Non-crossing marginal effects in additive quantile regression

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August 22, 2007

Abstract

We consider the nonparametric estimation problem of conditional regression quantiles with high-dimensional covariates. For the additive quantile regression mode, we propose a new procedure such that the estimated marginal effects of additive conditional quantile curves do not cross. The method is based on a combination of the marginal integration technique and non-increasing rearrangements, which were recently introduced in the context of estimating a monotone regression function. Asymptotic normality of the estimates is established with a one-dimensional rate of convergence and the finite sample properties are studied by means of a simulation study and a data example.

Keywords and Phrases: conditional quantiles, additive models, marginal integration, non-increasing rearrangements

1 Introduction

Regression techniques are widely used to quantify the relation between a response and a predictor. While ordinary least squares regression refers to the conditional mean, quantile regression was introduced by Koenker and Bassett (1978) to obtain a more sophisticated picture of the relation between the response and covariates. Since the seminal paper of these authors numerous scientists have worked on methodological and practical aspects of this method and the interested reader is refered to the recent monograph of Koenker (2005). Nonparametric methods for estimating conditional quantiles have lately found considerable interest in the literature [see e.g. Keming and Jones (1997), or Yu and Jones (1998)]. These authors concentrate on a univariate predictor and it is well known that for high-dimensional covariates nonparametric methods suffer from the curse of dimensionality, which does not allow precise estimation of conditional quantiles with reasonable sample sizes. For this reason several authors have recommended to use additive quantile models of the form

$$Q(\alpha|\mathbf{x}) = \sum_{k=1}^{d} Q_k(\alpha|x_k) + c(\alpha) \tag{1.1}$$

where $\alpha \in (0,1)$ [see Doksum and Koo (2000), Gooijer and Zerom (2003) and Horowitz and Lee (2005), among others]. In equation (1.1) the quantity $c(\alpha)$ denotes a constant, $\mathbf{x} = (x_1, \ldots, x_d)^T$ are the predictors, and $Q_k(\alpha|x_k)$ are functions relating the α-quantile of the conditional distribution functions to each coordinate of the predictor [note that these have to be normalized in order to make the model (1.1) identifiable – see the discussion in Section 2]. So far several authors have proposed methods for estimating the additive components in (1.1). Doksum and Koo (2000) suggest a spline estimate but do not provide rates of convergence of their estimator. De Gooijer and Zerom (2003) use a marginal integration estimate, while Horowitz and Lee (2005) propose a two step procedure, which fits a parametric model in the first step (with increasing dimension) for each coordinate and smooths in a second step by the local polynomial technique [see Fan and Gijbels (1996)]. In contrast to the estimate of De Gooijer and Zerom (2003), this method does not suffer from the curse of dimensionality [see below].

In the present paper we propose an alternative estimate of conditional quantiles in the additive model (1.1). Our investigations are motivated by the observation that in the one-dimensional case many nonparametric estimates of quantile curves are not monotone with respect to $\alpha \in (0,1)$ [see e.g. He (1997), Yu, Lu, and Stander (2003), or Koenker (2005), Chap. 7]. As a consequence quantile curves for different values of α may cross, which is an embarassing phenomenon in applications. In the context of estimating a conditional quantile curve in the additive model (1.1), the situation is similar, but the focus lies on the marginal effects of the conditional quantile function. The marginal effect of the conditional quantile function is the additive component $Q_k(\alpha|x_k)$ with respect to a certain covariate x_k plus the constant term $c(\alpha)$, which is denoted by $q_k(\alpha|x_k)$. Because Horowitz and Lee (2005) use a parametric fit based on the check function the resulting conditional quantile estimator is not necessarily monotone with respect to α, and a similar comment applies to the method proposed by Doksum and Koo (2000). On the other hand the procedure proposed by De Gooijer and Zerom (2003) yields non-crossing marginal effects of conditional quantile surfaces. However, this is only correct if the dimension of the covariates satisfies $d < 5$. When $d \geq 5$ the bias of the estimate has to be reduced by using negative kernels, which unfortunately destroys the monotonicity property [see Remark 3.3. in this paper].

As an alternative estimate of the additive conditional quantile model, we present an approach which is based on a combination of the marginal integration technique [see Linton and Nielsen (1995)] with the concept of non-increasing rearrangements [see Benett and Sharpley (1988)]. This methodology has been successfully applied by Dette and Volgushev (2007) and Chernozhukov, Fernandez-Val and Galichon (2007). The last named authors use the concept of non-increasing rearrangements to isotonize parametric (possible crossing) quantile estimates and study the weak convergence of the resulting statistics. Dette and Volgushev (2007) concentrate on the case of a one-dimensional covariate and isotonize and invert a nonparametric estimate of the conditional distribution function simultaneously in order to obtain nonparametric non-crossing estimates of quantile curves. The rest of the article is organized as follows. In Section 2, we describe the main concept of the method for estimating the marginal effects in an additive quantile regression model, which is based on a combination of these methods with the marginal integration technique. Our approach is applicable to any parametric or nonparametric estimate of the conditional distribution function. In Section 3, we state the asymptotic distributional properties of new statistic, if the the conditional distribution function is estimated by local constant or local linear techniques. The internalized marginal integration estimator is also investigated [see Hengartner and Sperlich (2005)], since in higher dimensional problems this estimator is interesting from a computational point of view. In Section 4 we present a small simulation study to compare the finite sample properties of the new method with the procedure introduced by De Gooijner and Zerom (2003), which is most similar in spirit with the estimate suggested in this paper. Finally, some of the technical details of the proofs of the asymptotic results are presented in Section 5.

2 Monotone rearrangements and marginal integration

Let $(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n)$ denote a sample of independent and identically distributed observations, where the d-dimensional random variable $\mathbf{X}_j = (X_{j1}, \ldots, X_{jd})^T$ has a q times continuously differentiable density, say p, with compact support $[0,1]^d$. Following Hall, Rodney, Wolff and Yao (1999), we introduce the random variable $Z_j = I\{Y_j \leq y\}$ with

$$E[Z_j|\mathbf{X}_j = \mathbf{x}] = P(Y_j \leq y|\mathbf{X}_j = \mathbf{x}) = F(y|\mathbf{x})$$

and the nonparametric regression model

$$Z_j = F(y|\mathbf{X}_j) + \sigma(y|\mathbf{X}_j)\varepsilon_j \quad j = 1, \ldots, n \tag{2.1}$$

where $E[\varepsilon_j] = 0$, $\mathrm{Var}(\varepsilon_j) = 1$, and $E[\varepsilon_j^4] \leq c < \infty$. The variance function $\sigma(y|\mathbf{x})$ can be further specified in terms of $F(y|\mathbf{x})$, i.e.

$$\sigma^2(y|\mathbf{x}) = E[(Z_j - F(y|\mathbf{x}))^2|\mathbf{X}_j = \mathbf{x}] = F(y|\mathbf{x})(1 - F(y|\mathbf{x})).$$

We consider the model (1.1) and add the conditions

$$E[Q_k(\alpha|X_{jk})] = 0, \quad k = 1, \ldots, d, \quad (j = 1, \ldots, n) \tag{2.2}$$

in order to make the components of the additive decomposition (1.1) identifiable. Let $\hat{F}(y|\mathbf{x})$ denote an estimate of the conditional distribution function $F(y|\mathbf{x}) = P(Y_j \leq y \mid \mathbf{X}_j = \mathbf{x})$, which will be specified below. Define $H : \mathbb{R} \to [0,1]$ as a strictly increasing distribution function, which will be used as a transformation to the compact interval $[0,1]$, since $F(\cdot|\mathbf{x})$ might have unbounded support. Note that $\hat{F}(y|\mathbf{x})$ is obtained by nonparametric methods and for this reason usually not increasing which yields some difficulties in the determination of the corresponding quantiles. In the following, we solve this problem and the problem of inversion simultaneously using the concept of monotone rearrangements [see Benett and Sharpley (1988)], which was introduced in the context of estimating a monotone regression function by Dette, Neumeyer and Pilz (2006). To be precise, let K_d denote a positive kernel function with compact support $[-1,1]$ and h_d denote a bandwidth, then we define

$$\hat{H}_I(\alpha|\mathbf{x}) = \frac{1}{Nh_d}\sum_{i=1}^{N}\int_{-\infty}^{\alpha} K_d\left(\frac{\hat{F}(H^{-1}(\frac{i}{N})|\mathbf{x})-u}{h_d}\right)du. \tag{2.3}$$

If $\hat{F}(y|\mathbf{x})$ is uniformly consistent and $N\to\infty$, $h_d\to 0$, it is intuitively clear that

$$\begin{aligned}\hat{H}_I(\alpha|\mathbf{x}) \;&\approx\; H_N(\alpha|\mathbf{x}) := \frac{1}{Nh_d}\sum_{i=1}^{N}\int_{-\infty}^{\alpha} K_d\left(\frac{F(H^{-1}(\frac{i}{N})|\mathbf{x})-u}{h_d}\right)du \\ &\approx\; \int I\{F(H^{-1}(s)|\mathbf{x})\le\alpha\}ds = H(Q(\alpha|\mathbf{x})),\end{aligned} \tag{2.4}$$

where $Q(\alpha|\mathbf{x}) = F^{-1}(\alpha|\mathbf{x})$. Consequently, we define

$$\hat{Q}_I(\alpha|\mathbf{x}) = H^{-1}(\hat{H}_I(\alpha|\mathbf{x})) \tag{2.5}$$

as the estimate of the conditional quantile $Q(y|\mathbf{x})$, and

$$Q_N(\alpha|\mathbf{x}) = H^{-1}(H_N(\alpha|\mathbf{x})) \tag{2.6}$$

as an approximation of the conditional quantile $Q(y|\mathbf{x})$. It will be demonstrated in the following section that the choice of the function H has no impact on the asymptotic properties of the estimate. Moreover, even for realistic sample sizes the impact of the choice of H is negligible and a practical recommendation regarding this choice will be given in Section 4.

Note that the estimate $\hat{H}_I$ is monotone with respect to α provided that the kernel K_d is positive on its support, which will be assumed throughout this paper. In the next step, we now apply the marginal integration technique [see Linton and Nielsen (1995), Chen, Härdle, Linton, Severence-Lossin (1996), or Hengartner and Sperlich (2005)] to obtain an estimator in the model (1.1). Without loss of generality, we focus on the problem of estimating the first component $Q_1(\alpha|x_1)$ in model (1.1) and the marginal effect of the first covariate, respectively. We introduce the following notations to be precise $\mathbf{X}_j = (X_{j1},\ldots,X_{jd})^T$, $X_{j\underline{1}} = (X_{j2},\ldots,X_{jd})^T$, and $\mathbf{x} = (x_1, x_{\underline{1}})^T$. Now we define the marginal integration estimator of the first marginal effect

$$\hat{q}_1(\alpha|x_1) = \frac{1}{n}\sum_{j=1}^{n}\hat{Q}_I(\alpha|x_1, X_{j\underline{1}}), \tag{2.7}$$

which can be regarded as the expection of $\hat{Q}_I(\alpha|\mathbf{X})$ with respect to the empirical distribution of $X_{\underline{1}} = (X_2,\ldots,X_d)^T$. This estimator is obviously monotone in α for fixed x_1. Note that by the strong law of large numbers and from the normalizing condition (2.2), we have

$$\frac{1}{n}\sum_{k=1}^{n} Q_N(\alpha|x_1, X_{k\underline{1}}) \xrightarrow{\text{a.s.}} \int Q(\alpha|\mathbf{x})p_{\underline{1}}(x_{\underline{1}})dx_{\underline{1}} = Q_1(\alpha|x_1) + c(\alpha) =: q_1(\alpha|x_1), \tag{2.8}$$

where $p_{\underline{1}}$ denotes the marginal density of $X_{\underline{1}} = (X_2, \ldots, X_d)^T$. Consequently, if $\hat{Q}_I(\alpha|\mathbf{x})$ is a (uniformly) consistent estimate of $Q(\alpha|\mathbf{x})$ it follows that $\hat{q}_1(\alpha|x_1)$ is a consistent estimate of $q_1(\alpha|x_1) := Q_1(\alpha|x_1) + c(\alpha)$. Finally,

$$\hat{q}_1(\alpha|x_1) - \frac{1}{n}\sum_{i=1}^{n} \hat{q}_1(\alpha|X_{i1}) \tag{2.9}$$

defines a consistent estimate of $Q_1(\alpha|x_1)$ (note that $\frac{1}{n}\sum_{i=1}^{n} Q_1(\alpha|X_{i1}) \overset{\text{a.s.}}{\to} E[Q_1(\alpha|X_{i1})] = 0$). The estimates of the other components are defined exactly in the same way, and the final estimate in the additive model (1.1) is given by

$$\hat{Q}_{\text{add}}(\alpha|\mathbf{x}) := \sum_{k=1}^{d} \hat{q}_k(\alpha|x_k) - \left(1 - \frac{1}{d}\right) \sum_{k=1}^{d} \frac{1}{n} \sum_{i=1}^{n} \hat{q}_k(\alpha|X_{ik}). \tag{2.10}$$

In the following section, we study the asymptotic properties of the estimate $\hat{q}_1(\alpha|x_1)$. The corresponding properties of the estimate $\hat{q}_k(\alpha|x_k)$ for $k = 2, \ldots, d$ follow in a straightforward manner.

3 Asymptotic properties

A precise statement of the main results requires the specification of an initial estimate of the conditional distribution function $F(y|\mathbf{x})$. For the sake of definiteness, we first consider a Nadaraya-Watson type estimator

$$\hat{F}(y|\mathbf{x}) = \hat{F}(y|x_1, x_{\underline{1}}) = \frac{\sum_{i=1}^{n} K_{h_1}(x_1 - X_{i1}) L_G(x_{\underline{1}} - X_{i\underline{1}}) I\{Y_i \le y\}}{\sum_{i=1}^{n} K_{h_1}(x_1 - X_{i1}) L_G(x_{\underline{1}} - X_{i\underline{1}})}. \tag{3.1}$$

The kernel K in (3.1) is a one-dimensional kernel with compact support, say $[-1, 1]$, and existing second moments satisfying

$$\int_{-1}^{1} xK(x)dx = 0, \ \frac{1}{2}\int_{-1}^{1} x^2 K(x)dx = \kappa_2(K). \tag{3.2}$$

Let $\nu_{\underline{1}} = (\nu_2, \ldots, \nu_d)$ be a multiindex of integers with $\nu_i \ge 0$, so that $x_{\underline{1}}^{\nu_{\underline{1}}} = x_2^{\nu_2} \ldots x_d^{\nu_d}$. Moreover, define $|\nu_{\underline{1}}| = \sum_{i=2}^{d} \nu_i$. The kernel L in (3.1) refers to a $(d-1)$-dimensional kernel of order q supported on $[-1, 1]^{d-1}$, i.e., L satisfies the conditions

(i) L is symmetric, i.e. $L(-x_{\underline{1}}) = L(x_{\underline{1}})$ for $x_{\underline{1}} \in [-1, 1]^{d-1}$,

(ii) $\int_{[-1,1]^{d-1}} L(x_{\underline{1}})dx_{\underline{1}} = 1$,

(iii) $\int_{[-1,1]^{d-1}} |x_{\underline{1}}^{\nu_{\underline{1}}}||L(x_{\underline{1}})|dx_{\underline{1}} < \infty$ for $|\nu_{\underline{1}}| \le q$,

(iv) $\int_{[-1,1]^{d-1}} x_{\underline{1}}^{\nu_{\underline{1}}} L(x_{\underline{1}})dx_{\underline{1}} = 0$ for $1 \le |\nu_{\underline{1}}| \le q - 1$,

(v) $\int_{[-1,1]^{d-1}} x_{\underline{1}}^{\nu_{\underline{1}}} L(x_{\underline{1}}) dx_{\underline{1}} \neq 0$ for some $|\nu_{\underline{1}}| = q$.

Note that the kernel L is for $q > 2$ not a probability density function anymore. The bandwidth h_1 corresponds to the first covariate. Denote $K_{h_1}(\cdot) = \frac{1}{h_1} K(\cdot/h_1)$ and

$$L_G(\mathbf{x}) = \frac{1}{\det(G)} L(G^{-1}\mathbf{x})$$

for the bandwidth matrix $G = \operatorname{diag}(g_2, \ldots, g_d) \in \mathbb{R}^{(d-1)\times(d-1)}$, where g_k refers to the bandwidth of the kth coordinate $(k = 2, \ldots, d)$.
Throughout this paper we make the following basic assumptions regarding the underlying model

(3.3) $\quad \mathbf{X}_j$ has a positive density p with supp $(p) = [0,1]^d, \; p \in C^q([0,1]^d)$,

(3.4) $\quad$ for any $y \in \mathbb{R} \quad F(y|\cdot) \in C^q([0,1]^d)$,

(3.5) $\quad F(\cdot|\mathbf{x}) \in C^1([0,1])$ and $Q'(\alpha|\mathbf{x}) > 0$,

(3.6) $\quad K_d'$ is Lipschitz continuous.

In (3.5) the function Q' denotes the derivative of the quantile function $Q(\alpha|\mathbf{x})$ with respect to the variable α (and its existence in a neighbourhood of the quantile of interest is assumed throughout this paper), while the partial derivatives with respect to the coordinates of the predictor $\mathbf{x} = (x_1, \ldots, x_d)^T$ are denoted by $\partial^s/\partial^s x_k$ $(s = 1, \ldots, q; k = 1, \ldots, d)$. Assumption (3.6) refers to the kernel used for the monotonizing inversion in (2.3).
In the following discussion we will investigate the asymptotic properties of the estimate $\hat{q}_1(\alpha|x_1)$ defined in (2.7). We focus on the marginal effect of the first component, but corresponding results for the other marginal effects can easily derived in the same way. For the sake of simplicity, we assume the same bandwidth for the remaining coordinates $x_{\underline{1}} = (x_2, \ldots, x_d)$, that is

$$g_2 = \ldots = g_d. \tag{3.7}$$

Regarding the bandwidths h_1, g_2 and h_d, we make the following assumptions

$$N = O(n) \tag{3.8}$$

$$nh_1 \to \infty, \;\; ng_2^{d-1} \to \infty, \;\; nh_1 g_2^{d-1} \to \infty, \;\; nh_d \to \infty \tag{3.9}$$

$$nh_1^5 = O(1) \tag{3.10}$$

$$ng_2^{2q+1} = O(1) \tag{3.11}$$

$$\frac{h_d}{h_1} = o(1) \tag{3.12}$$

$$\frac{1}{nh_1 g_2^{2(d-1)} h_d^2} = o(1) \tag{3.13}$$

Our first result specifies the asymptotic properties of the estimate $\hat{q}_1(\alpha|x_1)$ defined in (2.7) if the Nadaraya-Watson estimator is used for estimating the conditional distribution function. The case of the local linear estimate is briefly discussed in Remark 3.1. For a precise statement of the asymptotic properties, we recall the notation $x = (x_1, x_{\underline{1}})^T$ and obtain the following result.

Theorem 3.1. *If the assumptions (3.4) – (3.6) and (3.7) - (3.13) are satisfied, then we have for any* $\alpha \in (0,1)$

$$\sqrt{nh_1}(\hat{q}_1(\alpha|x_1) - q_1(\alpha|x_1) + b_1(\alpha|x_1)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, s^2(\alpha|x_1)),$$

where

$$\begin{aligned}
b_1(\alpha|x_1) &= \kappa_2(K)h_1^2 \int \Bigg[\frac{\partial^2}{\partial x_1^2} F(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}}) \\
&\quad + 2\frac{\frac{\partial}{\partial x_1} F(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}}) \frac{\partial}{\partial x_1} p(x_1, x_{\underline{1}})}{p(x_1, x_{\underline{1}})}\Bigg] \frac{1}{F'(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}})} p_{\underline{1}}(x_{\underline{1}}) dx_{\underline{1}}, \\
s^2(\alpha|x_1) &= \int K^2(v)dv \int \frac{\alpha(1-\alpha)p_{\underline{1}}^2(x_{\underline{1}})}{(F'(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}}))^2 p(x_1, x_{\underline{1}})} dx_{\underline{1}}, \\
\kappa_s(K) &= \frac{1}{s!} \int v^s K(v) dv.
\end{aligned}$$

Remark 3.1. There are numerous alternative estimates for the conditional distribution function which could be used as initial estimate. For example, if the conditional distribution function is estimated by a local linear techniques [see Masry and Fan (1997)], then asymptotic normality of the resulting estimate is still true but the bias term $b_1(\alpha|x_1)$ in Theorem 3.1 has to be replaced by

$$b_1(\alpha|x_1) = \kappa_2(K)h_1^2 \int \frac{\frac{\partial^2}{\partial x_1^2} F(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}})}{F'(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}})} p_{\underline{1}}(x_{\underline{1}}) dx_{\underline{1}}.$$

The local linear estimator for the conditional distribution function is not necessarily monotone increasing. Using our method, this does not pose a problem for the estimation of the marginal effects of the conditional quantile $q_1(\alpha|x_1)$, since the monotonizing inversion takes care for the monotonicity of the conditional quantile function with respect to α. The estimate for the marginal effect can be calculated as in the case of the Nadaraya-Watson estimator for the conditional distribution function.

Our second result of this section specifies the asymptotic properties of the estimate for $q_1(\alpha|x_1)$, if the internalized Nadaraya-Watson estimate is used for the estimation of the conditional distribution function, that is

$$\tilde{F}(y|\mathbf{x}) = \sum_{i=1}^{n} \frac{K_{h_1}(x_1 - X_{i1}) L_G(x_{\underline{1}} - X_{i\underline{1}}) I\{Y_i \le y\}}{\sum_{k=1}^{n} K_{h_1}(X_{k1} - X_{i1}) L_G(X_{k\underline{1}} - X_{i\underline{1}})} \tag{3.14}$$

[see Jones, Davies, and Park (1994) or Kim, Linton and Hengartner (1999)]. The internalized estimate is interesting from a computational point of view, since it can be regarded as a weighted sum

$$\tilde{F}(y|\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} K_{h_1}(x_1 - X_{i1}) L_G(x_{\underline{1}} - X_{i\underline{1}})\tilde{Y}_i$$

over the adjusted data $\tilde{Y}_i = I\{Y_i \le y\}/\hat{p}(X_{i1}, X_{i\underline{1}})$, where

$$\hat{p}(x_1, x_{\underline{1}}) = \frac{1}{n}\sum_{i=1}^{n} K_{h_1}(x_1 - X_{i1}) L_G(x_{\underline{1}} - X_{i\underline{1}}).$$

The corresponding estimate of $q_1(\alpha|x_1)$ is defined by

$$\tilde{q}_1(\alpha|x_1) = \frac{1}{n}\sum_{j=1}^{n} \tilde{Q}_I(\alpha|x_1, X_{j\underline{1}}), \tag{3.15}$$

where

$$\tilde{H}_I(\alpha|x_1, X_{j\underline{1}}) = \frac{1}{Nh_d}\sum_{i=1}^{N}\int_{-\infty}^{\alpha} K_d\left(\frac{\tilde{F}(H^{-1}(\frac{i}{N})|x_1, X_{j\underline{1}}) - u}{h_d}\right) du$$

and $\tilde{Q}_I(\alpha|x_1, X_{j\underline{1}}) = H^{-1}(\tilde{H}_I(\alpha|x_1, X_{j\underline{1}}))$ as in (2.3) and (2.5). The following result specifies the asymptotic behaviour of the estimate $\tilde{q}_1(\alpha|x_1)$.

Theorem 3.2. *If the assumptions of Theorem 3.1 are satisfied, then we have for any* $\alpha \in (0,1)$

$$\sqrt{nh_1}(\tilde{q}_1(\alpha|x_1) - q_1(\alpha|x_1) + \tilde{b}_1(\alpha|x_1) + \tilde{b}_2(\alpha|x_1)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{s}^2(\alpha|x_1)),$$

where

$$\begin{aligned}
\tilde{b}_1(\alpha|x_1) &= \kappa_2(K)h_1^2 \int \frac{\frac{\partial^2}{\partial x_1^2} F(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}})}{F'(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}})} p_{\underline{1}}(x_{\underline{1}}) dx_{\underline{1}}, \\
\tilde{b}_2(\alpha|x_1) &= \kappa_2(K)h_1^2 \int \frac{F(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}}) \frac{\partial^2}{\partial x_1^2} p(x_1, x_{\underline{1}})}{F'(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}}) p(x_1, x_{\underline{1}})} p_{\underline{1}}(x_{\underline{1}}) dx_{\underline{1}}, \\
\tilde{s}^2(\alpha|x_1) &= \int K^2(w)dw \int \frac{\left((F(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}}))^2 + \alpha(1-\alpha)\right) p_{\underline{1}}^2(x_{\underline{1}})}{p(x_1, x_{\underline{1}})(F'(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}}))^2} dx_{\underline{1}}.
\end{aligned}$$

To work with the internalized estimator as initial estimate yields an additional term to bias and variance, and is therefore less efficient. For a further discussion about the properties of external and internal estimation methods see Jones, Davies, and Park (1994). It is possible to get rid of the additional bias term by using a different bandwidth for the kernel density estimator in the denominator of (3.14). If the bandwidth of the kernel density estimator converges somewhat slower to 0 but still sufficiently fast, the term $\tilde{b}_2(\alpha|x_1)$ vanishes. See Mack and Müller (1989) for more details.

Remark 3.2. The asymptotic properties of the additive quantile function

$$\hat{Q}_{\text{add}}(\alpha|\mathbf{x}) = \sum_{k=1}^{d} \hat{q}_k(\alpha|x_k) - \left(1 - \frac{1}{d}\right) \sum_{k=1}^{d} \frac{1}{n} \sum_{j=1}^{n} \hat{q}_k(\alpha|X_{jk})$$

can be derived as well. The asymptotic bias of $\hat{Q}_{\text{add}}(\alpha|\mathbf{x})$ is

$$\sum_{k=1}^{d} b_k(\alpha|x_k) - \left(1 - \frac{1}{d}\right) \int b_k(\alpha|x_k) p_k(x_k) dx_k,$$

where $b_k(\alpha|x_k)$ is the bias of $\hat{q}_k(\alpha|x_k)$. But the asymptotic variance is just the sum of the variances of $\hat{q}_k(\alpha|x_k)$, since the terms in $Q_{\text{add}}(\alpha|\mathbf{x})$ are asymptotically uncorrelated.

Remark 3.3. At the end of this section, some remarks about the conditions regarding the bandwidths might be appropriate. A general drawback of the marginal integration method is that for higher dimensions $d \geq 5$ the third condition in (3.9), namely $nh_1 g_2^{d-1} \to \infty$, is not fulfilled using the bandwidth g_2 with the rate $n^{-1/5}$. In this case, the bias-term in the directions not of interest dominate the asymptotic properties of the estimate. A way out of this problem is to take L to be a higher order kernel. For our estimator $\hat{q}_1(\alpha|x_1)$, we have to deal with an additional bandwidth which yield that even for dimension $d = 2$ higher order kernel must be used or one has to weaken condition (3.12) and accept an extra bias-term. However, in the stated Theorems we over-smooth the variables not of interest by taking g_2 at the rate $n^{\frac{1}{2q+1}}$ for $q > 2$. This method still demonstrates simplicity and flexibility in its usage which is illustrated in more details in the following section. Furthermore, even for higher dimensions the estimator $\hat{q}_1(\alpha|x_1)$ of the marginal effects of x_1 is monotone in α.

Remark 3.4. Note that we can relax the assumption of independent data. In a more general setup, we assume that the process $\{(\mathbf{X}_j, Y_j)\}_{-\infty}^{\infty}$ is α-mixing or strongly mixing, that is

$$\sup_{A \in \mathcal{F}_{-\infty}^{0}, B \in \mathcal{F}_{k}^{\infty}} |P(AB) - P(A)P(B)| = \alpha(k) \to 0 \quad \text{as} k \to \infty \ ,$$

where $\mathcal{F}_a^b$ denotes the σ-algebra generated by the random variables $\{(\mathbf{X}_j, Y_j), a \leq j \leq b\}$ [Rosenblatt (1956)]. To retain the assertion of Theorem 3.1 for dependent data, we assume that the mixing coefficients $\alpha(k)$ fulfill

$$\sum_{j=1}^{\infty} j^a (\alpha(j))^{1/2} < \infty$$

for $a > \frac{1}{2}$ and that there exists a sequence $\{v_n\}$ of positive integers satisfying $v_n \to \infty$ and $v_n = o\left(\sqrt{nh_1^d}\right)$ such that

$$\sqrt{\frac{n}{h_1^d}} \alpha(v_n) \to 0, \quad \text{as } n \to \infty.$$

With these two additional assumptions, the assertion of Theorem 3.1 and 3.2 remain valid. For more details and a discussion of strongly mixing data in multivariate nonparametric regression settings see Masry (1996).

4 Finite sample properties and data analysis

In this section, we compare our estimator $\hat{q}_1(\alpha|x_1)$ with a procedure proposed by De Gooijner and Zerom (2003) in terms of finite sample properties. Their estimator, which they call an average quantile estimator, is the inverse of the reweighted Nadaraya-Watson estimator for a conditional distribution function introduced by Hall, Wolff, and Yao (1999). The reweighted estimator for the conditional distribution function has the following form

$$\breve{F}(y|\mathbf{x}) = \sum_{i=1}^{n} \frac{p_i(\mathbf{x}) K_{h_1}(x_1 - X_{i1}) L_G(x_{\underline{1}} - X_{i\underline{1}}) I\{Y_i \leq y\}}{\sum_{i=1}^{n} p_i(\mathbf{x}) K_{h_1}(x_1 - X_{i1}) L_G(x_{\underline{1}} - X_{i\underline{1}})},$$

where $p_i(\mathbf{x})$ denote weights depending on the data $\mathbf{X}_1, \ldots, \mathbf{X}_n$ with the properties that $p_i \geq 0$ for $i = 1, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$, and

$$\sum_{i=1}^{n} p_i(\mathbf{x})(X_{i1} - x_1) K_{h_1}(x_1 - X_{i1}) L_G(x_{\underline{1}} - X_{i\underline{1}}) = 0.$$

The weights p_i are chosen by maximizing $\prod_{i=1}^{n} p_i$. The estimator $\breve{F}(y|\mathbf{x})$ conserves the positivity and the monotonicity of the Nadaraya-Watson estimator with positive kernel functions K and L, but provides the more attractive bias of the local linear estimator. Since $\breve{F}(y|\mathbf{x})$ is monotone increasing, the inversion is easily obtained. For our approach, the monotonicity is not a constraint for the estimators of conditional distribution functions. Therefore local linear methods can also be used to estimate the conditional distribution function. In the following, we will compare three different estimators for the marginal effect of the additive quantile regression model. Instead of using the Nadaraya and Watson estimate for the conditional distribution function, we use the local linear estimator and apply afterwards the monotonizing inversion and the marginal integration technique. We call this estimator for the marginal effect of the first variable $\hat{q}_1(\alpha|x_1)$ and the estimate for the additive component $\hat{Q}_1(\alpha|x_1)$. The estimator proposed by De Gooijner and Zerom (2003) is denoted by $\breve{q}_1(\alpha|x_1)$ and $\breve{Q}_1(\alpha|x_1)$, respectively. Finally, we also investigate the finite sample properties of the internalized estimator discussed in the previous section $\tilde{q}_1(\alpha|x_1)$ and $\tilde{Q}_1(\alpha|x_1)$.
For the sake of practical convencience, we use a uniform distribution function on the interval

$$[\min(X_{j1}), \max(X_{j1})] \times \ldots \times [\min(X_{jd}), \max(X_{jd})]$$

for the function H to transform the data to $[0, 1]^d$.

Example 4.1 We consider the two-dimensional model

$$Y = 0.75X_1 + 1.5\sin(0.5\pi X_2) + 0.25\varepsilon, \tag{4.1}$$

where $\varepsilon \sim \mathcal{N}(0, 1)$. We assume that the covariates $(X_1, X_2)^T$ are bivariate normal with mean 0, variance 1, and correlation ρ. For the correlation, we distinguish two cases: a weak correlation

ρ	n	Component	$\hat{Q}_1(.5\|x_1)$ (local linear)	$\tilde{Q}_k(.5\|x_k)$ (internalized NW)	$\breve{Q}_k(.5\|x_k)$ (De Gooijer and Zerom)
.2	100	$.75x_1$	0.1176	0.2661	0.1374
		$1.5\sin(.5\pi x_2)$	0.2112	0.3543	0.1818
	200	$.75x_1$	0.0630	0.1971	0.1066
		$1.5\sin(.5\pi x_2)$	0.0969	0.1849	0.1272
	400	$.75x_1$	0.0474	0.1570	0.0734
		$\sin(.5\pi x_2)$	0.1169	0.2138	0.0936
.8	100	$.75x_1$	0.1939	0.4145	0.1365
		$1.5\sin(.5\pi x_2)$	0.2801	0.4611	0.4865
	200	$.75x_1$	0.1882	0.4385	0.1272
		$1.5\sin(.5\pi x_2)$	0.2305	0.3646	0.4350
	400	$.75x_1$	0.1829	0.4207	0.0985
		$\sin(.5\pi x_2)$	0.2152	0.3871	0.4009

Table 1: *The mean absolute deviation error of the different approaches.*

$\rho = 0.2$ and a strong correlation $\rho = 0.8$. This experiment was originally carried out by De Gooijer and Zerom (2003). The Epanechnikov kernel is used to estimate the conditional distribution function and to compute the monotonizing inversion, i.e.

$$K(x) = L(x) = K_d(x) = \frac{3}{4}(1 - x^2)I_{[-1,1]}(x).$$

We choose the bandwidths as in De Gooijer and Zerom (2003): $h_1 = 3\hat{\sigma}_1 n^{-1/5}$ for X_1 and $h_2 = \hat{\sigma}_2 n^{-1/5}$ for X_2, where $\hat{\sigma}_i$ is the standard deviation of the corresponding covariate. The quantile estimates are computed for $\alpha = 0.5$ and the sample sizes $n = 100, 200,$ and 400. Instead of using the mean squared error, the mean absolute deviation error (MADE) is collected, whereas observations outside of the square $[-2, 2]^2$ are disregarded to avoid boundary effects. In Table 1, we display the results of this finite sample study for model (4.1). In order to make our results comparable to De Gooijer and Zerom (2003) 41 simulation runs have been performed in each scenario. The results of the performance of $\breve{Q}_1(.5|\alpha)$ are extracted from De Gooijer and Zerom (2003). We observe that the internalized marginal integration estimate yields a larger MADE than the local linear approach in all cases. A comparison with the estimates of De Gooijer and Zerom (2003) shows only advantages for the internalized marginal integration estimate, if the second (more oscillating) component is estimated and the data is strongly correlated. In all other cases the estimate of De Gooijer and Zerom (2003) yields a smaller MADE. On the other hand the local linear estimate has a smaller MADE than the estimate of De Grooijer and Zerom (2003), except in the case $\rho = 0.8$, $n = 100, 200$ and 400 for $Q_1(.5|x_1) = 0.75x_1$.

Example 4.2 As a demonstration of the applicability of the presented method to estimate additive conditional quantile function in higher dimension than $d = 2$, we consider the model

$$Y = \sum_{k=1}^{4} \sin(X_k) + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, 1), \mathbf{X} \sim \mathcal{N}(0, \Sigma) \tag{4.2}$$

with two choices for Σ [for low and high correlation among the variables]

$$\Sigma_1 = \begin{pmatrix} 1.0 & 0.3 & 0.5 & 0.1 \\ 0.3 & 1.0 & 0.3 & 0.5 \\ 0.5 & 0.3 & 1.0 & 0.3 \\ 0.1 & 0.5 & 0.3 & 1.0 \end{pmatrix} \quad \text{and} \quad \Sigma_2 = \begin{pmatrix} 1.0 & 0.5 & 0.8 & 0.3 \\ 0.5 & 1.0 & 0.5 & 0.8 \\ 0.8 & 0.5 & 1.0 & 0.5 \\ 0.3 & 0.8 & 0.5 & 1.0 \end{pmatrix},$$

which was originally discussed by Hengartner and Sperlich (2005) in the context of traditional additive regression models. $n = 250$ observations are generated from this model for each of the 130 replications. Since the additive components and the marginal distributions are the same, we can average over all components at the same time. In the following table, the mean absolute deviation error and the mean squared error of the new estimate $\hat{Q}_k(.5|x_k)$ and the estimate $\breve{Q}_k(.5|x_k)$ proposed by De Gooijer and Zerom (2003) is recorded for the observations restricted to the square $[-2,2]^4$. Note that the estimates behave slightly better in the model with low correla-

	AADE($\hat{Q}_k(.5\|x_k)$)	AADE($\breve{Q}_k(.5\|x_k)$)	MSE($\hat{Q}_k(.5\|x_k)$)	MSE($\breve{Q}_k(.5\|x_k)$)
Σ_1	0.08459	0.14766	0.01017	0.05335
Σ_2	0.08473	0.15500	0.01269	0.05806

Table 2: *AADE and MSE of $\hat{Q}_k(.5|x_k)$ and $\breve{Q}_k(.5|x_k)$ in the low and high correlation model (4.2).*

tion among the covariates. Furthermore, a comparison of the two estimates with respect to both criteria shows that the new estimate $\hat{Q}_k(.5|x_k)$ performs substantially better than the estimate $\breve{Q}_k(.5|x_k)$ suggested by De Gooijner and Zerom (2003).

Example 4.3 To illustrate the performance on a real data set, we estimate the marginal effects for the Boston housing data. The Boston housing data contains the housing values of suburbs of Boston and 13 variables/criterias, which might have an influence on the housing prices like pollution, crime, and urban amenities. This dataset has been analyzed by several authors, also in the context of quantile regression. We focus on four covariates

- per capita crime rate (crime),
- average number of rooms per dwelling (rooms),
- weighted mean of distance to five Boston employment centres (distance),
- lower status of the population (econstatus),

and fit an additive conditional quantile model. We applied cross validation to determine the bandwidth for the four different variables. To simplify this problem, we set

$$h_1 = \hat{\sigma}_{\text{crime}} k, \;\; h_2 = \hat{\sigma}_{\text{rooms}} k, \;\; h_3 = \hat{\sigma}_{\text{distance}} k, \;\; h_4 = \hat{\sigma}_{\text{econstatus}} k,$$

where $\hat{\sigma}$ is the standard deviation of the corresponding variables. The cross validation criteria is minimized for $k \in [1/11, 30/11]$ for each marginal effect separately. Since the values are quite similar, we set $k = 1$ for all covariates. In Figure 1, we display five different curves of the marginal effects $Q(\alpha|X_k)$ for fixed $\alpha = 0.05, 0.25, 0.5, 0.75, 0.95$. Note that the marginal effects are monotone in α.

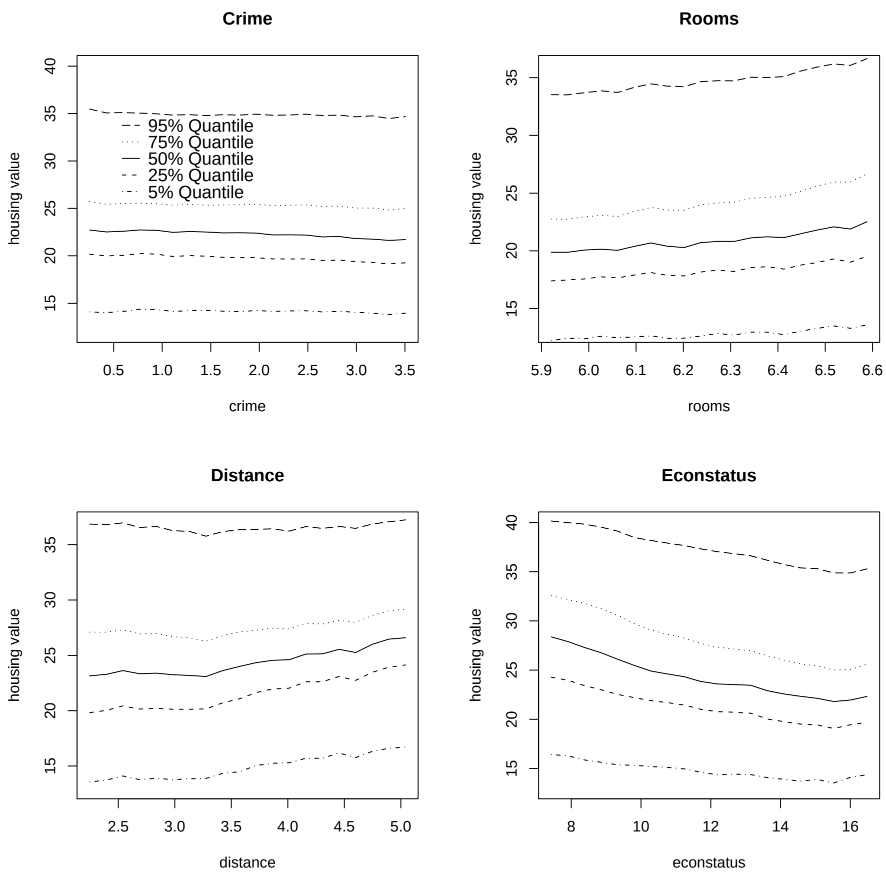

Figure 1: *Boston housing dataset: The marginal effects at five different levels of* α

5 Appendix: proof of main results

Proof of Theorem 3.1: For the sake of simplicity, we assume that $N = n$ and that the distribution function H corresponds to the uniform distribution. Recall the definition of q_1 in (2.8) and of $Q_n(\alpha|\mathbf{x})$ in (2.6), then it is easy to see that by the law of the iterated logarithm and the bandwidth conditions (3.12)

$$q_1(\alpha|x_1) = q_{1,n}(\alpha|x_1) + o\Big(\frac{1}{\sqrt{nh_1}}\Big)\ ,$$

where

$$q_{1,n}(\alpha|x_1) \quad = \quad \frac{1}{n}\sum_{i=1}^{n} Q_n(\alpha|x_1, X_{i\underline{1}}).$$

Now a straightforward argument shows that the assertion follows from the weak convergence

$$\sqrt{nh_1}(\hat{q}_1(\alpha|x_1) - q_{1,n}(\alpha|x_1) + b_1(\alpha|x_1)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, s^2(\alpha|x_1)), \tag{5.1}$$

where the bias $b_1(\alpha|x_1)$ and the variance $s^2(\alpha|x_1)$ are defined in Theorem 3.1. For a proof of (5.1) we use a Taylor expansion and obtain

$$\begin{aligned} \hat{q}_1(\alpha|x_1) - q_{1,n}(\alpha|x_1) \quad &= \quad \frac{1}{n}\sum_{j=1}^{n}\Big[\hat{Q}_I(\alpha|x_1, X_{j\underline{1}}) - Q_n(\alpha|x_1, X_{j\underline{1}})\Big] \\ &= \quad \Delta_n^{(1)}(\alpha|x_1) + \frac{1}{2}\Delta_n^{(2)}(\alpha|x_1), \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} \Delta_n^{(1)}(\alpha|x_1) \quad &= \quad -\frac{1}{n^2h_d}\sum_{j=1}^{n}\sum_{i=1}^{n} K_d\left(\frac{F(\frac{i}{n}|x_1, X_{j\underline{1}}) - \alpha}{h_d}\right)\left(\hat{F}(\frac{i}{n}|x_1, X_{j\underline{1}}) - F(\frac{i}{n}|x_1, X_{j\underline{1}})\right), \\ \Delta_n^{(2)}(\alpha|x_1) \quad &= \quad -\frac{1}{n^2h_d^2}\sum_{j=1}^{n}\sum_{i=1}^{n} K_d'\left(\frac{\xi_i - \alpha}{h_d}\right)\left(\hat{F}(\frac{i}{n}|x_1, X_{j\underline{1}}) - F(\frac{i}{n}|x_1, X_{j\underline{1}})\right)^2, \end{aligned}$$

where $\xi_i = \xi_i(\alpha, x_1, X_{j\underline{1}})$ satisfies $|\xi_i - F(\frac{i}{n}|x_1, X_{j\underline{1}})| \leq |\hat{F}(\frac{i}{n}|x_1, X_{j\underline{1}}) - F(\frac{i}{n}|x_1, X_{j\underline{1}})|$ for $i = 1, \ldots, n$. In the first step, we show that $\Delta_n^{(2)}(\alpha|x_1) = o_p\left(\frac{1}{\sqrt{nh_1}}\right)$. We obtain

$$\begin{aligned} |\Delta_n^{(2)}(\alpha|x_1)| \quad &= \quad \frac{1}{n^2h_d^2}\Big|\sum_{j=1}^{n}\sum_{i=1}^{n} K_d'\left(\frac{\xi_i - \alpha}{h_d}\right)\left(\hat{F}(\frac{i}{n}|x_1, X_{j\underline{1}}) - F(\frac{i}{n}|x_1, X_{j\underline{1}})\right)^2\Big| \\ &= \quad \frac{1}{n^2h_d^2}\Big|\sum_{j=1}^{n}\sum_{i=1}^{n} K_d'\left(\frac{F(\frac{i}{n}|x_1, X_{j\underline{1}}) - \alpha}{h_d}\right)\left[1 + \left(K_d'\left(\frac{F(\frac{i}{n}|x_1, X_{j\underline{1}}) - \alpha}{h_d}\right)\right)^{-1} \times\right. \\ &\qquad \left.\left(K_d'\left(\frac{\xi_i - \alpha}{h_d}\right) - K_d'\left(\frac{F(\frac{i}{n}|x_1, X_{j\underline{1}}) - \alpha}{h_d}\right)\right)\right]\left(\hat{F}(\frac{i}{n}|x_1, X_{j\underline{1}}) - F(\frac{i}{n}|x_1, X_{j\underline{1}})\right)^2\Big| \end{aligned}$$

$$(5.3) \quad = \frac{(1+o_p(1))}{n^2h_d^2}\left|\sum_{j=1}^n\sum_{i=1}^n K_d'\left(\frac{F(\frac{i}{n}|x_1,X_{j\underline{1}})-\alpha}{h_d}\right)\left(\hat F(\frac{i}{n}|x_1,X_{j\underline{1}})-F(\frac{i}{n}|x_1,X_{j\underline{1}})\right)^2\right|$$

$$= (1+o_p(1))\Delta_n^{(2.1)}(\alpha|x_k),$$

where the last equation defines the quantity $\Delta_n^{(2.1)}$ in an obvious manner. In line (5.3), we used the Lipschitz continuity of K_d' and the uniform convergence rate of $\hat F(\alpha|x_1,x_{\underline{1}})$ [see Collomb and Härdle (1986)], since

$$(5.4) \quad \left|K_d'\left(\frac{\xi_i-\alpha}{h_d}\right)-K_d'\left(\frac{F(\frac{i}{n}|x_1,X_{j\underline{1}})-\alpha}{h_d}\right)\right| \le L\left|\frac{\xi_i-F(\frac{i}{n}|x_1,X_{j\underline{1}})}{h_d}\right|$$

$$\le L\left|\frac{\hat F(\frac{i}{n}|x_1,X_{j\underline{1}})-F(\frac{i}{n}|x_1,X_{j\underline{1}})}{h_d}\right|$$

$$= O_p\left(\frac{\log n}{nh_1g_2^{d-1}h_d^2}\right)^{1/2} = o_p(1).$$

Using the bandwidth condition (3.13), it follows

$$E[|\Delta_n^{(2.1)}(\alpha|x_k)|X_j] \le \frac{1}{n^2h_d^2}\sum_{j=1}^n\sum_{i=1}^n\left|K_d'\left(\frac{F(\frac{i}{n}|x_1,X_{j\underline{1}})-\alpha}{h_d}\right)\right|\times$$

$$E\left[\left(\hat F(\frac{i}{n}|x_1,X_{j\underline{1}})-F(\frac{i}{n}|x_1,X_{j\underline{1}})\right)^2|X_j\right]$$

$$= O_p\left(\frac{1}{h_d}\left(\frac{1}{nh_1g_2^{d-1}}\right)\right) = o_p\left(\frac{1}{\sqrt{nh_1}}\right).$$

Now we can turn to the remaining term $\Delta_n^{(1)}(\alpha|x_1)$ which can be decomposed into bias- and variance-part. We obtain observing the representation (2.1)

$$(5.5) \quad \Delta_n^{(1)}(\alpha|x_1) = -\frac{(1+o_p(1))}{nh_d}\sum_{j=1}^n\int_0^1 K_d\left(\frac{F(t|x_1,X_{j\underline{1}})-\alpha}{h_d}\right)\left(\hat F(t|x_1,X_{j\underline{1}})-F(t|x_1,X_{j\underline{1}})\right)$$

$$= -\frac{(1+o_p(1))}{n^2h_d}\sum_{j=1}^n\sum_{k=1}^n\int_0^1 K_d\left(\frac{F(t|x_1,X_{j\underline{1}})-\alpha}{h_d}\right)K_{h_1}(x_1-X_{k1})\times$$

$$L_G(X_{j\underline{1}}-X_{k\underline{1}})\frac{I\{Y_k\le t\}-F(t|x_1,X_{j\underline{1}})}{p(x_1,X_{j\underline{1}})}dt$$

$$(5.6) \quad = (1+o_p(1))\left(\Delta_n^{(1.1)}(\alpha|x_1)+\Delta_n^{(1.2)}(\alpha|x_1)\right),$$

where

$$(5.7) \quad \Delta_n^{(1.1)}(\alpha|x_1) = -\frac{1}{n^2h_d}\sum_{j=1}^n\sum_{k=1}^n\int_0^1 K_d\left(\frac{F(t|x_1,X_{j\underline{1}})-\alpha}{h_d}\right)K_{h_1}(x_1-X_{k1})\times$$

$$L_G\left(X_{j\underline{1}} - X_{k\underline{1}}\right)\left(\frac{F(t|X_{k1}, X_{k\underline{1}}) - F(t|x_1, X_{j\underline{1}})}{p(x_1, X_{j\underline{1}})}\right) dt$$

$$\text{(5.8)} \qquad \Delta_n^{(1.2)}(\alpha|x_1) = -\frac{1}{n^2 h_d}\sum_{j=1}^n \sum_{k=1}^n \int_0^1 K_d\left(\frac{F(t|x_1, X_{j\underline{1}}) - \alpha}{h_d}\right) K_{h_1}\left(x_1 - X_{k1}\right) \times L_G\left(X_{j\underline{1}} - X_{k\underline{1}}\right)\left(\frac{\sigma(t|X_{k1}, X_{k\underline{1}})\varepsilon_k}{p(x_1, X_{j\underline{1}})}\right) dt.$$

The terms $\Delta_n^{(1.1)}$ and $\Delta_n^{(1.2)}$ are now investigated separately with an analysis similar as in Chen, Härdle, Linton and Severance-Lossin (1996). First of all, $\Delta_n^{(1.1)}(\alpha|x_1)$ can be written as

$$\Delta_n^{(1.1)}(y|x_1) = -\frac{1}{n}\sum_{j=1}^n \eta_j(\alpha|x_1),$$

where

$$\text{(5.9)} \qquad \eta_j(\alpha|x_1) = \frac{1}{n}\sum_{k=1}^n K_{h_1}\left(x_1 - X_{k1}\right) L_G\left(X_{j\underline{1}} - X_{k\underline{1}}\right) \frac{\left(F(Q(\alpha|x_1, X_{j\underline{1}})|X_{k1}, X_{k\underline{1}}) - F(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})\right)}{F'(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})p(x_1, X_{j\underline{1}})}.$$

We break $\eta_j(\alpha|x_1)$ into two uncorrelated parts

$$\eta_j(\alpha|x_1) = E[\eta_j(\alpha|x_1)|X_j] + \left(\eta_j(\alpha|x_1) - E[\eta_j(\alpha|x_1)|X_j]\right).$$

For the conditional expectation of $\eta_j(\alpha|x_1)$, we have

$$\begin{aligned}
E[\eta_j(\alpha|x_1)|X_j] &= \int K_{h_1}\left(x_1 - u_1\right) L_G\left(X_{j\underline{1}} - u_{\underline{1}}\right) \times \\
&\quad \frac{\left(F(Q(\alpha|x_1, X_{j\underline{1}})|u_1, u_{\underline{1}}) - F(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})\right)}{F'(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})p(x_1, X_{j\underline{1}})} p(u_1, u_{\underline{1}}) du_1 du_{\underline{1}} \\
&= \int K(v_1)L(v_{\underline{1}})p(x_1 - h_1 v_1, X_{j\underline{1}} - g_2 v_{\underline{1}}) \\
&\quad \frac{\left(F(Q(\alpha|x_1, X_{j\underline{1}})|x_1 - h_1 v_1, X_{j\underline{1}} - g_2 v_{\underline{1}}) - F(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})\right)}{F'(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})p(x_1, X_{j\underline{1}})} dv_1 dv_{\underline{1}} \\
&= (1 + o(1))h_1^2 \kappa_2(K)\Bigg(\frac{\frac{1}{2}\frac{\partial^2}{\partial x_1^2}F(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})}{F'(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})} \\
&\quad + \frac{\frac{\partial}{\partial x_1}F(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})}{F'(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})p(x_1, X_{j\underline{1}})}\Bigg) + O_p(g_2^q).
\end{aligned}$$

Now it is easy to see that

$$E[\Delta_n^{(1.1)}(\alpha|x_1)] = -(1 + o(1))b_1(\alpha|x_1).$$

To evaluate the variance of $\Delta_n^{(1.1)}(\alpha|x_1)$, we use the decomposition through $\eta_j(\alpha|x_1)$. Since

$$E[(\eta_j(\alpha|x_1) - E[\eta_j(\alpha|x_1)|X_j])^2|X_j] \leq O_p\left(\frac{1}{nh_1g_2^{d-1}}(h_1^2+g_2^2)\right)$$

we can estimate the variance of $\Delta_n^{(1.1)}(\alpha|x_1)$ as

$$\begin{aligned}\mathrm{Var}(\Delta_n^{(1.1)}(\alpha|x_1)) &\leq E\left[(\Delta_n^{(1.1)}(\alpha|x_1))^2\right] = \frac{1}{n}E\left[(E[\eta_j(\alpha|x_1)|X_j])^2\right] + O\left(\frac{(h_1+g_2)^2}{n^2h_1g_2^{d-1}}\right)\\ &= O\left(\frac{(h_1^2+g_2^q)^2}{n} + \frac{(h_1+g_2)^2}{n^2h_1g_2^{d-1}}\right) = o\left(\frac{1}{nh_1}\right),\end{aligned}$$

which shows

$$\Delta_n^{(1.1)}(\alpha|x_1) + b(\alpha|x_1) = o_p\Big(\frac{1}{\sqrt{nh_1}}\Big). \tag{5.10}$$

Now we consider the term $\Delta_n^{(1.2)}$ in (5.8), which has expectation $E[\Delta^{(1.2)}(\alpha|x_1)] = 0$. To calculate the variance, we use a similar analysis as for $\Delta_n^{(1.1)}$

$$\Delta_n^{(1.2)}(\alpha|x_1) \quad = \quad -\frac{1}{nh_1}\sum_{k=1}^{n} K\left(\frac{x_1 - X_{k1}}{h_1}\right)\varepsilon_k\beta_k(\alpha|x_1),$$

where

$$\beta_k(\alpha|x_1) = \frac{1}{n}\sum_{j=1}^{n}\frac{L_G\left(X_{j\underline{1}} - X_{k\underline{1}}\right)\sigma(Q(\alpha|x_1, X_{j\underline{1}})|X_{k1}, X_{k\underline{1}})}{p(x_1, X_{j\underline{1}})F'(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})}.$$

Now we treat $\beta_k(\alpha|x_1)$ as $\eta_k(\alpha|x_1)$ and split it up into

$$E[\beta_k(\alpha|x_1)|X_k] + (\beta_k(\alpha|x_1) - E[\beta_k(\alpha|x_1)|X_k]).$$

We calculate the conditional expectation of $\beta_k(\alpha|x_1)$ as

$$\begin{aligned}E[\beta_k(\alpha|x_1)|X_k] &= \int \frac{L_G\left(u_{\underline{1}} - X_{k\underline{1}}\right)\sigma(Q(\alpha|x_1, u_{\underline{1}})|X_{k1}, X_{k\underline{1}})p(u_1, u_{\underline{1}})}{p(x_1, u_{\underline{1}})F'(Q(\alpha|x_1, u_{\underline{1}})|x_1, u_{\underline{1}})}du_1du_{\underline{1}}\\ &= \frac{\sigma(Q(\alpha|x_1, X_{k\underline{1}})|X_{k1}, X_{k\underline{1}})p_{\underline{1}}(X_{k\underline{1}})}{p(x_1, X_{k\underline{1}})F'(Q(\alpha|x_1, X_{k\underline{1}})|x_1, X_{k\underline{1}})} + O_p(g_2^q)\end{aligned}$$

Furthermore we have

$$\begin{aligned}E\left[(\beta_k(\alpha|x_1) - E[\beta_k(\alpha|x_1)|X_k])^2|X_k\right] &\leq \frac{1}{n}\int\left(\frac{L_G\left(u_{\underline{1}} - X_{k\underline{1}}\right)\sigma(Q(\alpha|x_1, u_{\underline{1}})|X_{k1}, X_{k\underline{1}})}{p(x_1, u_{\underline{1}})F'(Q(\alpha|x_1, u_{\underline{1}})|x_1, u_{\underline{1}})}\right)^2 p(u_1, u_{\underline{1}})du_1du_{\underline{1}}\\ &= \frac{1}{ng_2^{d-1}}\left(\int L^2(v_{\underline{1}})dv_{\underline{1}}\right)\frac{\sigma(Q(\alpha|x_1, X_{k\underline{1}})|X_{k1}, X_{k\underline{1}})p_{\underline{1}}(X_{k\underline{1}})}{p^2(x_1, X_{k\underline{1}})\left(F'(Q(\alpha|x_1, X_{k\underline{1}})|x_1, X_{k\underline{1}})\right)^2}\\ &= O_p\left(\frac{1}{ng_2^{d-1}}\right).\end{aligned}$$

So basically $\Delta_n^{(1.2)}(\alpha|x_1)$ is of the form

$$\Delta_n^{(1.2)}(\alpha|x_1) = -\frac{1}{n}\sum_{k=1}^{n} K_{h_1}(x_1 - X_{k1})\, \frac{p_{\underline{1}}(X_{k\underline{1}})\sigma(Q(\alpha|x_1, X_{k\underline{1}})|X_{k1}, X_{k\underline{1}})\varepsilon_k}{p(x_1, X_{k\underline{1}})F'(Q(\alpha|x_1, X_{k\underline{1}})|x_1, X_{k\underline{1}})} + o_p\left(\frac{1}{\sqrt{nh_1}}\right),$$

and the variance of the dominating term on the right hand side, say $\hat{\Delta}_n^{(1.2)}(\alpha|x_1)$, can be easily calculated. i.e.

$$\text{(5.11)}\quad \mathrm{Var}(\sqrt{nh_1}\hat{\Delta}_n^{(1.2)}(\alpha|x_1)) = \left(\int K^2(v)dv\right)\int \frac{\alpha(1-\alpha)p_{\underline{1}}^2(x_{\underline{1}})dx_{\underline{1}}}{p(x_1, x_{\underline{1}})(F'(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}})} + o(1).$$

A similar calculation shows that Ljapunoff's condition is satisfied for the term $\Delta_n^{(1.2)}$, that is

$$\text{(5.12)}\quad \begin{aligned} &\sum_{j=1}^{n} E\Bigg[\frac{\sqrt{nh_1}}{nh_1}\sum_{k=1}^{n} K\left(\frac{x_1 - X_{k1}}{h_1}\right) \\ &\qquad\times \frac{p_{\underline{1}}(X_{k\underline{1}})\sigma(Q(\alpha|x_1, X_{k\underline{1}})|X_{k1}, X_{k\underline{1}})\varepsilon_k}{p(x_1, X_{k\underline{1}})F'(Q(\alpha|x_1, X_{k\underline{1}})|x_1, X_{k\underline{1}}))}\Bigg]^4 \\ &= \frac{E[\varepsilon_1^4]}{nh_1^2}\int K^4\left(\frac{x_1 - u_1}{h_1}\right)\left(\frac{\sigma(Q(\alpha|x_1, u_{\underline{1}})|u_1, u_{\underline{1}})p_{\underline{1}}(u_{\underline{1}})}{F'(Q(\alpha|x_1, u_{\underline{1}})|x_1, u_{\underline{1}})p(x_1, u_{\underline{1}})}\right)^4 p(u_1, u_{\underline{1}})du_1 du_{\underline{1}} \\ &= \frac{E[\varepsilon_1^4]}{nh_1}\left(\int K^4(v)dv\right)\int \frac{\alpha^2(1-\alpha)^2 p_{\underline{1}}^4(u_{\underline{1}})}{(F'(Q(\alpha|x_1, u_{\underline{1}})|x_1, u_{\underline{1}}))^4 p^3(x_1, u_{\underline{1}})}du_{\underline{1}} = O\left(\frac{1}{nh_1}\right), \end{aligned}$$

which establishes the weak convergence

$$\sqrt{nh_1}\Delta_n^{(1.2)}(\alpha|x_1) \xrightarrow{\mathcal{D}} \mathcal{N}(0, s^2(\alpha|x_1)).$$

A combination with (5.10) and (5.5) yields (5.1) which proves Theorem 3.1.

□

Proof of Theorem 3.2: Again we consider the uniform distribution H, and $N = n$. A similar argument as presented at the beginning of the proof of Theorem 3.1 shows that the assertion of the theorem follows from the weak convergence

$$\text{(5.13)}\quad \sqrt{nh_1}(\tilde{q}(\alpha|x_1) - q_{1,n}(\alpha|x_1) + \tilde{b}_1(\alpha|x_1) + \tilde{b}_2(\alpha|x_1)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{s}^2(\alpha|x_1)),$$

where

$$q_{1,n}(\alpha|x_1) = \frac{1}{n}\sum_{k=1}^{n} Q_n(\alpha|x_1, X_{k2})$$

and $Q_n(\alpha|x_1, X_{2k})$ is defined in (2.4). A Taylor expansion yields

$$\text{(5.14)}\quad \tilde{q}_1(\alpha|x_1) - q_{1,n}(\alpha|x_1) = \tilde{\Delta}_n^{(1)}(\alpha|x_1) + \frac{1}{2}\tilde{\Delta}_n^{(2)}(\alpha|x_1)\,,$$

where

$$(5.15)\ \tilde{\Delta}_n^{(1)}(\alpha|x_1) = -\frac{1}{n^2 h_d}\sum_{k,i=1}^{n} K_d\left(\frac{F(\frac{i}{n}|x_1, X_{k\underline{1}})-\alpha}{h_d}\right)(\tilde{F}(\frac{i}{n}|x_1, X_{k\underline{1}}) - F(\frac{i}{n}|x_1, X_{k\underline{1}})),$$

$$(5.16)\ \tilde{\Delta}_n^{(2)}(\alpha|x_1) = -\frac{1}{2}\frac{1}{n^2 h_d^2}\sum_{k,i=1}^{n} K_d'\left(\frac{\xi_i(\alpha, x_1, X_{k\underline{1}})-\alpha}{h_d}\right)(\tilde{F}(\frac{i}{n}|x_1, X_{k\underline{1}}) - F(\frac{i}{n}|x_1, X_{k\underline{1}}))^2.$$

As in the proof of Theorem 3.1, we can show that the term $\tilde{\Delta}_n^{(2)}$ is asymptotically negligible. The Term $\tilde{\Delta}_n^{(1)}$ requires a more sophisticated treatment and we use the decomposition

$$(5.17)\qquad \tilde{a}_n^{(1)}(\alpha|x_1) = (1+o(1))\{\tilde{\Delta}_n^{(1.1)}(\alpha|x_1) + \tilde{\Delta}_n^{(1.2)}(\alpha|x_1) + \tilde{\Delta}_n^{(1.3)}(\alpha|x_1)\},$$

where

$$(5.18)\quad \tilde{\Delta}_n^{(1.1)}(\alpha|x_1) = -\frac{1}{nh_d}\sum_{k=1}^{n}\int_0^1 K_d\left(\frac{F(t|x_1, X_{k\underline{1}})-\alpha}{h_d}\right) \times \left[\frac{1}{n}\sum_{j=1}^{n}\frac{K_{h_1}(x_1 - X_{j1})\, L_G(X_{k\underline{1}} - X_{j\underline{1}})\, \sigma(t|X_{j1}, X_{j\underline{1}})\varepsilon_j}{\frac{1}{n}\sum_{l=1}^{n} K_{h_1}(X_{j1} - X_{l1})\, L_G(X_{j\underline{1}} - X_{l\underline{1}})}\right] dt,$$

$$(5.19)\quad \tilde{\Delta}_n^{(1.2)}(\alpha|x_1) = -\frac{1}{nh_d}\sum_{k=1}^{n}\int_0^1 K_d\left(\frac{F(t|x_1, X_{k\underline{1}})-\alpha}{h_d}\right) \times \left[\frac{1}{n}\sum_{j=1}^{n}\frac{K_{h_1}(x_1 - X_{j1})\, L_G(X_{k\underline{1}} - X_{j\underline{1}})\,(F(t|X_{j1}, X_{j\underline{1}}) - F(t|x_1, X_{k\underline{1}}))}{\frac{1}{n}\sum_{l=1}^{n} K_{h_1}(X_{j1} - X_{l1})\, L_G(X_{j\underline{1}} - X_{l\underline{1}})}\right] dt$$

$$(5.20)\quad \tilde{\Delta}_n^{(1.3)}(\alpha|x_1) = -\frac{1}{nh_d}\sum_{k=1}^{n}\int_0^1 K_d\left(\frac{F(t|x_1, X_{k\underline{1}})-\alpha}{h_d}\right) F(t|x_1, X_{k\underline{1}}) \times \left[\frac{1}{n}\sum_{j=1}^{n}\frac{K_{h_1}(x_1 - X_{j1})\, L_G(X_{k\underline{1}} - X_{j\underline{1}})}{\frac{1}{n}\sum_{l=1}^{n} K_{h_1}(X_{j1} - X_{l1})\, L_G(X_{j\underline{1}} - X_{l\underline{1}})} - 1\right] dt.$$

Like in the proof of Theorem 3.1 we show that

$$\tilde{\Delta}_n^{(1.1)}(\alpha|x_1) = -\frac{1}{n}\sum_{j=1}^{n}\frac{K_{h_1}(x_1 - X_{j1})\, p_{\underline{1}}(X_{j\underline{1}})\sigma(Q(\alpha|x_1, X_{j\underline{1}})|X_{j1}, X_{j\underline{1}})\varepsilon_j}{p(X_{j1}, X_{j\underline{1}})F'(Q(\alpha|x_1, X_{j\underline{1}})|x_1, X_{j\underline{1}})} + o_p\left(\frac{1}{\sqrt{nh_1}}\right),$$

and it follows in a similar manner as in the last proof that

$$(5.21)\ \sqrt{nh_1}\tilde{\Delta}_n^{(1.1)}(\alpha|x_1) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \int K^2(w)dw \int \frac{\alpha(1-\alpha)p_{\underline{1}}^2(x_{\underline{1}})}{p(x_1, x_{\underline{1}})(F'(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}}))^2} dx_{\underline{1}}\right),$$

where the representation of the variance and the Ljapunoff condition are established by a straightforward calculation. We now investigate the random variable $\tilde{\Delta}_n^{(1.2)}(\alpha|x_1)$ in (5.19), for which we obtain

$$\tilde{\Delta}_n^{(1.2)}(\alpha|x_1) = \frac{1}{n}\sum_{k=1}^{n}\tilde{\beta}_k(\alpha|x_1) + o_p\left(\frac{1}{\sqrt{nh_1}}\right)$$

where

$$
\begin{aligned}
\tilde{\beta}_k(\alpha|x_1) &= -\frac{1}{n}\sum_{j=1}^{n} K_{h_1}(x_1 - X_{j1})\, L_G(X_{k\underline{1}} - X_{j\underline{1}}) \\
&\times \frac{(F(Q(\alpha|x_1, X_{k\underline{1}})|X_{j1}, X_{j\underline{1}}) - F(Q(\alpha|x_1, X_{k\underline{1}})|x_1, X_{k\underline{1}}))}{p(X_{j1}, X_{j\underline{1}})F'(Q(\alpha|x_1, X_{k\underline{1}})|x_1, X_{k\underline{1}})}
\end{aligned}
$$

The calculation of the expectation and variance of $\frac{1}{n}\sum_{k=1}^{n}\tilde{\beta}_k(\alpha|x_1)$ is straightforward and gives

$$
E\Big[\frac{1}{n}\sum_{k=1}^{n}\tilde{\beta}_k(\alpha|x_1)\Big] = -\tilde{b}_1(\alpha|x_1) + o\Big(\frac{1}{\sqrt{nh_1}}\Big) \tag{5.22}
$$

$$
\mathrm{Var}\Big[\frac{1}{n}\sum_{k=1}^{n}\tilde{\beta}_k(\alpha|x_1)\Big] = o\Big(\frac{1}{nh_1}\Big) \tag{5.23}
$$

For the remaining term $\tilde{\Delta}_n^{(1.3)}$ we obtain by similar arguments as in Kim, Linton, and Hengartner (1999),which yield

$$
\begin{aligned}
\tilde{\Delta}_n^{(1.3)}(\alpha|x_1) &= -\frac{(1+o_p(1))}{n}\sum_{k=1}^{n}\frac{F(Q(\alpha|x_1, X_{k\underline{1}})|x_1, X_{k\underline{1}})}{F'(Q(\alpha|x_1, X_{k\underline{1}})|x_1, X_{k\underline{1}})}\left[\frac{K_{h_1}(x_1 - X_{k1})p_{\underline{1}}(X_{k\underline{1}})}{\hat{p}(X_{k1}, X_{k\underline{1}})} - 1\right] \\
&\quad + o_p\left(\frac{1}{\sqrt{nh_1}}\right)
\end{aligned}
$$

Consequently we have

$$
\sqrt{nh_1}(\tilde{\Delta}_n^{(1.3)}(\alpha|x_1) + \tilde{b}_2(\alpha|x_1)) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \int K^2(w)dw \int \frac{(F(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}}))^2 p_{\underline{1}}^2(x_{\underline{1}})}{p(x_1, x_{\underline{1}})(F'(Q(\alpha|x_1, x_{\underline{1}})|x_1, x_{\underline{1}}))^2}dx_{\underline{1}}\right).
$$

Moreover, the terms $\tilde{\Delta}_n^{(1.1)}(\alpha|x_1)$ and $\tilde{\Delta}_n^{(1.3)}(\alpha|x_1)$ are uncorrelated which gives (by the Cramér-Wold device).

$$
\sqrt{nh_1}(\tilde{\Delta}_n^{(1.1)}(\alpha|x_1) + \tilde{\Delta}_n^{(1.3)}(\alpha|x_1) + \tilde{b}_2(\alpha|x_1)) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \tilde{s}^2(\alpha|x_1)\right) \tag{5.24}
$$

The assertion of the theorem now follows from (5.22), (5.23), and (5.24).

□

Acknowledgements The support of the Deutsche Forschungsgemeinschaft (SFB 475, "Komplexitätsreduktion in multivariaten Datenstrukturen"). is gratefully acknowledged. The work of H. Dette was supported in part by a NIH grant award IR01GM072876:01A1. The authors are grateful to Isolde Gottschlich, who typed parts of the paper with considerable technical expertise.

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