Random block matrices generalizing the classical ensembles

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Abstract

In this paper we consider random block matrices which generalize the classical Laguerre ensemble and the Jacobi ensemble. We show that the random eigenvalues of the matrices can be uniformly approximated by the roots of matrix orthogonal polynomials and obtain a rate for the maximum difference between the eigenvalues and the roots. This relation between the random block matrices and matrix orthogonal polynomials allows a derivation of the asymptotic spectral distribution of the matrices.

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1 Introduction

The three classical ensembles of random matrix theory are the Hermite, Laguerre and Jacobi ensembles. Associated with each ensemble there is a real positive parameter β which is usually considered for three values. The case $\beta = 1$ corresponds to real matrices, while the ensembles for

$\beta = 2$ and $\beta = 4$ arise from complex and quaternion random matrices, respectively, according to Dyson (1962) threefold classification. Dumitriu and Edelman (2002) provided tridiagonal random matrix models for the general β-Hermite and β-Laguerre ensembles for all $\beta > 0$. The development of a tridiagonal matrix model corresponding to the general β-Jacobi ensemble for all $\beta > 0$ was an open problem, which was recently considered by Killip and Nenciu (2004). The spectral distributions of large dimensional matrices of the three classical ensembles have been studied extensively in the literature, see Mehta (2004), Bai (1999) or Bai and Silverstein (1995). Several authors have extended the study of random matrices to the case of random block matrices and we refer to the works of Girko (2000) and Oraby (2007a,b), among others. Recently, Dette and Reuther (2009) considered random block matrices which generalize the tridiagonal model of the Hermite ensemble constructed by Dumitriu and Edelman (2002) and obtained the asymptotic spectral distribution. It is the purpose of the present paper to investigate the asymptotic properties of some random block tridiagonal matrices corresponding to the classical Laguerre and Jacobi ensemble. In Section 2 we revisit the Jacobi ensemble and introduce the random block matrices considered in this paper. In Section 3 we will review some facts on matrix orthogonal polynomials and the limiting distribution of their roots. In Section 4 we demonstrate that the eigenvalues of the random block matrices can be approximated uniformly (almost surely) by the deterministic roots of matrix orthogonal polynomials. Matrix polynomials have been studied by several authors, see Sinap and van Assche (1994), Duran and van Assche (1995), Duran (1995, 1996, 1999), Duran and Lopez-Rodriguez (1996, 1997), Grünbaum (2003) and Damanik et al. (2008). In particular, Duran (1999) and Dette and Reuther (2009) provided limit theorems for the empirical distribution of the roots of orthogonal matrix polynomials and these results are applied to obtain the asymptotic spectral distribution of the random block matrices. In Section 5 we introduce a generalization of the Laguerre ensemble to block matrices and study the corresponding limiting spectral distribution. Finally, the proofs of some technical results are deferred to an Appendix in Section 6.

2 The Jacobi ensemble and the corresponding block matrices

The Jacobi ensemble is defined by its density

$$f_{\beta,a,b}(\lambda) = c_{\beta,a,b} \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta \prod_{j=1}^{n} (2 - \lambda_j)^a (2 + \lambda_j)^b I_{(-2,2)}(\lambda_j), \tag{2.1}$$

where $\beta > 0,\ a, b > -1$ and the normalization constant $c_{\beta,a,b}$ is given by

$$c_{\beta,a,b} = 4^{-n(a+b+\frac{n-1}{2}\beta+1)} \prod_{j=0}^{n-1} \frac{\Gamma(1+\frac{\beta}{2})\Gamma(a+b+(n+j-1)\frac{\beta}{2}+2)}{\Gamma(1+\frac{\beta}{2}+\frac{\beta}{2}j)\Gamma(a+\frac{\beta}{2}j+1)\Gamma(b+\frac{\beta}{2}j+1)}.$$

Killip and Nenciu (2004) provided a tridiagonal random matrix model of the Jacobi ensemble where the entries are composed of independent random variables with beta distributions on the interval $[-1, 1]$. To be precise, note that the Beta distribution $Beta(a, b)$ on the interval $[-1, 1]$ is defined by the density

$$f(x) = 2^{1-a-b} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} (1-x)^{a-1}(1+x)^{b-1} I_{(-1,1)}(x). \tag{2.2}$$

Now let α_k for $k = 0, \ldots, 2n-1$ be independent random variables with

$$\alpha_k \sim \begin{cases} Beta\left(\frac{2n-k-2}{4}\beta_n + a + 1, \frac{2n-k-2}{4}\beta + b + 1\right) & \text{for } k \text{ even,} \\ Beta\left(\frac{2n-k-3}{4}\beta + a + b + 2, \frac{2n-k-1}{4}\beta\right) & \text{for } k \text{ odd,} \end{cases}$$

then the joint density of the eigenvalues of the tridiagonal matrix

$$J_n = J_n(\beta, a, b) = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \end{pmatrix} \in \mathbb{R}^{n \times n} \tag{2.3}$$

with entries

$$\begin{aligned} b_{k+1} &= (1-\alpha_{2k-1})\alpha_{2k} - (1+\alpha_{2k-1})\alpha_{2k-2}\ , \\ a_{k+1} &= \left\{(1-\alpha_{2k-1})(1-\alpha_{2k}^2)(1+\alpha_{2k+1})\right\}^{1/2} \end{aligned}$$

$(\alpha_{2n-1} = \alpha_{-1} = \alpha_{-2} = -1)$ is given by the Jacobi ensemble (2.1). As a generalization of the matrix J_n, we consider random tridiagonal block matrices of the form

$$J_n^{(p)} = J_n^{(p)}(\beta, a, b) := \begin{pmatrix} B_{0,n}^{(p)} & A_{1,n}^{(p)} & & & & \\ A_{1,n}^{(p)} & B_{1,n}^{(p)} & A_{2,n}^{(p)} & & & \\ & A_{2,n}^{(p)} & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & A_{\frac{n}{p}-1,n}^{(p)} \\ & & & & A_{\frac{n}{p}-1,n}^{(p)} & B_{\frac{n}{p}-1,n}^{(p)} \end{pmatrix} \in \mathbb{R}^{n \times n}, \tag{2.4}$$

where $n = mp$ with $m, p \in \mathbb{N}$ and the symmetric $p \times p$ blocks $A_{i,n}^{(p)}$ and $B_{i,n}^{(p)}$ are defined by

$$B_{i,n}^{(p)} := \begin{pmatrix} b_{ip+1}^{(1,n)} & a_{ip+1}^{(1,n)} & a_{ip+2}^{(2,n)} & \cdots & \cdots & a_{(i+1)p-1}^{(p-1,n)} \\ a_{ip+1}^{(1,n)} & b_{ip+2}^{(1,n)} & a_{ip+2}^{(1,n)} & & \cdots\cdots & a_{(i+1)p-1}^{(p-2,n)} \\ a_{ip+2}^{(2,n)} & \vdots & \vdots & \ddots & & \vdots \\ \vdots & & & & b_{(i+1)p-1}^{(1,n)} & a_{(i+1)p-1}^{(1,n)} \\ a_{(i+1)p-1}^{(p-1,n)} & \cdots & \cdots & \cdots & a_{(i+1)p-1}^{(1,n)} & b_{(i+1)p}^{(1,n)} \end{pmatrix}$$

and

$$A_{i,n}^{(p)} := \begin{pmatrix} a_{ip}^{(p,n)} & a_{ip}^{(p-1,n)} & a_{ip}^{(p-2,n)} & \cdots & \cdots & a_{ip}^{(1,n)} \\ a_{ip}^{(p-1,n)} & a_{ip+1}^{(p,n)} & a_{ip+1}^{(p-1,n)} & \cdots & \cdots & a_{ip+1}^{(2,n)} \\ a_{ip}^{(p-2,n)} & & \ddots & & & \vdots \\ \vdots & & & & a_{ip+p-2}^{(p,n)} & a_{ip+p-2}^{(p-1,n)} \\ a_{ip}^{(1,n)} & \cdots & \cdots & \cdots & a_{ip+p-2}^{(p-1,n)} & a_{ip+p-1}^{(p,n)} \end{pmatrix},$$

respectively. The entries in these matrices are given by

$$b_{k+1}^{(1,n)} = (1 - \alpha_{2k-1}^{(1,n)})\alpha_{2k}^{(1,n)} - (1 + \alpha_{2k-1}^{(1,n)})\alpha_{2k-2}^{(1,n)},$$

$$a_{k+1}^{(j,n)} = ((1 - \alpha_{2k-1}^{(j,n)})(1 - {\alpha_{2k}^{(j,n)}}^2)(1 + \alpha_{2k+1}^{(j,n)}))^{\frac{1}{2}},$$

where the random variables

$$\alpha_k^{(j,n)} \sim \begin{cases} Beta\left(\frac{2n-k-2}{4}\beta_n^{(j)} + a_n^{(j)} + 1, \frac{2n-k-2}{4}\beta_n^{(j)} + b_n^{(j)} + 1\right) & \text{for } k \text{ even} \\ Beta\left(\frac{2n-k-3}{4}\beta_n^{(j)} + a_n^{(j)} + b_n^{(j)} + 2, \frac{2n-k-1}{4}\beta_n^{(j)}\right) & \text{for } k \text{ odd} \end{cases}$$

are independent and for $j = 1, \ldots p$ the parameters $a_n^{(j)}, b_n^{(j)}, \beta_n^{(j)} \in \mathbb{R}$ satisfy $a_n^{(j)}, b_n^{(j)} > -1$ and $\beta_n^{(j)} > 0$. Note that in the case $p = 1$ the matrix $J_n^{(p)}$ reduces to the Jacobi matrix J_n given in (2.3). A similar extension of the Laguerre ensemble to block matrices will be defined in Section 5. In the following sections we investigate the limiting spectral behaviour of block matrices of the form as in (2.4).

3 Matrix orthogonal polynomials

Consider a sequence $(P_n)_{n\geq 0}$ of $p \times p$ matrix polynomials, i.e. of polynomials with matrix coefficients in $\mathbb{R}^{p\times p}$. A matrix measure Σ is a $p \times p$ matrix of signed Borel measures such that for each Borel set $A \subset \mathbb{R}$ the matrix $\Sigma(A)$ is symmetric and nonnegative definite. The sequence $(P_n)_{n\geq 0}$ of matrix polynomials is orthonormal with respect to the matrix measure Σ if

$$\int P_n(x) d\Sigma(x) P_m^T(x) = \delta_{nm} I_p, \tag{3.1}$$

where I_p denotes the $p \times p$ identity matrix. By Favard's Theorem (see Sinap and van Assche (1996)) a sequence of matrix orthonormal polynomials $(P_n)_{n\geq 0}$ can be characterized by a three term recurrence

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_n P_n(t) + A_n^T P_{n-1}(t), \quad n \geq 0, \tag{3.2}$$

with initial condition $P_{-1}(t) = 0_p,\ P_0(t) = A_0$. For example, the matrix Chebyshev polynomials of the first kind $(T_n^{A,B})_{n\geq 0}$ are defined recursively by

$$\begin{aligned} tT_1^{A,B}(t) &= AT_2^{A,B}(t) + BT_1^{A,B}(t) + \sqrt{2}AT_0^{A,B}(t), \\ tT_n^{A,B}(t) &= AT_{n+1}^{A,B}(t) + BT_n^{A,B}(t) + AT_{n-1}^{A,B}(t),\ n \geq 2, \end{aligned} \tag{3.3}$$

where A and B are symmetric $p \times p$ matrices, A is non-singular and $T_0^{A,B}(t) = I_p$, $T_1^{A,B}(t) = (\sqrt{2}A)^{-1}(tI_p - B)$. The Chebyshev polynomials $(T_n^{A,B})_{n\geq 0}$ are orthonormal with respect to a measure that is absolutely continuous with respect to the Lebesgue measure multiplied by the identity matrix. The corresponding density will be denoted by $X_{A,B}$. The theory of matrix orthogonal polynomials is substantially richer than the corresponding theory in the one-dimensional case and even the matrix Chebyshev polynomials have not been studied in full detail, see for example Duran (1999). In the following discussion we consider sequences of matrix orthonormal polynomials $(R_{n,k})_{n\geq 0}$ for $k \in \mathbb{N}$ defined by the recursion

$$tR_{n,k}(t) = A_{n+1,k}R_{n+1,k}(t) + B_{n,k}R_{n,k}(t) + A_{n,k}^T R_{n-1,k}(t),\ n \geq 0 \tag{3.4}$$

where $R_{-1,k}(t) = 0_p,\ R_{0,k}(t) = I_p$ and $B_{i,k} \in \mathbb{R}^{p\times p}$ are symmetric and $A_{i,k} \in \mathbb{R}^{p\times p}$ non-singular matrices which depend on the extra parameter k. The roots of the matrix polynomial $R_{n,k}$ are the roots of the scalar polynomial

$$\det R_{n,k}(t)$$

of degree np. It can be shown that the matrix polynomial $R_{n,k}$ has precisely np real roots, say $x_{n,k,j} (j = 1, \ldots, np)$, where each root has at most multiplicity p. The empirical distribution of the roots is defined by

$$\delta_{n,k} := \frac{1}{np} \sum_{j=1}^{np} \delta_{x_{n,k,j}},\ n, k \geq 1, \tag{3.5}$$

where δ_z denotes the Dirac measure at the point $z \in \mathbb{R}$. Of particular interest are the asymptotic properties of the empirical distribution $\delta_{n,k}$ if $n, k \to \infty$. For this purpose we consider sequences $(n_j)_{j\geq 0}$ and $(k_j)_{j\geq 0}$ of positive integers such that $\lim_{j\to\infty} n_j/k_j = u$ for some $u \in \mathbb{R}$ and we denote the corresponding limit as $\lim_{n/k\to u}$ (if it exists). The following Theorem by Dette and Reuther (2009) gives the limit distribution of the roots and will be one main tool in the study of the asymptotic eigenvalue distributions of the random block matrices defined in the previous section.

Theorem 3.1 *Consider a sequence of matrix orthonormal polynomials defined by the three term recursion* (3.4), *where for all $\ell \in \mathbb{N}_0$ and a given $u > 0$*

$$\lim_{\frac{n}{k}\to s} A_{n-\ell,k} = A(s) \tag{3.6}$$

$$\lim_{\frac{n}{k}\to s} B_{n-\ell,k} = B(s) \tag{3.7}$$

for all $s \in (0, u)$ with non-singular and symmetric matrices $\{A(s)|s \in (0, u)\}$ and symmetric matrices $\{B(s)|s \in (0, u)\}$. If there exists a number $M > 0$ such that

$$\bigcup_{k=1}^{\infty}\bigcup_{n=0}^{\infty} \{z \in \mathbb{R}|\det R_{n,k}(z) = 0\} \subset [-M, M], \tag{3.8}$$

then the empirical measure $\delta_{n,k}$ defined by (3.5) *converges weakly to a matrix measure which is absolutely continuous with respect to the Lebesgue measure multiplied with the identity matrix. The density of the limiting distribution is given by*

$$f(t) = \frac{1}{u}\int_0^u \operatorname{tr}\left[\frac{1}{p}X_{A(s),B(s)}(t)\right] ds, \tag{3.9}$$

where $X_{A(s),B(s)}$ is the density of the matrix measure corresponding to the matrix Chebyshev polynomials of the first kind.

Remark 3.2 If the matrices $\{A(s)|s \in (0, u)\}$ are positive definite, the density (3.9) can be given explicitly. In this case it can be shown that

$$\operatorname{tr}\left[\frac{1}{p}X_{A(s),B(s)}(t)\right] = \frac{1}{p}\sum_{j=1}^{p} \frac{-\frac{d}{dt}\lambda_j^{A(s),B(s)}(t)}{\pi\sqrt{4-\left(\lambda_j^{A(s),B(s)}(t)\right)^2}} I_{\{-2<\lambda_j^{A(s),B(s)}(t)<2\}}\ , \tag{3.10}$$

where $\lambda_j^{A(s),B(s)}(t)$ for $j = 1, \ldots, p$ denote the eigenvalues of the matrix $A^{-1}(s)(B(s) - tI_p)$.

4 Spectral asymptotics for the generalized Jacobi ensemble

In this section we study the weak asymptotics of the random eigenvalues $\lambda_1^{(n,p)}, \dots, \lambda_n^{(n,p)}$ of the block matrix $J_n^{(p)}$ defined in (2.4). The corresponding empirical distribution is given by the (random) measure

$$\sigma_n^{(p)} := \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j^{(n,p)}} \ . \tag{4.1}$$

We first show that the eigenvalues of the matrix $J_n^{(p)}$ can be approximated by the deterministic roots of the orthogonal matrix polynomials $(R_{m,n}^{(p)}(x))_{m>0}$ which are defined recursively by $R_{-1,n}^{(p)}(x) = 0_p$, $R_{0,n}^{(p)}(x) = I_p$ and

$$xR_{m,n}^{(p)}(x) = D_{m+1,n}^{(p)}R_{m+1,n}^{(p)}(x) + C_{m,n}^{(p)}R_{m,n}^{(p)}(x) + D_{m,n}^{(p)}R_{m-1,n}^{(p)}(x), \ m \geq 0. \tag{4.2}$$

The varying (matrix-valued) recursion coefficients in this recursion are given by

$$C_{i,n}^{(p)} := \begin{pmatrix} c_{ip+1}^{(1,n)} & d_{ip+1}^{(1,n)} & d_{ip+2}^{(2,n)} & \cdots & \cdots & d_{(i+1)p-1}^{(p-1,n)} \\ d_{ip+1}^{(1,n)} & c_{ip+2}^{(1,n)} & d_{ip+2}^{(1,n)} & \cdots & \cdots & d_{(i+1)p-1}^{(p-2,n)} \\ d_{ip+2}^{(2,n)} & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & c_{(i+1)p-1}^{(1,n)} & d_{(i+1)p-1}^{(1,n)} \\ d_{(i+1)p-1}^{(p-1,n)} & \cdots & \cdots & \cdots & d_{(i+1)p-1}^{(1,n)} & c_{(i+1)p}^{(1,n)} \end{pmatrix}$$

and

$$D_{i,n}^{(p)} := \begin{pmatrix} d_{ip}^{(p,n)} & d_{ip}^{(p-1,n)} & d_{ip}^{(p-2,n)} & \cdots & \cdots & d_{ip}^{(1,n)} \\ d_{ip}^{(p-1,n)} & d_{ip+1}^{(p,n)} & d_{ip+1}^{(p-1,n)} & \cdots & \cdots & d_{ip+1}^{(2,n)} \\ d_{ip}^{(p-2,n)} & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & d_{ip+p-2}^{(p,n)} & d_{ip+p-2}^{(p-1,n)} \\ d_{ip}^{(1,n)} & \cdots & \cdots & \cdots & d_{ip+p-2}^{(p-1,n)} & d_{ip+p-1}^{(p,n)} \end{pmatrix},$$

where the entries in these matrices are obtained from the entries of the matrix $J_n^{(p)}$ by essentially replacing each random variable by its expectation, that is

$$c_{k+1}^{(1,n)} = (1 - E[\alpha_{2k-1}^{(1,n)}])E[\alpha_{2k}^{(1,n)}] - (1 + E[\alpha_{2k-1}^{(1,n)}])E[\alpha_{2k-2}^{(1,n)}],$$

$$d_{k+1}^{(j,n)} = ((1 - E[\alpha_{2k-1}^{(j,n)}])(1 - E[\alpha_{2k}^{(j,n)}]^2)(1 + E[\alpha_{2k+1}^{(j,n)}]))^{\frac{1}{2}}.$$

The expectation of a random variable $X \sim Beta(p,q)$ with density (2.2) is $\frac{q-p}{q+p}$, which gives for the random variables under consideration

$$E[\alpha_k^{(j,n)}] = \begin{cases} \dfrac{b_n^{(j)} - a_n^{(j)}}{\frac{2n-k-2}{2}\beta_n^{(j)} + a_n^{(j)} + b_n^{(j)} + 2} & \text{for } k \text{ even,} \\ \dfrac{\frac{\beta_n^{(j)}}{2} - a_n^{(j)} - b_n^{(j)} - 2}{\frac{2n-k-2}{2}\beta_n^{(j)} + a_n^{(j)} + b_n^{(j)} + 2} & \text{for } k \text{ odd,} \end{cases} \tag{4.3}$$

for $0 \le k \le 2n-2$ and $E[\alpha_{-2}] = E[\alpha_{-1}] = E[\alpha_{2n-1}] = -1$. A straightforward calculation now yields that the entries of the matrices $C_{i,n}^{(p)}$ and $D_{i,n}^{(p)}$ are given by

$$c_{k+1}^{(1,n)} = \frac{2(b_n^{(1)} - a_n^{(1)})(a_n^{(1)} + b_n^{(1)} + 2)}{((n-k-1)\beta_n^{(1)} + a_n^{(1)} + b_n^{(1)} + 2)((n-k)\beta_n^{(1)} + a_n^{(1)} + b_n^{(1)} + 2)}$$

for $k \ge 1$ and

$$c_1^{(1,n)} = \frac{2(b_n^{(1)} - a_n^{(1)})}{(2n-2)\frac{\beta_n^{(1)}}{2} + a_n^{(1)} + b_n^{(1)} + 2}$$

and similarly for $k \ge 1$

$$\begin{aligned} d_{k+1}^{(j,n)} = &\Big(\frac{4((2n-2k+2)\beta_n^{(j)}/2 + 2a_n^{(j)} + 2b_n^{(j)} + 4)((n-k+1)\beta_n^{(j)}/2 + a_n^{(j)} + 1)}{((2n-2k+3)\beta_n^{(j)}/2 + a_n^{(j)} + b_n^{(j)} + 2)((2n-2k+2)\beta_n^{(j)}/2 + a_n^{(j)} + b_n^{(j)} + 2)^2} \\ &\times \frac{((n-k+1)\beta_n^{(j)}/2 + b_n^{(j)} + 1)(n-k+1)\beta_n^{(j)}}{((2n-2k+1)\beta_n^{(j)}/2 + a_n^{(j)} + b_n^{(j)} + 2)} \Big)^{\frac{1}{2}} \end{aligned}$$

and

$$d_1^{(j,n)} = \Big(8 \frac{((n-1)\beta_n^{(j)}/2 + a_n^{(j)} + 1)((n-1)\beta_n^{(j)}/2 + b_n^{(j)} + 1)(n-1)\beta_n^{(j)}}{((n-1)\beta_n^{(j)} + a_n^{(j)} + b_n^{(j)} + 2)^2((2n-3)\beta_n^{(j)}/2 + a_n^{(j)} + b_n^{(j)} + 2)} \Big)^{\frac{1}{2}}.$$

Now let $n = mp$ for some $m \in \mathbb{N}$ and choose the parameters $\beta_n^{(j)}, a_n^{(j)}, b_n^{(j)}$ such that the matrices $D_{i,n}^{(p)}$ are non-singular. Then the matrix polynomial $R_{n/p,n}^{(p)}(x)$ of degree $n/p \in \mathbb{N}$ has $(n/p)p = n$ real roots. Our next Theorem shows that the eigenvalues of the matrix $J_n^{(p)}$ can be approximated by the roots of the matrix polynomial $R_{n/p,n}^{(p)}(x)$ with high probability.

Theorem 4.1 *Let $\lambda_1^{(n,p)} \le \ldots \le \lambda_n^{(n,p)}$ be the ordered eigenvalues of the matrix $J_n^{(p)}$ and denote by $x_1^{(n,p)} \le \ldots \le x_n^{(n,p)}$ the ordered roots of the matrix polynomial $R_{n/p,n}^{(p)}(x)$. Then for all $\varepsilon \in (0,1]$ and $n \ge 1$ the inequality*

$$P\Big(\max_{1 \le j \le n} |\lambda_j^{(n,p)} - x_j^{(n,p)}| > \epsilon \Big) \tag{4.4}$$

$$\leq 4p(2n-1)\exp\Big(\big(\log\big(1+\frac{\epsilon^2}{648p^2+2\epsilon^2}\big)-\frac{\epsilon^2}{648p^2+2\epsilon^2}\big)(a_n+b_n+2)\Big),$$

holds, where we define

$$a_n := \min_{1\leq j\leq p} a_n^{(j)} \qquad \textit{and} \qquad b_n := \min_{1\leq j\leq p} b_n^{(j)}. \tag{4.5}$$

Proof: The recursion (4.2) of the matrix polynomial $R^{(p)}_{n/p,n}(x)$ implies that the roots $x_1^{(n,p)} \leq \ldots \leq x_n^{(n,p)}$ are the ordered eigenvalues of the block tridiagonal matrix

$$E_n^{(p)} := \begin{pmatrix} C^{(p)}_{0,n} & D^{(p)}_{1,n} & & & \\ D^{(p)}_{1,n} & C^{(p)}_{1,n} & D^{(p)}_{2,n} & & \\ & D^{(p)}_{2,n} & \ddots & \ddots & \\ & & \ddots & \ddots & D^{(p)}_{\frac{n}{p}-1,n} \\ & & & C^{(p)}_{\frac{n}{p}-1,n} & C^{(p)}_{\frac{n}{p}-1,n} \end{pmatrix},$$

which contains the recurrence coefficients of the matrix polynomials $(R^{(p)}_{m,n}(x))_{m>0}$. Now Weyl's inequality and Theorem 5.6.9 in Horn and Johnsohn (1985) yields

$$\max_{1\leq j\leq n} |\lambda_j^{(n,p)} - x_j^{(n,p)}| \leq \max_{1\leq k\leq n} \sum_{j=1}^{n} |\{\big(J_n^{(p)} - E_n^{(p)}\big)\}_{k,j}|, \tag{4.6}$$

which gives an upper bound for the difference between the eigenvalues of the matrices $J_n^{(p)}$ and $E_n^{(p)}$. By using essentially the same arguments as in Silverstein (1985) and Dette and Nagel (2009) we can show the inequality

$$\max_{1\leq k\leq n} \sum_{j=1}^{n} |\{\big(J_n^{(p)} - E_n^{(p)}\big)\}_{k,j}| \leq (3p-1)\sqrt{12X_n} + 6X_n, \tag{4.7}$$

where the random variable X_n is defined as

$$X_n := \max_{j=1,\ldots,p} \max_{0\leq k\leq 2n-2} |\alpha_k^{(j,n)} - E[\alpha_k^{(j,n)}]|. \tag{4.8}$$

This implies for the probability under consideration

$$P\Big(\max_{1\leq j\leq n} |\lambda_j^{(n,p)} - x_j^{(n,p)}| > \epsilon\Big) \leq P\left((3p-1)\sqrt{3X_n} + 3X_n > \frac{\epsilon}{2}\right)$$
$$\leq P\left(3p\sqrt{3X_n} > \frac{\epsilon}{2}\right) = P\Big(X_n > \frac{\epsilon^2}{108p^2}\Big).$$

The second inequality holds because $3X_n \leq \sqrt{3X_n}$ if $3X_n \leq 1$ and $3X_n > \frac{\epsilon}{2}$ if $3X_n > 1 \geq \epsilon$. Observing Lemma A.1 in Dette and Nagel (2009) it follows that

$$\begin{aligned}
P\Big(X_n > \frac{\epsilon^2}{108p^2}\Big) \leq & \sum_{j=1}^{p} \sum_{k=0}^{2n-2} P\Big(\Big|\alpha_k^{(j,n)} - E\left[\alpha_k^{(j,n)}\right]\Big| > \frac{\epsilon^2}{108p^2}\Big) \\
\leq & \sum_{j=1}^{p} \sum_{k=0}^{2n-2} 4 \exp\left(c\left(a_n^{(j)} + b_n^{(j)} + 2\right)\right) \\
\leq & 4p(2n-1) \exp\left(c\left(a_n + b_n + 2\right)\right) \ ,
\end{aligned}$$

where the constant c is given by

$$c = \log\Big(1 + \frac{\epsilon^2}{648p^2 + 2\epsilon^2}\Big) - \frac{\epsilon^2}{648p^2 + 2\epsilon^2}. \tag{4.9}$$

This proves the assertion of the Theorem. □

Note that the constant c given in (4.9) is negative and therefore the probablility (4.4) decays exponentially fast. This indicated that the random eigenvalues of the generalized Jacobi ensemble can be approximated uniformly by the roots of the matrix polynomials $R_{n/p,n}^{(p)}(x)$ almost surely. The following Theorem makes this statement more precise and provides a rate for the convergence. The proof follows by similar arguments as the proof of Theorem 2.2 in Dette and Imhof (2007) and is therefore omitted.

Theorem 4.2 *Let $\lambda_1^{(n,p)} \leq \ldots \leq \lambda_n^{(n,p)}$ be the ordered eigenvalues of the matrix $J_n^{(p)}$ and denote by $x_1^{(n,p)} \leq \ldots \leq x_n^{(n,p)}$ the ordered roots of the matrix polynomial $R_{n/p,n}^{(p)}$. Suppose*

$$\lim_{n\to\infty} \frac{(a_n + b_n)}{\log n} = \infty, \tag{4.10}$$

where a_n and b_n are defined in (4.5), then there exists an a.s. finite random variable S such that the inequality

$$\max_{1\leq j\leq n} |\lambda_j^{(n,p)} - x_j^{(n,p)}| \leq \Big(\frac{\log n}{a_n + b_n}\Big)^{\frac{1}{4}} S \tag{4.11}$$

holds for all $n \geq 2$.

Theorem 4.3 *Denote by $\lambda_1^{(n,p)} \leq \ldots \leq \lambda_n^{(n,p)}$ the ordered eigenvalues of the matrix $J_n^{(p)}$, where the parameters $\beta_n^{(j)}, a_n^{(j)}, b_n^{(j)}$, $j = 1, \ldots, p$ are chosen such that the matrices $D_i^{(p)}$, $i = 1, \ldots, n/p-1$ and the matrix $D^{(p)}(s)$ defined below is non-singular for $0 < s < 1/p$. Recall the notation (4.5), suppose that*

$$\lim_{n\to\infty} \frac{(a_n + b_n)}{\log n} = \infty, \tag{4.12}$$

and that the limits

$$\lim_{n\to\infty} \frac{a_n^{(j)}}{n\beta_n^{(j)}} =: a^{(j)} < \infty \text{ and } \lim_{n\to\infty} \frac{b_n^{(j)}}{n\beta_n^{(j)}} =: b^{(j)} < \infty \tag{4.13}$$

exist. Then, almost surely, the empirical distribution $\sigma_n^{(p)}$ of the eigenvalues of the matrix $J_n^{(p)}$ converges weakly towards a measure that is absolutely continuous with respect to the Lebesgue measure. The density of this measure is given by

$$f(t) = \int_0^{\frac{1}{p}} tr[X_{D^{(p)}(s),C^{(p)}(s)}(t)]ds, \tag{4.14}$$

where $X_{D^{(p)}(s),C^{(p)}(s)}(t)$ denotes the Lebesgue density of the matrix measure corresponding to the matrix Chebychev polynomials of the first kind defined in (3.3) *with matrices*

$$C^{(p)}(s) := \begin{pmatrix} c^{(1)}(s) & d^{(1)}(s) & d^{(2)}(s) & \cdots & \cdots & d^{(p-1)}(s) \\ d^{(1)}(s) & c^{(1)}(s) & d^{(1)}(s) & \cdots & \cdots & d^{(p-2)}(s) \\ d^{(2)}(s) & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & c^{(1)}(s) & d^{(1)}(s) \\ d^{(p-1)}(s) & \cdots & \cdots & \cdots & d^{(1)}(s) & c^{(1)}(s) \end{pmatrix}$$

and

$$D^{(p)}(s) := \begin{pmatrix} d^{(p)}(s) & d^{(p-1)}(s) & d^{(p-2)}(s) & \cdots & \cdots & d^{(1)}(s) \\ d^{(p-1)}(s) & d^{(p)}(s) & d^{(p-1)}(s) & \cdots & \cdots & d^{(2)}(s) \\ d^{(p-2)}(s) & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & d^{(p)}(s) & d^{(p-1)}(s) \\ d^{(1)}(s) & \cdots & \cdots & \cdots & d^{(p-1)}(s) & d^{(p)}(s) \end{pmatrix}$$

with entries

$$c^{(1)}(s) = \frac{2(b^{(1)^2} - a^{(1)^2})}{(1 - sp + a^{(1)} + b^{(1)})^2},$$

$$d^{(j)}(s) = \Big(\frac{4(1 - sp + 2a^{(j)} + 2b^{(j)})(\frac{1-sp}{2} + a^{(j)})(\frac{1-sp}{2} + b^{(j)})(1 - sp)}{(1 - sp + a^{(j)} + b^{(j)})^4}\Big)^{\frac{1}{2}}.$$

Proof: Under the conditions (4.13) the recursion coefficients $D_i^{(p)}$ and $C_i^{(p)}$ of the matrix polynomial $R_{n/p,n}^{(p)}(x)$ converge to the limiting matrices given in the theorem, i.e. for all $\ell \in \mathbb{N}$ and $s \in (0, 1/p)$

$$\lim_{\frac{i}{n} \to s} C_{i-l,n}^{(p)} = C^{(p)}(s) \text{ and } \lim_{\frac{i}{n} \to s} D_{i-l,n}^{(p)} = D^{(p)}(s).$$

By Geršgorin's Theorem [see Horn and Johnsohn (1985)] the convergence implies the existence of an $M > 0$ such that the roots $x_1^{(n,p)} \leq \ldots < x_n^{(n,p)}$ of $R_{n/p,n}^{(p)}(x)$ are elements of a compact interval $[-M, M]$. An application of Theorem 3.1 with $u = \lim_{n\to\infty} \frac{n}{p}/n = \frac{1}{p}$ yields that the empirical distribution $\delta_n^{(p)}$ of the roots converges weakly to a measure with density (4.14). Now consider the Lévy distance $L(\sigma_n^{(p)}, \delta_n^{(p)})$ between the empirical measures, then Bai (1999) gives the inequality

$$L^3(\sigma_n^{(p)}, \delta_n^{(p)}) \leq \frac{1}{n} \sum_{j=1}^{n} \left| \lambda_j^{(n,p)} - x_j^{(n,p)} \right|^2$$

and Theorem 4.2 shows that the right-hand side converges almost surely to 0, which gives the almost sure weak convergence of $\sigma_n^{(p)}$ with the same limit as $\delta_n^{(p)}$, that is the measure with Lebesgue density defined by (4.14). $\Box$

Remark 4.4 Note that condition (4.13) ensures the existence of the limiting matrices $C^{(p)}(s)$ and $D^{(p)}(s)$. The condition can be relaxed for some j, as long as the limits of the matrix entries still exist and the matrix $D^{(p)}(s)$ is non-singular.

We conclude this section with a few examples to illustrate the shape of the limit distribution of the eigenvalues. First note that in the case $p = 1$ the eigenvalues of the matrix $J_n^{(p)}$ are the eigenvalues of the classical Jacobi ensemble which have been considered by Collins (2005) and Dette and Nagel (2009). Now consider the case $p = 2$. In this case we have to choose the parameters $a_n^{(i)}, b_n^{(i)}, \beta_n^{(i)}$ for $i = 1, 2$ such that the matrices $D^{(p)}(s)$ defined in Theorem 4.3 are non-singular. In the examples considered here the matrix $D^{(p)}(s)$ is also positive definite and we can calculate the density of the limit distribution using formula (3.10).

The left part of Figure 1 displays a simulated histogram of the eigenvalues of $J_n^{(p)}$ for $n = 5000$ (i.e. $m = n/p = 2500$) and parameters $a_n^{(1)} = 20n, b_n^{(1)} = n, a_n^{(2)} = b_n^{(2)} = n$ and $\beta_n^{(1)} = \beta_n^{(2)} = 1$ while the right part shows the corresponding limit distribution obtained from Theorem 4.3.

Figure 2 displays the simulated histogram and the corresponding limit density for the parameters $a_n^{(1)} = b_n^{(1)} = n,\ \beta_n^{(1)} = 1,\ a_n^{(2)} = b_n^{(2)} = \sqrt{n},\ \beta_n^{(2)} = 1$. In this case some of the limits in the condition (4.13) are equal to 0.

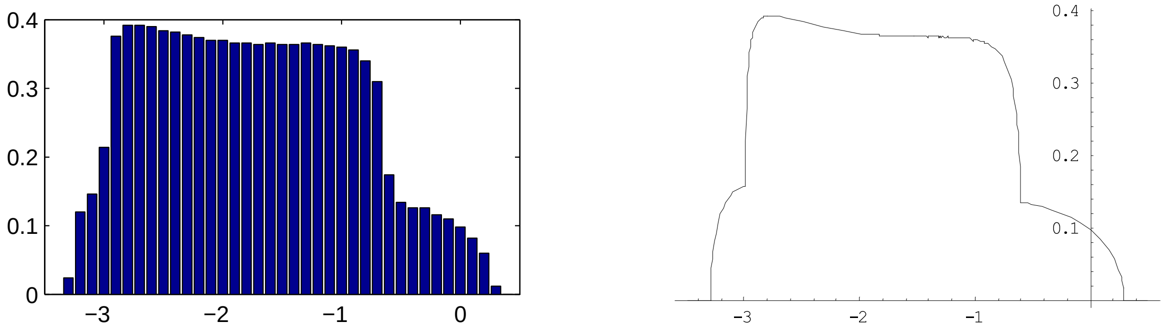

Figure 1: *Simulated and limiting spectral density of the random block matrix* $J_n^{(p)}$ *in the case* $p = 2,\ a_n^{(1)} = 20n, b_n^{(1)} = n,\ \beta_n^{(1)} = 1,\ a_n^{(2)} = b_n^{(2)} = n,\ \beta_n^{(2)} = 1$. *In the simulation the eigenvalues of a* 5000×5000 *matrix were calculated.*

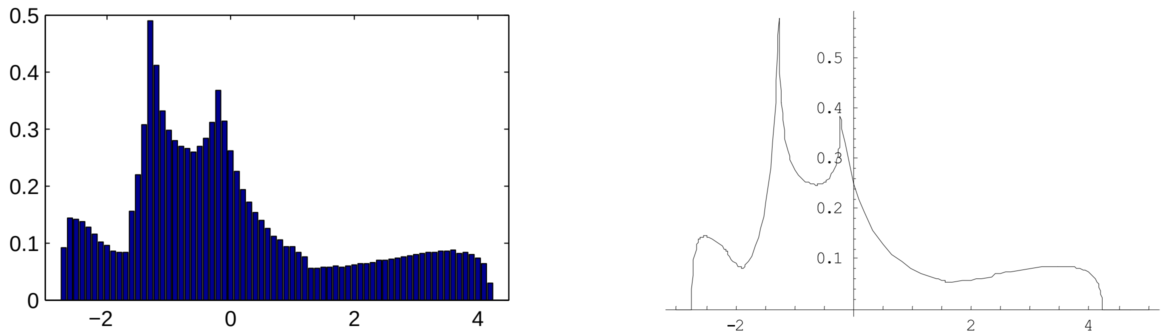

Figure 2: *Simulated and limiting spectral density of the random block matrix* $J_n^{(p)}$ *in the case* $p = 2,\ a_n^{(1)} = b_n^{(1)} = n,\ \beta_n^{(1)} = 1,\ a_n^{(2)} = b_n^{(2)} = \sqrt{n},\ \beta_n^{(2)} = 1$. *In the simulation the eigenvalues of a* 5000×5000 *matrix were calculated.*

As stated in Remark 4.4 we can calculate the limit distribution even if some parameters converge to infinity at a rate larger than n. Figure 3 illustrates the convergence in this case for the parameters $a_n^{(1)} = 2n,\ b_n^{(1)} = n^3,\ \beta_n^{(1)} = 1,\ a_n^{(2)} = \sqrt{n},\ b_n^{(2)} = n,\ \beta_n^{(2)} = 2$.

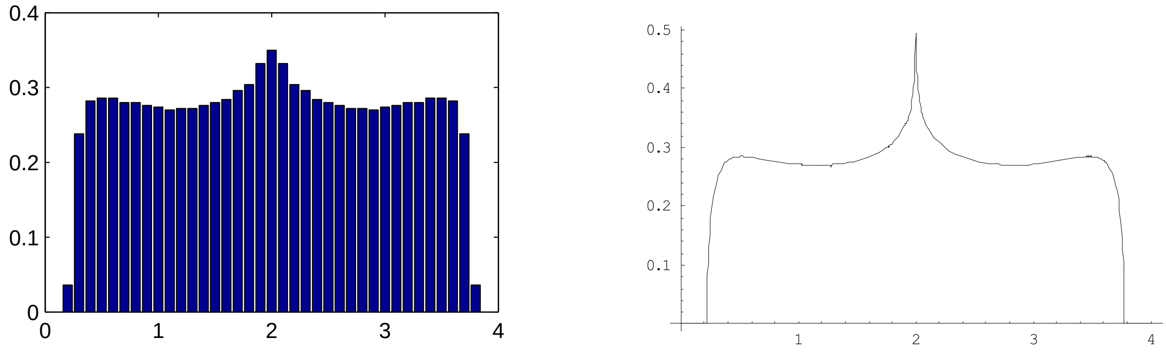

Figure 3: *Simulated and limiting spectral density of the random block matrix $J_n^{(p)}$ in the case $p = 2,\ a_n^{(1)} = 2n,\ b_n^{(1)} = n^3,\ \beta_n^{(1)} = 1,\ a_n^{(2)} = \sqrt{n},\ b_n^{(2)} = n,\ \beta_n^{(2)} = 2$. In the simulation the eigenvalues of a 5000×5000 matrix were calculated.*

5 Random block Laguerre ensembles

Following the idea of the previous paragraphs, we can define a generalization of the Laguerre ensemble. The tridiagonal matrix model of the Laguerre ensemble is replaced by a block tridiagonal matrix while maintaining the general structure of the entries. Recall that the density defining the Laguerre ensemble is defined by

$$f_{\beta,a}(\lambda) = c_{\beta,a} \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{n} \lambda_i^{a-(n-1)\frac{\beta}{2}-1} e^{-\sum_{i=1}^{n} \frac{\lambda_i}{2}} I_{(0,\infty)}(\lambda_j) \ , \tag{5.1}$$

where $\beta > 0,\ a > (n-1)\frac{\beta}{2} > 0$ and the normalization constant $c_{\beta,a}$ is given by

$$c_{\beta,a} = 2^{-na} \prod_{j=1}^{n} \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+j\frac{\beta}{2})\Gamma(a-(n-j)\frac{\beta}{2})} \ .$$

Dumitriu and Edelman (2002) provided a tridiagonal random matrix model of the Laguerre ensemble. For this purpose let $X_{2a}, X_{2a-\beta}, \dots, X_{2a-(n-1)\beta},\ Y_\beta, \dots, Y_{(n-1)\beta}$ be independent random variables, where $X_r^2, Y_r^2 \sim \chi^2(r)$ are chi-square distributed. Then the joint density of the eigenvalues $\lambda_1 \le \ldots \le \lambda_n$ of the matrix

$$L_n = L_n(a,\beta) = \begin{pmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_{n-1} \end{pmatrix} \in \mathbb{R}^{n\times n} \tag{5.2}$$

is given by (5.1), where the entries in the matrix L_n are defined by

$$\begin{aligned} b_0 &= \frac{1}{2a} X_{2a}^2, \\ b_{i-1} &= \frac{1}{2a}(X^2_{2a-(i-1)\beta} + Y^2_{(n+1-i)\beta}),\ i = 2,\ldots,n, \\ a_i &= \frac{1}{2a} X_{2a-(i-1)\beta} Y_{(n-i)\beta},\ i = 1,\ldots,n-1. \end{aligned}$$

Next we consider random tridiagonal block matrices of the form

$$L_n^{(p)} = L_n^{(p)}(a,\beta) := \begin{pmatrix} B_{0,n}^{(p)} & A_{1,n}^{(p)} & & & \\ A_{1,n}^{(p)} & B_{1,n}^{(p)} & A_{2,n}^{(p)} & & \\ & A_{2,n}^{(p)} & \ddots & \ddots & \\ & & \ddots & \ddots & A^{(p)}_{\frac{n}{p}-1,n} \\ & & & A^{(p)}_{\frac{n}{p}-1,n} & B^{(p)}_{\frac{n}{p}-1,n} \end{pmatrix} \in \mathbb{R}^{n\times n}, \tag{5.3}$$

where $n = mp$ with $m, p \in \mathbb{N}$ and the symmetric $p \times p$ blocks $A_{i,n}^{(p)}$ and $B_{i,n}^{(p)}$ are given by

$$B_{i,n}^{(p)} := \begin{pmatrix} b_{ip}^{(1,n)} & a_{ip+1}^{(1,n)} & a_{ip+2}^{(2,n)} & \cdots & a_{(i+1)p-1}^{(p-1,n)} \\ a_{ip+1}^{(1,n)} & b_{ip+1}^{(1,n)} & a_{ip+2}^{(1,n)} & \cdots & a_{(i+1)p-1}^{(p-2,n)} \\ a_{ip+2}^{(2,n)} & & \ddots & & \vdots \\ \vdots & & & b_{(i+1)p-2}^{(1,n)} & a_{(i+1)p-1}^{(1,n)} \\ a_{(i+1)p-1}^{(p-1,n)} & \cdots & \cdots & a_{(i+1)p-1}^{(1,n)} & b_{(i+1)p-1}^{(1,n)} \end{pmatrix}$$

and

$$A_{i,n}^{(p)} := \begin{pmatrix} a_{ip}^{(p,n)} & a_{ip}^{(p-1,n)} & a_{ip}^{(p-2,n)} & \cdots & a_{ip}^{(1,n)} \\ a_{ip}^{(p-1,n)} & a_{ip+1}^{(p,n)} & a_{ip+1}^{(p-1,n)} & \cdots & a_{ip+1}^{(2,n)} \\ a_{ip}^{(p-2,n)} & & \ddots & & \vdots \\ \vdots & & & a_{ip+p-2}^{(p,n)} & a_{ip+p-2}^{(p-1,n)} \\ a_{ip}^{(1,n)} & \cdots & \cdots & a_{ip+p-2}^{(p-1,n)} & a_{ip+p-1}^{(p,n)} \end{pmatrix}.$$

The entries are defined by

$$\begin{aligned}
b_0^{(1,n)} &= \frac{1}{2a_n^{(1)}} X^2_{2a_n^{(1)}},\\
b_{k-1}^{(1,n)} &= \frac{1}{2a_n^{(1)}} (X^2_{2a_n^{(1)}-(k-1)\beta_n^{(1)}} + Y^2_{(n+1-k)\beta_n^{(1)}}),\ k = 2, \dots, n,\\
a_k^{(j,n)} &= \frac{1}{2a_n^{(j)}} X_{2a_n^{(j)}-(k-1)\beta_n^{(j)}} Y_{(n-k)\beta_n^{(j)}},\ k = 1, \dots, n-1,
\end{aligned}$$

where for $j = 1, \dots p$ the parameters $a_n^{(j)}, \beta_n^{(j)} \in \mathbb{R}$ satisfy $a_n^{(j)} > \frac{\beta_n^{(j)}}{2}(n-1)$. For $p = 1$, the matrix $L_n^{(p)}$ reduces to the matrix L_n defined in (5.2) and the eigenvalues are distributed according to the density (5.1). We will show in this Section that the eigenvalues of the random block matrix $L_n^{(p)}$ can be almost surely approximated by the roots of the matrix polynomial $R_{m,n}^{(p)}(x)$ which is defined by the recurrence relation

$$xR_{m,n}^{(p)}(x) = D_{m+1,n}^{(p)} R_{m+1,n}^{(p)}(x) + C_{m,n}^{(p)} R_{m,n}^{(p)}(x) + D_{m,n}^{(p)} R_{m-1,n}^{(p)}(x),\ m \geq 0, \tag{5.4}$$

where we define the varying (matrix-valued) coefficients in the recursion by

$$C_{i,n}^{(p)} := \begin{pmatrix}
c_{ip}^{(1,n)} & d_{ip+1}^{(1,n)} & d_{ip+2}^{(2,n)} & \cdots & \cdots & d_{(i+1)p-1}^{(p-1,n)}\\
d_{ip+1}^{(1,n)} & c_{ip+1}^{(1,n)} & d_{ip+2}^{(1,n)} & \cdots & \cdots & d_{(i+1)p-1}^{(p-2,n)}\\
d_{ip+2}^{(2,n)} & & \ddots & & \vdots & \vdots\\
\vdots & & & \ddots & c_{(i+1)p-2}^{(1,n)} & d_{(i+1)p-1}^{(1,n)}\\
d_{(i+1)p-1}^{(p-1,n)} & \cdots & \cdots & \cdots & d_{(i+1)p-1}^{(1,n)} & c_{(i+1)p-1}^{(1,n)}
\end{pmatrix}$$

and

$$D_{i,n}^{(p)} := \begin{pmatrix}
d_{ip}^{(p,n)} & d_{ip}^{(p-1,n)} & d_{ip}^{(p-2,n)} & \cdots & \cdots & d_{ip}^{(1,n)}\\
d_{ip}^{(p-1,n)} & d_{ip+1}^{(p,n)} & d_{ip+1}^{(p-1,n)} & \cdots & \cdots & d_{ip+1}^{(2,n)}\\
d_{ip}^{(p-2,n)} & & \ddots & & \vdots & \vdots\\
\vdots & & & \ddots & d_{ip+p-2}^{(p,n)} & d_{ip+p-2}^{(p-1,n)}\\
d_{ip}^{(1,n)} & \cdots & \cdots & \cdots & d_{ip+p-2}^{(p-1,n)} & d_{ip+p-1}^{(p,n)}
\end{pmatrix}.$$

The entries in these matrices are given by

$$\begin{aligned}
c_0^{(1,n)} &= 1,\\
c_{k-1}^{(1,n)} &= \frac{2a_n^{(1)} + (n+2-2k)\beta_n^{(1)}}{2a_n^{(1)}},\\
d_k^{(j,n)} &= \frac{1}{2a_n^{(j)}} \sqrt{(2a_n^{(j)} - (k-1)\beta_n^{(j)})(n-k)\beta_n^{(j)}}.
\end{aligned}$$

The following results extend the results of Chapter 4 to the generalized Laguerre ensemble. The proofs are similar to those given in Section 4 and omitted for the sake of brevity.

Theorem 5.1 *Let $\lambda_1^{(n,p)}, \ldots, \lambda_n^{(n,p)}$ denote the ordered eigenvalues of the matrix $L_n^{(p)}$ defined in (5.3) and $x_1^{(n,p)} < \ldots < x_n^{(n,p)}$ the ordered roots of the matrix polynomial $R_{n/p,n}^{(p)}(x)$ Then for any $\varepsilon \in (0,1]$ the inequality*

$$P\Big(\max_{1\le j\le n} |\lambda_j^{(n,p)} - x_j^{(n,p)}| > \varepsilon\Big) \le 2p(2n-1)\exp\Big(-\frac{\epsilon^4 a_n}{4(6p-1)^4}\Big) \tag{5.5}$$

holds for $n \ge 1$, where $a_n := \min_{1\le j\le p} a_n^{(j)}$.

Theorem 5.2 *Let $\lambda_1^{(n,p)} \le \ldots \le \lambda_n^{(n,p)}$ be the ordered eigenvalues of the matrix $L_n^{(p)}$ and denote by $x_1^{(n,p)} \le \ldots \le x_n^{(n,p)}$ the ordered roots of the matrix polynomial $R_{n/p,n}^{(p)}$. Suppose that*

$$\lim_{n\to\infty} \frac{a_n}{\log n} = \infty \tag{5.6}$$

with $a_n := \min_{1\le j\le p} a_n^{(j)}$, then there exists an almost sure finite random variable S such that

$$\max_{1\le j\le n} |\lambda_j^{(n,p)} - x_j^{(n,p)}| \le \Big(\frac{\log n}{a_n}\Big)^{\frac{1}{4}} S \tag{5.7}$$

holds for all $n \ge 2$.

The rate given in (5.7) can be improved if we impose additional conditions of the parameters of the generalized Laguerre ensemble. The following Theorem makes this statement more precise and is a generalization of Theorem 2.5 of Dette and Imhof (2007) to the matrix case. We outline the proof in the Appendix.

Theorem 5.3 *Let $\lambda_1^{(n,p)} \le \ldots \le \lambda_n^{(n,p)}$ be the ordered eigenvalues of the matrix $L_n^{(p)}$ and denote by $x_1^{(n,p)} \le \ldots \le x_n^{(n,p)}$ the ordered roots of the matrix polynomial $R_{n/p,n}^{(p)}(x)$. Suppose that there exists a $K > 0$ such that*

$$n + K \ge \frac{2a_n^{(j)}}{\beta_n^{(j)}} \ge n - 1 + \frac{1}{\beta_n^{(j)}} \tag{5.8}$$

holds for all $n \ge 2$. Then there exists an almost sure finite random variable S such that the inequality

$$\max_{1\le j\le n} |\lambda_j^{(n,p)} - x_j^{(n,p)}| \le \Big(\frac{\log n}{n}\Big)^{\frac{1}{2}} S$$

holds for all $n \ge 2$.

Theorem 5.1 and Theorem 5.2 show that as n tends to infinity, the eigenvalues of the matrix $L_n^{(p)}$ can be almost surely approximated by the roots of the matrix polynomial $R_{n/p,n}^{(p)}(x)$. Using the same arguments as in the proof of Theorem 4.3 we can show that the limit distribution of the roots given by Theorem 3.1 can be transferred to the random eigenvalues.

Theorem 5.4 *Denote by $\lambda_1^{(n,p)} \leq \ldots \leq \lambda_n^{(n,p)}$ the ordered eigenvalues of the matrix $L_n^{(p)}$, where the parameters $\beta_n^{(j)}, a_n^{(j)},\ j = 1, \ldots, p$ are chosen such that the matrices $D_{i,n}^{(p)},\ i = 1, \ldots, n/p - 1$ and the matrix $D^{(p)}(s)$ defined below are non-singular for $0 < s < 1/p$. Suppose that*

$$\lim_{n\to\infty} \frac{a_n}{\log n} = \infty, \tag{5.9}$$

where we use the notation $a_n := \min\limits_{1\leq j\leq p} a_n^{(j)}$ and that the limits

$$\lim_{n\to\infty} \frac{a_n^{(j)}}{n\beta_n^{(j)}} =: a^{(j)} < \infty \quad (j = 1, \ldots, p). \tag{5.10}$$

exist. Then, almost surely, the empirical distribution of the eigenvalues of the matrix $L_n^{(p)}$ converges weakly to a measure which is absolutely continuous with respect to the Lebesgue measure. The density of this measure is given by

$$f(t) = \int_0^{\frac{1}{p}} tr[X_{D^{(p)}(s),C^{(p)}(s)}(t)]ds, \tag{5.11}$$

where $X_{D^{(p)}(s),C^{(p)}(s)}(t)$ denotes the Lebesgue density of the matrix measure corresponding to the matrix Chebychev polynomials of the first kind defined in (3.3) with matrices

$$C^{(p)}(s) := \begin{pmatrix} c^{(1)}(s) & d^{(1)}(s) & d^{(2)}(s) & \cdots & \cdots & d^{(p-1)}(s) \\ d^{(1)}(s) & c^{(1)}(s) & d^{(1)}(s) & \cdots & \cdots & d^{(p-2)}(s) \\ d^{(2)}(s) & & & & \ddots & \vdots \\ \vdots & & & & c^{(1)}(s) & d^{(1)}(s) \\ d^{(p-1)}(s) & \cdots & \cdots & \cdots & d^{(1)}(s) & c^{(1)}(s) \end{pmatrix}$$

$$D^{(p)}(s) := \begin{pmatrix} d^{(p)}(s) & d^{(p-1)}(s) & d^{(p-2)}(s) & \cdots & \cdots & d^{(1)}(s) \\ d^{(p-1)}(s) & d^{(p)}(s) & d^{(p-1)}(s) & \cdots & \cdots & d^{(2)}(s) \\ d^{(p-2)}(s) & & & \ddots & & \vdots \\ \vdots & & & & d^{(p)}(s) & d^{(p-1)}(s) \\ d^{(1)}(s) & \cdots & \cdots & \cdots & d^{(p-1)}(s) & d^{(p)}(s) \end{pmatrix}$$

and entries

$$c^{(1)}(s) = \frac{2a^{(1)} + 1 - 2sp}{2a^{(1)}},$$
$$d^{(j)}(s) = \frac{1}{2a^{(j)}}\sqrt{2a^{(j)} - 2a^{(j)}sp - sp + (sp)^2}.$$

We conclude this section with some examples for the case $p = 2$. Figure 4 and Figure 5 show a simulated histogram of the eigenvalues of $L_n^{(p)}$ for $n = 5000$ (left panels) and the corresponding limiting density (right panels). Note that the parameters are chosen such that the matrix $C^{(p)}(u)$ defined in Theorem 5.4 is positive definite and the limiting density can be obtained according to formula (3.10).

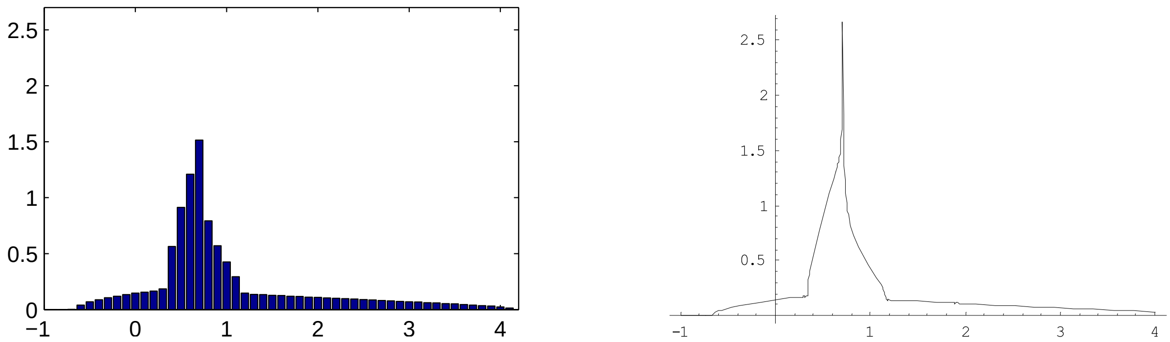

Figure 4: *Simulated and limiting spectral density of the matrix* $L_n^{(p)}$ *for* $p = 2,\ a_n^{(1)} = 2n,\ \beta_n^{(1)} = 1,\ a_n^{(2)} = n,\ \beta_n^{(2)} = 1$. *In the simulation the eigenvalues of a* 5000×5000 *matrix were calculated.*

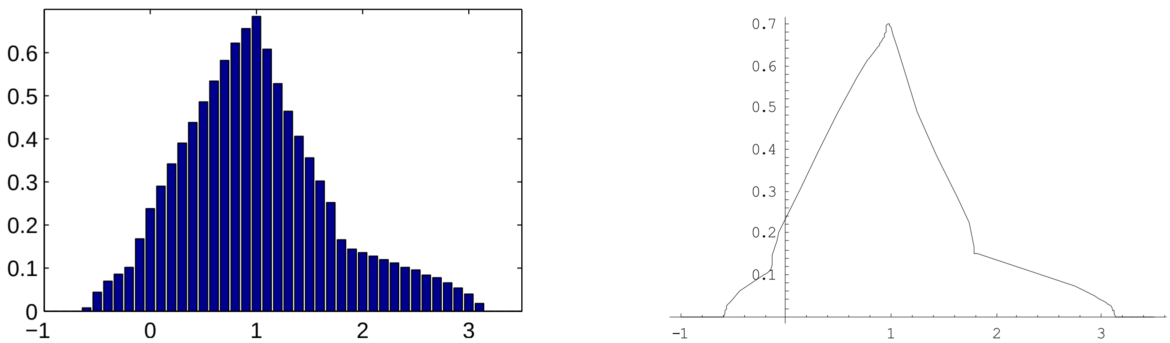

Figure 5: *Simulated and limiting spectral density of the matrix* $L_n^{(p)}$ *for* $p = 2,\ a_n^{(1)} = 10n,\ \beta_n^{(1)} = 1,\ a_n^{(2)} = n,\ \beta_n^{(2)} = 1$. *In the simulation the eigenvalues of a* 5000×5000 *matrix were calculated.*

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ant random matrix ensembles).

6 Appendix: Proof of Theorem 5.3

The assertion of Theorem 5.3 follows from a sharper upper bound for the probability in (5.5). First we define the random variables

$$
\begin{aligned}
Z_n^{(1,j)} &= \max_{0\le i\le n-1} \frac{\big|X_{2a_n^{(j)}-i\beta_n^{(j)}} - (2a_n^{(j)} - i\beta_n^{(j)})\big|}{2a_n^{(j)}},\\
Z_n^{(2,j)} &= \max_{1\le i\le n-1} \frac{\big|Y_{i\beta_n^{(j)}} - i\beta_n^{(j)}\big|}{2a_n^{(j)}},\\
Z_n^{(3,j)} &= \max_{1\le i\le n-1} \frac{\big|X_{2a_n^{(j)}-(i-1)\beta_n^{(j)}}Y_{(n-i)\beta_n^{(j)}} - \sqrt{(2a_n^{(j)}-(i-1)\beta_n^{(j)})(n-i)\beta_n^{(j)}}\big|}{2a_n^{(j)}},\\
Z_n^{(1)} &= \max_{1\le j\le p}\{Z_n^{(1,j)}\},\ Z_n^{(2)} = \max_{1\le j\le p}\{Z_n^{(2,j)}\}\ ,Z_n^{(3)} = \max_{1\le j\le p}\{Z_n^{(3,j)}\},\\
Z_n &= \max\{Z_n^{(1)}, Z_n^{(2)}, Z_n^{(3)}\}.
\end{aligned}
$$

Similar to the proof of Theorem 4.1 we can show for the maximum difference between the eigenvalues $\lambda_j^{(n,p)}$ and the roots $x_j^{(n,p)}$ the inequality

$$\max_{1\le j\le n} |\lambda_j^{(n,p)} - x_j^{(n,p)}| \le 4pZ_n$$

and therefore

$$P\Big(\max_{1\le j\le n}|\lambda_j^{(n,p)} - x_j^{(n,p)}| > \epsilon\Big) \le P\Big(\max\{Z_n^{(1)}, Z_n^{(2)}\} > \frac{\epsilon}{4p}\Big) + P\Big(Z_n^{(3)} > \frac{\epsilon}{4p}\Big). \tag{6.1}$$

Now the two probabilities in (6.1) can be considered separately and the arguments in Dette and Imhof (2007) show that

$$P\Big(\max\{Z_n^{(1)}, Z_n^{(2)}\} > \frac{\epsilon}{4p}\Big) \le \sum_{j=1}^{p}(2n-1)p_1(j)$$

and

$$P\Big(Z_n^{(3)} > \frac{\epsilon}{4p}\Big) \le \sum_{j=1}^{p}(n-1)(p_1(j)+p_2(j)),$$

where

$$p_1(j) = 2\Big(\Big(1+\frac{\epsilon}{4p\sqrt{K+2}}\Big)\exp\Big(-\frac{\epsilon}{4p\sqrt{K+2}}\Big)\Big)^{a_n^{(j)}},$$
$$p_2(j) = 2\exp\Big(\frac{(K+1)^2\beta_n^{(j)}}{8}\Big)\Big(1-\frac{\epsilon}{4p}\Big)^{(n-1)\beta_n^{(j)}}\exp\Big(\frac{a_n^{(j)}\epsilon}{2p}\Big).$$

This gives the upper bound

$$P\Big(\max_{1\le j\le n}|\lambda_j^{(n,p)} - x_j^{(n,p)}| > \epsilon\Big) \le \sum_{j=1}^{p}(3n-2)p_1(j) + (n-1)p_2(j)$$

which is a sharper inequality than (5.5). The arguments presented in Dette and Imhof (2007) now show that the difference between the eigenvalues $\lambda_j^{(n,p)}$ and the roots $x_j^{(n,p)}$ satisfies the inequality in Theorem 5.3 and conclude the proof.

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