Single-index quantile regression models for censored data

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Abstract

When the dimension of the covariate space is high, semiparametric regression models become indispensable to gain flexibility while avoiding the curse of dimensionality. These considerations become even more important for incomplete data. In this work, we consider the estimation of a semiparametric single-index model for conditional quantiles with right-censored data. Iteratively applying the local-linear smoothing approach, we simultaneously estimate the linear coefficients and the link function. We show that our estimating procedure is consistent and we study its asymptotic distribution. Numerical results are used to show the validity of our procedure and to illustrate the finite-sample performance of the proposed estimators.

Keywords and Phrases: Weak convergence, kernel smoothing, conditional quantiles, multivariate data, survival analysis, local-polynomial smoothing, bandwidth, asymptotic analysis.

1 Introduction

Quantile regression is a very attractive alternative to the classical mean regression model based on the quadratic loss. While the latter provides only information about the central behavior of the data, by varying the quantile level, the former provides a more complete picture, both in the center and in the tails. At the same time, one does not need to impose restrictive assumptions about the unknown data generating process. There are many cases where studying the conditional mean is uninformative compared to the conditional upper or lower quantiles representing more extreme situations. A nice illustration can be found in Elsner et al. (2008), where the interest lies in the lifetime-maximum wind speeds of tropical cyclones. The authors found that trends are near zero for the mean and lower quantiles (median and below), but are upward for higher quantiles.

With the objective of providing a robust yet easily computable alternative to linear mean models, Koenker and Bassett (1978) propose a method to estimate a linear quantile model using the so-called check loss function. This seminal work inspired many researchers from different fields and the method has been generalized and adapted to a wide range of statistical applications including fully nonparametric methods like local-polynomial or spline smoothing;

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see e.g., Yu and Jones (1998), Koenker et al. (1994). Although a completely nonparametric approach is flexible, its application requires a large amount of data in order to overcome the curse of dimensionality. While retaining much flexibility, semiparametric models avoid the curse of dimensionality by imposing some structure on the model. One such structure is the single-index model in which one assumes that the objective function depends linearly on the covariates through an unknown link function. Many widely used parametric models can be seen as particular cases of the single-index model. Examples are the linear regression model and the generalized linear model. In a single-index model, no matter the number of covariates, the curse of dimensionality is avoided because the nonparametric part (link function) is of dimension one. This model was investigated and successfully applied to many objective functions, including the conditional mean and conditional quantiles. For some related papers, see, for example, Ichimura (1993), Klein and Spady (1993), Härdle et al. (1993), Carroll et al. (1997), Delecroix et al. (2003), Wu et al. (2010), Kong and Xia (2012) to cite just some of the relevant papers.

The majority of the available literature is devoted to the case where the variable of interest, say Y, is completely observed. This is not the case in many interesting applications including survival analysis where censoring prevents the direct application of "classical" semi-parametric methods because instead of observing Y, we only observe the minimum of Y and the censoring variable. Compared to the uncensored case, the literature on single-index models dealing with censoring is very sparse and, to the best of our knowledge, it only considers the case of the conditional mean; see for example Lopez et al. (2013) and the references therein.

In this paper, we study the single index-model for the conditional quantile function when the data are right-censored. We estimate the parameters of interest by constructing a weighted check function in a way similar to the method of El Ghouch and Van Keilegom (2009). The main difficulties here are the non-differentiability of the check loss function and the fact that the weight function depends on the censoring distribution, which is unknown and needs to be estimated and then plugged-in in the estimating equation. Our proposed local-linear estimation method is based on an iterative procedure involving a $\sqrt{n}$-consistent estimator of the single-index parameters. In every iteration, we need to maximize a large number of local equations. We derive the asymptotic properties of the resulting quantile regression function under some suitable sufficient conditions. The practical performance of the proposed method is examined via Monte Carlo experiments. The estimator is shown to perform very well for data of moderate size, even when the percentage of censoring is relatively high.

The remainder of the paper is organized as follows. Section 2 describes the estimation procedure. The asymptotic properties such as the consistency and the asymptotic normality of our semiparametric estimator are obtained in Section 3. The problem of selecting the bandwidth parameter is tackled in Section 4. Simulation studies are presented in Section 5. We conclude with two appendices containing the proofs and technical details.

2 Model and estimation

Suppose that Y is a non-negative response depending on a d-dimensional covariate X. The object of interest in this paper is the τth conditional quantile of Y given $X = x$, $\tau \in (0, 1)$, which we denote by $Q_\tau(x)$. We impose a single-index structure on Q_τ, i.e., we suppose that

$$Q_\tau(x) = m_\tau(x^T \beta_{0,\tau}), \tag{2.1}$$

where $m_\tau : \mathbb{R} \to \mathbb{R}$ is an unknown smooth link function and where $\beta_{0,\tau}$ is a vector of unknown coefficients in the unit sphere $S^{d-1} = \{\beta \in \mathbb{R}^d : \|\beta\| = 1\}$. For identifiability reasons, we suppose that the first coordinate of $\beta_{0,\tau}$ is positive, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. As long as it will not cause any ambiguity, we suppress the index τ and write $m = m_\tau$ and $\beta_0 = \beta_{0,\tau}$. In model (2.1), estimating Q_τ boils down to estimating m and β_0.

For $u \in \mathbb{R}$, let $\rho_\tau(u) = u\{\tau - \mathbb{1}(u < 0)\}$ denote the check function. Then, it is well known that β_0 is given by

$$\beta_0 = \underset{\beta \in \mathbb{R}^d}{\operatorname{argmin}}\ \mathbb{E}[\rho_\tau\{Y - m(X^T\beta)\}] = \underset{\beta \in \mathbb{R}^d}{\operatorname{argmin}}\ \mathbb{E}\left[\mathbb{E}[\rho_\tau\{Y - m(X^T\beta)\}|X^T\beta]\right]. \tag{2.2}$$

The expressions $\mathbb{E}[\rho_\tau\{Y - m(X^T\beta)\}]$ and $\mathbb{E}[\rho_\tau\{Y - m(X^T\beta)\}|X^T\beta]$ can be interpreted as the expected and the conditional expected loss, respectively.

For the moment, let us suppose that there is no censoring and that we observe an i.i.d. sample $(X_i, Y_i)_{i=1}^n$ from (X, Y). The following procedure for estimating β_0 and $m(v)$, where $v \in \mathbb{R}$ is arbitrary, stems from Wu et al. (2010). The main idea is to define an empirical analogue of the expected loss in (2.2), which can be minimized subsequently. Let $\beta \in S^{d-1}$ be given. Then, assuming that m is sufficiently smooth and that $X_i^T\beta$ is close to v, a Taylor expansion yields

$$m(X_i^T\beta) \approx m(v) + m'(v)(X_i^T\beta - v) = a + b(X_i^T\beta - v),$$

where $a = m(v)$ and $b = m'(v)$. Thus,

$$\sum_{i=1}^n \rho_\tau\left\{Y_i - a - b(X_i^T\beta - v)\right\} K\{(X_i^T\beta - v)/h\} \tag{2.3}$$

with some kernel function K and a bandwidth h, represents an empirical analogue of the conditional expected loss in (2.2). Note that, for given $\beta = \beta_0$, minimizing (2.3) with respect to a and b yields oracle estimators for $m(v)$ and $m'(v)$, respectively. To get an empirical analogue of $\mathbb{E}[\rho_\tau\{Y - m(X^T\beta)\}]$, we need to average (2.3) over v. Hence, setting $v = v_j = X_j^T\beta$, we obtain

$$\sum_{j=1}^n \sum_{i=1}^n \rho_\tau\left\{Y_i - a_j - b_j(X_{ij}^T\beta)\right\} w_{ij}(\beta), \tag{2.4}$$

where $X_{ij} = X_i - X_j$ and where

$$w_{ij}(\beta) = \left\{\sum_{i=1}^n K\left(\frac{X_{ij}^T\beta}{h}\right)\right\}^{-1} K\left(\frac{X_{ij}^T\beta}{h}\right).$$

By minimizing the expression in (2.4) with respect to $(a_j, b_j)_{j=1}^n$ and β we obtain estimators of $(m(v_j), m'(v_j))_{j=1}^n$ and β_0. As this joint minimization is not feasible, Wu et al. (2010) proposed an iterative procedure based on successive estimation of β_0 and $(m(v), m'(v))$, for any given $v \in \mathbb{R}$. In the present paper, we adapt their approach to the case where the observations of the response variable may be censored.

In the presence of censoring, we do not fully observe the response variables Y_i. Instead, we observe a sequence of i.i.d. triplets $(X_i, Z_i, \Delta_i)_{i=1}^n$ from (X, Z, Δ), where $Z = \min(Y, C)$, $\Delta = \mathbb{1}(Y \le C)$ and $C \ge 0$ denotes a censoring variable. In the present paper we will assume that

(C) C is independent of (X, Y).

Let F_C denote the cumulative distribution function (c.d.f.) of C. Then, some simple calculations based on the tower property of conditional expectations show that, for any measurable functions $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}^p$ ($p \geq 1$),

$$\mathbb{E}[h(Y) \,|\, g(X)] = \mathbb{E}\left[\frac{h(Z)\Delta}{1 - F_C(Z-)} \,\middle|\, g(X)\right]. \tag{2.5}$$

Therefore, we can write $\mathbb{E}\left[\rho_\tau\{Y - a - b(X^T\beta - v)\} \,|\, X^T\beta\right]$ as

$$\mathbb{E}\left[\tau(Y - Z)|X^T\beta\right] + \mathbb{E}\left[\{Z - a - b(X^T\beta - v)\}\left[\tau - Q\mathbb{1}\{Z < a + b(X^T\beta - v)\}\right] \middle| X^T\beta\right],$$

where $Q = \Delta/\{1 - F_C(Z-)\}$, and since the first term does not depend on a and b and only minimization is concerned, this suggests to replace (2.3) by

$$\sum_{i=1}^n \{Z_i - a - b(X_i^T\beta - v)\}\left[\tau - Q_i\mathbb{1}\{Z_i < a + b(X_i^T\beta - v)\}\right] K\left(\frac{X_i^T\beta - v}{h}\right),$$

where $Q_i = \Delta_i/\{1 - F_C(Z_i-)\}$. Since the c.d.f. F_C of the censoring variable is unknown, we need to replace it by some suitable estimator $\hat{F}_C$. Given $\beta \in S^{d-1}$, this suggests to estimate $m(v)$ and $m'(v)$ by $\hat{m}(v, \beta) = \hat{a}(v, \beta)$ and $\hat{m}(v, \beta) = \hat{b}(v, \beta)$, where

$$\begin{aligned}(\hat{a}(v, \beta), \hat{b}(v, \beta)) = \underset{a,b\in\mathbb{R}}{\operatorname{argmin}} \sum_{i=1}^n &\{Z_i - a - b(X_i^T\beta - v)\} \\ &\times \left[\tau - \hat{Q}_i\mathbb{1}\{Z_i < a + b(X_i^T\beta - v)\}\right] K\left(\frac{X_i^T\beta - v}{h}\right),\end{aligned} \tag{2.6}$$

where $\hat{Q}_i = \Delta_i/\{1 - \hat{F}_C(Z_i-)\}$. Still, it remains to construct an estimator for β_0. To do so, we proceed as in the uncensored case and define the following empirical analogue of (2.4):

$$\sum_{j=1}^n \sum_{i=1}^n \{Z_i - a_j - b_j(X_{ij}^T\beta)\}\left[\tau - \hat{Q}_i\mathbb{1}\{Z_i < a_j + b_j(X_{ij}^T\beta)\}\right] w_{ij}(\beta).$$

As indicated above, the joint minimization of the resulting expression with respect to $(a_j, b_j)_{j=1}^n$ and β is unfeasible, hence we propose the following iterative procedure adapted from Wu et al. (2010).

Step 1. Start with an initial estimator $\hat{\beta}^{(0)}$ of β_0 and set $\beta_{iter} = \hat{\beta}^{(0)}$ (see below for a suitable example on how to obtain $\hat{\beta}^{(0)}$).

Step 2. For $j = 1, \ldots, n$, let

$$(\hat{a}_j, \hat{b}_j) = \underset{a,b\in\mathbb{R}}{\operatorname{argmin}} \sum_{i=1}^n \{Z_i - a - b(X_{ij}^T\beta_{iter})\}\left[\tau - \hat{Q}_i\mathbb{1}\{Z_i < a + b(X_{ij}^T\beta_{iter})\}\right] w_{ij}(\beta_{iter}).$$

Step 3. Using the estimates $(\hat{a}_j, \hat{b}_j)_{j=1}^n$, set

$$\beta^\star = \underset{\beta\in\mathbb{R}^d}{\operatorname{argmin}} \sum_{j=1}^n \sum_{i=1}^n \{Z_i - \hat{a}_j - \hat{b}_j(X_{ij}^T\beta)\}\left[\tau - \hat{Q}_i\mathbb{1}\{Z_i < \hat{a}_j + \hat{b}_j(X_{ij}^T\beta)\}\right] w_{ij}(\beta_{iter})$$

and update β_{iter} by setting $\beta_{iter} = \beta^\star/\|\beta^\star\|$.

Step 4. Repeat Steps 2 and 3 until the difference between two consecutive estimations of β is smaller than a given threshold and define the final estimate $\hat{\beta}$ by setting $\hat{\beta} = \beta_{iter}$.

Step 5. For any desired index value $v \in \mathbb{R}$, estimate $m(v)$ and $m'(v)$ by $\hat{m}(v, \hat{\beta}) = \hat{a}(\hat{\beta})$ and $\hat{m}'(v, \hat{\beta}) = \hat{b}(\hat{\beta})$, the latter estimators being defined in (2.6). For any desired index value $x \in \mathbb{R}^d$, estimate $Q_\tau(x)$ by $\hat{m}(x^T\hat{\beta}, \hat{\beta})$.

Step 1 requires an initial estimator for β_0. We propose to use an estimator adapted from the OPG (outer product of gradients)-method in the mean regression context in Xia et al. (2002). The underlying idea is as follows: For any $x \in \mathbb{R}^d$, we have $\partial m(x^T\beta_0)/\partial x = m'(x^T\beta_0)\beta_0$. Hence, the partial derivatives of $m(x^T\beta_0)$ with respect to x are parallel to β_0. For $j = 1, \ldots, n$, let $b_j = m'(X_j^T\beta_0)\beta_0$. One can easily see that the (standardized) eigenvector corresponding to the largest eigenvalue of $V_n = n^{-1}\sum_{i=1}^n b_j b_j^T$ is given by β_0, which suggests to estimate β_0 by replacing b_j in the definition of V_n by suitable estimators $\hat{b}_j$, that is, we define $\hat{\beta}_0$ as the (standardized) eigenvector corresponding to the largest eigenvalue of $\hat{V}_n = n^{-1}\sum_{j=1}^n \hat{b}_j\hat{b}_j^T$. For the estimation of b_j, we propose to use the local-polynomial estimators

$$(\hat{a}_j, \hat{b}_j^T) = \underset{(a,b^T)\in\mathbb{R}^{d+1}}{\operatorname{argmin}} \sum_{i=1}^n \{Z_i - a - b^T X_{ij}\}\left[\tau - \hat{Q}_i \mathbb{1}\{Z_i < a + b^T X_{ij})\}\right] K(X_{ij}/h),$$

where K denotes a d-dimensional kernel.

3 Asymptotic results

For fixed $v \in \mathbb{R}$, suppose that there exist neighborhoods $U_{\beta_0}, U_{m(v)}$ and U_v of $\beta_0, m(v)$ and v, respectively, such that the following regularity conditions hold:

(A1) The kernel K is a density function on $\mathbb{R}$ which is symmetric around 0, has a compact support denoted by $\mathrm{supp}(K)$ and is differentiable with a bounded derivative.

(A2) The function m is twice continuously differentiable on U_v with bounded derivatives.

(A3) (i) The support of X, denoted by $\mathrm{supp}(X)$, is contained in a compact subset D_X of $\mathbb{R}^d$.

(ii) For any $\beta \in U_{\beta_0}$, the random variable $X^T\beta$ has a density $f_{X^T\beta}$. The function $U_{\beta_0} \times U_v \to \mathbb{R}, (\beta, u) \mapsto f_{X^T\beta}(u)$ is bounded and Lipschitz-continuous at (β_0, v).

(A4) (i) The conditional distribution $F_{Y|X}$ of Y given X has a conditional density $f_{Y|X}(\cdot|\cdot)$ that is bounded on $U_{m(v)} \times \mathrm{supp}(X)$.

(ii) For any $\beta \in U_{\beta_0}$, the conditional distribution of Y given $X^T\beta$ has a conditional density $f_{Y|X^T\beta}(\cdot|\cdot)$. The function $U_{\beta_0} \times U_{m(v)} \times U_v \to \mathbb{R}, (\beta, y, u) \mapsto f_{Y|X^T\beta}(y \mid u)$ is bounded and Lipschitz-continuous at $(\beta_0, m(v), v)$.

(iii) $U_{\beta_0} \times U_{m(v)} \times U_v \to \mathbb{R}, (\beta, y, u) \mapsto f_{Y|X^T\beta}(y|u)$ is partially differentiable with respect to y and the derivative, denoted by $f'_{Y|X^T\beta}(y|u)$, is bounded.

(A5) The point $v \in \mathbb{R}$ satisfies $F_Z\{m(v)\} < 1$, where F_Z denotes the c.d.f. of Z.

Before we formulate the main results, let us introduce some additional notation. For $\beta \in \mathbb{R}^d$ and $u \in \mathbb{R}$, let $\mathcal{X}_i(\beta, u) = \left(1, (X_i^T\beta - u)/h\right)^T$, $\mathcal{Z}_i(\beta, u) = Z_i - m(u) - m'(u)(X_i^T\beta - u)$ and

$\mathcal{K}_i(\beta, u) = K\{(X_i^T\beta - u)/h\}$. Moreover, set $\bar{K}_j = \int_{\mathbb{R}} u^j K(u)\,du$ and $\bar{K}'_j = \int_{\mathbb{R}} u^j K^2(u)\,du$ for $j \in \{0, 1, 2, 3\}$ and let

$$\bar{K} = \begin{pmatrix} \bar{K}_0 & \bar{K}_1 \\ \bar{K}_1 & \bar{K}_2 \end{pmatrix}, \qquad \bar{K}' = \begin{pmatrix} \bar{K}'_0 & \bar{K}'_1 \\ \bar{K}'_1 & \bar{K}'_2 \end{pmatrix}.$$

For some constant $M > 0$, let U_M denote the closed d-dimensional ball of radius M with center 0, i.e., $U_M = \{\gamma \in \mathbb{R}^d : \|\gamma\| \le M\}$. Finally, for $\beta \in \mathbb{R}^d$ and $u \in \mathbb{R}$ (usually considered to be close to β_0 and v), let

$$\mathbb{M}_n(u, \beta) = \sqrt{nh}\left\{\begin{pmatrix} \hat{m}(u, \beta) - m(v) \\ h\{\hat{m}'(u, \beta) - m'(v)\} \end{pmatrix} - \frac{h^2}{2}\bar{K}^{-1}\begin{pmatrix} \bar{K}_2 \\ \bar{K}_3 \end{pmatrix} m''(v)\right\}$$

with $\hat{m}(u, \beta)$ and $\hat{m}'(u, \beta)$ as defined in (2.6).

Theorem 3.1. *Suppose that $h = h(n) \to 0$ satisfies $\lim_{n\to\infty} nh^3 = \infty$ and $nh^5 = O(1)$ as $n \to \infty$. Then, for any $v \in \mathbb{R}$ that satisfies conditions (A1)–(A5) and for any $M > 0$,*

$$\sup_{(\gamma,\kappa)\in U_M\times[-M,M]} \left\| \mathbb{M}_n(v_n^\kappa, \beta_n^\gamma) - V^{-1}\frac{1}{\sqrt{nh}}\sum_{i=1}^n \Big[\tau - Q_i \mathbb{1}\left\{Z_i < m(X_i^T\beta_0)\right\}\Big] \times \mathcal{X}_i(\beta_0, v)\mathcal{K}_i(\beta_0, v) \right\| = o_P(1),$$

where $v_n^\kappa = v + \kappa/\sqrt{n}$ and $\beta_n^\gamma = \beta_0 + \gamma/\sqrt{n}$ and where $V = \left[f_{Y|X^T\beta_0}\{m(v)\,|\,v\}\, f_{X^T\beta_0}(v)\right] \times \bar{K}$.

Note that the sum between the norm signs in Theorem 3.1 consists of centered summands as a consequence of (2.5). The uniformity in γ and κ in Theorem 3.1 is essential for the next corollary which can be regarded as the main result of this paper: it states that the final estimator for $Q_\tau(x)$ in Step 5 is asymptotically normally distributed.

Corollary 3.2. *Let $\hat{\beta}_n \in S^{d-1}$ be an estimator for β_0 such that $\hat{\gamma}_n = \sqrt{n}(\hat{\beta}_n - \beta_0) = O_P(1)$. Suppose that the conditions on the bandwidth of Theorem 3.1 are met. Then, for any $v \in \mathbb{R}$ that satisfies conditions (A1)-(A5) and for any $x \in \mathbb{R}^d$ such that $v = x^T\beta_0$ satisfies conditions (A1)–(A5),*

$$\mathbb{M}_n(v, \hat{\beta}_n) \rightsquigarrow \mathcal{N}_2\big(0, \sigma^2(v)\bar{K}^{-1}\bar{K}'\bar{K}^{-1}\big) \quad \textit{and} \quad \mathbb{M}_n(x^T\hat{\beta}_n, \hat{\beta}_n) \rightsquigarrow \mathcal{N}_2\big(0, \sigma^2(x^T\beta_0)\bar{K}^{-1}\bar{K}'\bar{K}^{-1}\big),$$

where, for any $v \in \mathbb{R}$,

$$\sigma^2(v) = \frac{\Phi_{\beta_0}\{m(v)\,|\,v\} - \tau^2}{f^2_{Y|X^T\beta_0}\{m(v)\,|\,v\}\, f_{X^T\beta_0}(v)}$$

and where, for any $u, v \in \mathbb{R}$,

$$\Phi_{\beta_0}(u\,|\,v) = \mathbb{E}\left[\frac{\mathbb{1}(Y < u)}{1 - F_C(Y-)}\,\middle|\, X^T\beta_0 = v\right].$$

4 Bandwidth selection

The practical performance of any nonparametric regression technique depends crucially on the choice of smoothing parameters. A (theoretical) local optimal bandwidth can be derived from the result in Corollary 3.2 by minimizing the asymptotic mean squared error of $\hat{m}(v, \hat{\beta})$ with respect to h, yielding

$$h_n^{opt} = h_n^{opt}(v) = \left\{\frac{\sigma^2(v)\bar{K}_0}{\{m''(v)\}^2\bar{K}_2^2}\right\}^{1/5} n^{-1/5}.$$

Unfortunately, this expression is not directly applicable in practice, since it depends on several unknown quantities. Even in the simpler non-censored case, the derivation of reliable estimators for the respective quantities is delicate. For that reason, alternative procedures for the bandwidth selection have been proposed, see, e.g., Yu and Jones (1998) or Kong and Xia (2012) for procedures relying on the mean-regression case. However, these procedures are not directly applicable in the presence of censoring. For that reason, we propose to use the following leave-one out cross validation (CV) procedure (see also Zheng and Yang, 1998; Leung, 2005; El Ghouch and Van Keilegom, 2009):

(CV1) For a given h, estimate $\hat{\beta} = \hat{\beta}(h)$ as in Step 1-4 in Section 2. In particular, store the values for $\hat{Q}_i$.

(CV2) For any $j = 1, \ldots, n$, set $\hat{m}_{-j,h}(X_j^T\hat{\beta}) = \hat{a}_{-j}(X_j^T\hat{\beta}, \hat{\beta})$, where, for any $v \in \mathbb{R}$ and $\beta \in S^{d-1}$,

$$\big(\hat{a}_{-j}(v,\beta), \hat{b}_{-j}(v,\beta)\big) = \underset{a,b\in\mathbb{R}}{\operatorname{argmin}} \sum_{\substack{i=1,\ldots,n\\ i\neq j}} \{Z_i - a - b(X_i^T\beta - v)\} \times \left[\tau - \hat{Q}_i \mathbb{1}\{Z_i < a + b(X_i^T\beta - v)\}\right] K\left(\frac{X_i^T\beta - v}{h}\right)$$

denotes the estimator based on all observations except the jth.

(CV3) For $j = 1, \ldots, n$, set $\widehat{cv}_{-j,h} = |\hat{m}_{-j,h}(X_j^T\hat{\beta}) - Z_j\hat{Q}_j|$ and let $CV(h)$ denote the sample median of $\widehat{cv}_{-1,h}, \ldots, \widehat{cv}_{-n,h}$.

(CV4) Repeat the first three steps for several bandwidths and set $h_n^{CV} = \operatorname{argmin}_h CV(h)$.

We use here the absolute CV instead of the classical and very popular least squares CV because it leads to a more robust smoothing estimator; see Wang and Scott (1994). This is particularly interesting in our case since the transformed working data $Z_j\Delta_j/(1 - \hat{F}_C(Z_j-))$ may be noisy and because robustness is one of the motivations behind considering quantile regression.
In Section 5, we will show that the estimator based on the cross-validation method has a good finite-sample performance.

5 Numerical results

In this section, we assess the finite-sample performance of the 5-step estimator for $m(v)$. For reasons of numerical stability we constrain the minimization in Step 2 to a compact set $[-M, M]^2$, with $M = 20$. Additionally, we stop the algorithm in Step 4 after at most 15 iterations, if convergence has not occurred until then. We perform 1,000 repetitions for two different models, two sample sizes ($n = 100$ and $n = 200$), two levels of censoring (on average 25% and 50%), three values of $\tau \in \{0.3, 0.5, 0.7\}$, 15 different bandwidths $h \in \{0.1, 0.15, \ldots, 0.75, 0.8\}$ and 32 values for $v \in \{0.05, 0.1, \ldots, 1.55, 1.6\}$. Additionally, we investigate the performance of the cross-validation method described in Section 4. The results are reported only partially. The considered models are as follows.

Model 1 (location model).
For $i = 1, \ldots, n$, we consider

$$Y_i = \exp(X_i^T\beta_0) + \varepsilon_i, \qquad X_i = (X_{i,1}, \ldots, X_{i,d}),$$

where $X_{i,j}$ is i.i.d. uniform on $(0,1)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, d$, and where ε_i is i.i.d. standard exponential. During the simulation study, we fix $d = 3$ and $\beta_0 = 14^{-1/2} \times (1, 2, 3)$.

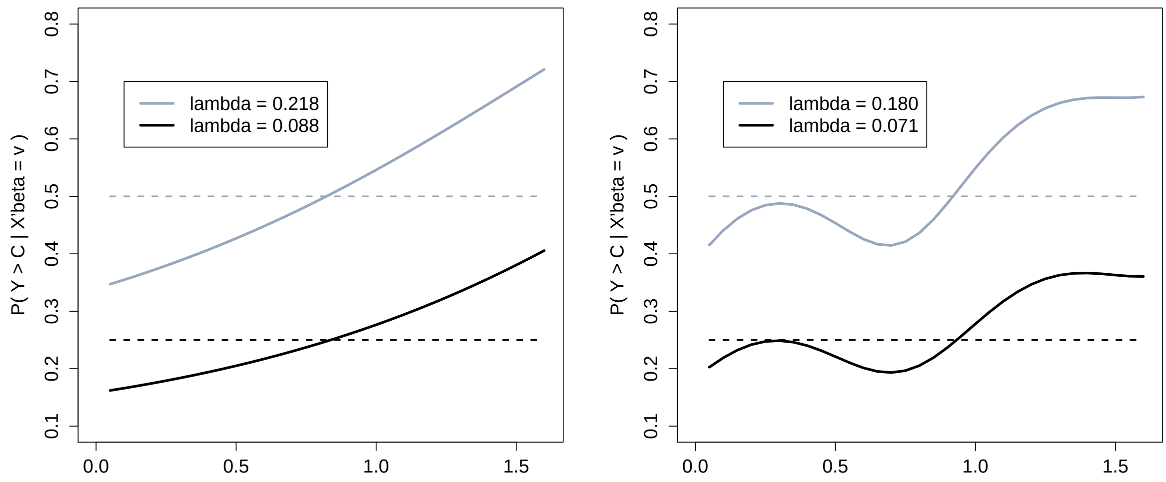

Figure 1: Probability of censoring $v \mapsto \Pr(Y > C \,|\, X^T\beta_0 = v)$ for Model 1 (left figure) and Model 2 (right figure). The average probability of censoring $\Pr(Y > C)$ is 25% for the black curve ($\lambda = 0.088$ or $\lambda = 0.071$) and 50% for the grey curve ($\lambda = 0.218$ or $\lambda = 0.180$).

Note that the τth conditional quantile of Y_i given $X_i = x$ is given by $\exp(x^T\beta_0) + q_\varepsilon(\tau)$, where $q_\varepsilon(\tau) = -\log(1-\tau)$ denotes the τth quantile of the standard exponential distribution. In particular, the conditional quantile curves for different values of τ are parallel.

The censoring variables are supposed to be i.i.d. exponential with parameter λ, independent of X_i and ε_i. We consider two settings for the parameter: $\lambda = 0.218$ which corresponds to a proportion of censoring of about 50%, and $\lambda = 0.088$ which corresponds to a proportion of censoring of about 25%. Note that the probability of censoring at a given $X = x$ is given by

$$\Pr(Y > C \,|\, X = x) = \Pr\{C - \varepsilon < \exp(x^T\beta_0)\} = 1 - \frac{e^{-\lambda e^{x^T\beta_0}}}{1+\lambda}.$$

The corresponding curve $v \mapsto \Pr(Y > C \,|\, X^T\beta_0 = v)$ is depicted in Figure 1 for $\lambda \in \{0.088, 0.218\}$. From these graphs, we expect the estimator $\hat{m}(v, \hat\beta)$ to have worse performance for large values of v, i.e., for values that are close to $6/\sqrt{14} \approx 1.6036$, the upper bound of the support of $X^T\beta_0$.

Model 2 (location-scale model).

For $i = 1, \ldots, n$, we consider

$$Y_i = \exp(X_i^T\beta_0) + \left\{\sin\left(2\pi X_i^T\beta_0\right) + 2\right\}\varepsilon_i, \qquad X_i = (X_{i,1}, \ldots, X_{i,d}),$$

where $X_{i,j}$, β_0 and ε_i are as in Model 1. Note that the τth conditional quantile of Y_i given $X_i = x$ is given by $\exp(x^T\beta_0) + \{\sin(2\pi x^T\beta_0) + 2\} \times q_\varepsilon(\tau)$, where $q_\varepsilon(\tau) = -\log(1-\tau)$ denotes the τth quantile of the standard exponential distribution.

Again, the censoring variables are supposed to be i.i.d. exponential with parameter λ, independent of X_i and ε_i. In this case, the probability of censoring at a given $X = x$ is given by

$$\Pr(Y > C \,|\, X = x) = 1 - \frac{e^{-\lambda e^{x^T\beta_0}}}{1 + \lambda\{\sin(2\pi x^T\beta_0) + 2\}}.$$

The corresponding curve $v \mapsto \Pr(Y > C \,|\, X^T\beta_0 = v)$ is depicted in Figure 1 for $\lambda = 0.071$ (25% of censoring on average) and $\lambda = 0.180$ (50% of censoring on average).

The results of our simulation study are partially reported in Figures 2–4 and in Table 1. Figure 2 serves to illustrate the performance of the 4-step estimator for β_0. We restrict ourselves to Model 1 with $n = 200$, $\tau = 0.5$ and an average censoring proportion of 25%. The left picture shows boxplots of the estimator $\hat{\beta}$ for various bandwidths. The right picture shows the trimmed squared bias, variance and MSE, summed over the coordinates of $\hat{\beta}$. Here, trimming means that we discard both the largest and the smallest 2.5% of the simulation outcomes, which results in a more robust measure of the quality of the estimator $\hat{\beta}$. This appears to be advisable in this specific framework, since the $O(n)$ optimization routines in each iteration step happen to yield very unlikely results with a small probability. The results are quite promising and show that the estimator $\hat{\beta}$ is very robust with respect to the bandwidth, provided it is chosen larger than 0.2.

Under the same setting as for Figure 2, Figure 3 illustrates the performance of the estimator $\hat{m}(v, \hat{\beta})$ for fixed $v = 0.8$ and various bandwidths. Again, we opt for robust measures of the quality of the estimator: the left picture shows boxplots of the absolute estimation error, whereas the right one shows the trimmed squared bias, variance and MSE (solid lines). As expected, the bias increases with h, whereas the variance decreases. The minimum value of the MSE is attained at $h = 0.35$. Finally, the dashed lines in the right picture depict trimmed squared bias, variance and MSE for the oracle estimator of $m(v)$ which is based on the true value of β_0; see equation (2.6). Note that the oracle estimator is unfeasible but may serve as a benchmark. We observe that it clearly performs better than the 5-step estimator $\hat{m}(v, \hat{\beta})$, while at the same time showing a similar qualitative behavior in terms of the MSE, variance and bias curves.

Figure 4 shows results on the performance of $\hat{m}$ for a fixed bandwidth and for various values of v. The left picture concerns Model 1 with bandwidth $h = 0.35$ as suggested by the results in Figure 3. In the right picture, we consider Model 2 with $n = 200,\ \tau = 0.5$, a proportion of 25% of censoring and with $h = 0.2$ chosen by the same criteria as applied for Model 1 (the results are not shown here for the sake of brevity). Again, the results are as expected: the bandwidth is chosen optimally for $v = 0.8$, hence the results are better at that point. As we get away from this center point, the performance gets worse and this phenomena is more pronounced for those values of v for which the probability of censoring depicted in Figure 1 is higher.

A global picture of the impact of the choice of n, τ, the censoring proportion and the model can be drawn from the results in Table 1. For various settings, we state the minimal median absolute estimation error (MAE) multiplied by 10, where "minimal" refers to minimization over the bandwidths $h \in \{0.1, 0.15 \ldots, 0.75, 0.8\}$. As expected, results get better for the simpler model (Model 1), larger sample sizes, smaller values of τ and for those v for which the probability of censoring $\Pr(Y > C \mid X^T\beta_0 = v)$ as plotted in Figure 1 is smaller.

Finally, Table 2 shows simulation results on the cross-validation method for choosing the optimal bandwidth as described in Section 4. For the sake of brevity, we only consider Model 1 and a proportion of censoring of 25%. In this case, the results in the first column of Table 1 may serve as a benchmark. We measure the quality of the cross validation methods in terms of the "relative efficiency" with respect to the minimal median absolute error, defined as

$$RE = \frac{\min_{h\in\{0.1,0.15,\ldots,0.8\}} MAE\{\hat{m}(v,\hat{\beta},h)\}}{MAE\{\hat{m}(v,\hat{\beta},h_n^{CV})\}}. \tag{5.1}$$

The results in Table 2 show that, overall, the cross-validation method has a good performance, with values of RE not falling below 0.77. For high levels of τ, the estimator based on h_n^{CV} even outperforms the estimator based on any fixed bandwidth $h \in \{0.1, 0.15, \ldots, 0.8\}$.

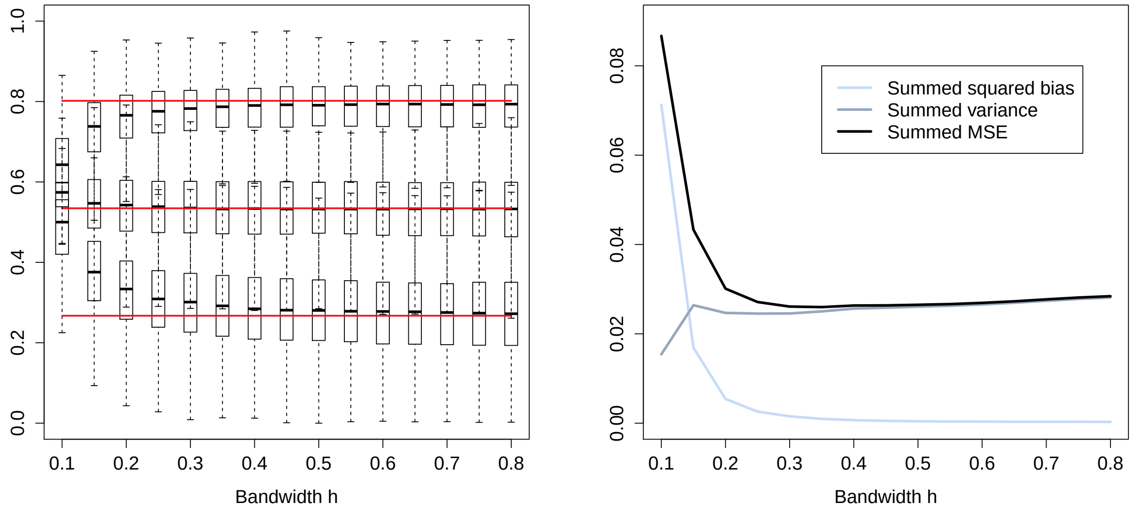

Figure 2: Simulation results for the estimation of β_0 in Model 1, for $\tau = 0.5$, $n = 200$, 25% of censoring on average and based on 1000 repetitions. Left: boxplots on the estimation of each coordinate of β_0 (without depicting outliers), the red lines are the true values of $\beta_0 = (1, 2, 3)/\sqrt{14}$. Right: sum of trimmed squared bias (black), sum of trimmed variance (red) and sum of trimmed MSE (green) over the three coordinates. Here, "trimmed" means that both the largest and the smallest 2.5% of the simulation outcomes are discarded.

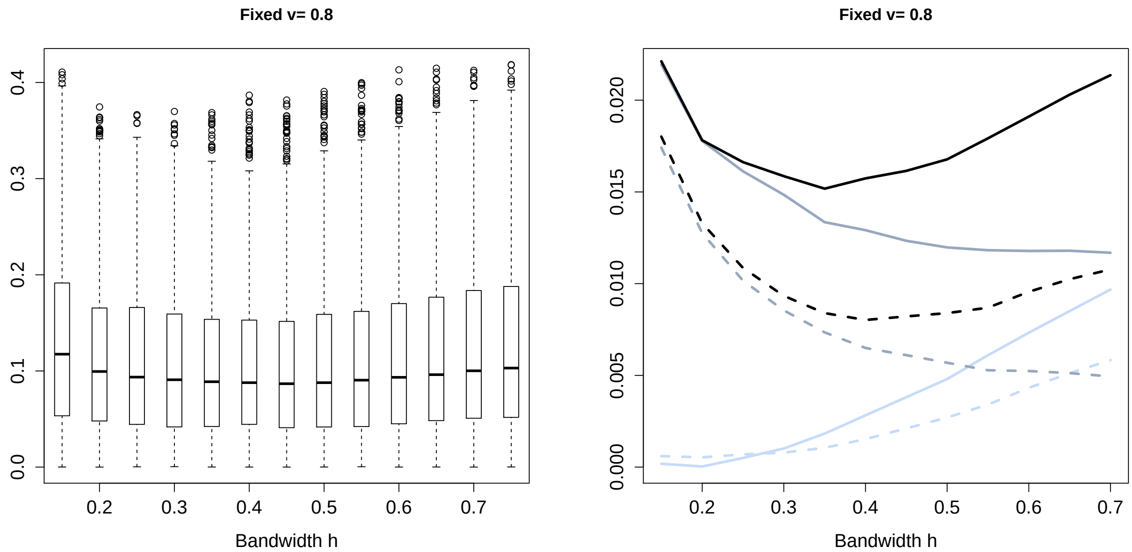

Figure 3: Simulation results for the estimation of $m(v)$ in Model 1, for $\tau = 0.5$, $n = 200$, 25% of censoring on average, fixed $v = 0.8$ and based on 1000 repetitions. Left: boxplots of the absolute estimation error. Right: trimmed squared bias (light grey lines), variance (dark grey) and MSE (black), based on discarding both the largest and the smallest 2.5% of the simulation outcomes. The dotted lines correspond to the oracle estimator of m which is based on the true value of β_0 instead of its estimator $\hat{\beta}$.

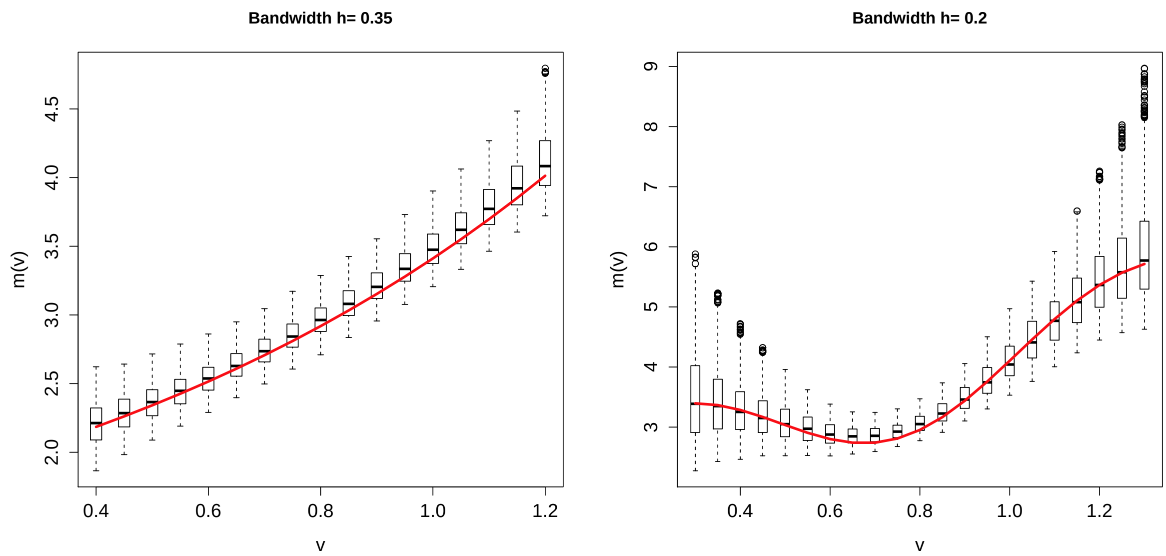

Figure 4: Boxplots of the estimator $\hat{m}(v, \hat{\beta})$ for various v. Both pictures are based on $\tau = 0.5$, $n = 200$, 25% of censoring on average and 1000 repetitions. Left: Model 1 with fixed $h = 0.35$, right: Model 2 with fixed $h = 0.2$.

			MAE Model 1			MAE Model 2		
n	Av. Cens.	τ	$v = 0.6$	$v = 0.8$	$v = 1$	$v = 0.6$	$v = 0.8$	$v = 1$
100	25%	0.3	0.782	0.758	0.943	1.339	1.163	1.695
		0.5	1.218	1.286	1.598	2.094	2.041	2.860
		0.7	2.144	2.212	3.036	3.858	3.416	5.624
	50%	0.3	0.991	1.015	1.212	1.606	1.621	2.163
		0.5	1.651	1.840	2.292	2.766	3.018	3.983
		0.7	3.448	3.997	5.601	5.742	6.025	10.066
200	25%	0.3	0.532	0.561	0.695	0.939	0.817	1.222
		0.5	0.877	0.922	1.113	1.528	1.267	1.973
		0.7	1.549	1.618	2.048	2.621	2.474	3.797
	50%	0.3	0.627	0.684	0.845	1.056	0.977	1.490
		0.5	1.082	1.266	1.579	1.860	1.984	2.693
		0.7	2.323	2.727	3.719	3.959	4.381	6.915

Table 1: Simulation results for the estimation of $m(v)$ for $v \in \{0.6, 0.8, 1\}$, $n \in \{100, 200\}$, an average censoring rate in $\{25\%, 50\%\}$, $\tau \in \{0.3, 0.5, 0.7\}$ and for Model 1 and 2 based on 1000 repetitions each. Stated is the minimal value of the median average error (MAE) over bandwidths in $\{0.1, 0.15, \ldots, 0.75, 0.8\}$, multiplied by 10.

τ	$n=100$			$n=200$		
	$v=0.6$	$v=0.8$	$v=1$	$v=0.6$	$v=0.8$	$v=1$
0.3	0.820	0.772	0.857	0.780	0.814	0.864
0.5	0.834	0.821	0.887	0.829	0.826	0.916
0.7	0.951	0.992	1.101	0.992	1.013	1.145

Table 2: Relative efficiency RE (see (5.1)) of our estimator with a bandwidth chosen by the cross-validation method as given in Section 4. Results are for $v \in \{0.6, 0.8, 1\}$, $n \in \{100, 200\}$, an average censoring rate of 25%, $\tau \in \{0.3, 0.5, 0.7\}$ and for Model 1 based on 1000 repetitions each.

A Appendix A : Proofs of the main results

Proof of Theorem 3.1. Recall that $Q_i = \Delta_i/\{1 - F_C(Z_i-)\}$ and $\hat{Q}_i = \Delta_i/\{1 - \hat{F}_C(Z_i-)\}$ and, for $t \in \mathbb{R}$, set

$$\zeta_i(t) = t\big\{\tau - Q_i \mathbb{1}(t < 0)\big\}, \qquad \hat{\zeta}_i(t) = t\big\{\tau - \hat{Q}_i \mathbb{1}(t < 0)\big\}.$$

Note that $(\hat{m}(u,\beta), \hat{m}'(u,\beta)) = (\hat{a}(u,\beta), \hat{b}(u,\beta))$ as defined in (2.6) can be written as the minimizer of the expression

$$\sum_{i=1}^{n} \hat{\zeta}_i\{Z_i - a - b(X_i^T\beta - u)\}\mathcal{K}_i(\beta, u) \tag{A.1}$$

with respect to a and b, where $\mathcal{K}_i(\beta,u) = K\{(X_i^T\beta - u)/h\}$. Furthermore, for $\beta \in \mathbb{R}^d$ and $u \in \mathbb{R}$, set

$$\Theta_n(\beta, u) = \sqrt{nh}\begin{pmatrix} \hat{m}(u,\beta) - m(u) \\ h\{\hat{m}'(u,\beta) - m'(u)\} \end{pmatrix}.$$

Note that $\Theta_n(\beta_n^\gamma, v_n^\kappa)$ is, up to the bias term and up to a negligible term arising from replacing $m(v_n^\kappa)$ and $m'(v_n^\kappa)$ by $m(v)$ and $m'(v)$, respectively, the expression of primary interest in Theorem 3.1. Recall that $\mathcal{X}_i(\beta,u) = \big(1, (X_i^T\beta - u)/h\big)^T$ and $\mathcal{Z}_i(\beta,u) = Z_i - m(u) - m'(u)(X_i^T\beta - u)$ and define, for $\Theta \in \mathbb{R}^2$,

$$L_n(\Theta, \beta, u) = \sum_{i=1}^{n}\left[\zeta_i\left\{\mathcal{Z}_i(\beta,u) - \frac{\Theta^T\mathcal{X}_i(\beta,u)}{\sqrt{nh}}\right\} - \zeta_i\{\mathcal{Z}_i(\beta,u)\}\right]\mathcal{K}_i(\beta,u), \tag{A.2}$$

$$\hat{L}_n(\Theta, \beta, u) = \sum_{i=1}^{n}\left[\hat{\zeta}_i\left\{\mathcal{Z}_i(\beta,u) - \frac{\Theta^T\mathcal{X}_i(\beta,u)}{\sqrt{nh}}\right\} - \hat{\zeta}_i\{\mathcal{Z}_i(\beta,u)\}\right]\mathcal{K}_i(\beta,u). \tag{A.3}$$

Observing (A.1), one can easily show that, for any $\beta \in \mathbb{R}^d$ and $u \in \mathbb{R}$,

$$\Theta_n(\beta, u) = \underset{\Theta \in \mathbb{R}^2}{\operatorname{argmin}}\ \hat{L}_n(\Theta, \beta, u).$$

We are going to prove Theorem 3.1 by an application of Theorem 2 in Kato (2009) and begin by showing that $\Theta \mapsto \hat{L}_n(\Theta, \beta_n^\gamma, v_n^\kappa)$ is convex, for any $\gamma \in U_M$ and $\kappa \in [-M, M]$.

Since sums of convex functions are convex, it suffices to show that the ith summand in the definition of $\hat{L}_n(\Theta, \beta_n^\gamma, v_n^\kappa)$ is convex. Since $\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)$ is independent of Θ, it is even sufficient to prove convexity of

$$\Theta \mapsto \hat{L}_{n,i}(\Theta) = \hat{\zeta}_i\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) - \frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right\} - \hat{\zeta}_i\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa)\}.$$

Now, we can decompose

$$\hat{L}_{n,i}(\Theta) = -\hat{Q}_i \left\{ \mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) - \frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}} \right\} \mathbb{1} \left\{ \mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < \frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}} \right\} + \hat{Q}_i \times \mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) \mathbb{1}\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < 0\} - \tau \frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}.$$

Since $t \mapsto -t\mathbb{1}(t<0)$ is convex and since $\Theta \mapsto \mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) - \{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\}/\sqrt{nh}$ is linear, we see that the first summand on the right of this decomposition is convex. The second one does not depend on Θ and the third one is linear, and hence also convex. This proves the desired convexity.

Now, let us show (10) in Kato (2009); more precisely, that

$$\sup_{(\gamma,\kappa)\in U_M \times [-M,M]} \left| \hat{L}_n(\Theta, \beta_n^\gamma, v_n^\kappa) + \Theta^T U_n - \frac{1}{2}\Theta^T V \Theta \right| = o_P(1), \tag{A.4}$$

where

$$U_n = \frac{1}{\sqrt{nh}} \sum_{i=1}^n \left[\tau - Q_i \mathbb{1}\left\{Z_i < m(X_i^T\beta_0)\right\}\right] \mathcal{X}_i(\beta_0)\mathcal{K}_i(\beta_0) + \frac{\sqrt{nh^5}}{2} f_{Y|X^T\beta_0}\{m(v)\,|\,v\}\, m''(v) f_{X^T\beta_0}(v) \begin{pmatrix} \bar{K}_2 \\ \bar{K}_3 \end{pmatrix}. \tag{A.5}$$

First of all, as a consequence of Lemma B.1 and the fact that $nh^5 = O(1)$ for $n\to\infty$, we get

$$\sup_{(\gamma,\kappa)\in U_M \times [-M,M]} |\hat{L}_n(\Theta, \beta_n^\gamma, v_n^\kappa) - L_n(\Theta, \beta_n^\gamma, v_n^\kappa)| = o_P(1). \tag{A.6}$$

Moreover, Lemma B.2 yields

$$\sup_{(\gamma,\kappa)\in U_M \times [-M,M]} |L_n(\Theta, \beta_n^\gamma, v_n^\kappa) - L_n(\Theta, \beta_0, v)| = o_P(1). \tag{A.7}$$

Finally, observe that we can write

$$L_n(\Theta, \beta_0, v) = \Theta^T W_n(\beta_0, v) + \mathbb{E}[L_n(\Theta, \beta_0, v) - \Theta^T W_n(\beta_0, v) \mid \mathcal{A}_n(\beta_0)] + R_n(\Theta, \beta_0, v),$$

where, for $\beta \in \mathbb{R}^d$, $\mathcal{A}_n(\beta)$ denotes the sigma-field generated by $X_1^T\beta, \dots, X_n^T\beta$ and where, for $u \in \mathbb{R}$,

$$W_n(\beta, u) = -\frac{1}{\sqrt{nh}} \sum_{i=1}^n \left[\tau - Q_i \mathbb{1}\left\{\mathcal{Z}_i(\beta, u) < 0\right\}\right] \mathcal{X}_i(\beta, u)\mathcal{K}_i(\beta, u) \tag{A.8}$$

and

$$R_n(\Theta, \beta, u) = L_n(\Theta, \beta, u) - \mathbb{E}[L_n(\Theta, \beta, u)\,|\,\mathcal{A}_n(\beta)] - \left\{\Theta^T W_n(\beta, u) - \mathbb{E}[\Theta^T W_n(\beta, u) \mid \mathcal{A}_n(\beta, u)]\right\}. \tag{A.9}$$

Then, it follows from Lemma B.6, Lemma B.4 and Lemma B.5 that

$$L_n(\Theta, \beta_0, v) = -\Theta^T U_n + \frac{1}{2}\Theta^T V \Theta + o_P(1). \tag{A.10}$$

Hence, (A.4) is a mere consequence of (A.6), (A.7) and (A.10).

Finally, observing that U_n is constant in (γ, κ), that its dominating sum converges to a normal distribution and noting that $nh^5 = O(1)$, we easily obtain that U_n satisfies (11) in Kato (2009). Therefore, an application of Kato's Theorem 2 finalizes the proof. □

Proof of Corollary 3.2. Let us first show that $\|\mathbb{M}_n(v,\hat{\beta}_n) - A_n(v)\| = o_P(1)$, where

$$A_n(v) = V^{-1}\frac{1}{\sqrt{nh}}\sum_{i=1}^{n}\Big[\tau - Q_i\mathbb{1}\left\{Z_i < m(X_i^T\beta_0)\right\}\Big] \times \mathcal{X}_i(\beta_0,v)\mathcal{K}_i(\beta_0,v).$$

Let $\varepsilon,\eta > 0$ be given. Since $\hat{\gamma}_n = O_P(1)$, we can choose $M > 0$ large enough such that $\Pr\{\hat{\gamma}_n \notin U_M\} < \eta/2$ for all $n \in \mathbb{N}$. Therefore, since $\beta_n^{\hat{\gamma}_n} = \hat{\beta}_n$,

$$\begin{aligned}\Pr(\|\mathbb{M}_n(v,\hat{\beta}_n) - A_n(v)\| > \varepsilon) &\le \Pr\{\|\mathbb{M}_n(v,\beta_n^{\hat{\gamma}_n}) - A_n(v)\| > \varepsilon, \hat{\gamma}_n \in U_M\} + \eta/2\\ &\le \Pr(\sup_{\gamma\in U_M}\|\mathbb{M}_n(v,\beta_n^{\gamma}) - A_n(v)\| > \varepsilon) + \eta/2 \le \eta\end{aligned}$$

for sufficiently large n by Theorem 3.1. One can proceed analogously to prove that $\|\mathbb{M}_n(x^T\hat{\beta}_n,\hat{\beta}_n) - A_n(x^T\beta_0)\| = o_P(1)$.

Now, let us show that $A_n(v) \rightsquigarrow \mathcal{N}_2\big(0,\sigma^2(v)\bar{K}^{-1}\bar{K}'\bar{K}^{-1}\big)$. By iterated expectation, each entry of the covariance matrix $\mathrm{Var}\{VA_n(v)\}$ can be written as

$$D_j = h^{-1}\mathbb{E}\left[\mathbb{E}\left[\left[\tau - Q_i\mathbb{1}\{Z_i < m(X_i^T\beta_0)\}\right]^2 \mid X_i^T\beta_0\right]\left(\frac{X_i^T\beta_0 - v}{h}\right)^j K^2\left(\frac{X_i^T\beta_0 - v}{h}\right)\right],$$

for some $j \in \{0,1,2\}$. Note that (2.5) implies

$$\begin{aligned}&\mathbb{E}[Q_i\mathbb{1}\{Z_i < m(X_i^T\beta_0)\} \mid X_i^T\beta_0] = F_{Y|X^T\beta_0}\{m(X_i^T\beta_0) \mid X_i^T\beta_0\} = \tau,\\ &\mathbb{E}[Q_i^2\mathbb{1}\{Z_i < m(X_i^T\beta_0)\} \mid X_i^T\beta_0] = \Phi_{\beta_0}\{m(X_i^T\beta_0) \mid X_i^T\beta_0\},\end{aligned}$$

where the last equality in the first line follows from the continuity of the map $y \mapsto F_{Y|X^T\beta_0}(y \mid x^T\beta_0)$. Therefore, we can write

$$\begin{aligned}D_j &= h^{-1}\int\left[\Phi_{\beta_0}\{m(u) \mid u\} - \tau^2\right]\left(\frac{u-v}{h}\right)^j K\left(\frac{u-v}{h}\right) f_{X^T\beta_0}(u)\,du\\ &= \int\left[\Phi_{\beta_0}\{m(v+hw) \mid v+hw\} - \tau^2\right] w^j K(w)\, f_{X^T\beta_0}(v+hw)\,dw\\ &= \left[\Phi_{\beta_0}\{m(v) \mid v\} - \tau^2\right] \times f_{X^T\beta_0}(v) \times \bar{K}_j' + o(1),\end{aligned}$$

where the last equality follows from the continuity of $f_{X^T\beta_0}$ and Φ_{β_0} (in both arguments). Then, the assertion of the corollary follows from the classical central limit theorem for row-wise independent triangular arrays. $\square$

B Appendix B : Auxiliary results for the proof of Theorem 3.1

Lemma B.1. *Under the conditions of Theorem 3.1,*

$$\sup_{(\gamma,\kappa)\in U_M\times[-M,M]}\left|\hat{L}_n(\Theta,\beta_n^{\gamma},v_n^{\kappa}) - L_n(\Theta,\beta_n^{\gamma},v_n^{\kappa})\right| = O_P(\sqrt{h\log n}).$$

Proof. We can write $\hat{L}_n(\Theta,\beta_n^{\gamma},v_n^{\kappa}) - L_n(\Theta,\beta_n^{\gamma},v_n^{\kappa}) = R_{n1} + R_{n2}$, where

$$\begin{aligned}R_{n1} = &\sum_{i=1}^{n}\left\{\mathcal{Z}_i(\beta_n^{\gamma},v_n^{\kappa}) - \frac{\Theta^T\mathcal{X}_i(\beta_n^{\gamma},v_n^{\kappa})}{\sqrt{nh}}\right\}\\ &\times\left[\mathbb{1}\left\{\mathcal{Z}_i(\beta_n^{\gamma},v_n^{\kappa}) < \frac{\Theta^T\mathcal{X}_i(\beta_n^{\gamma},v_n^{\kappa})}{\sqrt{nh}}\right\} - \mathbb{1}\left\{\mathcal{Z}_i(\beta_n^{\gamma},v_n^{\kappa}) < 0\right\}\right] \times (Q_i - \hat{Q}_i)\mathcal{K}_i(\beta_n^{\gamma},v_n^{\kappa})\\ R_{n2} = &-\sum_{i=1}^{n}\frac{\Theta^T\mathcal{X}_i(\beta_n^{\gamma},v_n^{\kappa})}{\sqrt{nh}}\mathbb{1}\left\{\mathcal{Z}_i(\beta_n^{\gamma},v_n^{\kappa}) < 0\right\}(Q_i - \hat{Q}_i)\mathcal{K}_i(\beta_n^{\gamma},v_n^{\kappa}).\end{aligned}$$

Let us begin by considering R_{n1}. Clearly,

$$\left|\mathbb{1}\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < \frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right\} - \mathbb{1}\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < 0\right\}\right|$$
$$\leq \mathbb{1}\left\{-\left|\frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right| \leq \mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < \left|\frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right|\right\}, \quad \text{(B.1)}$$

whence we can bound $|R_{n1}|$ from above by

$$\sum_{i=1}^n 2\left|\frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right| \mathbb{1}\left\{|\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa)| \leq \left|\frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right|\right\} |Q_i - \hat{Q}_i| \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa).$$

Let $C_0 = \sqrt{1 + |\sup(\operatorname{supp} K)|^2}$. Then, the expression on the right-hand side of the last formula can be bounded by

$$\frac{2C_0\|\Theta\|}{\sqrt{nh}} \sum_{i=1}^n \mathbb{1}\left\{|\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa)| \leq \frac{C_0\|\Theta\|}{\sqrt{nh}}\right\} |Q_i - \hat{Q}_i| \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa).$$

Furthermore, for $(X_i^T \beta_n^\gamma - v_n^\kappa)/h \in \operatorname{supp}(K)$, the inequality $|\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa)| \leq C_0\|\Theta\|/\sqrt{nh}$ implies that $Z_i \leq m(v) + C_0' h + C_0\|\Theta\|/\sqrt{nh}$, with some universal constant C_0'. Therefore, we can proceed to bound $|R_{n1}|$ by

$$\frac{2C_0\|\Theta\|}{\sqrt{nh}} \sum_{i=1}^n \mathbb{1}\left\{Z_i \leq m(v) + C_0' h + \frac{C_0\|\Theta\|}{\sqrt{nh}}\right\} |Q_i - \hat{Q}_i| \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)$$
$$= \frac{2C_0\|\Theta\|}{\sqrt{nh}} \sum_{i=1}^n \mathbb{1}\left\{Z_i \leq m(v) + C_0' h + \frac{C_0\|\Theta\|}{\sqrt{nh}}\right\} \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)$$
$$\times \Delta_i \frac{|\hat{F}_C(Z_i-) - F_C(Z_i-)|}{\{1 - F_C(Z_i-)\}\{1 - \hat{F}_C(Z_i-)\}}.$$

As $F_Z\{m(v)\} < 1$ by condition (A5), we may choose $\mathcal{T}$ such that $m(v) < \mathcal{T} < \inf\{x \in \mathbb{R} : F_Z(x) = 1\}$. Then, for n large enough, we have $m(v) + C_0' h + \frac{C_0\|\Theta\|}{\sqrt{nh}} \leq \mathcal{T}$. Finally, since

$$\sup_{x \leq \mathcal{T}} \frac{|\hat{F}_C(x-) - F_C(x-)|}{\{1 - F_C(x-)\}\{1 - \hat{F}_C(x-)\}} = O_P\left(\sqrt{\frac{\log n}{n}}\right)$$

by the results in Lo and Singh (1986), and since $\mathbb{E}K\left\{(X_i^T \beta_n^\gamma - v_n^\kappa)/h\right\} = O(h)$ uniformly in γ and κ, we obtain that $\sup_{(\gamma,\kappa) \in U_M \times [-M,M]} |R_{n1}|$ is of order $O_P(\sqrt{h \log n})$, as asserted.

To see that also $\sup_{(\gamma,\kappa) \in U_M \times [-M,M]} |R_{n2}| = O_P(\sqrt{h \log n})$, note that $(X_i^T \beta_n^\gamma - v_n^\kappa)/h \in \operatorname{supp} K$ and $\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < 0$ implies that $Z_i \leq m(v) + C_0' h$, with some universal constant C_0'. The remaining argumentation is similar as for R_{n1}. □

Lemma B.2. *Under the conditions of Theorem 3.1,*

$$\sup_{(\gamma,\kappa) \in U_M \times [-M,M]} |L_n(\Theta, \beta_n^\gamma, v_n^\kappa) - L_n(\Theta, \beta_0, v)| = o_P(1),$$

where L_n is defined in (A.2).

Proof. Write $H_n(\Theta,\gamma,\kappa) := L_n(\Theta,\beta_n^\gamma,v_n^\kappa) - L_n(\Theta,\beta_0,v) = \sum_{i=1}^n H_{n,i}(\Theta,\gamma,\kappa)$, where

$$\begin{aligned} H_{n,i}(\Theta,\gamma,\kappa) = \left[\zeta_i\left\{\mathcal{Z}_i(\beta_n^\gamma,v_n^\kappa) - \frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma,v_n^\kappa)}{\sqrt{nh}}\right\} - \zeta_i\{\mathcal{Z}_i(\beta_n^\gamma,v_n^\kappa)\}\right]\mathcal{K}_i(\beta_n^\gamma,v_n^\kappa) \\ - \left[\zeta_i\left\{\mathcal{Z}_i(\beta_0,v) - \frac{\Theta^T\mathcal{X}_i(\beta_0,v)}{\sqrt{nh}}\right\} - \zeta_i\{\mathcal{Z}_i(\beta_0,v)\}\right]\mathcal{K}_i(\beta_0,v). \end{aligned}$$

In the following, let C denote some universal constant that is independent of $(\gamma,\kappa)\in U_M\times[-M,M]$ and $i=1,\dots,n$ and that may vary from line to line. Suppose we have shown that

$$\begin{aligned} &\text{(i) } |\mathbb{E}\left[H_n(\Theta,\gamma,\kappa)\right]| \le C\sqrt{h}, \\ &\text{(ii) } \operatorname{Var}\{H_n(\Theta,\gamma,\kappa)\} \le Ch, \\ &\text{(iii) } |H_{n,i}(\Theta,\gamma,\kappa)| \le C/\sqrt{nh}, \\ &\text{(iv) } |H_n(\Theta,\gamma,\kappa) - H_n(\Theta,\gamma',\kappa')| \le C\times(\sqrt{n}/h)\times\big(\|\gamma-\gamma'\|+|\kappa-\kappa'|\big). \end{aligned}$$

Then, assertion (iv) will allow to transfer the supremum $\sup_{(\gamma,\kappa)\in U_M\times[-M,M]}|H_n(\Theta,\gamma,\kappa)|$ into a maximum over a finite subset $U'\subset U_M\times[-M,M]$, whereas (i)-(iii) will allow to treat that finite maximum.

For transferring the supremum into a maximum, let U' be an $n^{-5/6}$-cover of $U_M\times[-M,M]$, that is, a finite collection of points in $U_M\times[-M,M]$ such that, for each $(\gamma,\kappa)\in U_M\times[-M,M]$, one can find a $(\gamma',\kappa')\in U'$ with $\{\|\gamma-\gamma'\|+|\kappa-\kappa'|\}\le n^{-5/6}$. Note that U' can be chosen in such a way that its cardinality is less than $C\times n^r$, where $r>0$ depends only on the dimension d. Then, connecting each $(\gamma,\kappa)\in U_M\times[-M,M]$ with some $(\gamma',\kappa')\in U'$ and using (iv), we get that

$$\sup_{(\gamma,\kappa)\in U_M\times[-M,M]}|H_n(\Theta,\gamma,\kappa)| \le C\times(\sqrt{n}/h)\times n^{-5/6} + \max_{(\gamma,\kappa)\in U'}|H_n(\Theta,\gamma,\kappa)|.$$

Since $(\sqrt{n}/h)\times n^{-5/6} = (nh^3)^{-1/3} = o(1)$, it suffices to show that the maximum on the right-hand side of the last display is $o_P(1)$.

To treat the finite maximum on the right-hand side of the last displayed formula, let $p\ge 3$. By Rosenthal's inequality and (ii) and (iii) above we obtain

$$\begin{aligned} \mathbb{E}\left[(H_n-\mathbb{E}H_n)^{2p}\right] &\le C_p\left[\{\operatorname{Var}(H_n)\}^p + \sum_{i=1}^n \mathbb{E}\big[(H_{n,i}-\mathbb{E}H_{n,i})^{2p}\big]\right] \\ &\le C_p\left\{h^p + n/(nh)^p\right\} \le C_p\times h^p, \end{aligned}$$

where the constant C_p depends on p and where we suppressed the arguments Θ,γ and κ. Since also $\{\mathbb{E}(H_n)\}^{2p}\le C_p\times h^p$ by (i), we can use convexity of $x\mapsto|x|^{2p}$ to obtain that $\mathbb{E}[H_n^{2p}]\le C_p\times h^p$. With $U'\subset U_M\times[-M,M]$ as chosen above such that $\#(U')\le Cn^r$, we obtain

$$\begin{aligned} \Pr\big(\max_{(\gamma,\kappa)\in U'}|H_n(\Theta,\gamma,\kappa)|>\varepsilon\big) &\le C\times n^r\max_{(\gamma,\kappa)\in U'}\Pr(|H_n(\Theta,\gamma,\kappa)|>\varepsilon) \\ &\le C\times n^r\mathbb{E}[H_n^{2p}]/\varepsilon^{2p} \le C_p\times n^r h^p = C_p\times(nh^5)^r\times h^{p-5r}. \end{aligned}$$

Choosing $p=p(r)$ sufficiently large, the latter expression converges to 0 for $n\to\infty$.

To finalize the proof, we have to prove (i)-(iv) above. Regarding (i) and (ii), we write

$$\begin{aligned} H_n(\Theta,\gamma,\kappa) = \Theta^T\{W_n(\beta_n^\gamma,v_n^\kappa) - W_n(\beta_0,v)\} \\ + \mathbb{E}[L_n(\Theta,\beta_n^\gamma,v_n^\kappa) - \Theta^T W_n(\beta_n^\gamma,v_n^\kappa)\mid\mathcal{A}_n(\beta_n^\gamma)] + R_n(\Theta,\beta_n^\gamma,v_n^\kappa) \\ - \mathbb{E}[L_n(\Theta,\beta_0,v) - \Theta^T W_n(\beta_0,v)\mid\mathcal{A}_n(\beta_0)] - R_n(\Theta,\beta_0,v), \end{aligned}$$

where W_n and R_n are defined in (A.8) and (A.9), respectively. Then, (i) and (ii) are mere consequences of Lemma B.3, B.4 and B.5.

For the proof of (iii), we can make use of the fact that, for any $a, b \in \mathbb{R}$,

$$|\zeta_i(a) - \zeta_i(b)| \leq \tau \times |a-b| + Q_i \times |a-b| \times \mathbb{1}(a < 0 \text{ or } b < 0),$$

which easily follows from the definition of ζ_i. Then,

$$\begin{aligned}
&\left|\zeta_i\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) - \frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right\} - \zeta_i\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa)\}\right| \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) \\
&\quad \leq \; Q_i \left|\frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right| \mathbb{1}\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < 0 \text{ or } \mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < \frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right\} \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) \\
&\qquad\qquad + \tau \left|\frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right| \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa).
\end{aligned}$$

Note that the same bound holds with $(\beta_n^\gamma, v_n^\kappa)$ replaced by (β_0, v). Now, the second summand on the right of the previous display is obviously bounded by $C/\sqrt{nh}$. Regarding the first summand, use condition (A5) to choose $\mathcal{T}$ such that $m(v) < \mathcal{T} < \inf\{x \in \mathbb{R} : F_Z(x) = 1\}$. For $(X_i^T \beta_n^\gamma - v_n^\kappa)/h \in \text{supp}(K)$, we have $|m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)| \leq C \times h$. Therefore, we can choose n_0 sufficiently large and independent of i and $\gamma \in U_M$ such that $m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa) \leq \mathcal{T}$ for all $n \geq n_0$. This implies

$$Q_i \times \mathbb{1}\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < 0\} \times \mathbb{1}\{(X_i^T\beta_n^\gamma - v_n^\kappa)/h \in \text{supp}(K)\} \leq \frac{\mathbb{1}\{Z_i < \mathcal{T}\}}{1 - F_C(Z_i-)} \leq \{1 - F_Z(\mathcal{T})\}^{-1} < \infty.$$

For later reference, by a simple adaptation of the preceding argument, we even have

$$Q_i \times \mathbb{1}\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < 0\} \times \mathbb{1}\{(X_i^T\beta_n^{\gamma'} - v_n^{\kappa'})/h \in \text{supp}(K)\} \leq C, \tag{B.2}$$

$$Q_i \times \mathbb{1}\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < \frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right\} \times \mathbb{1}\{(X_i^T\beta_n^{\gamma'} - v_n^{\kappa'})/h \in \text{supp}(K)\} \leq C \tag{B.3}$$

for any $(\beta_n^\gamma, v_n^\kappa) = (\beta_0 + \gamma/\sqrt{n}, v + \kappa/\sqrt{n})$ and $(\beta_n^{\gamma'}, v_n^{\kappa'}) = (\beta_0 + \gamma'/\sqrt{n}, v + \kappa'/\sqrt{n})$ with arbitrary $(\gamma, \kappa), (\gamma', \kappa') \in U_M \times [-M, M]$ and for sufficiently large n. Apply these bounds to prove (iii).

The proof of (iv) follows after some tedious but straightforward calculations for each summand of the decomposition

$$H_n(\Theta, \gamma, \kappa) - H_n(\Theta, \gamma', \kappa') = L_n(\Theta, \beta_n^\gamma, v_n^\kappa) - L_n(\Theta, \beta_n^{\gamma'}, v_n^{\kappa'}) = \sum_{i=1}^n (A_{n,i} + B_{n,i} + C_{n,i}),$$

where

$$\begin{aligned}
A_{n,i} &= \tau \frac{\Theta^T \mathcal{X}_i(\beta_n^{\gamma'}, v_n^{\kappa'})}{\sqrt{nh}} \mathcal{K}_i(\beta_n^{\gamma'}, v_n^{\kappa'}) - \tau \frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}} \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) \\
B_{n,i} &= \lambda_i \left\{\mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'}) - \frac{\Theta^T \mathcal{X}_i(\beta_n^{\gamma'}, v_n^{\kappa'})}{\sqrt{nh}}\right\} \mathcal{K}_i(\beta_n^{\gamma'}, v_n^{\kappa'}) \\
&\qquad\qquad - \lambda_i \left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) - \frac{\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right\} \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) \\
C_{n,i} &= \lambda_i \{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa)\} \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \lambda_i \left\{\mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'})\right\} \mathcal{K}_i(\beta_n^{\gamma'}, v_n^{\kappa'}),
\end{aligned}$$

and where, for $a \in \mathbb{R}$, $\lambda_i(a) = aQ_i\mathbb{1}(a < 0)$. Note that this function satisfies $|\lambda_i(a) - \lambda_i(b)| \leq Q_i|a-b|\mathbb{1}(a < 0 \text{ or } b < 0)$. For the sake of brevity, we only discuss how to bound the summand $C_{n,i}$. We have

$$|C_{n,i}| \leq \left|\lambda_i\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa)\right\} - \lambda_i\left\{\mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'})\right\}\right| \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) \\ + \left|\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{K}_i(\beta_n^{\gamma'}, v_n^{\kappa'})\right| \times \left|\lambda_i\left\{\mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'})\right\}\right|. \tag{B.4}$$

The first summand on the right can be bounded by

$$Q_i\left|\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'})\right| \times \mathbb{1}\{\mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'}) < 0 \text{ or } \mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < 0\} \times \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa),$$

which in turn is bounded by $(C/\sqrt{n}) \times \{\|\gamma - \gamma'\| + |\kappa - \kappa'|\}$ thanks to (B.2) and the fact that

$$\left|\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'})\right| \leq |m(v_n^\kappa) - m(v_n^{\kappa'})| + |m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa) - m'(v_n^{\kappa'})(X_i^T\beta_n^{\gamma'} - v_n^{\kappa'})| \\ \leq C\{\|\beta_n^\gamma - \beta_n^{\gamma'}\| + |v_n^\kappa - v_n^{\kappa'}|\} = (C/\sqrt{n}) \times \{\|\gamma - \gamma'\| + |\kappa - \kappa'|\},$$

by boundedness of $\operatorname{supp} X$ and by condition (A2). Regarding the second summand on the right of (B.4) recall that $|\lambda_i\{\mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'})\}| = Q_i|\mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'})|\mathbb{1}\{\mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'}) < 0\}$. Since $Z_i \geq 0$, the condition $\mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'}) < 0$ implies that

$$|\mathcal{Z}_i(\beta_n^{\gamma'}, v_n^{\kappa'})| \leq |Z_i| + |m(v_n^{\kappa'}) + m'(v_n^{\kappa'})(X_i^T\beta_n^{\gamma'} - v_n^{\kappa'})| \leq 2|m(v_n^{\kappa'}) + m'(v_n^{\kappa'})(X_i^T\beta_n^{\gamma'} - v_n^{\kappa'})| \leq C.$$

Finally,

$$\begin{aligned} &\left|\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{K}_i(\beta_n^{\gamma'}, v_n^{\kappa'})\right| \\ =& \left|\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{K}_i(\beta_n^{\gamma'}, v_n^{\kappa'})\right| \mathbb{1}\{(X_i^T\beta_n^\gamma - v_n^\kappa)/h \in \operatorname{supp} K \text{ or } (X_i^T\beta_n^{\gamma'} - v_n^{\kappa'})/h \in \operatorname{supp} K\} \\ \leq& C\{\|\beta_n^\gamma - \beta_n^{\gamma'}\| + |v_n^\kappa - v_n^{\kappa'}|\}/h \times \mathbb{1}\{(X_i^T\beta_n^\gamma - v_n^\kappa)/h \in \operatorname{supp} K \text{ or } (X_i^T\beta_n^{\gamma'} - v_n^{\kappa'})/h \in \operatorname{supp} K\}, \end{aligned}$$

which implies that the second summand in (B.4) is bounded by $C\{\|\beta_n^\gamma - \beta_n^{\gamma'}\| + |v_n^\kappa - v_n^{\kappa'}|\}/h = (C/\sqrt{nh^2}) \times \{\|\gamma - \gamma'\| + |\kappa - \kappa'|\}$ as a consequence of (B.2). This proves that $\sum_{i=1}^n |C_{n,i}| \leq C \times (\sqrt{n}/h) \times \{\|\gamma - \gamma'\| + |\kappa - \kappa'|\}$, and the sums over $A_{n,i}$ and $B_{n,i}$ are treated similarly. The details are omitted for the sake of brevity. □

Lemma B.3. *Under the conditions of Theorem 3.1,*

$$\mathbb{E}[W_n(\beta_n^\gamma, v_n^\kappa) - W_n(\beta_0, v)] = O(\sqrt{h}), \qquad \operatorname{Var}[W_n(\beta_n^\gamma, v_n^\kappa) - W_n(\beta_0, v)] = O(h),$$

where the O-terms are uniform in $\gamma \in U_M$ and $\kappa \in [-M, M]$.

Proof. We can decompose $W_n(\beta_0, v) - W_n(\beta_n^\gamma, v_n^\kappa) = B_{n1} + B_{n2}$, where

$$B_{n1} = \frac{1}{\sqrt{nh}}\sum_{i=1}^n \left[\tau - Q_i\mathbb{1}\{\mathcal{Z}_i(\beta_0, v) < 0\}\right] \times \{\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{X}_i(\beta_0, v)\mathcal{K}_i(\beta_0, v)\}$$

$$B_{n2} = \frac{1}{\sqrt{nh}}\sum_{i=1}^n Q_i\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) \times \left[\mathbb{1}\{\mathcal{Z}_i(\beta_0, v) < 0\} - \mathbb{1}\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < 0\}\right].$$

We can treat both terms separately and begin with the treatment of B_{n2}. Regarding its expected value, iterated expectation allows to write

$$\begin{aligned}\mathbb{E}[B_{n2}] &= \sqrt{\frac{n}{h}} \times \mathbb{E}\Big[\mathbb{E}\left[Q_i\big(\mathbb{1}\{\mathcal{Z}_i(\beta_0,v)<0\} - \mathbb{1}\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa)<0\}\big) \,\big|\, X_i\right] \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)\Big] \\ &= \sqrt{\frac{n}{h}} \times \mathbb{E}\Big[\Big(F_{Y|X}\{m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\,|\,X_i\} \\ &\qquad\qquad - F_{Y|X}\{m(v) + m'(v)(X_i^T\beta_0 - v)\,|\,X_i\}\Big)\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)\Big].\end{aligned}$$

An application of the mean value theorem shows that the difference between the conditional c.d.f.s in the preceding display can be written as

$$f_{Y|X}(\varepsilon_i^*\,|\,X_i)\left\{m(v_n^\kappa) - m(v) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa) - m'(v)(X_i^T\beta_0 - v)\right\},$$

where ε_i^* denotes some intermediate point lying between $m(v_n^\kappa)+m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)$ and $m(v)+m'(v)(X_i^T\beta_0 - v)$. Since $\|\beta_n^\gamma - \beta_0\| \le Mn^{-1/2}$ and $|v_n^\kappa - v| \le Mn^{-1/2}$, we can use conditions (A2) and (A4) to bound $\|\mathbb{E}[B_{n2}]\|$ by a constant multiple of

$$h^{-1/2}\mathbb{E}\left[\|\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\|\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)\right] = O(h^{1/2}),$$

uniformly in $\gamma \in U_M$ and $\kappa \in [-M, M]$.

Now, let us consider the covariance matrix of B_{n2}, for which we have

$$\mathrm{Var}(B_{n2}) = \frac{1}{h}\,\mathrm{Var}\Big[Q_i \times \left[\mathbb{1}\{\mathcal{Z}_i(\beta_0,v)<0\} - \mathbb{1}\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa)<0\}\right] \times \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\ \mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)\Big].$$

Each entry of the matrix on the right can be bounded by

$$\frac{1}{h}\mathbb{E}\Big[\mathbb{E}\left[Q_i^2 \times \left|\mathbb{1}\{\mathcal{Z}_i(\beta_0,v_n^\kappa)<0\} - \mathbb{1}\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa)<0\}\right| \,\big|\, X_i\right]\left(\tfrac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\right)^j K^2\left(\tfrac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\right)\Big] \tag{B.5}$$

for some $j \in \{0,1,2\}$. Exploiting that $|\mathbb{1}(A) - \mathbb{1}(B)| = \mathbb{1}(A\cup B) - \mathbb{1}(A\cap B)$ for any two events A, B, we can write the conditional expectation in the previous display as

$$\begin{aligned}&J\left[\max\left\{m(v) + m'(v)(X_i^T\beta_0 - v), m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\right\}\,|\,X_i\right] \\ &\qquad - J\left[\min\left\{m(v) + m'(v)(X_i^T\beta_0 - v), m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\right\}\,|\,X_i\right] \\ =\ &\left|J\{m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\,|\,X_i\} - J\{m(v) + m'(v)(X_i^T\beta_0 - v)\,|\,X_i\}\right|,\end{aligned} \tag{B.6}$$

where $J(\,\cdot\,|\,\cdot\,)$ is defined for $t \in \mathbb{R}, x \in \mathbb{R}^d$ as

$$J(t\,|\,x) = \mathbb{E}\left[Q_i^2\mathbb{1}(Z_i - t < 0)\,|\,X_i = x\right] = \mathbb{E}\left[\{1 - F_C(Y_i-)\}^{-1}\mathbb{1}(Y_i - t < 0)\,|\,X_i = x\right]. \tag{B.7}$$

Note that the partial derivative $J'(t\,|\,x) = \frac{\partial}{\partial t}J(t\,|\,x) = \{1 - F_C(t-)\}^{-1}f_{Y|X}(t\,|\,x)$ is bounded over a neighborhood of $m(v)$ and the support of X, by condition (A4) and (A5). The mean value theorem implies that the right-hand side of (B.6) can be bounded by a constant multiple of $\|\beta_n^\gamma - \beta_0\| + |v_n^\kappa - v|$, where the constant only depends on the bound on J' and on m' and and on the support of X. Since $\|\beta_n^\gamma - \beta_0\| + |v_n^\kappa - v| \le 2Mn^{-1/2}$ we obtain that the expression in (B.5) can be bounded by

$$\mathrm{const}\,\frac{1}{\sqrt{nh^2}} \times \mathbb{E}\Big[\left(\tfrac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\right)^j K^2\left(\tfrac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\right)\Big] = O(n^{-1/2}) = o(h).$$

Now, let us consider B_{n1} and let us begin by considering its expected value. We exemplarily only deal with its first coordinate which can be written as

$$
\begin{aligned}
&\sqrt{\frac{n}{h}}\mathbb{E}\Big[\mathbb{E}\big[\tau - Q_i\mathbb{1}\{\mathcal{Z}_i(\beta_0, v) < 0\} \,|\, X_i\big] \times \{\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{K}_i(\beta_0, v)\}\Big] \\
&\qquad = \sqrt{\frac{n}{h}}\mathbb{E}\Big[\big[\tau - F_{Y|X}(m(v) + m'(v)(X_i^T\beta_0 - v) \,|\, X_i)\big] \times \{\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{K}_i(\beta_0, v)\}\Big]. \quad \text{(B.8)}
\end{aligned}
$$

Since $\tau = F_{Y|X}\{m(X_i^T\beta_0) \,|\, X_i\}$, two Taylor expansions show that the term in square brackets on the right can be written as

$$
f_{Y|X}(\varepsilon_i^* \,|\, X_i)\{m(X_i^T\beta_0) - m(v) - m'(v)(X_i^T\beta_0 - v)\} = f_{Y|X}(\varepsilon_i^* \,|\, X_i)m''(v_i^*)(X_i^T\beta_0 - v)^2/2,
$$

where ε_i^* and v_i^* denote two intermediate points. Now, if either $(X_i^T\beta_n^\gamma - v_n^\kappa)/h$ or $(X_i^T\beta_0 - v)/h$ lies in $\mathrm{supp}(K)$, we can use conditions (A1), (A2) and (A4) to see that the last display is of order $O\{(h + n^{-1/2})^2\} = O(h^2)$. Thus, the absolute value of the right-hand side of (B.8) can be bounded by a constant multiple of

$$
\sqrt{nh^3}\ \mathbb{E}|\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{K}_i(\beta_0, v)| = \sqrt{nh^3}\ \mathbb{E}\left|K'(\xi_i)\left(\tfrac{X_i^T(\beta_n^\gamma - \beta_0) - (v_n^\kappa - v)}{h}\right)\right| = O(\sqrt{h}),
$$

where ξ_i denotes some intermediate value between $(X_i^T\beta_n^\gamma - v_n^\kappa)/h$ and $(X_i^T\beta_0 - v)/h$.

It remains to consider the covariance matrix of B_{n1}, which is given by

$$
\begin{aligned}
\mathrm{Var}(B_{n1}) &= \frac{1}{h}\,\mathrm{Var}\Big[\big[\tau - Q_i\mathbb{1}\{\mathcal{Z}_i(\beta_0, v) < 0\}\big] \times \{\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{X}_i(\beta_0, v)\mathcal{K}_i(\beta_0, v)\}\Big] \\
&\leq_L \frac{1}{h}\mathbb{E}\Big[\big[\tau - Q_i\mathbb{1}\{\mathcal{Z}_i(\beta_0, v) < 0\}\big]^2 \times \{\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{X}_i(\beta_0, v)\mathcal{K}_i(\beta_0, v)\} \\
&\qquad\qquad \times \{\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{X}_i(\beta_0, v)\mathcal{K}_i(\beta_0, v)\}^T\Big],
\end{aligned}
$$

where the inequality sign denotes the ordering in the Loewner order. As a consequence of (B.2), for sufficiently large n, we can find a uniform bound on $[\tau - Q_i\mathbb{1}\{\mathcal{Z}_i(\beta_0, v) < 0\}]^2$ both for $(X_i^T\beta_n^\gamma - v_n^\kappa)/h \in \mathrm{supp}(K)$ or for $(X_i^T\beta_0 - v)/h \in \mathrm{supp}(K)$. Therefore, it is sufficient to show that

$$
\begin{aligned}
&\mathbb{E}\Big[\{\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{X}_i(\beta_0, v)\mathcal{K}_i(\beta_0, v)\} \\
&\qquad\qquad \times \{\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{X}_i(\beta_0, v)\mathcal{K}_i(\beta_0, v)\}^T\Big] = O(h^2).
\end{aligned}
$$

Let us exemplarily indicate how to deal with the hardest entry, given by

$$
\mathbb{E}\Big[\Big\{\Big(\tfrac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\Big)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \Big(\tfrac{X_i^T\beta_0 - v}{h}\Big)\mathcal{K}_i(\beta_0, v)\Big\}^2\Big].
$$

By adding and subtracting $(X_i^T\beta_n^\gamma - v_n^\kappa)/h \times \mathcal{K}_i(\beta_0, v)$ inside the curly brackets, we can bound the last display by

$$
2\,\mathbb{E}\Big[\Big(\tfrac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\Big)^2\{\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) - \mathcal{K}_i(\beta_0, v)\}^2\Big] + 2\,\mathbb{E}\Big[\Big\{\Big(\tfrac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\Big) - \Big(\tfrac{X_i^T\beta_0 - v}{h}\Big)\Big\}^2\mathcal{K}_i(\beta_0, v)^2\Big].
$$

Exploiting that $\|\beta_n^\gamma - \beta_0\| + |v_n^\kappa - v| \leq 2Mn^{-1/2}$ and that $nh^3 \to \infty$, both summands can be seen to be of order $o(h^2)$, uniformly in $\gamma \in U_M$ and $\kappa \in [-M, M]$. □

Lemma B.4. *Under the conditions of Theorem 3.1,*

$$\sup_{(\gamma,\kappa)\in U_M\times[-M,M]} \mathbb{E}\,|S_n(\Theta,\beta_n^\gamma,v_n^\kappa)| = O(h), \tag{B.9}$$

$$\sup_{(\gamma,\kappa)\in U_M\times[-M,M]} \mathrm{Var}\,\{S_n(\Theta,\beta_n^\gamma,v_n^\kappa)\} = O\{(nh)^{-1}\}, \tag{B.10}$$

where

$$S_n(\Theta,\beta_n^\gamma,v_n^\kappa) = \mathbb{E}[L_n(\Theta,\beta_n^\gamma,v_n^\kappa) - \Theta^T W_n(\beta_n^\gamma,v_n^\kappa) \mid \mathcal{A}_n(\beta_n^\gamma)] - \frac{1}{2}\Theta^T V\Theta.$$

Proof. Set

$$V_n(\beta_n^\gamma,v_n^\kappa) = \frac{1}{nh}\sum_{i=1}^n f_{Y|X^T\beta_0}\{m(v)\,|\,v\}\,\mathcal{X}_i(\beta_n^\gamma,v_n^\kappa)\{\mathcal{X}_i(\beta_n^\gamma,v_n^\kappa)\}^T\mathcal{K}_i(\beta_n^\gamma,v_n^\kappa).$$

We begin the proof by showing that (B.9) and (B.10) hold with V replaced by $V_n(\beta_n^\gamma,v_n^\kappa)$. We can write $\mathbb{E}[L_n(\Theta,\beta_n^\gamma,v_n^\kappa)\,|\,\mathcal{A}_n(\beta_n^\gamma)] = A_{n1}+A_{n2}$, where

$$\begin{aligned} A_{n1} = \sum_{i=1}^n \Bigg\{ & \mathbb{E}\left[Q_i\mathcal{Z}_i(\beta_n^\gamma,v_n^\kappa)\mathbb{1}\left\{\mathcal{Z}_i(\beta_n^\gamma,v_n^\kappa)<0\right\}\,\middle|\,X_i^T\beta_n^\gamma\right] \\ & -\mathbb{E}\Bigg[Q_i\left\{\mathcal{Z}_i(\beta_n^\gamma,v_n^\kappa) - \frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma,v_n^\kappa)}{\sqrt{nh}}\right\} \\ & \qquad \mathbb{1}\left\{\mathcal{Z}_i(\beta_n^\gamma,v_n^\kappa) < \frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma,v_n^\kappa)}{\sqrt{nh}}\right\}\,\Bigg|\,X_i^T\beta_n^\gamma\Bigg]\Bigg\}\mathcal{K}_i(\beta_n^\gamma,v_n^\kappa), \\ A_{n2} = -\frac{\tau}{\sqrt{nh}} & \sum_{i=1}^n \Theta^T\mathcal{X}_i(\beta_n^\gamma,v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma,v_n^\kappa). \end{aligned}$$

For $\beta\in\mathbb{R}^d$, let $\varphi_\beta(\,\cdot\,|\,\cdot\,):\mathbb{R}^2\to\mathbb{R}$ be defined as

$$\varphi_\beta(t\,|\,u) = \mathbb{E}\left[Q_i(Z_i-t)\mathbb{1}(Z_i-t<0)\mid X_i^T\beta=u\right] = \mathbb{E}\left[(Y_i-t)\mathbb{1}(Y_i-t<0)\mid X_i^T\beta=u\right].$$

One can easily see that, in a neighborhood of $(\beta_0,m(v),v)$, φ_β is three times differentiable with respect to t, where the first derivative is given by $\varphi'_\beta(t\,|\,u) = \frac{\partial}{\partial t}\varphi_\beta(t\,|\,u) = -F_{Y|X^T\beta}(t\,|\,u)$. Moreover, the three derivatives are bounded in that neighborhood by condition (A4). Using the definition of $\mathcal{Z}_i(\beta,u)$ and of φ_β, we can write A_{n1} as

$$\begin{aligned}\sum_{i=1}^n \Bigg[& \varphi_{\beta_n^\gamma}\left\{m(v_n^\kappa)+m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\,\middle|\,X_i^T\beta_n^\gamma\right\} \\ & -\varphi_{\beta_n^\gamma}\left\{m(v_n^\kappa)+m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa) + \frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma,v_n^\kappa)}{\sqrt{nh}}\,\middle|\,X_i^T\beta_n^\gamma\right\}\Bigg]\mathcal{K}_i(\beta_n^\gamma,v_n^\kappa).\end{aligned}$$

A Taylor expansion of $t\mapsto\varphi_{\beta_n^\gamma}(t\,|\,X_i^T\beta_n^\gamma)$ shows that each summand can be written as

$$\begin{aligned} & F_{Y|X^T\beta_n^\gamma}\left\{m(v_n^\kappa)+m'(v_n^\kappa)(X_i^T\beta_n^\gamma-v_n^\kappa)\,|\,X_i^T\beta_n^\gamma\right\}\frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma,v_n^\kappa)}{\sqrt{nh}}\mathcal{K}_i(\beta_n^\gamma,v_n^\kappa) \\ & + f_{Y|X^T\beta_n^\gamma}\left\{m(v_n^\kappa)+m'(v_n^\kappa)(X_i^T\beta_n^\gamma-v_n^\kappa)\,|\,X_i^T\beta_n^\gamma\right\}\frac{\{\Theta^T\mathcal{X}_i(\beta_n^\gamma,v_n^\kappa)\}^2}{2nh}\mathcal{K}_i(\beta_n^\gamma,v_n^\kappa) \\ & + f'_{Y|X^T\beta_n^\gamma}\left\{m_i^*\,|\,X_i^T\beta_n^\gamma\right\}\frac{\{\Theta^T\mathcal{X}_i(\beta_n^\gamma,v_n^\kappa)\}^3}{6(nh)^{3/2}}\mathcal{K}_i(\beta_n^\gamma,v_n^\kappa), \end{aligned}$$

where m_i^* denotes some intermediate point at distance of at most $|\Theta^T \mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)|/\sqrt{nh}$ of $m(v_n^\kappa)+ m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)$. Also note that (2.5) implies that

$$\begin{aligned}\mathbb{E}\left[Q_i \mathbb{1}\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < 0\} \,\middle|\, X_i^T\beta_n^\gamma\right] &= \mathbb{E}\left[\mathbb{1}\{Y_i < m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\} \,\middle|\, X_i^T\beta_n^\gamma\right] \\ &= F_{Y|X^T\beta_n^\gamma}\left\{m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\,|\,X_i^T\beta_n^\gamma\right\}, \end{aligned} \tag{B.11}$$

for any $\gamma \in U_M$, where the last equality exploits continuity of the conditional distribution of Y given $X_i^T\beta_n^\gamma$. Then, recalling the definition of W_n in (A.8) and exploiting (B.11), we obtain

$$\begin{aligned} &\mathbb{E}[L_n(\Theta, \beta_n^\gamma, v_n^\kappa) - \Theta^T W_n(\beta_n^\gamma, v_n^\kappa)\,|\,\mathcal{A}_n(\beta_n^\gamma)] \\ &= A_{n1} + A_{n2} - \mathbb{E}\left[\Theta^T W_n(\beta_n^\gamma, v_n^\kappa)\mid \mathcal{A}_n(\beta_n^\gamma)\right] \\ &= \frac{1}{2nh}\sum_{i=1}^n f_{Y|X^T\beta_n^\gamma}\left\{m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\,\middle|\, X_i^T\beta_n^\gamma\right\}\{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\}^2\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) \\ &\qquad + \frac{1}{6(nh)^{3/2}}\sum_{i=1}^n f'_{Y|X^T\beta_n^\gamma}\left\{m_i^*\,\middle|\, X_i^T\beta_n^\gamma\right\}\{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\}^3\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) =: B_{n1} + B_{n2}. \end{aligned}$$

Clearly, $\mathbb{E}|B_{n2}| \le C \times \mathbb{E}[\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)]/\sqrt{nh^3} = o(h)$ and $\mathrm{Var}(B_{n2}) \le C \times \mathbb{E}[\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)^2]/(n^2h^3)$, which is of order $O\{(nh)^{-2}\}$. Hence, it remains to consider the expected value and variance of

$$\begin{aligned} B_{n1} - \frac{1}{2}\Theta^T V_n(\beta_n^\gamma, v_n^\kappa)\Theta = \frac{1}{2nh}\sum_{i=1}^n \Big\{ & f_{Y|X^T\beta_n^\gamma}\left\{m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\,\middle|\, X_i^T\beta_n^\gamma\right\} \\ & - f_{Y|X^T\beta_0}\{m(v)\,|\,v\}\Big\}\{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\}^2\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa). \end{aligned}$$

A Taylor expansion of $t \mapsto f_{Y|X^T\beta_n^\gamma}(t\,|\,X_i^T\beta_n^\gamma)$ yields the existence of some intermediate point $\bar{m}_i$ between $m(v_n^\kappa)$ and $m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)$ such that

$$\begin{aligned} & f_{Y|X^T\beta_n^\gamma}\left\{m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\,\middle|\, X_i^T\beta_n^\gamma\right\}\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) \\ &= f_{Y|X^T\beta_n^\gamma}\left\{m(v_n^\kappa)\,|\,X_i^T\beta_n^\gamma\right\}\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) + f'_{Y|X^T\beta_n^\gamma}\left(\bar{m}_i\,|\,X_i^T\beta_n^\gamma\right)m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa). \end{aligned}$$

By boundedness of $f'_{Y|X^T\beta_n^\gamma}$ from condition (A4) and by the fact that $(X_i^T\beta_n^\gamma - v_n^\kappa)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa) = O(h)\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)$, uniformly in i, the right-hand side of the last displayed formula can be written as

$$\left[f_{Y|X^T\beta_n^\gamma}\left\{m(v_n^\kappa)\,|\,X_i^T\beta_n^\gamma\right\} + O(h)\right]\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa),$$

where the $O(h)$-term is uniformly in $i = 1, \dots, n$, $\gamma \in U_M$ and $\kappa \in [-M, M]$. Moreover, by Lipschitz-continuity of $(\beta, y, u) \mapsto f_{Y|X^T\beta}\{y \mid u\}$ at $(\beta_0, m(v), v)$ from condition (A4), we get that the last displayed formula can be written as

$$\left[f_{Y|X^T\beta_0}\left\{m(v)\,|\,v\right\} + O(h)\right]\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa),$$

uniformly in $i = 1, \dots, n$, $\gamma \in U_M$ and $\kappa \in [-M, M]$. Hence,

$$\mathbb{E}|B_{n1} - \Theta^T V_n(\beta_n^\gamma, v_n^\kappa)\Theta/2| \le C \times \mathbb{E}[\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)] = O(h)$$

and similarly,

$$\mathrm{Var}\{B_{n1} - \Theta^T V_n(\beta_n^\gamma, v_n^\kappa)\Theta/2\} \le C \times \mathbb{E}[\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)^2]/n = O(h/n).$$

This completes the proof of (B.9) and (B.10) with V replaced by $V_n(\beta_n^\gamma, v_n^\kappa)$.

To finish the proof of the Lemma, denote the coordinates of $V_n(\beta_n^\gamma, v_n^\kappa)$ and V by

$$V_n(\beta_n^\gamma, v_n^\kappa) = \begin{pmatrix} V_{n0} & V_{n1} \\ V_{n1} & V_{n2} \end{pmatrix}, \qquad V = \begin{pmatrix} V_0 & V_1 \\ V_1 & V_2 \end{pmatrix}$$

and note that, since

$$\mathbb{E}|V_{nj} - V_j| \le \mathbb{E}|V_{nj} - \mathbb{E}(V_{nj})| + |\mathbb{E}V_{nj} - V_j| \le \sqrt{\operatorname{Var}(V_{nj})} + |\mathbb{E}V_{nj} - V_j|$$

for any $j \in \{0,1,2\}$, it suffices to show that $\mathbb{E}V_{nj} = V_j + O(h)$ and $\operatorname{Var} V_{nj} = O\{(nh)^{-1}\}$. Now,

$$\begin{aligned}
\mathbb{E}V_{nj} &= \frac{f_{Y|X^T\beta_0}\{m(v)\,|\,v\}}{h}\mathbb{E}\left[\left(\frac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\right)^j K\left(\frac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\right)\right] \\
&= \frac{f_{Y|X^T\beta_0}\{m(v)\,|\,v\}}{h}\int_{\mathbb{R}}\left(\frac{u - v_n^\kappa}{h}\right)^j K\left(\frac{u - v_n^\kappa}{h}\right) f_{X^T\beta_n^\gamma}(u)\,du \\
&= f_{Y|X^T\beta_0}\{m(v)\,|\,v\}\int_{\mathbb{R}} x^j K(x)\, f_{X^T\beta_n^\gamma}(v_n^\kappa + xh)\,dx.
\end{aligned}$$

By Lipschitz-continuity of $(\beta, u) \mapsto f_{X^T\beta}(u)$ in (β_0, v) as assumed in condition (A3), we can write the right-hand side of the last displayed formula as $V_j + O(h) + O(n^{-1/2}) = V_j + O(h)$.

Regarding the variance, split the variance into a sum of variances and bound each summand by its second moment to obtain

$$\begin{aligned}
\operatorname{Var}(V_{nj}) &\le \frac{f_{Y|X^T\beta_0}\{m(v)\,|\,v\}^2}{nh^2}\mathbb{E}\left[\left(\frac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\right)^{2j} K\left(\frac{X_i^T\beta_n^\gamma - v_n^\kappa}{h}\right)^2\right] \\
&= \frac{f_{Y|X^T\beta_0}\{m(v)\,|\,v\}^2}{nh}\int_{\mathbb{R}} x^{2j}K(x)^2\, f_{X^T\beta_n^\gamma}(v_n^\kappa + xh)\,dx = O\{(nh)^{-1}\},
\end{aligned}$$

uniformly in $\gamma \in U_M$ and $\kappa \in [-M, M]$. This proves the lemma. □

Lemma B.5. *Under the conditions of Theorem 3.1, we have*

$$\sup_{(\gamma,\kappa)\in U_M\times[-M,M]} \mathbb{E}[R_n(\Theta, \beta_n^\gamma, v_n^\kappa)^2] = O\{(nh)^{-1/2}\} = o(h).$$

Proof. Obviously, $R_n(\Theta, \beta_n^\gamma, v_n^\kappa)$ is centered, whence $\mathbb{E}[R_n(\Theta, \beta_n^\gamma, v_n^\kappa)^2] = \operatorname{Var}\{R_n(\Theta, \beta_n^\gamma, v_n^\kappa)\}$. Note that we can write $R_n(\Theta, \beta_n^\gamma, v_n^\kappa) = \sum_{i=1}^n\{R_i - \mathbb{E}[R_i\,|\,X_i^T\beta_n^\gamma]\}$, where

$$\begin{aligned}
R_i = Q_i&\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) - \frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right\} \\
&\left[\mathbb{1}\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < 0\right\} - \mathbb{1}\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) - \frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}} < 0\right\}\right]\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa).
\end{aligned}$$

Then, by iterated expectation,

$$\begin{aligned}
\operatorname{Var}\{R_n(\Theta, \beta_n^\gamma, v_n^\kappa)\} &= n\operatorname{Var}(R_i - \mathbb{E}[R_i\,|\,X_i]) \\
&\le n\mathbb{E}\left[\{R_i - \mathbb{E}[R_i\,|\,X_i]\}^2\right] = n\left\{\mathbb{E}R_i^2 - \mathbb{E}\left[\mathbb{E}[R_i\,|\,X_i]^2\right]\right\} \le n\mathbb{E}R_i^2.
\end{aligned}$$

Using (B.1), we can bound $n\mathbb{E}R_i^2$ from above by

$$
\begin{aligned}
& n\mathbb{E}\left[Q_i^2\left\{\mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) - \frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right\}^2\right. \\
& \qquad\qquad \left.\times\mathbb{1}\left\{-\left|\frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right| \le \mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < \left|\frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right|\right\}\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)^2\right] \\
& \le \|\Theta\|^2\frac{4}{h}\mathbb{E}\left[\mathbb{E}\left[Q_i^2\mathbb{1}\left\{-\left|\frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right| \le \mathcal{Z}_i(\beta_n^\gamma, v_n^\kappa) < \left|\frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right|\right\} \,\Big|\, X_i\right]\right. \\
& \qquad\qquad\qquad\qquad \left.\times\|\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\|^2\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)^2\right] \\
& = \|\Theta\|^2\frac{4}{h}\mathbb{E}\left[\left(J\left\{m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa) + \left|\frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right| \,\Big|\, X_i\right\}\right.\right. \\
& \qquad \left.\left. - J\left\{m(v_n^\kappa) + m'(v_n^\kappa)(X_i^T\beta_n^\gamma - v_n^\kappa) - \left|\frac{\Theta^T\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)}{\sqrt{nh}}\right| \,\Big|\, X_i\right\}\right)\|\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\|^2\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)^2\right],
\end{aligned}
\tag{B.12}
$$

where $J(\cdot\,|\,\cdot)$ is defined in (B.7). By the mean value theorem, there exists some ε_i^* between the arguments of $J(\cdot\,|\,X_i)$ in (B.12) such that its right-hand side equals

$$
\|\Theta\|^2\frac{8}{\sqrt{nh^3}}\mathbb{E}\left[J'(\varepsilon_i^*\,|\,X_i)\|\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\|^3\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)^2\right].
$$

As $J'(\cdot\,|\,\cdot)$ is bounded as explained after its definition in (B.7) and as $\mathbb{E}[\|\mathcal{X}_i(\beta_n^\gamma, v_n^\kappa)\|^3\mathcal{K}_i(\beta_n^\gamma, v_n^\kappa)^2] = O(h)$, we can finally conclude that $\mathrm{Var}\{R_n(\Theta, \beta_n^\gamma, v_n^\kappa)\}$ is of order $O\{(nh)^{-1/2}\} = o(h)$, uniformly in $\gamma \in U_M$ and $\kappa \in [-M, M]$. ☐

Lemma B.6. *Under the conditions of Theorem 3.1, we have*

$$
W_n(\beta_0, v) = -U_n + o_P(\sqrt{nh^5}),
$$

where U_n is defined in (A.5).

Proof. It suffices to consider the expected value and the variance of $W_n(\beta_0, v) + U_n$. Regarding the expected value, note that the sum in the definition of U_n has mean zero. Write $W_n(\beta_0, v) = (W_{n1}, W_{n2})^T$ and let us first consider its first coordinate. By iterated expectation and (B.11), we get that $\mathbb{E}[W_{n1}]$ is equal to

$$
\begin{aligned}
& \sqrt{\frac{n}{h}} \times \mathbb{E}\left[\left[Q_i\mathbb{1}\{\mathcal{Z}_i(\beta_0, v) < 0\} - \tau\right]K\{(X_i^T\beta_0 - v)/h\}\right] \\
& = \sqrt{\frac{n}{h}} \times \mathbb{E}\left[\left[F_{Y|X^T\beta_0}\left\{m(v) + m'(v)(X_i^T\beta_0 - v)\,|\,X_i^T\beta_0\right\} - \tau\right]K\{(X_i^T\beta_0 - v)/h\}\right] \\
& = \sqrt{\frac{n}{h}}\int_{\mathbb{R}}\left[F_{Y|X^T\beta_0}\left\{m(v) + m'(v)(u - v)\,|\,u\right\} - \tau\right]K\left(\frac{u - v}{h}\right)f_{X^T\beta_0}(u)\,du \\
& = \sqrt{nh}\int_{\mathbb{R}}\left[F_{Y|X^T\beta_0}\left\{m(v) + m'(v)hw\,|\,v + hw\right\} - \tau\right]K(w)\,f_{X^T\beta_0}(v + hw)\,dw \\
& = \sqrt{nh}\int_{\mathbb{R}}\left[F_{Y|X^T\beta_0}\left\{m(v + hw) - m''(\tilde{v})h^2w^2/2\,|\,v + hw\right\} - \tau\right]K(w)\,f_{X^T\beta_0}(v + hw)\,dw,
\end{aligned}
$$

where $\tilde{v} \in \mathbb{R}$ denotes some intermediate point lying strictly between v and $v + hw$. Now, note that, for any $x \in \mathbb{R}$,

$$
\tau = F_{Y|X^T\beta_0}\{Q_\tau(x)\,|\,x^T\beta_0\} = F_{Y|X^T\beta_0}\left\{m(x^T\beta_0)\,|\,x^T\beta_0\right\}
$$

as a consequence of continuity of the mapping $y \mapsto F_{Y|X^T\beta_0}(y \,|\, x^T\beta_0)$. In particular, we have $\tau = F_{Y|X^T\beta_0}\{m(v+hw)\,|\,v+hw\}$, whence the mean value theorem applied to the function $y \mapsto F_{Y|X^T\beta_0}\{y\,|\,v+hw\}$ implies that $\mathbb{E}[W_{n1}]$ is equal to

$$-\frac{\sqrt{nh^5}}{2}\int_{\mathbb{R}} f_{Y|X^T\beta_0}(\tilde{m}\,|\,v+hw)m''(\tilde{v})w^2K(w)\,f_{X^T\beta_0}(v+hw)\,dw,$$

where $\tilde{m} \in \mathbb{R}$ denotes a second intermediate point lying between $m(v+hw)$ and $m(v+hw) - m''(\tilde{v})h^2w^2/2$. Four further applications of the mean value theorem show that the last display can be written as

$$-\frac{\sqrt{nh^5}}{2} f_{Y|X^T\beta_0}\{m(v)\,|\,v\}\,m''(v)f_{X^T\beta_0}(v)\bar{K}_2 + o(\sqrt{nh^5}).$$

Completely analogous calculations for the second coordinate show that

$$\mathbb{E}[W_{n2}] = -\frac{\sqrt{nh^5}}{2} f_{Y|X^T\beta_0}\{m(v)\,|\,v\}\,m''(v)f_{X^T\beta_0}(v)\bar{K}_3 + o(\sqrt{nh^5}),$$

which implies that $\mathbb{E}[W_n(\beta_0,v) + U_n] = o(\sqrt{nh^5})$.

Now, consider the variance. To prove the lemma, it is sufficient to show that

$$\frac{1}{h}\operatorname{Var}\Big(Q_i\big[\mathbb{1}\{Z_i < m(v) + m'(v)(X_i^T\beta_0 - v)\} - \mathbb{1}\left\{Z_i < m(X_i^T\beta_0)\right\}\big]\mathcal{X}_i(\beta_0,v)\mathcal{K}_i(\beta_0,v)\Big)$$

is of order $O(h^2) = o(nh^5)$, as $n \to \infty$. Each entry of this matrix can be bounded by

$$\frac{1}{h}\mathbb{E}\Big[\mathbb{E}\left[Q_i^2\big|\mathbb{1}\{Z_i < m(v) + m'(v)(X_i^T\beta_0 - v)\} - \mathbb{1}\left\{Z_i < m(X_i^T\beta_0)\right\}\big|\,\big|\,X_i\right] \\ \times \left(\tfrac{X_i^T\beta_0 - v}{h}\right)^j K^2\left(\tfrac{X_i^T\beta_0 - v}{h}\right)\Big]$$

for some $j \in \{0,1,2\}$. Proceeding analogously as in the proof of Lemma B.3, it is sufficient to show that the conditional expectation in the last display is of order $O(h^2)$, uniformly. Again similarly as in the proof of Lemma B.3, it can be bounded by

$$\big|J\{m(v) + m'(v)(X_i^T\beta_0 - v)\,|\,X_i\} - J\{m(X_i^T\beta_0)\,|\,X_i\}\big| \\ \le \text{const}\times|m(v) + m'(v)(X_i^T\beta_0 - v) - m(X_i^T\beta_0)| \le \text{const}\times(X_i^T\beta_0 - v)^2,$$

where the estimations are based on Taylor expansions of $J(\cdot\,|\,X_i)$ and m, respectively. The assertion of the lemma follows upon noting that the support of X is bounded. □

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