On Self-normalization For Censored Dependent Data ☆

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Abstract

The paper is concerned with confidence interval construction for functionals of the survival distribution for censored dependent data. We adopt the recently developed self-normalizion approach (Shao, 2010), which does not involve consistent estimation of the asymptotic variance, as implicitly used in the blockwise empirical likelihood approach of El Ghouch et al. (2011). We also provide a rigorous asymptotic theory to derive the limiting distribution of the self-normalized quantity for a wide range of parameters. Additionally, finite sample properties of the SN-based intervals are carefully examined and a comparison with the empirical likelihood based counterparts is made.

Key words and phrases: censored data, dependence, empirical likelihood, quantile, self-normalization, survival analysis.

1. Introduction and Motivation

Censored data are frequently encountered in a spectrum of areas such as medical follow-up studies, engineering life-testing, economics and social sciences. A huge amount of literature is devoted to the inference for censored data that are independent and identically distributed (iid); see for example Kalbfleisch and Prentice (2002). However, dependence arises naturally in real applications when the data are collected sequentially in time or are

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observed in space. For example, in environmental research, concentration measurements are often subject to the measurement limit of the equipment; if the measurement is lower or greater than certain detection limit, it is reported as non-detects. When such data are collected over time, it naturally gives rise to a censored time series, see e.g., Zeger and Brookmeyer (1986), Glasbey and Nevison (1997) and Eastoe et al. (2006) among many others for such examples. In finance, prices subject to price limits imposed in stock markets, commodity future exchanges, and foreign exchange futures markets have been treated as censored variables. In economics, durations of unemployment may be right censored and correlated. In the field of clinical trials and population-based biomedical studies, censored data collected over adjacent neighbourhoods tend to produce more similar outcomes than distant ones due to similar environmental and social factors. The prevalence of censored dependent data calls for a rigorous treatment with dependence taken into account, since the existing procedures developed for iid censored data may not be applicable. However, the work in this direction that is available so far mainly focuses on deriving properties of the Kaplan-Meier estimator under various dependence settings. For example, consistency and asymptotic normality of the Kaplan-Meier (KM) estimator were obtained under ϕ-mixing conditions by Ying and Wei (1994); under α-mixing conditions by Cai (1998); and under the so-called positive or negative association by Cai and Roussas (1998). Cai (2001) obtained the uniform convergence rate of the KM estimator and proposed a consistent estimator of the asymptotic variance of the KM estimator.

In practice, we are often mainly interested in certain functionals of the survival distribution function, such as the median survival time, survival mean, mean residual life time, etc. To the best of our knowledge, there are very few results on the asymptotic distribution of a general functional of the KM estimator when the underlying data are dependent. Similarly, not much is known about conducting practical inference for the above-mentioned quantities. The only paper that we are aware of is El Ghouch et al. (2011), who applied block-wise empirical likelihood (BEL) method to construct confidence intervals for quantities that can be expressed as an integral with respect to the distribution function. One drawback of the BEL approach is that there seems no good guidance on the choice of block size, which can affect the finite sample coverage to a great degree. Also the framework of that paper excludes the quantile of survival distribution function (with median survival time as a special case), which is often of practical interest.

In this article, we aim to provide an alternative approach to confidence interval construction for censored time series. Our approach is an extension of the so-called self-normalized (SN) approach developed by Shao (2010) for a weakly dependent stationary time series. Unlike the traditional inference approaches, which involve consistent estimation of the asymptotic variance using a bandwidth-dependent procedure, or re-sampling methods and variants (say, sub-sampling, block bootstrap or BEL), the SN approach uses an inconsistent estimator of the asymptotic variance, which does not involve any bandwidth or smoothing parameter. Since the limiting distribution of the self-normalized quantity is pivotal, a confidence interval can be conveniently constructed. The extension to the censored time series is however nontrivial. The complication mainly arises in two aspects. First, the self-normalizer used in Shao (2010) is a functional of the estimators based on all the recursive sub-samples, i.e. $\{(X_1), (X_1, X_2), \cdots, (X_1, \cdots, X_n)\}$. For censored data, we do not observe the failure time series X_t, and for the first few subsamples, it may occur that all or most of the data points are censored, which makes estimation impossible or unstable. To attenuate this issue, we propose to use recursive subsamples with the first subsample having sample size $\lfloor \epsilon n \rfloor$, where $\epsilon \in (0, 1)$ is called the trimming parameter. Second, the theoretical arguments used in Shao (2010) seems not directly applicable to censored data, as the high level conditions on the remainder terms of the influence function based expansion are difficult to verify. To circumvent the difficulty, we build on recent results of Volgushev and Shao (2013), who provide a general approach to the asymptotic analysis of statistics which are functionals of (recursive) subsample estimators, and provide a rigorous asymptotic theory for the limiting distribution of the SN quantity in the censored time series setting. It is worth noting that our framework allows quantiles of the survival distribution and is thus considerably wider than that in El Ghouch et al. (2011). Additionally, our theory is developed under rather general assumptions that allow to incorporate many different types of weak dependence such as α-mixing or physical dependence. This is in contrast to the approach of El Ghouch et al. (2011) who only derive results under the assumption of β-mixing.

The rest of the paper is organized as follows. In Section 2, we describe the estimation and SN-based inference methodology. A rigorous theoretical derivation of the limiting distribution of the SN quantity is provided in Section 3. In Section 4, simulations are carried out to examine the finite sample performance of the SN-based CI and compare with the BEL approach in El Ghouch et al (2011). Section 5 concludes.

2. Methodology

Following El Ghouch et al (2011), we shall restrict our attention to censored time series. To fix the idea, let $X_1, \cdots, X_n$ be a sequence of failure times that might not be mutually independent, but share the same (marginal) distribution function F_X. Let $Y_1, \cdots, Y_n$ be the censoring time with a common (marginal) distribution function F_Y. The observations are given by $\{(Z_i, \delta_i)\}_{i=1}^n$ where $Z_i = \min(X_i, Y_i)$ and $\delta_i = 1_{X_i \le Y_i}$, namely, $\delta_i = 1$ if the i'th observation is not censored. Let $F_Z(t) = 1 - (1 - F_X(t))(1 - F_Y(t))$ be the distribution function of Z_i. The term *failure time* is a generic term inherited from survival analysis, but it may refer to the duration time, the concentration measurement, the rainfall amount, etc in different applications. Also notice that although only right censoring is discussed here, the framework can be applied to left-censored data by flipping the signs of X_i and Y_i.

In survival analysis, it is of primary interest to investigate functionals of the survival distribution function, or equivalently, the marginal distribution function of the unobserved X_i. In the i.i.d. setting, the nonparametric maximum likelihood estimator of the survival function $1 - F_X(t)$ is given by the product-limit (PL) estimator [see Kaplan and Meier (1958)] which takes the form

$$1 - \hat{F}_{X,n}(t) = \prod_{i: Z_i \le t} \left(1 - \frac{\delta_i}{A(Z_i)}\right),$$

where $A(t) = \sum_{i=1}^n 1_{Z_i \ge t}$ is the number of censored or uncensored observations that has a survival time no less than t. An equivalent form that is also frequently used is

$$1 - \hat{F}_{X,n}(t) = \prod_{Z_{(i)} \le t} \left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}},$$

where $Z_{(1)} \le Z_{(2)} \le \cdots < Z_{(n)}$ are the ordered observations Z_i, and $\delta_{(i)}$'s are the corresponding censoring indicators. Similarly, let $\hat{F}_{Y,n}$ denote the KM estimator of F_Y, then

$$1 - \hat{F}_{Y,n}(t) = \prod_{Z_{(i)} \le t} \left(\frac{n-i}{n-i+1}\right)^{1-\delta_{(i)}}.$$

We consider parameters that can be represented in the general form,

$$\theta = \phi(F_X) \tag{1}$$

where F_X is the distribution function of X_i and ϕ is a smooth mapping from the set of distribution functions to $\mathbb{R}^d$. This form provides a general framework for a large class of quantities that are of interest in practice. For example, letting

$$\theta = \int \xi(x) F_X(dx) \tag{2}$$

for some given measurable function ξ, the map $\phi_\xi : F_X \mapsto \int \xi(x) F_X(dx)$ is the form considered in El Ghouch et al. (2011). The parameter is reduced to the Kaplan-Meier (KM) estimator at time t if $\xi = 1_{(-\infty,t]}$; and it is the mean residual life time if $\xi(x) = (x-t)1_{x>t}$ and $F_X(t) < 1$; see Stute and Wang (1993) for some other examples. Another example of the form given in (1) is obtained by denoting by ϕ the 'quantile mapping', that is

$$\theta(F_X) = F_X^{-1}(q), \text{ for some given } q \in (0,1).$$

Note that this map is not included in the framework of (2).

The KM estimator can be naturally regarded as the counterpart of empirical distribution function F_n under censorship and an estimator of θ can then be obtained by the plug-in method, i.e., $\hat{\theta}_n = \phi(\hat{F}_{X,n})$. To construct a confidence interval for θ using normal approximation, one needs a consistent estimator of the asymptotic variance. A direct consistent estimation involves the derivation of an approximate formula for the asymptotic variance, followed by consistent estimation of unknown nuisance quantities using bandwidth-dependent procedures (e.g. blockwise jackknife); see El Ghouch et al. (2011) for a detailed discussion. The BEL approach adopted in El Ghouch et al. (2011) was originally proposed by Kitamura (1997) as an extension of the empirical likelihood (Owen, 2001) method to the time series context. Empirical likelihood is well known to provide an internal studentization so the empirical log-likelihood ratio evaluated at the true parameter (up to multiplication of a constant factor) has a limiting χ^2 distribution. The confidence interval for θ is then constructed as the set of θ such that the empirical log-likelihood ratio at θ is no greater than a given upper quantile of the χ^2 distribution. The BEL approach applies the EL to the blockwise smoothed moment conditions (or estimating equations), which corresponds to an implicit consistent long run variance (or asymptotic variance) estimation of the moment conditions. The theory is elegant in that the blockwise empirical log-likelihood ratio (upon multiplication of a constant factor) evaluated at the true parameter still converges to a χ^2 distribution, but a practical difficulty is the choice of block size, which seems largely unexplored even in the uncensored time series setting.

To alleviate the problem, we adopt the self-normalized approach (Lobato 2001, Shao 2010) which avoids consistent estimation of the asymptotic variance, is free of the choice of block size, and is also applicable to time series data. The main idea of the SN approach is to use recursive sub-sample estimates of θ to form a self-normalized quantity, which has a pivotal asymptotic distribution. To this end, we use $\hat{\theta}_k$ to denote the estimator of θ based on the sub-sample $\{(Z_1, \delta_1), \cdots, (Z_k, \delta_k)\}$. This estimator is stable when the size of the sub-sample is not too small, thus we introduce a trimming parameter to control the minimal sub-sample size. We denote ϵ as the fraction of the initial subsample size to the whole sample size.

When θ is a scalar, i.e. $d = 1$, the following result holds as a simple corollary to Theorem 1 stated in Section 3.

Corollary **1**. *Let $D_n^2 := n^{-2}\sum_{j=\lfloor \epsilon n \rfloor}^{n}[j(\hat{\theta}_j - \hat{\theta}_n)]^2$. Under the conditions specified in Theorem 1 in Section 3,*

$$T_n := \frac{n(\hat{\theta}_n - \theta)^2}{n^{-2}\sum_{j=\lfloor \epsilon n \rfloor}^{n}[j(\hat{\theta}_j - \hat{\theta}_n)]^2} \xrightarrow{\mathcal{D}} \frac{\mathbb{B}(1)^2}{\int_\epsilon^1 (\mathbb{B}(r) - r\mathbb{B}(1))^2 dr} := U_{1,\epsilon}. \tag{3}$$

The proposed SN-based $100\alpha\%$ confidence interval is given by

$$\left\{\theta : \theta_n \pm \sqrt{U_{1,\epsilon}(\alpha) \times n^{-3} \sum_{j=\lfloor \epsilon n \rfloor}^{n} [j(\hat{\theta}_j - \hat{\theta}_n)]^2}\right\} \tag{4}$$

where $U_{1,\epsilon}(\alpha)$ is the 100αth percentile of the distribution for $U_{1,\epsilon}$.

Note that the normalizing factor D_n^2 is an inconsistent estimator of the long run variance of $\hat{\theta}_n$, but is (asymptotically) proportional to the asymptotic variance, so the limiting distribution is pivotal for a given ϵ. The upper critical values of the distribution of $U_{1,\epsilon}$ can be easily approximated following Lobato (2001) by approximating a Brownian motion with the standardized partial sum process of iid N(0,1) random variables. We thus generate approximate critical values for the distribution of $U_{1,\epsilon}$, for $\epsilon = 0, 0.01, 0.02, \cdots, 0.5$ in R based on $500,000$ independent runs. The upper critical values of the distribution of $U_{1,\epsilon}$ turn out to be approximately a quadratic function of ϵ for several αs of practical interest, and the coefficients correspodning to the slope, the linear, and the quadratic terms for the fitted quadratic polynomial is given in Table 1 as well as the R^2 values (close to 1). The formulas provide a convenient way to get the upper critical values of $U_{1,\epsilon}$ for any $\epsilon \in [0, 0.5]$.

Please insert Table 1 here!

3. Asymptotic theory

In this section, we derive the asymptotic distribution of self-normalized statistics such as T_n defined in Corollary 1 in a general setting. To this end, recall that the Kaplan-Meier estimator $\hat{F}_{X,n}$ can be represented as a function of the two quantities

$$\hat{F}_Z(z) = \hat{F}_{Z,n}(z) := \frac{1}{n}\sum_{i=1}^{n} I\{Z_i \le z\}, \quad \hat{H}_0(z) = \hat{H}_{0,n}(z) := \frac{1}{n}\sum_{i=1}^{n} I\{Z_i \le z\}\delta_i.$$

More precisely,

$$\hat{F}_Z(z) = 1 - \prod_{x \le z}(1 - d\hat{\Lambda}(x)), \quad \hat{\Lambda}(z) := \int_{-\infty}^{z} \frac{1}{1 - \hat{F}_Z(x-)} d\hat{H}_0(x),$$

and the same representation holds for F_X in terms of F_Z and $H_0(y) := P(Z \le y, \delta = 1)$, see Chapter 3.9 in van der Vaart and Wellner (1996) for details. Here, $\prod_{x \le z}(1 - d\hat{\Lambda}(x))$ stands for the product-integral, see Chapter 3.9 in van der Vaart and Wellner (1996) for a precise definition. In other words,

$$\hat{F}_{X,n}(y) = \xi(\hat{F}_Z(\cdot), \hat{H}_0(\cdot))(y) \tag{5}$$

where ξ denotes the map $(\hat{F}_Z(\cdot), \hat{H}_0(\cdot)) \mapsto \hat{F}_{X,n}(\cdot)$ implicitly defined above. By the results in Section 3.9.4 of van der Vaart and Wellner (1996) the map ξ is compactly differentiable. In what follows, denote its derivative evaluated at the point (F_Z, H_0) by ξ'.

For the self-normalized approach, we need to consider the estimators

$$\hat{F}_{Z,k}(y) := \frac{1}{k}\sum_{i=1}^{k} I\{Z_i \le y\}, \quad \hat{H}_{0,k}(y) := \frac{1}{k}\sum_{i=1}^{k} I\{Z_i \le y\}\delta_i.$$

Additionally, let

$$\hat{F}_{X,k}(y) := \xi(\hat{F}_{Z,k}(\cdot), \hat{H}_{0,k}(\cdot))(y).$$

Note that the quantity $\hat{F}_{X,k}(y)$ is simply the Kaplan-Meier estimator computed from the sub-sample $(Z_1, \delta_1), ..., (Z_k, \delta_k)$. A natural way to estimate θ from the sub-sample $(Z_1, \delta_1), ..., (Z_k, \delta_k)$ is to define $\hat{\theta}_k := \phi(\hat{F}_{X,k}(\cdot))$.

One difficulty arising in the analysis of censored data lies in the fact that the distribution function F_X of the survival times is only identified [in a general non-parametric sense] up to the upper support point of the distribution F_Z, that is on the interval $(-\infty, \tau_Z)$ where $\tau_Z := \inf\{t | F_Z(t) = 1\}$. In what follows, we assume that there exists a $\tau < \tau_Z$ such that the quantity of interest, say θ, is $\mathbb{R}^d$-valued and depends only on the values of F_X on the interval $(-\infty, \tau)$ for some τ, τ_Z. This definition ensures that the parameter θ is identifiable from the observable data. Of course, the upper bound τ_Z is not known in practice. However, in many applications it suffices to assume that, we have $\theta = \phi(\hat{F}_X(\cdot)|_{(-\infty,\tau)})$ for some $\tau < \tau_Z$. One example is the estimation of $F_X(t)$ for $t < \tau_Z$. Another example is the estimation of $F_X^{-1}(\tau)$ for $\tau < F_X(\tau_Z)$. Note that a similar approach was taken by El Ghouch et al (2011).

In order to construct confidence intervals for possibly vector-valued parameters θ, we need to consider the following quantity

$$\hat{T}_n(\epsilon) := n(\hat{\theta}_n - \theta)^T \Big(n^{-2} \sum_{j=\lfloor \epsilon n \rfloor}^{n} j^2 (\hat{\theta}_j - \hat{\theta}_n)(\hat{\theta}_j - \hat{\theta}_n)^T \Big)^{-1} (\hat{\theta}_n - \theta).$$

In order to derive the limiting distribution of $\hat{T}_n(\epsilon)$, we make the following assumptions. Assume that for some $\tau_U < \tau_Z$ we have

- (F) The distribution functions F_Y, F_X are continuous on the support of F_Z and their support is contained in $[0, \infty)$.
- (C) The map $\phi : \ell^\infty([0, \tau_U]) \supset D_\phi \to \mathbb{R}^d$ is compactly differentiable at $F_X(\cdot)|[0, \tau_U]$ tangentially to the vector space W and its derivative is ϕ'.
- (W) Let $\mathcal{Z} := [0, \tau_U]$ and define

$$\begin{aligned} \mathbb{G}_{n,1} &:= t\sqrt{n}\Big(\hat{F}_{Z,\lfloor nt \rfloor}(z) - F_Z(z)\Big)_{t\in[0,1], z\in\mathcal{Z}}, \\ \mathbb{G}_{n,2} &:= t\sqrt{n}\Big(\hat{H}_{0,\lfloor nt \rfloor}(z) - H_0(z)\Big)\Big)_{t\in[0,1], z\in\mathcal{Z}}. \end{aligned}$$

Assume that for a separable, centered Gaussian process $\mathbb{G}$ on $\ell^\infty([0,1] \times [0, \tau_U]) \times \ell^\infty([0,1] \times [0, \tau_U])$ we have

$$\mathbb{G}_n := (\mathbb{G}_{n,1}, \mathbb{G}_{n,2}) \rightsquigarrow (\mathbb{G}^{(1)}, \mathbb{G}^{(2)}) = \mathbb{G}.$$

Additionally, assume that the sample paths of $\xi'\mathbb{G}$ [recall that ξ' was defined after (5)] are, with probability one, contained in the set

$$U := \Big\{ (h_t)_{t\in[0,1]} \Big| h_t \in W \quad \forall t, \ \sup_t \|h_t\|_\infty < \infty \Big\}$$

where W is from condition (C).

(G) Each component of the limit process $\mathbb{G}$ from condition (W) has a covariance function of the form $\mathbb{E}[\mathbb{G}^{(i)}(s,t)\mathbb{G}^{(j)}(s',t')] = (s \wedge s')K_{ij}(t,t')$ for $i,j = 1,2$ where K_{ij} is a non-degenerate, uniformly bounded covariance kernel.

Before we proceed, let us briefly discuss the conditions stated above.

Remark **1**. Assumption (F) is not very strong since in most applications of censored data the variables of interest X are canonically non-negative. Moreover, by a coordinate transformation it can be weakened to distributions with arbitrary finite lower support point.

Remark **2**. Assumption (C) is the compact differentiability assumption. It is satisfied for many examples of practical interest. First, it applies to the map $F \mapsto (F(y_1), ..., F(y_d))$ as long as $y_1, ..., y_d < \tau_Z$. Second, it is satisfied for a collection of quantiles. More precisely, denote by $\tau_1, ..., \tau_d$ a collection of numbers in $(0,1)$. Under the additional assumptions $F_Z(F_X^{-1}(\tau_j)) < 1$ for each $j = 1, ..., d$, if F_X has a positive density at the points $F_X^{-1}(\tau_1), ..., F_X^{-1}(\tau_d)$, the map $F \mapsto (F^{-1}(\tau_1), ..., F^{-1}(\tau_d))$ is compactly differentiable, see Section 3.9.4 in van der Vaart and Wellner (1996) for details. Finally, it is easy to see that the results also apply to the map $F \mapsto \int g(u)dF(u)$ as long as g is of bounded variation and its support is contained in $[0, \tau]$ for some $\tau < \tau_Z$.

Remark **3**. Assumption (W) is satisfied for many kinds of dependent data. To see this, observe that the processes $\mathbb{G}_{n,1}, \mathbb{G}_{n,2}$ defined there can be viewed as sequential empirical processes indexed by the classes of functions $\mathcal{F}_1 := \{y \mapsto I\{y \le z\} | z \in [0, \tau_U]\}$ and $\mathcal{F}_2 := \{(y, \delta) \mapsto \delta I\{y \le z\} | z \in [0, \tau_U]\}$, respectively. Under the additional assumption that F_Z has a uniformly bounded density, the bracketing numbers of those classes of functions [see van der Vaart and Wellner (1996) for a definition] are of the form $N_{[\,]}(\epsilon, \mathcal{F}_k, L^2(P_{Y,\delta})) \leq C\epsilon^{-1}$ for some finite constant C and $k = 1, 2$. Thus Theorem 2.16 in Volgushev and Shao (2013) and the findings in Andrews and Pollard (1994) show that (W) is satisfied for α-mixing sequences with $\alpha(k) \leq k^{-(2+\epsilon)}$ for some $\epsilon > 0$ [set $Q = 2$ and $\gamma = 2/(1 + \epsilon/2)$ in Andrews and Pollard (1994)]. Similarly, Theorem 2.16 in Volgushev and Shao (2013) and the results in Hagemann (2012) imply that (W) holds for sequences satisfying a geometric moment contraction assumption, see Wu and Shao (2004) for more details. Finally, note that condition (G) is also satisfied in both settings discussed above.

We now are ready to state our main result.

THEOREM **1**. *Let conditions (F), (C), (W), (G) hold. Denote by* $\mathbb{B}$ *a vector of independent standard Brownian motions on* $[0, 1]$*. Then for any fixed* $\epsilon \in (0, 1)$

$$\hat{T}_n(\epsilon) \ \rightsquigarrow \ \mathbb{B}(1)^T \Big(\int_{[\epsilon,1]} \Big(\mathbb{B}(s) - s\mathbb{B}(1)\Big)\Big(\mathbb{B}(s) - s\mathbb{B}(1)\Big)^T ds \Big)^{-1} \mathbb{B}(1).$$

Proof of Theorem 1. The proof relies on general results in Volgushev and Shao (2013), hereafter VS. More precisely, we will apply Proposition 3.1 in VS after setting the measure H defined there to be given by $H(A) := \lambda(\{t \in [\epsilon, 1] : (0, t) \in A\})$ with λ denoting the one-dimensional Lebesgue measure [note that here $(0, t)$ denotes a point in $\mathbb{R}^2$]. Now note that condition (C) implies (C) in VS. Moreover, (W) and (G) yield (W') and (A1') in VS, and by Proposition 2.12 in VS conditions (W), (A1) in VS follow. Similarly, (A2) in VS is a direct consequence of (W) in the present paper. Now we see that Proposition 3.1 in VS and the discussion thereafter imply the weak convergence of $\hat{T}_n(\epsilon)$ to

$$\begin{aligned} & \mathbb{V}_{0,1}^T \Big(\int_\Delta \Big(\mathbb{V}_{s,t} - (t-s)\mathbb{V}_{0,1}\Big)\Big(\mathbb{V}_{s,t} - (t-s)\mathbb{V}_{0,1}\Big)^T dH(s,t) \Big)^{-1} \mathbb{V}_{0,1} \\ = \ & \mathbb{V}_{0,1}^T \Big(\int_{[\epsilon,1]} \Big(\mathbb{V}_{0,t} - t\mathbb{V}_{0,1}\Big)\Big(\mathbb{V}_{0,t} - t\mathbb{V}_{0,1}\Big)^T dt \Big)^{-1} \mathbb{V}_{0,1} \end{aligned}$$

where $\mathbb{V}_{s,t} := \phi'(\xi'(\mathbb{G}^{(1)}(t, \cdot), \mathbb{G}^{(2)}(t, \cdot))(\cdot)) - \phi'(\xi'(\mathbb{G}^{(1)}(s, \cdot), \mathbb{G}^{(2)}(s, \cdot))(\cdot))$. It thus remains to show that the process $\mathbb{V}_{s,t}$ can be represented as

$$\mathbb{V}_{s,t} = \Sigma^{1/2}(\mathbb{B}(t) - \mathbb{B}(s))$$

with $\Sigma^{1/2}$ denoting a non-degenerate matrix and $\mathbb{B}$ a vector of independent standard Brownian motions on [0,1]. To see this, start by observing that the special structure of the derivative map ξ' together with the conditions on $\mathbb{G}$ implies that the process $\mathbb{F} := (\phi'(\xi'(\mathbb{G}^{(1)}(t,\cdot),\mathbb{G}^{(2)}(t,\cdot))(y)))_{(t,y)\in[0,1]\times\mathcal{Z}}$ is a centered Gaussian process and has a covariance structure of the form $\mathbb{E}[\mathbb{F}(t,y)\mathbb{F}(s,z)] = (s\wedge t)\kappa(y,z)$ for a uniformly bounded covariance kernel κ. Next observe that $\phi' = (\phi'_1,...,\phi'_d)$ with each ϕ'_j being a continuous, linear map on $W \subset \ell^\infty([0,\tau_U])$. By the Riesz representation theorem [see the discussion in the proof of Lemma 3.9.8 in van der Vaart and Wellner (1996)], there exist signed Borel measures $\mu_i, i=1,...,d$ on $\mathcal{Z}$ such that for $i=1,...,d$

$$(\phi' h)_i = \int h(s)d\mu_i(s).$$

We thus see that $\phi'\mathbb{F}$ is a vector of centered Gaussian processes that are also jointly Gaussian and that additionally

$$\mathbb{E}[(\phi'\mathbb{F}(s,\cdot))_i(\phi'\mathbb{F}(s',\cdot))_j] = \int (s\wedge s')\kappa(z,z')d\mu_i(z)d\mu_j(z') = (s\wedge s')\int \kappa(z,z')d\mu_i(z)d\mu_j(z').$$

The claim follows with $(\Sigma)_{i,j} = (\int \kappa(z,z')d\mu_i(z)d\mu_j(z'))_{i,j}$, and the proof of the theorem is thus complete. □

4. Simulations

In this section, a simulation study is carried out to compare the performance of three types of confidence intervals (EL, BEL and SN) in terms of coverage probability, interval length and computational time. Let *blk1* be the block size used in the BEL approach to divide the time series into overlapping blocks, and *blk2* is the one used to estimate the long run variance. Note that *blk1* equals one in the EL approach. Recall that no block size is needed for the SN approach but a trimming parameter ϵ is involved. Following the simulation design presented in El Ghouch et al. (2011), we generate time series data in the form of ARMA models $A_t = \sum_i \alpha_i A_{t-i} + \sum_j \gamma_j \epsilon_{t-j} + \epsilon_t$ with ϵ_i being Gaussian white noise. We then transform the data to have a pre-specified marginal distribution F_X and F_Y by the probability integral transformation. The sample size in each series is fixed at $n = 300$.

Model 1. The data are generated from $X_i \sim$ MA(3) with uniform censoring. The MA coefficients are $(\gamma_1,\gamma_2,\gamma_3) = (4.5,-3.1,2.7)$. For both this model and Model 2 below, the survival distribution is assumed to be standard exponential and the censoring distribution is uniform on $[0, c]$ where c is determined by the censoring percentage. The cut-off value decreases as censoring percentage increases, for example, the value of c is 3.921, 1.594 and 0.761 corresponding to censoring percentage of 25, 50 and 70.

Model 2. The data are generated from $X_i \sim \text{ARMA}(3, 3)$ with uniform censoring. The AR coefficients are $(\alpha_1, \alpha_2, \alpha_3) = (1.7, -1.3, 0.45)$ and MA coefficients $(\gamma_1, \gamma_2, \gamma_3) = (4.5, -3.1, 2.7)$. Note that the dependence is stronger under Model 2 than Model 1.

Model 3. Consider a bimodal mixture of the form $f = 0.8f_1 + 0.2f_2$, where f_1 is the density of $\exp(Z/2)$, with Z being $N(0, 1)$, and f_2 is the density of $N(0, 0.17^2)$. Let the censoring distribution be $Exp(\lambda)$ with the parameter λ determined by the censoring percentage. Then we simulate data from an AR(1) model with $\gamma = 0.8$ and transform the resulting time series using the marginal probability integral transform.

4.1. Estimating distribution function at a point $F_X(t_0)$

The first example is $\theta = F_X(t_0)$, namely, $\xi(t) = 1(t \le t_0)$ in (2). Talbe 2 presents the comparison of three methods in terms of the coverage percentage and average length of the 95% confidence interval at $t_0 = F_X^{-1}(p_0)$ for $p_0 = 0.2, 0.5$ and 0.7. For Model 1, the simulation time of 1000 runs is 1.8 hours for the SN method on a Dell PC with Intel Core 2 Duo E8400 processor. In contrast, the BEL method with the optimal block size selected from *blk1*$\times$ *blk2* $\in \{1, 2, 3, 5, 10, 15, 20\} \times \{1, 2, 3, 4, 5, 10, 15, 20\}$ takes 12.85 hours on average. The optimal block size is chosen to minimize the empirical coverage error and is actually an infeasible one. Here we perform the optimal block size selection following El Ghouch et al. (2011) to make a comparison with the SN method. Note that the required computational time for the BEL method would be more demanding if we perform the optimal block size selection on a finer grid.

Please insert Table 2 here!

Compared with the (B)EL approach, the SN-based CI is wider in its length, but is often closer to the nominal coverage level. Especially when (B)EL undercovers the true parameter even with the optimal block sizes, the SN approach tends to cover the parameter with higher probability, at the sacrifice of a longer interval; see e.g. the performance for Model 2 in the middle of Table 2 when dependence is strong. In Model 1 when the dependence is weak, BEL is competitive to SN in terms of the coverage probability but the comparison presented here is unfair to the SN approach as the optimal block size is empirically determined and is in fact not possible for a given time series in practice. Also, note that we chose the same cutoff-parameter $\epsilon = 0.2$ in all simulations for the SN method, the optimal block sizes for BEL were chosen differently for each model and estimation scenario. In Model 2 when the dependence is positively stronger and Model 3 when the distribution function is non-standard, the SN approach outperforms (B)EL in almost all the cases in the sense that coverage probability is closer to the nominal level. As mentioned in El Ghouch et al. (2011) and also from our own experience, the confidence interval based on (B)EL may over-cover or undercover the parameter with different combinations of block sizes and the coverage probability varies a lot with respect to block sizes.

4.2. Estimating the quantiles

A second example is quantile estimation when $\theta = F_X^{-1}(q)$. The median survival time corresponds to $q = 0.5$. It is often a quantity of practical interest and may be preferred to the mean for it is robust to long tails in the estimated survival distribution, while mean might not be estimable for a right censored variable with bounded support. In the setting of censored i.i.d. data, some literature regarding inference of the median survival time does exist. For example, Brookmeyer and Crowley (1982) proposed to construct an interval by inverting a generalized sign test for right censored data. Efron (1981) suggested a bootstrap-based CI, which was further extended by Cai and Kim (2003) to correlated censored data. Note that Cai and Kim (2003) dealt with clustered data, where the dependence exists within each cluster, the survival time and censoring are independent across clusters, the number of observation within a cluster is bounded and the number of clusters grows to infinity. Their setting is quite different from ours since for a time series, the number of clusters can be regarded as one but the number of observations in this cluster is increasing as more data become available. Given the differences in the two settings, we therefore do not present a comparison between the SN method and the approach used in Cai and Kim (2003).

Naturally we would expect the estimating procedure to break down if q is large relative to the censoring percentage since the q-th quantile of the unobserved data is poorly estimated in most of the SN subsamples. In some situations the sub-samples may not be able to produce an estimate, even when not all the data in the initial sub-sample are censored.

And the resulting NA output from the initial sub-samples further affects the inconsistent estimation for asymptotic variance, rendering the confidence interval length NA. In the simulation when summarizing for the empirical coverage probability and CI length, we choose to discard the NA values.

In Models 1 and 2, uniform censoring is employed with an upper bound which cuts off the value at some particular point c, the exact values are 3.921, 1.594 and 0.761 for censoring percentage of 25, 50 and 70, respectively, they corresponds to 0.980, 0.800, and 0.533 cut-off quantile of standard exponential distribution. Essentially it is impossible to draw meaningful inference for any quantile higher than the cut-off points. In practice, the SN approach both results in high NA output for the interval length, and low coverage probability after removing the NA values. Also it is extremely difficult to estimate the quantile near the cut-off points. For example, the associated NA count of median under 70% censoring is more than 600 out of the 1000 independent runs. Such performance is expected since any nonparametric method will fail given insufficient data, hence the result for that cell is not presented in Table 3. For the results shown in Table 3, the number of associated NA counts is zero for most cells and negligible for others, and is omitted from presentation. On the up side, for such cells, the SN method performs quite well delivering a reasonably accurate coverage probability. Comparing Model 1 to Model 2, we find that, when the dependence is positive and gets stronger, the interval gets longer, which agrees with intuition.

In Model 3, the coverage probability is consistently high for different q values. The reason is that in Model 3, exponential censoring is used instead of the uniform censoring. Since the exponential distribution is unbounded and light tailed with a decreasing density, the censoring affects the estimation of quantiles in a different way. When censoring percentage increases, it appears that the length of CI also increases while preserving proper coverage probability. As a side note, the computation time for 1000 runs of one model with size $n = 300$ is about 30 minutes for all the presented q values at a specific censoring level for the SN method. If we increase sample size from 300 to 1000, the CI length shortens by around $\sqrt{3/10}$ in most cases and coverage probability gets closer to the nominal level.

Please insert Table 3 here!

4.3. Estimating the mean of survival time

Another example of smooth function is the mean life, or mean survival function $\theta = \int_0^\infty t dF(t)$. It is also related to another basic parameter of interest called the *mean residual life* or *remaining life expectancy* function at time t which is defined as $E(X - t|X > t)$. The mean residual life is the area under the survival curve to the right of t divided by $1 - F_X(t)$, while the mean life is the total area under the survival curve by taking $t = 0$ in the mean residual life function. The presence of censoring prevents us from accurately estimating the mean survival function, hence a proper truncation is necessary. To this end, we estimate instead $\theta = \int_0^\tau t dF(t)$ for some given τ . A standard procedure is to choose a truncation with respect to the censoring rate. Following El Ghouch et al (2011), we choose $\tau = F^{-1}(0.79)$ at 25% censoring and $\tau = F^{-1}(0.65)$ at 50% censoring. The results are summarized in Table 4. As we can see, the SN method performs very competitively relative to (B)EL approach in Models 1 and 2 and the coverage probability is greatly improved by using the SN approach in Model 3, and the SN method delivers a longer interval in all cases. Again the reported values for BEL and EL are based on the infeasible optimal block size chosen by optimizing over a grid of block sizes.

Please insert Table 4 here!

4.4. The effect of the trimming parameter ϵ

In this subsection, we investigate the effect of ϵ on the performance of the proposed approach. In finite samples, the SN method does not work with an extremely small ϵ value in the presence of censoring since the subsample estimates cannot be obtained if all the data points in a subsample are censored. A similar trimming issue also comes up in Zhou and Shao (2013), who extended the SN approach to the time series regression problem with fixed regressors. In the latter paper, a rule of thumb is to use $\epsilon = 0.1$, which was found to lead to satisfactory performance for a number of models.

Table 5 illustrates the effect of ϵ on the coverage probability and interval length when the parameter is $F(t_0)$ or quantiles. When ϵ ranges from 0.05 to 0.5 and the censoring percentage is 0.25, smaller ϵs correspond to more accurate coverage and shorter intervals in most cases, although the difference is not substantial in some cases. To give a theoretical explanation of this phenomenon, we note that the confidence interval constructed by SN is given by

$$\hat{\theta}_n \pm \sqrt{U_{1,\epsilon}(\alpha) \times D_n(\epsilon)^2/n}$$

where $D_n^2(\epsilon) = n^{-2}\sum_{j=\lfloor \epsilon n \rfloor}^{n}[j(\hat{\theta}_j - \hat{\theta}_n)]^2$ is a function of ϵ. The expected 95% interval length is $2\sqrt{U_{1,\epsilon}(0.95)/n}ED_n(\epsilon)$. We shall look into the ratio of the expected interval length compared to the $\epsilon = 0$ case. That is,

$$\text{Ratio}_n(\epsilon) = \frac{\sqrt{U_{1,\epsilon}(0.95)}ED_n(\epsilon)}{\sqrt{U_{1,0}(0.95)}ED_n(0)},$$

which converges to

$$\text{Ratio}(\epsilon) := \frac{\sqrt{U_{1,\epsilon}(0.95)}E(\sqrt{\int_\epsilon^1(\mathbb{B}(r) - r\mathbb{B}(1))^2dr})}{\sqrt{U_{1,0}(0.95)}E(\sqrt{\int_0^1(\mathbb{B}(r) - r\mathbb{B}(1))^2dr})}$$

under suitable conditions, where the latter can be approximated numerically. Figure 1 presents the plot of $\text{Ratio}(\epsilon)$ as a function of ϵ. Interestingly it can be seen that choosing ϵ close to 0.1 yields a shortest confidence interval, which provides some theoretical support to the suggestion made in Zhou and Shao (2013). On the other hand, it should be noted that the length of the CI is not overly sensitive to the choice of ϵ, with the ratio bounded between 0.985 and 1.085 when $\epsilon \in [0, 0.5]$. This provides a partial explanation why the interval gets slightly longer when ϵ increases from 0.1 to 0.5 in Table 5. As to the coverage accuracy with respect to ϵ, we would need to resort to Edgeworth expansion of the studentized quantity, which seems very challenging for censored dependent case.

Please insert Figure 1 here!

Overall, the choice of ϵ appears to be less influential, and its impact on the inference is captured by the limiting distribution anyway. By contrast, the block size has a sizable impact on the BEL approach when χ^2 approximation is used and its choice is not captured by the χ^2 limiting distribution.

Please insert Table 5 here!

5. Conclusion

In this paper we extend the SN approach in Shao (2010) to the inference of censored time series. A rigorous asymptotic theory is provided to justify the limiting distribution of the SN quantity. Compared to the work of El Ghouch et al. (2011), our approach is much easier to implement as recursive subsample estimates are very easy to calculate and no sophisticated algorithm needs to be developed. Computationally speaking, the cost of the SN approach can be considerably cheaper than the BEL approach if the optimal block size selection is pursued. Statistically speaking, the SN-based interval appears to have more accurate coverage in most cases with a longer length. This is not surprising given empirical findings in Shao (2010), which also contains theoretical explanations. Furthermore, the SN method has a wider applicability than the BEL approach for the inference of censored data, as the latter was developed in El Ghouch et al. (2011) in a framework that excludes the quantiles of survival distribution.

To conclude, we mention a few possible topics for future research. As this work seems to be the first attempt to generalize the SN method to censored time series data, a closely related topic is to consider censored spatial data. The key difficulty lies in the fact that there is no natural ordering for spatial observations. Recently, Zhang et al. (2013) made an extension of the SN approach to spatial setting by artificially ordering the data. It might be possible to combine the approach in Zhang et al. (2013) and the one developed in this paper. Furthermore, the choice of trimming parameter ϵ, although captured in the first order limiting distribution, may still lead to different finite sample results for different ϵs. The optimal choice presumably depends on the given loss function and seems very difficult to derive as it hinges on the high order Edgeworth expansion of the finite sample distribution of the SN quantity; see Zhang and Shao (2013) for recent findings on the distribution of studentized sample mean of a Gaussian weakly dependent time series. Finally, it seems possible to extend the SN approach to the inference of the regression parameter in censored quantile regression models. Further research along this direction is well underway.

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α	Intercept	ϵ	ϵ^2	R^2
90%	29.230 (0.311)	-17.661 (3.289)	192.141 (6.655)	99.674%
95%	46.947 (0.550)	-26.935 (5.850)	324.576 (11.839)	99.653%
97.5%	68.736 (0.779)	-38.774 (8.231)	499.149 (16.659)	99.715%
99%	103.290 (1.365)	-52.317 (14.415)	776.136 (29.172)	99.657%
99.5%	134.871 (1.758)	-73.261 (18.567)	1049.470 (37.576)	99.685%

Table 1: Regression output of upper critical values of $U_{1,\epsilon}$ as a quadratic function of ϵ at different α levels with associated R^2 values. Values inside parentheses are the corresponding standard errors.

			Model 1			Model 2			Model 3		
p_0	%cens	Var	EL	BEL	SN	EL	BEL	SN	EL	BEL	SN
0.2	25	coverage	0.953	0.953	0.951	0.903	0.913	0.937	0.926	0.932	0.953
		length	0.091	0.091	0.120	0.179	0.091	0.281	0.100	0.105	0.144
		parameters	1	(1,1)	$\epsilon = 0.2$	15	(5,20)	$\epsilon = 0.2$	10	(30,10)	$\epsilon = 0.2$
	50	coverage	0.955	0.958	0.952	0.904	0.909	0.931	0.930	0.939	0.943
		length	0.093	0.093	0.123	0.184	0.185	0.285	0.116	0.122	0.171
		parameters	1	(1,1)	$\epsilon = 0.2$	20	(5,20)	$\epsilon = 0.2$	10	(30,15)	$\epsilon = 0.2$
	70	coverage	0.950	0.950	0.956	0.895	0.902	0.921	0.912	0.914	0.933
		length	0.097	0.097	0.131	0.187	0.188	0.292	0.157	0.158	0.229
		parameters	1	(1,1)	$\epsilon = 0.2$	20	(5,20)	$\epsilon = 0.2$	15	(5,20)	$\epsilon = 0.2$
0.5	25	coverage	0.947	0.950	0.959	0.902	0.911	0.943	0.937	0.943	0.951
		length	0.093	0.097	0.129	0.237	0.237	0.376	0.090	0.094	0.115
		parameters	3	(15,4)	$\epsilon = 0.2$	20	(20,15)	$\epsilon = 0.2$	5	(15,5)	$\epsilon = 0.2$
	50	coverage	0.951	0.950	0.960	0.873	0.877	0.947	0.948	0.950	0.956
		length	0.107	0.109	0.153	0.237	0.241	0.393	0.144	0.148	0.212
		parameters	10	(5,10)	$\epsilon = 0.2$	20	(20,20)	$\epsilon = 0.2$	1	(15,10)	$\epsilon = 0.2$
	70	coverage	0.949	0.950	0.955	0.891	0.898	0.924	0.934	0.934	0.955
		length	0.190	0.193	0.266	0.308	0.303	0.496	0.245	0.245	0.363
		parameters	10	(5,2)	$\epsilon = 0.2$	15	(15,20)	$\epsilon = 0.2$	15	(1,15)	$\epsilon = 0.2$
0.7	25	coverage	0.949	0.949	0.948	0.870	0.871	0.940	0.941	0.942	0.950
		length	0.099	0.099	0.140	0.204	0.207	0.353	0.113	0.114	0.148
		parameters	2	(1,2)	$\epsilon = 0.2$	20	(15,30)	$\epsilon = 0.2$	1	(5,1)	$\epsilon = 0.2$
	50	coverage	0.951	0.951	0.952	0.846	0.850	0.943	0.921	0.921	0.951
		length	0.143	0.143	0.197	0.222	0.223	0.404	0.161	0.161	0.238
		parameters	2	(1,2)	$\epsilon = 0.2$	15	(5,20)	$\epsilon = 0.2$	10	(1,10)	$\epsilon = 0.2$

Table 2: Simulation 1 result of 95% CI for $F(t_0)$ at $t_0 = F^{-1}(p_0)$ for Model 1 (left), Model 2 (middle) and Model 3 (right). In the table, *coverage* is the empirical coverage percentage; *length* is the mean CI length over $B = 1000$ simulated confidence intervals, sample size is $n = 300$ in each run. The result for EL and BEL is selected according to the average *minimum coverage error* and the corresponding combination of block sizes are reported in *parameters*. The user chosen parameter(s) for EL refer to the block size used in estimating long run variance; for BEL refers to (blk1,blk2) where *blk1* is the block size used in determining subgroups in BEL and *blk2* is block size used in estimating long run variance. The parameter for SN refers to the initial fraction of the data included in the sub-sample and is fixed at $\epsilon = 0.2$ in the simulation.

	%cens	q	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	25	coverage	0.928	0.934	0.931	0.931	0.933	0.942	0.927	0.929	/
		length	0.112	0.157	0.188	0.221	0.272	0.351	0.486	0.768	/
Model 1	50	coverage	0.933	0.940	0.928	0.927	0.936	0.934	/	/	/
$n = 300$		length	0.114	0.162	0.199	0.251	0.330	0.455	/	/	/
	70	coverage	0.920	0.940	0.916	/	/	/	/	/	/
		length	0.117	0.178	0.237	/	/	/	/	/	/
	25	coverage	0.944	0.933	0.941	0.945	0.948	0.935	0.927	0.933	0.915
		length	0.060	0.085	0.104	0.122	0.145	0.187	0.256	0.397	0.716
Model 1	50	coverage	0.941	0.940	0.940	0.942	0.949	0.931	/	/	/
$n = 1000$		length	0.061	0.087	0.113	0.135	0.176	0.245	/	/	/
	70	coverage	0.945	0.940	0.940	0.944	/	/	/	/	/
		length	0.062	0.093	0.130	0.181	/	/	/	/	/
	25	coverage	0.919	0.932	0.931	0.932	0.939	0.929	0.924	0.932	/
		length	0.244	0.377	0.498	0.623	0.778	0.947	1.177	1.520	/
Model 2	50	coverage	0.915	0.932	0.930	0.929	0.918	/	/	/	/
$n = 300$		length	0.247	0.384	0.510	0.627	0.759	/	/	/	/
	70	coverage	0.917	/	/	/	/	/	/	/	/
		length	0.249	/	/	/	/	/	/	/	/
	25	coverage	0.936	0.943	0.941	0.938	0.939	0.936	0.942	0.936	0.930
		length	0.126	0.197	0.268	0.334	0.411	0.507	0.637	0.846	1.234
Model 2	50	coverage	0.942	0.939	0.937	0.944	0.936	0.934	/	/	/
$n = 1000$		length	0.127	0.199	0.274	0.343	0.430	0.539	/	/	/
	70	coverage	0.939	0.935	0.933	/	/	/	/	/	/
		length	0.128	0.204	0.278	/	/	/	/	/	/
	25	coverage	0.919	0.94	0.933	0.922	0.926	0.914	0.912	0.917	0.925
		length	0.214	0.207	0.203	0.200	0.247	0.374	0.429	0.286	0.310
Model 3	50	coverage	0.918	0.934	0.923	0.936	0.930	0.910	0.916	0.934	0.926
$n = 300$		length	0.232	0.251	0.282	0.337	0.451	0.650	0.677	0.482	0.547
	70	coverage	0.921	0.938	0.933	0.939	0.926	/	/	/	/
		length	0.300	0.366	0.465	0.619	0.857	/	/	/	/
	25	coverage	0.933	0.937	0.945	0.938	0.941	0.944	0.94	0.942	0.931
		length	0.116	0.112	0.111	0.107	0.122	0.206	0.247	0.158	0.170
Model 3	50	coverage	0.934	0.950	0.958	0.940	0.941	0.926	0.926	0.940	0.917
$n = 1000$		length	0.129	0.137	0.152	0.179	0.235	0.362	0.386	0.242	0.26
	70	coverage	0.940	0.940	0.935	0.932	0.942	0.922	0.895	/	/
		length	0.161	0.188	0.232	0.301	0.429	0.644	0.672	/	/

Table 3: Simulation result of 95% CI for $F^{-1}(q)$ for different q values based on the SN method, where $q = 0.5$ corresponds to the median survival time. In the table, *coverage* is the empirical coverage percentage; *length* is the mean CI length over $B = 1000$ simulated confidence intervals after removing NA values. The existence of NA values is due to censoring when no valid estimate can be obtained from the subsample, typically when the quantile is high relative to the censoring rate. The counts of NA values associated with each result presented are small (< 58), most of them zero. Here we choose $\epsilon = 0.2$ for all Models.

		Model 1			Model 2			Model 3		
% cens		EL	BEL	SN	EL	BEL	SN	EL	BEL	SN
25	coverage	0.950	0.950	0.953	0.950	0.950	0.950	0.928	0.932	0.942
	length	0.131	0.131	0.177	0.178	0.178	0.25	0.197	0.197	0.285
50	coverage	0.950	0.950	0.948	0.941	0.941	0.939	0.934	0.936	0.949
	length	0.116	0.116	0.159	0.153	0.153	0.212	0.193	0.195	0.288

Table 4: Simulation result of 95% CI for the (truncated) survival mean for Model 1 (left), Model 2 (middle) and Model 3 (right). In the table, *coverage* is the empirical coverage percentage; *length* is the mean CI length over $B = 1000$ simulated confidence intervals, sample size is $n = 300$ in each run. The result for EL and BEL is selected according to the *minimum coverage error* (the optimal combination of block sizes are not reported here). The initial fraction used for SN is $\epsilon = 0.2$ for all models.

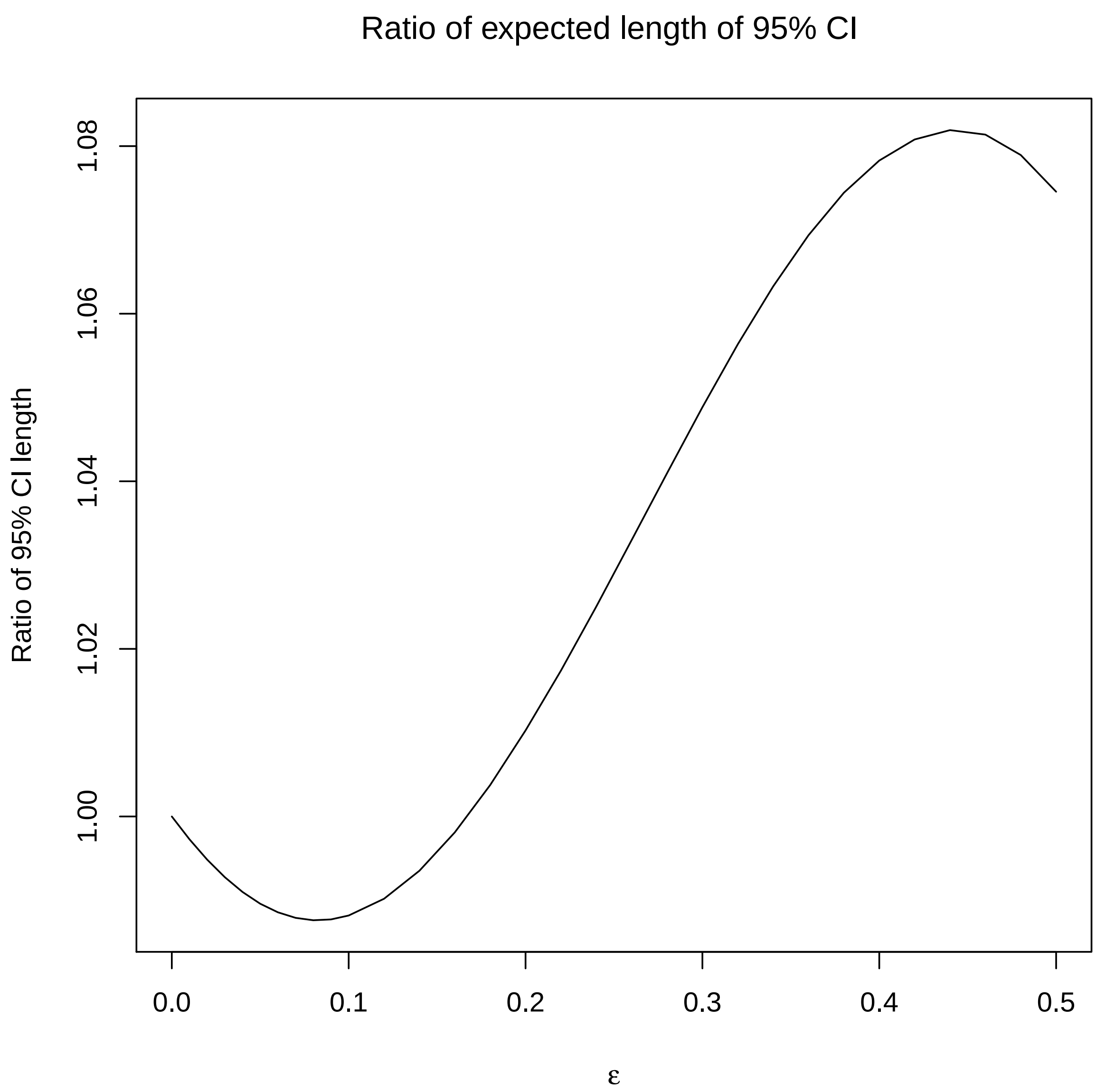

Figure 1: Ratio of expected 95% CI length at different ϵ levels to the one at 0.

Model 3 with 25% censoring and sample size $n = 300$

		$F(t_0)$			$F^{-1}(q)$								
ϵ		p_0= 0.2	0.5	0.7	q=0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	coverage	0.950	0.955	0.955	0.919	0.944	0.940	0.929	0.935	0.921	0.917	0.927	0.936
	length	0.138	0.111	0.142	0.204	0.198	0.194	0.194	0.241	0.361	0.409	0.280	0.300
0.1	coverage	0.950	0.955	0.955	0.922	0.945	0.937	0.926	0.931	0.918	0.915	0.926	0.935
	length	0.141	0.113	0.145	0.209	0.202	0.198	0.197	0.244	0.367	0.418	0.283	0.310
0.2	coverage	0.953	0.951	0.950	0.919	0.940	0.933	0.922	0.926	0.914	0.912	0.917	0.925
	length	0.144	0.115	0.148	0.214	0.207	0.203	0.200	0.247	0.374	0.429	0.286	0.310
0.3	coverage	0.952	0.952	0.949	0.920	0.939	0.926	0.917	0.924	0.913	0.912	0.918	0.920
	length	0.148	0.118	0.151	0.221	0.212	0.207	0.205	0.252	0.381	0.441	0.290	0.313
0.4	coverage	0.952	0.956	0.944	0.913	0.933	0.929	0.905	0.921	0.909	0.912	0.918	0.914
	length	0.151	0.120	0.154	0.227	0.216	0.212	0.210	0.258	0.389	0.454	0.296	0.317
0.5	coverage	0.954	0.950	0.938	0.907	0.937	0.930	0.909	0.926	0.905	0.904	0.919	0.908
	length	0.155	0.122	0.158	0.234	0.222	0.217	0.215	0.265	0.397	0.467	0.303	0.324
Model 3 with 25% censoring and sample size $n = 1000$													
ϵ		p_0=0.2	0.5	0.7	q=0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	coverage	0.948	0.949	0.946	0.933	0.932	0.934	0.936	0.943	0.944	0.930	0.934	0.944
	length	0.078	0.060	0.080	0.115	0.111	0.108	0.104	0.122	0.196	0.236	0.152	0.166
0.1	coverage	0.948	0.949	0.948	0.933	0.933	0.936	0.934	0.942	0.944	0.931	0.934	0.944
	length	0.079	0.061	0.080	0.116	0.112	0.109	0.105	0.123	0.198	0.239	0.153	0.167
0.2	coverage	0.945	0.947	0.945	0.933	0.937	0.945	0.938	0.941	0.944	0.94	0.942	0.931
	length	0.081	0.062	0.082	0.116	0.112	0.111	0.107	0.122	0.206	0.247	0.158	0.170
0.3	coverage	0.958	0.957	0.953	0.932	0.933	0.933	0.933	0.937	0.950	0.935	0.938	0.944
	length	0.081	0.063	0.085	0.122	0.118	0.115	0.110	0.128	0.208	0.253	0.161	0.173
0.4	coverage	0.951	0.955	0.951	0.932	0.928	0.936	0.925	0.939	0.949	0.934	0.936	0.944
	length	0.083	0.064	0.087	0.125	0.121	0.118	0.113	0.131	0.214	0.261	0.165	0.176
0.5	coverage	0.943	0.954	0.954	0.931	0.925	0.937	0.927	0.937	0.943	0.931	0.939	0.946
	length	0.085	0.066	0.088	0.129	0.125	0.122	0.116	0.134	0.219	0.267	0.169	0.180

Table 5: Effect of initial fraction ϵ on simulation result of 95% CI for $F(t_0)$ at $t_0 = F^{-1}(p_0)$ and for $F^{-1}(q)$ based on the SN method. The data are simulated from Model 3 with 25% censoring at sample size $n = 300, 1000$, and the result is based on 1000 independent runs.