi(u) \mathrm{tr}(\beta_g^*; \mathrm{H}_c^0(Y_{\ddot{w}}^m)[\chi]).$$

The right action of $I_{m,\underline{w}_m}/I^m$ does not affect the a-coordinate of a point $\dot{x}I \in Y_{\ddot{w}}^m$, thus we see from the Lemma 5.5 that $\mathrm{tr}(\beta_g^*; \mathrm{H}_c^0(Y_{\ddot{w}}^m)[\chi]) = 0$, unless $\beta_g^*(a) = a$. A simple computation shows that this can only be the case if g is conjugate to an element in $E^\times U_{\mathfrak{J}}^n$. This shows (ii)$'$. If g is conjugate to an element of $E^\times U_{\mathfrak{J}}$, then we can assume $g \in E^\times U_{\mathfrak{J}}$ and (i)$'$ can be shown as in the proof of Proposition 4.22. ☐

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