

# Stochastic multi-scale selection of the stopping index in PET

Nicolai Bissantz<sup>1,\*</sup>, Bernard Mair<sup>2</sup>, and Axel Munk<sup>3</sup>

<sup>1</sup> Fakultät für Mathematik, Ruhr-Universität Bochum, Universitätsstraße 150, NA 3/70, 44801 Bochum, Germany

<sup>2</sup> Department of Mathematics, University of Florida, Gainesville, USA

<sup>3</sup> Institut für Mathematische Stochastik, Universität Göttingen, Germany

In this paper we present a fully data-driven selection algorithm for the stopping criterion for MLEM reconstructions in PET. The method can be generalized to various other reconstruction algorithms, and is based on a statistical analysis of the residuals between projected model and data.

To this end we test whether the residuals are consistent with the hypothesis of being solely due to Poisson noise. Moreover, our method includes a multiresolution approach, i.e. we test whether the residuals are consistent with pure Poisson noise for all possible re-binning of the data into increasing bin sizes in the detector space, at all positions. Unlike previous methods, this method is robust in the presence of model uncertainty in the system matrix.

Technically, our method is based on the almost sure limiting behaviour of partial sums of the residuals. The test statistic includes a rate function which appears in the corresponding almost sure limit theorem for Poisson noise, and which is different from the Gaussian noise case. Finally, we present results from a Monte Carlo study which demonstrates the performance of the method.

Copyright line will be provided by the publisher

## 1 Introduction

It has been shown that positron emission tomography (PET) images can be reconstructed by MLEM (maximum likelihood expectation maximization) and OSEM (ordered subsets EM) algorithms in an effective way, if the iterations are stopped early. Such an early termination is necessary, since it has been observed, for example, that MLEM iterates initialized with a uniform image initially improve, but after a certain point become more noisy and lose resolution with increasing iterations [8]. Several methods have been proposed so far for the selection of feasible stopping iterations [3–6].

Here, we present a new method for the selection of the stopping iteration for the MLEM algorithm. Our approach is based on a stochastic multi-scale analysis of the differences between projected images and the observed detector data. The method considers sums of the normalized residuals for each row of the sinogram on all possible scales. This is in contrast to previous methods which only involved the single total sum of (normalized) residuals for all detectors. From a theoretical point-of-view, our method is founded in recent results from probability theory on the almost sure behaviour of the maximum of the partial sum process for Poisson data. Simulation results indicate that our method yields robust predictions (with respect to modeling errors in the system matrix) for the optimal stopping iteration, which are somewhat less noisy than the iterate with maximal signal-to-noise ratio for given data.

## 2 The stochastic multi-scale selection method

We suppose that at our disposal is PET data which can be modelled as

$$Y_i \sim \text{Poiss}([\mathbf{A}\mathbf{x}]_i), \quad i = 1, \dots, n, \quad (1)$$

where  $\mathbf{A}$  is the matrix of probabilities representing the scanner,  $\mathbf{x}$  is the  $m$ -dimensional vector of emission intensities, and  $[\mathbf{A}\mathbf{x}]_i$  is the  $i^{\text{th}}$  entry of the vector  $\mathbf{A}\mathbf{x}$ . If  $\hat{\mathbf{x}}$  is an estimator of  $\mathbf{x}$ , we aim to decide whether or not  $\hat{\mathbf{x}}$  is a “good” estimator. To this end we consider the normalized residuals

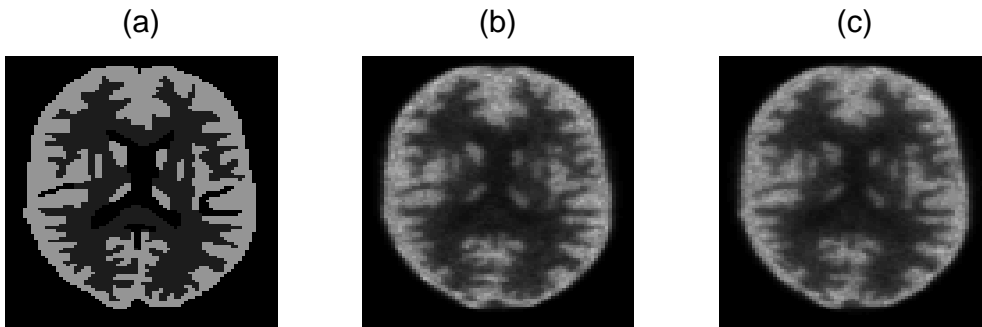
$$R_i = \frac{Y_i - [\mathbf{A}\hat{\mathbf{x}}]_i}{\sqrt{[\mathbf{A}\hat{\mathbf{x}}]_i}}, \quad i = 1, \dots, n. \quad (2)$$

Observe that if  $\hat{\mathbf{x}}$  is a good approximation to  $\mathbf{x}$ , we expect the residuals to have mean zero and unit variance. Moreover, they should, in distribution, neither be “too large” (which would indicate substantial remaining signal in the residuals) nor “too small” (which would indicate overfitting).

The stochastic multi-scale selection method is based on the test statistic

$$\mathcal{D}_n = \max_{0 \leq j < n} \max_{1 \leq k \leq n-j} \frac{|S_{j+k} - S_j|}{k\alpha(k/\log(n))}, \quad (3)$$

\* Corresponding author E-mail: nicolai.bissantz@ruhr-uni-bochum.de, Phone: +49 234 23 291



**Fig. 1** True image of Hoffmann's phantom (a) and typical results from deprojection in the exact and inexact case (w.r.t. the operator matrix  $\mathbf{A}$  known and ill-known, respectively). The iterates shown in (b) and (c) have been selected with our multiresolution criterion, and are  $I_s = 17$  and  $I_s = 15$ , respectively. All plots are for  $10^6$  total counts and use the same color scaling.

which is the maximum of all partial sums  $|S_{j+k} - S_j| = \left| \sum_{i=j+1}^{j+k} R_i \right|$  of residuals  $R_i$ . Here,  $k\alpha(\cdot)$  is a scaling function defined below, and we consider the maximum of partial sums of adjacent (normalized) residuals at all scales  $k$ , at all positions  $j$ . In other words, the test statistic  $\mathcal{D}_n$  tests simultaneously on all positions whether the residuals are consistent with the distribution of the noise - or whether systematic deviations exist somewhere, on any scale.

Our method is theoretically founded on following result adapted from Steinebach [7].

**Theorem 2.1** *If  $\{R_i\}$  are i.i.d. and  $\alpha(\cdot)$  is the "inverse Chernoff function" of the distribution of  $R_1$ , where  $\alpha(\cdot)$  depends on the Poisson parameter, then  $\lim_{n \rightarrow \infty} \mathcal{D}_n = 1$  a.s.*

Observe that for the residuals to be identically distributed the function  $\mathbf{x}$  has to be a constant vector. Hence, we suggest to use a modification  $\mathcal{B}_n$  of the test statistic  $\mathcal{D}_n$  based on simulations of the test statistic [1]. The stochastic multi-scale parameter selection rule is now given by selecting the stopping index such that the test statistic  $\mathcal{B}_n \approx 1$ . An extension of the method is to attribute all iterates for which the test statistic  $\mathcal{B}_n$  is within certain quantiles of the simulated test statistic.

### 3 Results

We now discuss briefly the results from our simulations of the stochastic multi-scale selection method with simulated data based on a slice of the Hoffman brain phantom consisting of  $128 \times 128$  pixels and one million total counts. The data (sinogram) was arranged in 192 angles with 160 detector bins for each angle. First, we performed tests in the exact case, where the same matrix  $\mathbf{A}$  was used both to generate the data and in the MLEM reconstruction of the images. However, in practice it is not possible to accurately model the forward projection operator that generates the scanner data. Hence, we also performed tests of the method on data obtained from an approximate matrix  $\tilde{\mathbf{A}}$ . This approximation was obtained by assuming uniformly distributed detector gains between 95% and 105%.

In Fig. 1 we compare the true image of Hoffmann's phantom with typical results from deprojection with the true matrix  $\mathbf{A}$  and with the approximate matrix  $\tilde{\mathbf{A}}$ . Overall, the main features of the image are recovered well by the iterate  $I_s$  predicted from the stochastic multi-scale criterion. It turns out that these iterates are less noisy than the iterates  $I_{\text{snr}}$  which are selected such that they have maximal signal-to-noise ratio. For more details on the simulation results we refer to [1] and [2].

### References

- [1] N. Bissantz, B. Mair, and A. Munk, "A multi-scale stopping criterion for MLEM reconstructions in PET," *IEEE Nucl. Sci. Symp. Conf. Rec.* **6**, October/November 2006, 3376–3379 (2006).
- [2] N. Bissantz, B. Mair, and A. Munk, "A stochastic multi-scale selection of the stopping iteration for MLEM reconstructions in PET," *in preparation*.
- [3] K. J. Coakley, "A cross-validation procedure for stopping the EM algorithm and deconvolution of neutron depth profiling spectra," *IEEE Trans. Nuclear Science*, **38**, 9–15 (1991).
- [4] T. Hebert, R. Leahy, and M. Singh, "Fast MLE for SPECT using an intermediate polar representation and a stopping criterion," *IEEE Trans. Nucl. Science*, **35**, 615–619 (1988).
- [5] G. N. Kontaxakis, and G. S. Tzanakos, "Practical application of a new stopping criterion for the EM-MLE image reconstruction for PET," in *Proc. Internat. Conf. IEEE Engineering in Medicine and Biology Soc.*, October/November 1996, 848–849 (1996).
- [6] J. Llacer, and E. Veklerov, "Feasible images and practical stopping rules for iterative algorithms in emission tomography," *IEEE Trans. Med. Imag.*, **8**, 186–193 (1989).
- [7] J. Steinebach, "On a conjecture of Revesz and its analogue for renewal processes," in *Asymptotic Methods in Probability and Statistics (ICAMPS 1997)*, July 1997, Carleton Univ., Ottawa, Ontario, Canada, 311–322 (1997).
- [8] Y. Vardi, L. A. Shepp, and L. Kaufman, "A statistical model for positron emission tomography," *J. Amer. Statist. Assoc.* **80**, 8–37 (1985).