Limit theorems for bipower variation of semimartingales

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Abstract

This paper presents limit theorems for certain funtionals of semimartingales observed at high frequency. In particular, we extend results from Jacod [4] to the case of bipower variation, showing under standard assumptions that one obtains a limiting variable, which is in general different from the case of a continuous semimartingale. In a second step a truncated version of bipower variation is constructed, which has a similar asymptotic behaviour as standard bipower variation for a continuous semimartingale and thus provides a feasible central limit theorem for the estimation of the integrated volatility even when the semimartingale exhibits jumps.

Keywords: bipower variation, central limit theorem, high frequency observations, semimartingale, stable convergence.

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1 Introduction

A key issue in financial econometrics is to use discrete observations to draw inference about certain characteristics of an underlying stochastic process X, typically separated into two substantially different cases: One either has low-frequency data, thus a fixed lag between the observations and a time horizon tending to infinity, or high-frequency data with a fixed time horizon and lags converging to zero. Throughout this paper, we assume to be in the high frequency situation with a time horizon $[0,T]$, say, and regular observations times $\frac{i}{n}$, $i = 0, \dots, \lfloor nT \rfloor$. X is typically regarded as a one-dimensional semimartingale (satisfying some mild additional assumptions) living on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and in this case it is classical to focus on the estimation of the entire quadratic variation of X or parts thereof, namely the integrated volatility or the sum of squared jumps.

Let us be more specific: The basic assumption on the latent price process X is that it is an *Itô semimartingale*, which means that its characteristics are absolutely continuous with respect to Lebesgue measure. Equivalently, we have the representation

$$X_t \;=\; X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s + (\delta 1_{\{|\delta| \le 1\}}) \star (\underline{\mu} - \underline{\nu})_t + (\delta 1_{\{|\delta| > 1\}}) \star \underline{\mu}_t, \tag{1.1}$$

where W is a Brownian motion and $\underline{\mu}$ and $\underline{\nu}$ are a Poisson random measure on $\mathbb{R}_+ \times E$ and its compensator $\underline{\nu}(dt, dz) = dt \otimes \lambda(dz)$, where $(E, \mathcal{E})$ is an auxiliary space and λ a σ-finite measure. Some classical assumptions on the coefficients b, σ and δ will be given later. For all unexplained (but standard) notation see Jacod and Shiryaev [9].

In this case, the quadratic variation $[X,X]_t$ of X is almost surely finite and has the representation (with $\Delta X_s = X_s - X_{s-}$)

$$[X,X]_t = \int_0^t \sigma_s^2 \, ds + \sum_{s \le t} |\Delta X_s|^2,$$

where the first and the second term on the right hand side are the afore-mentioned integrated volatility and the sum of squared jumps, respectively. In terms of semimartingale theory, they constitute the quadratic variations of the continuous and the purely discontinuous martingale part of X. For financial applications, the integrated volatility is the most important quantity that has to be estimated, and over the last years several methods have been developed to tackle this task. At least for some of these estimators it is important, whether the underlying semimartingale is continuous or exhibits jumps; others, however, are robust to jumps and thus work in a discontinuous framework as well.

Let us give a brief overview on the four most prominent estimators for the integrated volatility. When the underlying process is known to be continuous (and when some mild assumptions on the processes (b_t) and (σ_t) are satisfied as well), thus having the representation

$$X_t \;=\; X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s,$$

then the quadratic variation of X and the integrated volatility coincide, and necessarily any estimator for the quadratic variation of a semimartingale becomes an estimator for the integrated volatility. Quite naturally, one chooses the (in some sense optimal) realized variance $RV(X)^n$, for which we have with the notation $\Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$ and for each $t > 0$:

$$RV(X)_t^n := \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n X|^2 \xrightarrow{P} \int_0^t \sigma_s^2 \, ds,$$

and one can prove a stable central limit theorem of the form

$$\sqrt{n} \left(RV(X)_t^n - \int_0^t \sigma_s^2 \, ds \right) \xrightarrow{\mathcal{L}-(s)} \sqrt{2} \int_0^t \sigma_s^2 \, dW_s'. \tag{1.2}$$

Here, W' denotes a second Brownian motion, defined on an appropriate extension of the original probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and being independent of $\mathcal{F}$, and $\xrightarrow{\mathcal{L}-(s)}$ denotes ($\mathcal{F}$-)stable convergence in law. See Barndorff-Nielsen and Shephard [2] for some econometric applications and consult Jacod and Shiryaev [9] for a definition of stable convergence and various properties.

As a matter of fact, the realized variance becomes inconsistent when the process exhibits jumps, and thus one would like to find estimators for the integrated volatility that hold in the more comprehensive model (1.1) as well. Basically three types of estimators owning this property have been discussed throughout the last years. A threshold estimator of the form

$$TV(X, \alpha, \varpi)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n X|^2 1_{\{|\Delta_i^n X| < \alpha n^{-\varpi}\}}$$

with $\alpha > 0$ and $\varpi \in (0, \frac{1}{2})$ was introduced by Mancini for some special settings (see Mancini [10] for a review) and extended in Jacod [4] to the general semimartingale case. The intuition behind this estimator is to cut off large increments over small intervals as it is likely that they are due to a jump of the underlying process X. It has been shown that each $TV(X, \alpha, \varpi)_t^n$ is consistent for $\int_0^t \sigma_s^2 \, ds$, and under some additional conditions on the measure ν (stating that the jumps of X are of finite variation with an index of activity smaller than one) and for an appropriate choice of ϖ one obtains a stable central limit theorem of the form (1.2). Alternatively, it has become popular to use multipower variations, which are defined by

$$MV(X, \mathbf{r})_t^n = n^{|\mathbf{r}|/2-1} \sum_{i=1}^{\lfloor nt \rfloor - q+1} \prod_{j=1}^{q} |\Delta_{i+j-1}^n X|^{r_j}.$$

Here, $\mathbf{r}$ denotes a q-dimensional vector $\mathbf{r} = (r_1, \ldots, r_q)$ with non-negative components r_j, and we set $|\mathbf{r}| = r_1 + \ldots + r_q$, $\mathbf{r}_- = \min(r_1, \ldots, r_q)$ and $\mathbf{r}_+ = \max(r_1, \ldots, r_q)$. The intuition behind multipower variations is that increments over intervals with large jumps are typically paired with small increments and therefore (depending on the choice of $\mathbf{r}$) do not play a role in the asymptotics. Precisely, we define m_p to be the p-th absolute moment of a standard normal distribution, set $m_{\mathbf{r}} = \prod_{j=1}^q m_j$ and obtain

$$MV(X, \mathbf{r})_t^n \xrightarrow{P} m_{\mathbf{r}} \int_0^t \sigma_s^{|\mathbf{r}|} \, ds,$$

as long as any component r_j of $\mathbf{r}$ is smaller than two. Central limit theorems can be obtained as well, but again one needs further restrictions on $\mathbf{r}$ and X. Basically, one has to suppose that $\frac{s}{2-s} < \mathbf{r}_- \leq \mathbf{r}_+ < 1$ holds, where s denotes the index of jump activity of X. Then we have (again under further assumptions on the processes (b_t) and (σ_t))

$$\sqrt{n}\,\left(MV(X,\mathbf{r})_t^n - m_{\mathbf{r}} \int_0^t \sigma_s^{|\mathbf{r}|}\,ds\right) \xrightarrow{\mathcal{L}-(s)} \sqrt{p(\mathbf{r})} \int_0^t \sigma_s^{|\mathbf{r}|}\,dW'_s, \tag{1.3}$$

where W' has the same properties as in (1.2) and

$$p(\mathbf{r}) = \prod_{j=1}^{q} m_{2r_j} - (2q-1)\prod_{j=1}^{q} m_{r_j}^2 + 2\sum_{k=1}^{q-1}\prod_{j=1}^{k} m_{r_j} \prod_{j=q-k+1}^{q} m_{r_j} \prod_{j=1}^{q-k} m_{r_j+r_{j+k}}.$$

When X is continuous, all these results hold regardless of $\mathbf{r}$. See Barndorff-Nielsen et al. [3], Woerner [11] or Jacod [5] for details.

The two most prominent examples for multipower variation are $MV(X,(1,1))_t^n$ and $MV(X,(2/3,2/3,2/3))_t^n$, in the following simply called bipower and tripower variation, respectively. Both estimators are (up to a proper scaling) consistent for the integrated volatility, but only for the latter one we have a feasible central limit theorem, since we know from (1.3) the precise form of the conditional variance and are thus able to prove weak convergence of the standardised tripower variation to a standard normal distribution.

The aim of this paper is twofold. On the one hand, we will indeed prove that a central limit theorem for bipower variation holds in the discontinuous case as well, but which is of a substantially different form than for a continuous Itô semimartingale. This result is an extension of the theory developed in Jacod [4] for certain power variations to the multipower case. On the other hand, we will introduce a truncated version of bipower variation and prove for this quantity a central limit theorem of the same type as before. In contrast to $TV(X,\alpha,\varpi)_t^n$ and $MV(X,(2/3,2/3,2/3))_t^n$ this result does also hold for an index of jump activity of X which is equal to one, and thus truncated bipower variation is a more comprehensive alternative to estimate the integrated volatility in the presence of jumps.

2 Assumptions and Notation

It is well-known from Barndorff-Nielsen et al. [1] or Jacod [4] that we need some additional regularity conditions on the coefficients b, σ and δ in order to derive a central limit theorem for certain bipower variation processes. All of these are gathered into the following hypothesis:

Hypothesis (H) : The process X has the form (1.1) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and we have further:

a) The process (b_t) is optional and left-continuous with right limits.

b) The function δ is predictable and locally bounded by a family (γ_k) of non-negative (deterministic) functions on $\mathbb{L}^2(E,\mathcal{E},\lambda)$, such that $\int_E \Phi_1 \circ \gamma_k(z)\;\lambda(dz) < \infty$ with $\Phi_s(z) = 1 \wedge |z|^s$.

c) The process (σ_t) is an Itô semimartingale itself and admits the representation

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_s \, ds + \int_0^t \widetilde{\sigma}_s \, dW_s + M_t + \sum_{s \le t} \Delta\sigma_s \, 1_{\{|\Delta\sigma_s| > v\}}, \tag{2.4}$$

where M is a local martingale orthogonal to W and with bounded jumps. Furthermore, we have $\langle M, M \rangle_t = \int_0^t a_s ds$, and the compensator of $\sum_{s \le t} 1_{\{|\Delta\sigma_s| > v\}}$ is $\int_0^t a'_s ds$. $(\widetilde{b}_t)$, (a_t) and (a'_t) are assumed to be optional and locally bounded, whereas $(\widetilde{\sigma}_t)$ is optional and left-continuous with right limits.

d) $\sigma_t > 0$ for all t almost surely. □

Some of the assumptions in Hypothesis (H) are rather weak (and sometimes just as strict as necessary in order to have the various integrals occuring in the definition of X well-defined), while others are more restrictive.

For the third one regarding the structure of the volatility process σ, note that even in Barndorff-Nielsen et al. [1], where the authors have derived a central limit theorem for bipower variation when the underlying process X is a *continuous* Itô semimartingale, such an additional assumption is necessary. (In contrast, when one only wants to draw inference about the realised quadratic variation, this condition can be removed by an application of Itô's formula, which is not available in this setting.) Thus, it is natural to have a similar condition involved here. Furthermore, the representation of σ_t in c) (which is the same as in Jacod et al. [7]) is similar to the one in Barndorff-Nielsen et al. [1], where the martingale M_t has been spilt up into a purely discontinuous part and a Brownian part, whose driving Wiener process is orthogonal to W.

The condition in b) implies that the process $X - X^c$ (and in particular the jump part of X) is of finite variation, where X^c denotes the continuous martingale part of X. It is well-known that in this case one has an equivalent representation of X as follows:

$$X_t \;=\; X_0 + B_t + \int_0^t \sigma_s \, dW_s + \sum_{s \le t} \Delta X_s, \tag{2.5}$$

where $B_t = \int_0^t b_s \, ds - (\delta 1_{\{|\delta| \le 1\}}) \star \underline{\nu}_t$ is another drift process of finite variation.

Before we come to the results, we have to introduce some auxiliary quantities, all defined on an extension of the original probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. For convenience, we assume that we have a second probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t), \mathbb{P}')$ supporting two sequences of normally distributed random variables $(U_{m+})_{m \ge 1}$ and $(U_{m-})_{m \ge 1}$, each having mean zero and variance one, and a Brownian motion W'. All random variables defined above are assumed to be mutually independent. Then we set

$$\widetilde{\Omega} = \Omega \times \Omega', \qquad \widetilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}', \qquad \widetilde{\mathbb{P}} = \mathbb{P} \times \mathbb{P}',$$

and we extend all quantities defined on the original probability spaces to the product space in the standard way. Any expectation with respect to $\widetilde{\mathbb{P}}$ will further be denoted by $\widetilde{E}$. In order to construct a filtration on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ we let $(T_m)_{m \ge 1}$ be any sequence of stopping times that exhausts the jump times of the process X, that is for any ω we have

$T_m(\omega) \neq T_{m'}(\omega)$ for $m \neq m'$ and $\underline{\mu}(\omega, \{t\} \times E) = 1$ if and only if $t = T_m(\omega)$ for some m. Then $(\widetilde{\mathcal{F}}_t)$ is defined to be the smallest right-continuous filtration containing $(\mathcal{F}_t)$ and such that W' is adapted and U_{m+} and U_{m-} are $\widetilde{\mathcal{F}}_{T_m}$-measurable for each m.

With these definitions at hand we define

$$U'_t(\delta) = \sum_m |\Delta X_{T_m}| \left(\sigma_{T_m-} |U_{m-}| + \sigma_{T_m} |U_{m+}| \right) \tag{2.6}$$

for each t, emphasising its dependency on the function δ through the jumps of X. Under assumption (H), $U'(\delta)$ is a process of finite variation, hence it is absolutely summable and does not depend on the particular choice of the stopping times T_m. We set further

$$U''_t = \sqrt{1 + 2m_1^2 - 3m_1^4} \int_0^t \sigma_s^2 \, dW'_s, \tag{2.7}$$

where W' is another Brownian motion, which is defined on an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and is independent of $\mathcal{F}$ as well. Here, the scalar $\sqrt{1 + 2m_1^2 - 3m_1^4}$ corresponds with $\sqrt{p(\mathbf{r})}$ from (1.3) for this specific choice of $\mathbf{r}$.

Bipower variation for given n and t was defined as $MV(X, (1,1))_t^n$, but for brevity of notation we will simply use

$$V(X)_t^n = \sum_{i=1}^{\lfloor nt \rfloor - 1} |\Delta_i^n X| |\Delta_{i+1}^n X|.$$

3 Results

In the following we will prove a (stable) central limit theorem for

$$\overline{V}(X)_t^n = \sqrt{n} \left(V(X)_t^n - m_1^2 \int_0^t \sigma_s^2 \, ds \right), \tag{3.8}$$

which holds pointwise for each $t \geq 0$, but not for the entire process $\overline{V}(X)^n$, unless the process X has continuous paths almost surely, in which case the corresponding result is well-known from Barndorff-Nielsen et al. [1].

Theorem 3.1 *Assume (H). Then for each t the random variable $\overline{V}(X)_t^n$ converges stably in law with limiting variable $U'_t(\delta) + U''_t$.*

Remark 3.2 In the continuous case the limiting variable is simply U''_t, and there is a simple intuition, where the additional component $U'_t(\delta)$ in the general setting comes from: Each increment of X containing a jump appears twice in $V(X)_t^n$ (forget about border effects), one time paired with the previous increment and one time paired with the subsequent one. If we suppose for a moment that the underlying semimartingale exhibits only finitely many jumps on the interval $[0, t]$, then each of the two adjacent increments does not contain a jump with a probability converging to one. Thus, if the jump lies within $[\frac{i-1}{n}, \frac{i}{n}]$, it obviously dominates $\Delta_i^n X$, whereas the asymptotic behaviour of $\Delta_{i-1}^n X$ and $\Delta_{i+1}^n X$ is driven by the Brownian part as for a continuous semimartingale. Terms in $V(X)_t^n$ not affected by jumps behave in the same way as before, and this gives the result. The extension to the case of infinite activity can be obtained by standard methods, if one cuts off jumps smaller than $\frac{1}{q}$ first (for any $q > 0$ there are only finitely many jumps remaining) and then letting q tend to infinity. $\square$

Note that the limiting process is not a martingale as for (1.3) unless X is continuous, since $U_t'(\delta)$ is an increasing process of finite variation. A natural way to remove this bias is to subtract an estimator for $U_t'(\delta)$ from $\sqrt{n}\ V(X)_t^n$. Intuitively, such an estimator is given by $\sqrt{n}\ V^*(X, \alpha, \varpi)_t^n$ with

$$V^*(X,\alpha,\varpi)_t^n = \sum_{i=1}^{\lfloor nt \rfloor - 1} |\Delta_i^n X||\Delta_{i+1}^n X| \cdot \Big(1_{\{|\Delta_i^n X| \geq \alpha n^{-\varpi}\}} 1_{\{|\Delta_{i+1}^n X| < \alpha n^{-\varpi}\}} + 1_{\{|\Delta_i^n X| < \alpha n^{-\varpi}\}} 1_{\{|\Delta_{i+1}^n X| \geq \alpha n^{-\varpi}\}} \Big)$$

for $\alpha > 0$ and $\varpi \in (0, \frac{1}{2})$. Alternatively, we can develop the asymptotic theory for the direct analogue of $TV(X, \alpha, \varpi)_t^n$, namely

$$TV^*(X,\alpha,\varpi)_t^n = \sum_{i=1}^{\lfloor nt \rfloor - 1} |\Delta_i^n X| 1_{\{|\Delta_i^n X| < \alpha n^{-\varpi}\}} |\Delta_{i+1}^n X| 1_{\{|\Delta_{i+1}^n X| < \alpha n^{-\varpi}\}}.$$

The intuition behind both estimators is the same: A large value of $|\Delta_i^n X|$ indicates the existence of a big jump in $[\frac{i-1}{n}, \frac{i}{n}]$, whereas a small value of $|\Delta_i^n X|$ suggests that the continuous martingale part of X is dominating. With

$$\overline{V}^*(X)_t^n = \sqrt{n}\ \Big((V(X)_t^n - V^*(X,\alpha,\varpi)_t^n) - m_1^2 \int_0^t \sigma_s^2\ ds \Big)$$

and

$$\overline{TV}^*(X)_t^n = \sqrt{n}\ \Big(TV^*(X,\alpha,\varpi)_t^n - m_1^2 \int_0^t \sigma_s^2\ ds \Big)$$

we end up with the following theorem.

Theorem 3.3 *Assume (H) and let $\alpha > 0$ and $\varpi \in (0, \frac{1}{2})$ be arbitrary. Then for each t the random variables $\overline{V}^*(X)_t^n$ and $\overline{TV}^*(X)_t^n$ converge stably in law with the same limiting variable U_t''.*

Note that we only need hypothesis (H) in order to derive the stable convergence in Theorem 3.3, and thus we have a less restrictive result than the corresponding limit theorems for $TV(X, \alpha, \varpi)_t^n$ and $MV(X, (2/3, 2/3, 2/3))_t^n$, which need a stronger condition than assumption b) in (H). Precisely, both claims rely on the fact that the familiy (γ_k) from b) satisfies $\int_E \Phi_s \circ \gamma_k(z)\ \lambda(dz) < \infty$ for some $s < 1$.

In order to derive a classical central limit theorem for the estimation of the integrated volatility recall that the stochastic convergence

$$MV(X,(4/3,4/3,4/3))_t^n \xrightarrow{P} m_{4/3}^3 \int_0^t \sigma_s^4 \, ds$$

holds. Thus, the following corollary can be derived easily.

Corollary 3.4 *Assume (H) and let $\alpha > 0$ and $\varpi \in (0, \frac{1}{2})$ be arbitrary. Then for each t we have*

$$\sqrt{n} \, \frac{m_1^{-2} \; TV^*(X,\alpha,\varpi)_t^n - \int_0^t \sigma_s^2 \, ds}{\sqrt{\frac{1+2m_1^2-3m_1^4}{m_1^4 \; m_{4/3}^3} MV(X,(4/3,4/3,4/3))_t^n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1),$$

where $\xrightarrow{\mathcal{L}}$ denotes convergence in law.

A similar result holds for $V(X)_t^n - V^*(X,\alpha,\varpi)_t^n$ as well.

Remark 3.5 Obviously, it is not necessary to restrict oneself to the case of bipower variation as similar results hold for any $MV(X,\mathbf{1})_t^n$, where $\mathbf{1}$ is q-dimensional having components equal to 1 ($q \geq 2$). In this case the limiting variable for the untruncated version (connected with the CLT in Theorem 3.1) is $\widetilde{U}_t' + U_t''$, where

$$\widetilde{U}_t' = \sum_m |\Delta X_{T_m}| \left(\sum_{j=0}^{q-1} \left\{ \sigma_{T_m -} \sum_{k=1}^{j} |U_{m,-k}| + \sigma_{T_m +} \sum_{k=j+1}^{q-1} |U_{m,k-j}| \right\} \right),$$

for a family of mutually independent standard normally distributed random variables

$$(U_{m,-(q-1)}, \ldots U_{m,-1}, U_{m,1}, \ldots U_{m,(q-1)})_{m \geq 1}$$

on $(\Omega', \mathcal{F}', (\mathcal{F}_t'), \mathbb{P}')$. For the truncated versions (connected with the CLTs in Theorem 3.3) we have the same result as in (1.3). $\square$

4 Appendix

As usual, a localisation procedure as for example explained in Jacod [6] allows us to assume that the processes a_t, a_t', b_t, $\widetilde{b}_t$ and $\widetilde{\sigma}_t$ as well as σ_t, W_t and X_t are all bounded themselves. Moreover, we may replace the family (γ_k) by a bounded function γ having the same properties. Constants appearing in the proofs are usually denoted by C or C_p, if we want to emphasise their dependency on an additional parameter p.

Let us start with a lemma proving the claim on $U'(\delta)$.

Lemma 4.1 *Assume (H). Then*

$$U'_t(\delta) = \sum_{m:\ T_m \le t} |\Delta X_{T_m}| \ \Big(\sigma_{T_m -}|U_{m-}| + \sigma_{T_m}|U_{m+}|\Big) \tag{4.9}$$

defines an increasing process of finite variation and does thus not depend on the particular choice of the sequence $(T_m)_{m\ge1}$.

Proof of Lemma 4.1. We have

$$\widetilde{E}\Big[\sigma_{T_m -}|U_{m-}| + \sigma_{T_m}|U_{m+}| \ \Big|\mathcal{F}\Big] < C\Big(\sigma_{T_m -} + \sigma_{T_m}\Big) < C$$

by assumption, and from the condition on the jump activity of X

$$\begin{aligned}
\widetilde{E}\Big[U'_t(\delta)\Big] &= \widetilde{E}\Big(\sum_{m:\ T_m\le t} |\Delta X_{T_m}| \ \widetilde{E}\Big[\sigma_{T_m -}|U_{m-}| + \sigma_{T_m}|U_{m+}| \ \Big|\mathcal{F}\Big] \ \Big) \\
&\le C\ E\Big[\sum_{m:\ T_m\le t} |\Delta X_{T_m}|\Big] = C\ E\Big[\int_0^t \int_E |\delta(s,z)| \ \lambda(dz)\ ds\Big] \\
&\le C\ t \int_E \gamma(z)\ \lambda(dz) < \infty
\end{aligned}$$

follows. The claim is obvious now. □

Lemma 4.1 allows us to choose any sequence $(T_m)_{m\ge1}$ of stopping times with pairwise disjoint graphs such that $\Delta X_t \neq 0$ implies that $t = T_m$ for some integer m. For our purposes it is convenient to choose that sequence as follows: For any integer q (and with $1/0 = \infty$) let $T_{m,q}$ denote the successive jump times of the Poisson process $\underline{\mu}((0,t] \times \{z \mid 1/(q-1) \ge \gamma(z) > 1/q\})$, where γ is the function occuring in (H). Obviously, the graphs of these stopping times are pairwise disjoint (both in m and q), and we may define $(T_m)_{m\ge1}$ to be any reordering of $\{T_{m,q} \mid m, q \ge 1\}$. Moreover, we denote by P_q the set of all m such that $T_m = T_{m,r}$ for some $r \le q$.

Recalling (2.5) and condition (H) we define several auxiliary processes. We set

$$X'_t \ = \ X_0 + B_t + \int_0^t \sigma_s dW_s, \qquad J_t = \sum_{s\le t} \Delta X_s, \tag{4.10}$$

and define for any integer $q > 1$

$$J(q)_t = (\delta\ 1_{\{\gamma(z)>1/q\}}) \star \underline{\mu}_t, \qquad X(q)_t = X'_t + J(q)_t, \qquad X'(q)_t = X_t - X(q)_t. \tag{4.11}$$

We have to introduce some further additional notation. Define $\underline{\mu}(q)$ to be the random measure given by the restriction of $\underline{\mu}$ to $\{x \mid \gamma(x) > 1/q\}$, thus it is associated with the large jumps of X. It follows that if we denote by $(\overline{\mathcal{F}'_t})$ the smallest filtration containing $(\mathcal{F}_t)$ and making $\underline{\mu}(q)$ $\overline{\mathcal{F}'_0}$-measurable, W remains a Wiener process with respect to this new filtration, and X' has the representation in (4.10), both with respect to $(\mathcal{F}_t)$ and $(\overline{\mathcal{F}'_t})$.

As in Jacod et al. [7] we denote by $\Omega_n(t,q)$ the set of all ω in Ω such that the following properties are satisfied for any $m \neq m'$ in P_q with $T_m(\omega), T_{m'}(\omega) \leq t$:

$$\frac{1}{n} \leq T_m(\omega), T_{m'}(\omega) \leq t - \frac{2}{n}, \qquad |T_m(\omega) - T_{m'}(\omega)| \geq \frac{4}{n}, \qquad nT_m \text{ is not an integer.}$$

Since $\Omega_n(t,q) \to \Omega$ almost surely (in n) for any q, we will assume in the following quite often that ω belongs to $\Omega_n(t,q)$, once we are working with a fixed parameter q.

Before we come to the proof of the two main results of this paper, we state a set of inequalities for increments of the various processes in (4.10) and (4.11). Most of them are well-known from the theory of semimartingales, but at least for one of these we will give a short proof.

Lemma 4.2 *Assume (H). Then the following inequalities hold for arbitrary i and n and any $r > 0$:*

$$E\Big[|\Delta_i^n X|^r \Big| \mathcal{F}_{\frac{i-1}{n}}\Big] < C\ n^{-(\frac{r}{2}\wedge 1)}, \qquad E\Big[|\Delta_i^n J|^r \Big| \mathcal{F}_{\frac{i-1}{n}}\Big] < C\ n^{-(r\wedge 1)},$$
$$E\Big[|\Delta_i^n X'|^r \Big| \mathcal{F}_{\frac{i-1}{n}}\Big] < C\ n^{-\frac{r}{2}}, \qquad E\Big[|\Delta_i^n W|^r \Big| \mathcal{F}_{\frac{i-1}{n}}\Big] < C\ n^{-\frac{r}{2}}.$$

Moreover, with

$$e_q = \int_{\{\gamma(z)\leq 1/q\}} \gamma(z)\ \lambda(dz) \tag{4.12}$$

we have for any q, i and n:

$$E\Big[|\Delta_i^n X'(q)| \Big| \mathcal{F}_{\frac{i-1}{n}}\Big] \leq e_q\ n^{-1}.$$

Proof of Lemma 4.2. We will only prove the last assertion. It holds

$$\begin{aligned} E\Big[|\Delta_i^n X'(q)|\ \Big| \mathcal{F}_{\frac{i-1}{n}}\Big] &\leq E\Big[\sum_{m\in P_q^c:\ T_m\in[\frac{i-1}{n},\frac{i}{n}]} |\Delta X_{T_m}|\ \Big| \mathcal{F}_{\frac{i-1}{n}}\Big] \\ &= E\Big[\int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\{\gamma(z)\leq 1/q\}} |\delta(u,z)|\ \lambda(dz)\ du\ \Big| \mathcal{F}_{\frac{i-1}{n}}\Big] \\ &\leq \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\{\gamma(z)\leq 1/q\}} \gamma(z)\ \lambda(dz)\ du = \frac{e_q}{n}. \end{aligned}$$

□

We end this preliminary part with an auxiliary result on stable convergence, for which we have to introduce some further notation. For any m the stopping time T_m is contained in exactly one interval of size $\frac{i}{n}$, and we set $I_m^n = \min(i : \frac{i}{n} \geq T_m)$, thus $T_m \in (\frac{I_m^n - 1}{n}, \frac{I_m^n}{n}]$. Moreover, we define a couple of random variables depending on m and n, namely

$$\begin{array}{lll} U_{m-}^n = \sqrt{n}\ \Delta_{I_m^n-1}^n W, & U_{m+}^n = \sqrt{n}\ \Delta_{I_m^n+1}^n W, & U^n = (U_{m-}^n, U_{m+}^n)_{m\geq 1}, \\ \rho_{m-}^{\prime n} = \sqrt{n}\ \sigma_{\frac{I_m^n-2}{n}}\ \Delta_{I_m^n-1}^n W, & \rho_{m+}^{\prime n} = \sqrt{n}\ \sigma_{I_m^n}\ \Delta_{I_m^n+1}^n W, & \rho^{\prime n} = (\rho_{m-}^{\prime n}, \rho_{m+}^{\prime n})_{m\geq 1}, \\ \rho_{m-}^n = \sqrt{n}\ \Delta_{I_m^n-1}^n X', & \rho_{m+}^n = \sqrt{n}\ \Delta_{I_m^n+1}^n X', & \rho^n = (\rho_{m-}^n, \rho_{m+}^n)_{m\geq 1}, \\ \rho_{m-} = \sigma_{T_m-}\ U_{m-}, & \rho_{m+} = \sigma_{T_m}\ U_{m+}, & \rho = (\rho_{m-}, \rho_{m+})_{m\geq 1}, \end{array}$$

where U_{m-} and U_{m+} are the random variables introduced in Section 2. We start with a claim, which is similar to a lemma from Jacod [6].

Lemma 4.3 *The sequence ρ^n converges stably in law to the sequence ρ.*

Proof of Lemma 4.3. We will only give a sketch of the proof, since main parts are analogous to the one of the corresponding result in Jacod [6].

The first step is to establish the convergence $U^n \xrightarrow{\mathcal{L}-(s)} U$, where $U = (U_{m-}, U_{m+})_{m\geq 1}$. This result can be shown by similar methods as in Jacod [6] or Jacod and Protter [8], and we will only give the basic idea. Note that one has to prove

$$E[g(U^n)Z] \to \widetilde{E}[g(U)Z]$$

for a bounded $\mathcal{F}$-measurable Z and a bounded and continuous function g. Without loss of generality it can be assumed that Z is measurable with respect to the σ-algebra $\mathcal{G}$ which is generated by W and $\underline{\mu}$, as U^n and U are $\mathcal{G}$- and $\mathcal{G} \times \mathcal{F}'$-measurable, respectively. Since $\underline{\mu}$ has the form $\underline{\mu} = \sum_{m\geq 1} \delta_{\{T_m, V_m\}}$, where δ denotes the Dirac measure, and the V_m are suitable E-valued variables, and by a density argument it is sufficient to show (4.13) for the specific choice

$$Z = f(W) \prod_{m=1}^{M} h_m(T_m) h'_m(V_m), \qquad g((y_m)_{m\geq 1}) = \prod_{m=1}^{M} g_m(y_m),$$

any integer M. Here, f, h_m, h'_m, g_m denote bounded and continuous functions on the obvious spaces connected with W, T_m, V_m, U^n (and U). Note further that $W_t^n = W_t - \sum_{m=1}^{M} (W_{T_m + \frac{2}{n}} - W_{(T_m - \frac{2}{n})^+})$ converges uniformly to W_t, which allows us to focus on W^n only. For given M,

$$\Omega_n(M) = \bigcap_{m,m' \in \{1,\ldots M\}, m \neq m'} \Big\{ \omega : \ |T_m(\omega) - T_{m'}(\omega)| \geq \frac{4}{n} \Big\}$$

converges to Ω almost surely, and by construction W^n, the family (V_m) and the family $(h_m(T_m)\ g_m(U^n_{m-}, U^n_{m+}))$ are mutually independent on $\Omega_n(M)$. We are thus left to prove

$$E\Big[\prod_{m=1}^{M} h_m(T_m)\ g_m(U^n_{m-}, U^n_{m+})\ 1_{\Omega_n(M)} \Big] \to \widetilde{E}\Big[\prod_{m=1}^{M} h_m(T_m)\ g_m(U_{m-}, U_{m+}) \Big],$$

and the first claim follows easily.

For the second step, note that

$$\Big(\sigma_{\frac{I^n_m - 2}{n}}, \sigma_{\frac{I^n_m}{n}} \Big)_{m\geq 1} \xrightarrow{P} \Big(\sigma_{T_m -}, \sigma_{T_m} \Big)_{m\geq 1},$$

since σ is càdlàg and bounded, and the first claim plus the properties of stable convergence yield ${\rho'}^n \xrightarrow{\mathcal{L}-(s)} \rho$. It remains to prove that ${\rho'}^n_{m-} - \rho^n_{m-} \xrightarrow{P} 0$ holds for each m, as the result with $m-$ replaced by $m+$ can be shown analogously. However, this is an easy consequence of Lemma 4.2 and the assumptions on σ. $\square$

Proof of Theorem 3.1. The proof will basically consist of five steps, and we will quite often refer to details given in Barndorff-Nielsen et al. [1] or Jacod [6]. The main step is the following decomposition of $\overline{V}(X)^n_t$, which holds for any fixed integer q. We have

$$\begin{aligned}\overline{V}(X)^n_t &= \sqrt{n}\,\Big(V(X')^n_t - m_1^2\int_0^t \sigma_s^2\,ds\Big) + \sqrt{n}\,\Big(V(X(q))^n_t - V(X')^n_t\Big)\\ &+ \sqrt{n}\,\Big(V(X)^n_t - V(X(q))^n_t\Big) =: \overline{V}_1(X)^n_t + \overline{V}_2(X)^n_t + \overline{V}_3(X)^n_t.\end{aligned}$$

Step 1. Here, we simply show that the conditions for an application of Theorem 2.4. in Barndorff-Nielsen et al. [1] are fulfilled, from which we conclude that

$$\overline{V}_1(X)^n_t \overset{\mathcal{L}-(s)}{\longrightarrow} U''_t. \tag{4.13}$$

To this end, note that B_t in (2.5) can be written as

$$B_t = \int_0^t \overline{b}_s\,ds \qquad \text{with} \qquad \overline{b}_s = b_s - \int \delta(s,z)1_{\{|\delta(s,z)|\le 1\}}\,\lambda(dz),$$

and following Jacod [4] it has the same properties as the original drift process. Moreover, both conditions on the volatility process (σ_t) are similar to the ones in Barndorff-Nielsen et al. [1]. Thus (4.13) holds. $\square$

Step 2. We set

$$\delta(q)(\omega,s,z) = \delta(\omega,s,z)\;1_{\{\gamma(z)\ge 1/q\}}$$

and prove

$$\overline{V}_2(X)^n_t = \sqrt{n}\,\Big(V(X(q))^n_t - V(X')^n_t\Big) \overset{\mathcal{L}-(s)}{\longrightarrow} U'_t(\delta(q)) \tag{4.14}$$

for any fixed integer q. It can easily be seen that only those summands in

$$V(X(q)) - V(X') = \sum_{i=1}^{\lfloor nt\rfloor-1}\Big(|\Delta_i^n X(q)||\Delta_{i+1}^n X(q)| - |\Delta_i^n X'||\Delta_{i+1}^n X'|\Big)$$

are non-vanishing, for which we have $i = I^n_m$ or $i = I^n_m + 1$ with m in P_q. Thus

$$\begin{aligned}V(X(q)) - V(X') &= \sum_{m\in P_q}\Big\{\Big(|\Delta^n_{I^n_m}X(q)||\Delta^n_{I^n_m+1}X(q)| - |\Delta^n_{I^n_m}X'||\Delta^n_{I^n_m+1}X'|\Big)\\ &\quad+\Big(|\Delta^n_{I^n_m}X(q)||\Delta^n_{I^n_m+1}X(q)| - |\Delta^n_{I^n_m}X'||\Delta^n_{I^n_m+1}X'|\Big)\Big\}.\end{aligned}$$

On $\Omega_n(t,q)$ we have $\Delta^n_{I^n_m-1}X(q) = \Delta^n_{I^n_m-1}X'$ as well as $\Delta^n_{I^n_m+1}X(q) = \Delta^n_{I^n_m+1}X'$, so by an application of Lemma 4.2 we end up with

$$V(X(q)) - V(X') = \sum_{m \in P_q} |\Delta^n_{I^n_m} X(q)| \left(|\Delta^n_{I^n_m-1}X'| + |\Delta^n_{I^n_m+1}X'| \right) + O_p(n^{-1}),$$

since there are only finitely many elements in P_q. From

$$\Big| |\Delta^n_{I^n_m} X(q)| - |\Delta X_{T_m}| \Big| \le |\Delta^n_{I^n_m} X'|$$

we conclude finally that

$$\begin{aligned} \sqrt{n}\ (V(X(q)) - V(X')) &= \sqrt{n} \sum_{m \in P_q} |\Delta X_{T_m}| \left(|\Delta^n_{I^n_m-1}X'| + |\Delta^n_{I^n_m+1}X'| \right) + O_p(n^{-\frac{1}{2}}) \\ &= \sum_{m \in P_q} |\Delta X_{T_m}| \left(|\rho^n_{m-}| + |\rho^n_{m+}| \right) + O_p(n^{-\frac{1}{2}}). \end{aligned}$$

Using Lemma 4.3, (2.6) and the continuity theorem for stable convergence the result follows. □

Step 3. Here, we prove the joint stable convergence

$$\left(\overline{V}_1(X)^n_t, \overline{V}_2(X)^n_t \right) \stackrel{\mathcal{L}-(s)}{\longrightarrow} \left(U''_t, U'_t(\delta(q)) \right) \tag{4.15}$$

for any fixed integer q. Set $\beta^n_i = \sigma_{\frac{i-1}{n}}\ \Delta^n_i W$ and $\beta'^n_i = \sigma_{\frac{i-1}{n}}\ \Delta^n_{i+1} W$ and let

$$\zeta^n_i = |\beta^n_i||\beta'^n_i| - E\Big[|\beta^n_i||\beta'^n_i| \ \Big| \mathcal{F}_{\frac{i-1}{n}} \Big].$$

As in Barndorff-Nielsen et al. [1] we have

$$\overline{V}_1(X)^n_t = \sqrt{n} \sum_{i=1}^{\lfloor nt \rfloor -1} \zeta^n_i + o_p(1),$$

uniformly in t. Denote the sum on the right hand side by $\overline{U}^n_t$. Our aim is to show the stable convergence

$$\left(\overline{U}^n, (\rho^n_{m-}, \rho^n_{m+})_{m \in P_q} \right) \stackrel{\mathcal{L}-(s)}{\longrightarrow} \left(U'', (\rho_{m-}, \rho_{m+})_{m \in P_q} \right), \tag{4.16}$$

from which (using the last part in the proof of (4.14)) the result in (4.15) can be concluded. Again, this result has a similar expression in Jacod [4].

Note that we have to show

$$E\Big[h(\overline{U}^n) \prod_{m=1}^r g_m(\rho^n_{m-}, \rho^n_{m+})\ Y \Big] \to \widetilde{E}\Big[h(U'') \prod_{m=1}^r g_m(\rho_{m-}, \rho_{m+})\ Y \Big],$$

each r, for an arbitrary bounded and Lipschitz h (on the Skorokhod space of functions on $\mathbb{R}_+$ endowed with an appropriate metric for the Skorokhod topology) and any familiy of bounded and continuous functions g_m, and where the $T_1, T_2, \ldots$ are the jump times of $\underline{\mu}(q)$. As before, it suffices to prove this assertion for a bounded and $\mathcal{H}$-measurable Y, where $\mathcal{H}$ is generated by the measure $\underline{\mu}(q)$ and the processes σ, W and X.

First, the same argument as in the proof of Lemma 4.3 allows us to replace ρ^n and ρ by U^n and U, so we are left to prove

$$E\Big[h(\overline{U}^n)\ \prod_{m=1}^{r} g_m(U^n_{m-}, U^n_{m+})\ Y\Big] \to \widetilde{E}\Big[h(U'')\ \prod_{m=1}^{r} g_m(U_{m-}, U_{m+})\ Y\Big], \tag{4.17}$$

each r.

For a given integer p, we set $T_m^{p-} = (T_m - \frac{1}{p})^+$ and $T_m^{p+} = (T_m + \frac{1}{p})$ and define $B_p = \bigcup_{m\geq 1}(T_m^{p-}, T_m^{p+}]$. Moreover, we let $\overline{\mathcal{F}'}^p$ be the smallest filtration, which contains $\overline{\mathcal{F}'}$ and with respect to which the process $W_t^p = \int_0^t 1_{B_p}(s)\ dW_s$ is $\overline{\mathcal{F}'_0}^p$-measurable. Since B_p is $\overline{\mathcal{F}'_0}$-measurable by construction and as it decreases to the union of the graphs of the stopping times T_m as p grows, it is likely that we can replace the processes $\overline{U}^n_t$ and U''_t in (4.17) by

$$\overline{U}^{n,p}_t = \sum_{i\in\Gamma_n(p,t)} \zeta^n_i, \qquad U''^p_t = \sqrt{1 + 2m_1^2 - 3m_1^4}\ \int_0^t \sigma_s^2\ 1_{B_p^c}(s)\ dW'_s,$$

respectively, where $\Gamma_n(p,t)$ is the set of all integers i such that $[\frac{i-2}{n}, \frac{i+1}{n}] \cap B_p = \emptyset$. Indeed, it is easy to see that for $p \to \infty$ both

$$\sup_{s\leq t} |U''_s - U''^p_s| \xrightarrow{P} 0 \quad \text{and} \quad \sup_{s\leq t} |\overline{U}^n_s - \overline{U}^{n,p}_s| \xrightarrow{P} 0$$

hold, the latter result uniformly in n. (4.17) follows, once we have shown

$$E\Big[h(\overline{U}^{n,p})\ \prod_{m=1}^{r} g_m(U^n_{m-}, U^n_{m+})\ Y\Big] \to \widetilde{E}\Big[h(U''^p)\ \prod_{m=1}^{r} g_m(U_{m-}, U_{m+})\ Y\Big] \tag{4.18}$$

for each p.

Finally, a close look at the proof of Theorem 2.4. in Barndorff-Nielsen et al. [1] shows that Step 1) still holds for $\overline{U}^{n,p}$ and U''^p (conditionally on $\overline{\mathcal{F}'_0}^p$), that is

$$E\Big[h(\overline{U}^{n,p})\ Y\ \Big|\overline{\mathcal{F}'_0}^p\Big] \to \widetilde{E}\Big[h(U''^p)\ Y\ \Big|\overline{\mathcal{F}'_0}^p\Big].$$

Since any $g_m(U^n_{m-}, U^n_{m+})$ is bounded and measurable with respect to $\overline{\mathcal{F}'_0}^p$,

$$\begin{aligned} E\Big[h(\overline{U}^{n,p})\ \prod_{m=1}^{r} g_m(U^n_{m-}, U^n_{m+})\ Y\Big] &= E\Big[\prod_{m=1}^{r} g_m(U^n_{m-}, U^n_{m+})\ E\Big[h(\overline{U}^{n,p})\ Y\ \Big|\overline{\mathcal{F}'_0}^p\Big]\Big] \\ &= E\Big[\prod_{m=1}^{r} g_m(U^n_{m-}, U^n_{m+})\ \widetilde{E}\Big[h(U''^p)\ Y\ \Big|\overline{\mathcal{F}'_0}^p\Big]\Big] + o(1) \end{aligned}$$

follows. However, $\widetilde{E}\Big[h(U''^p)\ Y\ \Big|\overline{\mathcal{F}'_0}^p\Big]$ is bounded and $\overline{\mathcal{F}'_0}^p$-measurable, so proving (4.18) simply means proving $U^n \stackrel{\mathcal{L}-(s)}{\longrightarrow} U$, and thus we are done. $\square$

Step 4. In this step we show that the third term in the decomposition of $\overline{V}(X)^n_t$ is asymptotically negligible, i.e. we have for each $\eta, t \geq 0$

$$\lim_{q\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(|\overline{V}_3(X)^n_t| > \eta\Big) = 0. \tag{4.19}$$

We have

$$\begin{aligned} & |\Delta_i^n X||\Delta_{i+1}^n X| - |\Delta_i^n X(q)||\Delta_{i+1}^n X(q)| \\ = \ & |\Delta_i^n X|\ \Big(|\Delta_{i+1}^n X| - |\Delta_{i+1}^n X(q)|\Big) + \Big(|\Delta_i^n X| - |\Delta_i^n X(q)|\Big)\ |\Delta_{i+1}^n X(q)| \end{aligned}$$

and may thus conclude (with $\Big||\Delta_i^n X| - |\Delta_i^n X(q)|\Big| \leq |\Delta_i^n X'(q)|$) that

$$|\overline{V}_3(X)^n_t| \leq \sqrt{n} \sum_{i=1}^{\lfloor nt\rfloor -1} \Big(|\Delta_i^n X||\Delta_{i+1}^n X'(q)| + |\Delta_i^n X'(q)||\Delta_{i+1}^n X(q)|\Big) =: A_n + B_n.$$

Regarding A_n, we obtain from Lemma 4.2 that

$$E\Big[|\Delta_i^n X||\Delta_{i+1}^n X'(q)|\Big] = E\Big[|\Delta_i^n X|\ E\Big[|\Delta_{i+1}^n X'(q)|\ \Big|\mathcal{F}_{\frac{i}{n}}\Big]\Big] \leq C\ e_q\ n^{-\frac{3}{2}}$$

and thus $E[|A_n|] \leq C\ e_q$. From Lebesgue's Theorem we have $\lim_{q\to\infty} e_q = 0$ as well (note that $\int_E \gamma(z)\ \lambda(dz) < \infty$ by assumption), and hence we conclude

$$\lim_{q\to\infty} \limsup_{n\to\infty} \mathbb{P}(A_n > \eta) = 0.$$

(4.19) obviously follows, once we have shown a similar result for the quantity B_n. To this end, note that $\Delta_{i+1}^n X(q)$ equals $\Delta_{i+1}^n X'$ unless $i+1 = I_m^n$ for some m. Using

$$|\Delta_{i+1}^n X(q)| \leq |\Delta_{i+1}^n X'| + |\Delta_{i+1}^n J(q)|$$

we obtain the following decomposition:

$$\begin{aligned} B_n &= \sqrt{n} \sum_{i=1}^{\lfloor nt\rfloor -1} |\Delta_i^n X'(q)||\Delta_{i+1}^n X(q)| \\ &\leq \sqrt{n} \sum_{i=1}^{\lfloor nt\rfloor -1} |\Delta_i^n X'(q)||\Delta_{i+1}^n X'| + \sqrt{n} \sum_{m\in P_q} |\Delta_{I_m^n-1}^n X'(q)||\Delta_{I_m^n}^n J(q)| \end{aligned}$$

The first sum on the right hand side can be treated in a similar way as A_n. For the second quantity note that we have only finitely many summands, each of which is of small order. Precisely, recalling the filtration $(\overline{\mathcal{F}'_t})$ we have

$$E\Big[|\Delta_{I_m^n-1}^n X'(q)||\Delta_{I_m^n}^n J(q)|\ \Big|\overline{\mathcal{F}'}_{\frac{I_m^n-2}{n}}\Big] = |\Delta_{I_m^n}^n J(q)|\ E\Big[|\Delta_{I_m^n-1}^n X'(q)|\ \Big|\overline{\mathcal{F}'}_{\frac{I_m^n-2}{n}}\Big]$$

and

$$E\Big[|\Delta^n_{I^n_m-1}X'(q)|\ \Big|\mathcal{F}'_{\frac{I^n_m-2}{n}}\Big] \le \frac{e_q}{n}$$

in a similar way as in Lemma 4.2. Thus

$$E\Big[\sqrt{n}\sum_{m\in P_q}|\Delta^n_{I^n_m}J(q)||\Delta^n_{I^n_m-1}X'(q)|\ \Big|\mathcal{F}'_0\Big] \le e_q\ n^{-\frac{1}{2}}\sum_{s\le t}|\Delta^n_i X|1_{\{|\Delta^n_i X|>1/q\}}$$

and (4.19) follows. □

Step 5. Here, we finally show that

$$\lim_{q\to\infty}\widetilde{E}\Big[\sup_{s\le t}|U'_s(\delta)-U'_s(\delta(q))|\Big]=0, \tag{4.20}$$

which finishes the proof of Theorem 3.1. From (4.9) we obtain

$$\begin{aligned}\widetilde{E}\Big[\sup_{s\le t}|U'_s(\delta)-U'_s(\delta(q))|\Big] &\le \widetilde{E}\Big[\ \widetilde{E}\Big[|U'_t(\delta)-U'_t(\delta(q))|\ \Big|\mathcal{F}\Big]\ \Big]\\ &\le C\ E\Big[\sum_{s\le t}|\Delta X_s|1_{\{|\Delta X_s|\le 1/q\}}\Big],\end{aligned}$$

which by the same arguments as in Step 4) converges to zero as q tends to infinity. Thus, (4.20) follows and we are done. □

Proof of Theorem 3.3. We show first that we can focus on $\overline{TV}^*(X)^n_t$ only and use for the latter steps the decomposition

$$\overline{TV}^*(X)^n_t \;=\; \sqrt{n}\ \Big(TV^*_1(X)^n_t - m_1^2\int_0^t\sigma_s^2\ ds\Big)+\sqrt{n}\ TV^*_2(X)^n_t,$$

where

$$TV^*_1(X)^n_t=\sum_{i=1}^{\lfloor nt\rfloor-1}|\Delta^n_i X'|1_{\{|\Delta^n_i X|<\alpha n^{-\varpi}\}}|\Delta^n_{i+1}X'|1_{\{|\Delta^n_{i+1}X|<\alpha n^{-\varpi}\}}$$

and

$$TV^*_2(X)^n_t=\sum_{i=1}^{\lfloor nt\rfloor-1}\Big(|\Delta^n_i X||\Delta^n_{i+1}X|-|\Delta^n_i X'||\Delta^n_{i+1}X'|\Big)1_{\{|\Delta^n_i X|<\alpha n^{-\varpi}\}}1_{\{|\Delta^n_{i+1}X|<\alpha n^{-\varpi}\}}.$$

Step 1. For any $\eta, t>0$ we have

$$\lim_{n\to\infty}\mathbb{P}\Big(\Big|\overline{V}^*(X)^n_t-\overline{TV}^*(X)^n_t\Big|>\eta\Big)=0.$$

Obviously, the relation

$$\overline{V}^*(X)^n_t - \overline{TV}^*(X)^n_t = \sqrt{n} \sum_{i=1}^{\lfloor nt \rfloor -1} |\Delta^n_i X| 1_{\{|\Delta^n_i X| \geq \alpha n^{-\varpi}\}} |\Delta^n_{i+1} X| 1_{\{|\Delta^n_{i+1} X| \geq \alpha n^{-\varpi}\}}$$

holds, and with the notation (4.10) we have

$$\begin{aligned} |\Delta^n_i X||\Delta^n_{i+1} X| &= |\Delta^n_i X'||\Delta^n_{i+1} X'| + \Big(|\Delta^n_i X| - |\Delta^n_i X'|\Big)|\Delta^n_{i+1} X'| \\ &+ |\Delta^n_i X' + \Delta^n_i J|\Big(|\Delta^n_{i+1} X| - |\Delta^n_{i+1} X'|\Big) \\ &\leq |\Delta^n_i X'||\Delta^n_{i+1} X'| + |\Delta^n_i J||\Delta^n_{i+1} X'| + |\Delta^n_i X'||\Delta^n_{i+1} J| + |\Delta^n_i J||\Delta^n_{i+1} J|. \end{aligned}$$

Since $E\Big[|\Delta^n_i J| \,\Big|\mathcal{F}_{\frac{i-1}{n}}\Big] \leq C\; n^{-1}$ and by taking successive conditional expectations and using Markov's inequality the proof amounts to showing that

$$E\Big[|\Delta^n_i X'| 1_{\{|\Delta^n_i X| \geq \alpha n^{-\varpi}\}} \,\Big|\mathcal{F}_{\frac{i-1}{n}}\Big] \leq C\; n^{-\gamma} \tag{4.21}$$

for some $\gamma > 3/4$. For later purposes we will prove an even stronger result, namely that (4.21) holds with $\gamma > 1$.

To this end, note that we have $1_{\{|\Delta^n_i X| \geq \alpha n^{-\varpi}\}} \leq 1_{\{|\Delta^n_i X'| \geq \frac{1}{2}\alpha n^{-\varpi}\}} + 1_{\{|\Delta^n_i J| \geq \frac{1}{2}\alpha n^{-\varpi}\}}$. Then for any $r > 0$ it holds

$$E\Big[|\Delta^n_i X'| 1_{\{|\Delta^n_i X'| \geq \frac{1}{2}\alpha n^{-\varpi}\}} \,\Big|\mathcal{F}_{\frac{i-1}{n}}\Big] \leq C\; n^{r\varpi} E\Big[|\Delta^n_i X'|^{r+1} \,\Big|\mathcal{F}_{\frac{i-1}{n}}\Big] \leq C\; n^{r(\varpi - \frac{1}{2}) - \frac{1}{2}},$$

and since $\varpi < \frac{1}{2}$ we have the desired result by choosing r large enough. On the other hand, for $0 < \rho < 1$ Hölder's inequality yields

$$\begin{aligned} &E\Big[|\Delta^n_i X'| 1_{\{|\Delta^n_i J| \geq \frac{1}{2}\alpha n^{-\varpi}\}} \,\Big|\mathcal{F}_{\frac{i-1}{n}}\Big] \leq C\; n^{\rho\varpi} E\Big[|\Delta^n_i X'||\Delta^n_i J|^{\rho} \,\Big|\mathcal{F}_{\frac{i-1}{n}}\Big] \\ \leq\;& C\; n^{\rho\varpi} E\Big[|\Delta^n_i X'|^{\frac{1}{1-\rho}} \,\Big|\mathcal{F}_{\frac{i-1}{n}}\Big]^{1-\rho} E\Big[|\Delta^n_i J| \,\Big|\mathcal{F}_{\frac{i-1}{n}}\Big]^{\rho} \leq C\; n^{\rho(\varpi - 1) - \frac{1}{2}}, \end{aligned}$$

and by choosing ρ sufficiently close to one we are done. □

Step 2. We come now to $\overline{TV}^*(X)^n_t$ and show first that the stable convergence

$$\sqrt{n}\, \Big(\overline{TV}^*_1(X)^n_t - m_1^2 \int_0^t \sigma_s^2\; ds\Big) \;\xrightarrow{\mathcal{L}-(s)}\; U''_t$$

holds. Using (4.13) this claim can be reduced to

$$\lim_{n\to\infty} \mathbb{P}\Big(\sqrt{n}\, \Big|V_1(X)^n_t - TV^*_1(X)^n_t\Big| > \eta\Big) = 0$$

for any $\eta, t > 0$. A standard calculation with indicator functions forces us to show

$$\lim_{n\to\infty} \mathbb{P}\Big(\sqrt{n} \sum_{i=1}^{\lfloor nt \rfloor -1} |\Delta^n_i X'||\Delta^n_{i+1} X'| 1_{A(j)^n_i} > \eta\Big) = 0,$$

where j runs from one to three and

$$\begin{array}{rcl} A(1)_i^n &=& \{\omega:\ |\Delta_i^n X| \geq \alpha n^{-\varpi}\} \cap \{\omega:\ |\Delta_{i+1}^n X| \geq \alpha n^{-\varpi}\}, \\ A(2)_i^n &=& \{\omega:\ |\Delta_i^n X| \geq \alpha n^{-\varpi}\} \cap \{\omega:\ |\Delta_{i+1}^n X| < \alpha n^{-\varpi}\}, \\ A(3)_i^n &=& \{\omega:\ |\Delta_i^n X| < \alpha n^{-\varpi}\} \cap \{\omega:\ |\Delta_{i+1}^n X| \geq \alpha n^{-\varpi}\}. \end{array}$$

However, since we know from Step 1) that (4.21) holds with $\gamma > 1$, and upon observing that $E\Big[|\Delta_i^n X'|\ \Big|\mathcal{F}_{\frac{i-1}{n}}\Big] < Cn^{-\frac{1}{2}}$ and by taking successive conditional expectations, all claims are obvious. $\square$

Step 3. It remains to show that for any $\eta, t > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\Big(\sqrt{n}\ TV_2^*(X)_t^n > \eta\Big) = 0.$$

As before, we have

$$\Big||\Delta_i^n X||\Delta_{i+1}^n X| - |\Delta_i^n X'||\Delta_{i+1}^n X'|\Big| \leq |\Delta_i^n X||\Delta_{i+1}^n J| + |\Delta_i^n J||\Delta_{i+1}^n X'|,$$

and hence we can use similar arguments as in Step 4) of the preceding proof. Without loss of generality let us focus on the second summand only. We fix $q > 0$ again, and thus we have with the notation from (4.11)

$$|\Delta_i^n J||\Delta_{i+1}^n X'| = |\Delta_i^n J(q)||\Delta_{i+1}^n X'| + |\Delta_i^n X'(q)||\Delta_{i+1}^n X'|. \tag{4.22}$$

Obviously, $1_{\{|\Delta_i^n X| < \alpha n^{-\varpi}\}} \leq 1_{\{|\Delta_i^n J(q)| < 2\alpha n^{-\varpi}\}} + 1_{\{|\Delta_i^n J(q)| \geq 2\alpha n^{-\varpi},\ |\Delta_i^n X'(q)| \geq \alpha n^{-\varpi}\}}$. For some n_q we have $1/q > 2\alpha n^{-\varpi}$ for all $n > n_q$, and thus for n large enough it holds that

$$\sqrt{n} \sum_{i=1}^{\lfloor nt \rfloor - 1} |\Delta_i^n J(q)||\Delta_{i+1}^n X'| 1_{\{|\Delta_i^n J(q)| < 2\alpha n^{-\varpi}\}} 1_{\{|\Delta_{i+1}^n X| < \alpha n^{-\varpi}\}} = 0$$

identically (on $\Omega_n(t,q)$). For the second indicator recall the filtration $(\overline{\mathcal{F}'_t})$. Then we have

$$\begin{array}{rcl} E\Big[|\Delta_i^n J(q)| 1_{\{|\Delta_i^n X'(q)| \geq \alpha n^{-\varpi}\}}\ \Big|\mathcal{F}_{\frac{i-1}{n}}\Big] &<& C\ n^{\varpi} E\Big[|\Delta_i^n J(q)||\Delta_i^n X'(q)|\ \Big|\mathcal{F}_{\frac{i-1}{n}}\Big] \\ &=& C\ n^{\varpi} E\Big[|\Delta_i^n J(q)|\ E\Big[|\Delta_i^n X'(q)|\ \Big|\overline{\mathcal{F}'}_{\frac{i-1}{n}}\Big]\ \Big|\mathcal{F}_{\frac{i-1}{n}}\Big]. \end{array}$$

Since both

$$E\Big[|\Delta_i^n X'(q)|\ \Big|\overline{\mathcal{F}'}_{\frac{i-1}{n}}\Big] < C\ n^{-1} \qquad \text{and} \qquad E\Big[|\Delta_i^n J(q)|\mathcal{F}_{\frac{i-1}{n}}\Big] < C\ n^{-1}$$

and upon observing that $\varpi < \frac{1}{2}$ and by taking successive expectations it is easy to show the desired result for the first summand in (4.22). For the latter one, the same theory as for A_n in Step 4) of Theorem 3.1 applies, and thus we are done. $\square$

References

[1] Barndorff-Nielsen, O., Graversen, S., Jacod, J., Podolskij, M., Shephard N. (2006): A central limit theorem for realised power and bipower variations of continuous semimartingales. In: Yu. Kabanov, R. Liptser and J. Stoyanov (eds.), *From Stochastic Calculus to Mathematical Finance. The Shiryaev Festschrift.* Springer-Verlag, Berlin, 33-68.

[2] Barndorff-Nielsen, O., Shephard, N. (2002). Econometric analysis of realised volatility and its use in estimating stochastic volatility models. J. Roy. Statist. Soc. Ser. B 64, 253-280.

[3] Barndorff-Nielsen, O., Shephard, N., Winkel, M. (2006): Limit theorems for multipower variation in the presence of jumps. *Stoch. Proc. Appl.*, **116**, 796-806.

[4] Jacod, J. (2008): Asymptotic properties of realized power variations and related functionals of semimartingales. *Stoch. Proc. Appl.*, **118**, 517-559.

[5] Jacod, J. (2006): Asymptotic properties of realized power variations and related functionals of semimartingales: Multipower Variations. *Unpublished Paper*

[6] Jacod, J. (2007): Statistics and high-frequency data. *Unpublished Paper*

[7] Jacod, J., Podolskij, M., Vetter, M. (2008): Limit theorems for moving averages of discretized processes plus noise. *Working Paper*

[8] Jacod, J., Protter, P. (1998): Asymptotic error distributions for the Euler method for stochastic differential equations. *Ann. Probab.*, **26**, 267-307.

[9] Jacod, J., Shiryaev, A. (2003): *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin.

[10] Mancini, C. (2009): Non-parametric estimation for models with stochastic diffusion coefficients and jumps. *Scand. J. Statist.*, **36**, 270-296.

[11] Woerner, J. (2006): Power and multipower variation: inference for high-frequency data. In: M. do Rosário Grossinho, A. Shiryaev, M. Esquivel, P. Oliveira (eds.), *Stochastic Finance.* Springer-Verlag, Berlin, 343-364.